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Extension closed properties on generalized topological groups

Abstract In this paper, we continue to investigate some important results in generalized topological groups and we prove extension closed property for connectedness, compactness, and separability of generalized topological groups. Last, we define generalized topological group actions on generalized topological spaces and we establish a homeomorphism between action group and action space.

الملخص

نواصل في هذه الورقة البحث في بعض النتائج الهامة في الزمر التوبولوجية المُعَمَّمَة. نثبت خاصية تمديد مغلقة للترابط، والتراص، والانفصال للزمر التوبولوجية المُعَمَّمَة. أخيراً، نعرف أفعال زمر توبولوجية مُعَمَّمَة على فضاءات توبولوجية مُعَمَّمَة، ونثبت وجود تشاكل مستمر بين زمرة الفعل وفضاء الفعل.

Mathematics Subject Classification General topology · Topological groups

1 Introduction

In [2], Császár introduced and extensively studied the notion of generalized open sets discarding finite intersection axiom from the general topology. Since then he and many other authors in the literature have shown that important properties and results still hold, with some or no modification.

In [8], we defined the generalized topological group structure and we proved some basic results. Especially, we examined generalized connectedness property in [8].

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In [9], we defined the ultra Hausdorff property of spaces and we gave some basic characterizations and we investigated the relation between generalized compactness and ultra Hausdorfness.

In this paper, we continue to investigate some important results in generalized topological groups and prove extension closed property for connectedness, compactness, first countability and separability of generalized topological groups. In the last section, we define generalized topological group actions on generalized topological spaces and we establish a homeomorphism between action group and action space.

2 More results on generalized topological groups, extension closed properties of quotient spaces

From [6], we know that Cartesian product of $\mathcal{G}$-compact sets is $\mathcal{G}$-compact.

Theorem 2.1 *For any two $\mathcal{G}$-compact subsets E and F of a $\mathcal{G}$-topological group G, their product EF in G is a $\mathcal{G}$-compact subspace of G.*

Proof Since multiplication is $\mathcal{G}$-continuous, the subspace EF of G is a $\mathcal{G}$-continuous image of the Cartesian product $E \times F$ of the spaces E and F. Since $E \times F$ is $\mathcal{G}$-compact, the space EF is $\mathcal{G}$-compact. □

Theorem 2.2 *Let G be a $\mathcal{G}$-ultra Hausdorff normal $\mathcal{G}$-compact $\mathcal{G}$-topological group, F a $\mathcal{G}$-compact subset of G, and P a $\mathcal{G}$-closed subset of G. Then, the sets FP and PF are $\mathcal{G}$-closed.*

Proof Since G is $\mathcal{G}$-compact then $P \times F$ is $\mathcal{G}$-compact and by previous result, PF and FP are $\mathcal{G}$-compact. Since G is $\mathcal{G}$-ultra Hausdorff and $\mathcal{G}$-normal, FP and PF are $\mathcal{G}$-closed. □

From [6], we know that Cartesian product of $\mathcal{G}$-connected sets is $\mathcal{G}$-connected.

Theorem 2.3 *For any two $\mathcal{G}$-connected subsets E and F of a $\mathcal{G}$-topological group G, their product EF in G is a $\mathcal{G}$-connected subspace of G.*

Proof Since multiplication is $\mathcal{G}$-continuous, the subspace EF of G is a $\mathcal{G}$-continuous image of the Cartesian product $E \times F$ of the spaces E and F. Since $E \times F$ is $\mathcal{G}$-connected from [6], the space EF is $\mathcal{G}$-connected. □

Definition 2.4 A $\mathcal{G}$-closed $\mathcal{G}$-continuous mapping with $\mathcal{G}$-compact preimages of points is called $\mathcal{G}$-**perfect**.

Theorem 2.5 *The $\mathcal{G}$-quotient mapping π of G onto the $\mathcal{G}$-quotient space G/H is $\mathcal{G}$-perfect where H is a $\mathcal{G}$-compact subgroup of a $\mathcal{G}$-ultra Hausdorff normal $\mathcal{G}$-topological group G.*

Proof Take any $\mathcal{G}$-closed subset P of G. Then, by Theorem 2.2, PH is $\mathcal{G}$-closed in G. However, PH is the union of cosets that is $PH = \pi^{-1}\pi(P)$. It follows by definition of a quotient mapping that the set $\pi(P)$ is $\mathcal{G}$-closed in the quotient space G/H. Thus, π is a $\mathcal{G}$-closed mapping. In addition, if $y \in G/H$ and $\pi(x) = y$ for some $x \in G$, then $\pi^{-1}(y) = xH$ is a $\mathcal{G}$-compact subset of G. Hence, the fibers of π are $\mathcal{G}$-compact and π is $\mathcal{G}$-perfect. □

Corollary 2.6 *Let H be a $\mathcal{G}$-compact subgroup of a $\mathcal{G}$-ultra Hausdorff normal $\mathcal{G}$-topological group G such that the $\mathcal{G}$-quotient space G/H is $\mathcal{G}$-compact. Then, G is also $\mathcal{G}$-compact.*

Definition 2.7 (*discrete group*) Let G be a $\mathcal{G}$-topological group. Then, a subgroup K is called **discrete group** if $K \cap V$ is at most one singleton set, for any $\mathcal{G}$-open set V in G.

In [8], we proved the following result which we need in the proof of the next theorem.

Lemma 2.8 *Let U be an arbitrary $\mathcal{G}$-open neighborhood of the neutral element e of a $\mathcal{G}$-connected $\mathcal{G}$-topological group G. Then, $G = \bigcup_{n=1}^{\infty} U^n$.*

Theorem 2.9 *Let G be a $\mathcal{G}$-connected $\mathcal{G}$-topological group and e its identity element. If U is any $\mathcal{G}$-open neighborhood of e, then G is generated by U.*

Proof Let U be a $\mathcal{G}$-open neighborhood of e. For each $n \in \mathbb{N}$, we denote by U^n the set of elements of the form $u_1 \ldots u_n$, where each $u_i \in U$. Let $W := \bigcup_{n \in \mathbb{N}} U^n$. Since each U^n is $\mathcal{G}$-open, we have that W is a $\mathcal{G}$-open set. We now see that it is also $\mathcal{G}$-closed.

Let g be an element of generalized closure of W. That is, $g \in Cl_{\mathcal{G}} W$. Since gU^{-1} is a $\mathcal{G}$-open neighborhood of g, it must intersect W. Thus, let $h \in W \cap gU^{-1}$.

Since $h \in gU^{-1}$, then $h = gu^{-1}$ for some elements $u \in U$. Since $h \in W$, then $h \in U^n$ for some $n \in \mathbb{N}$, i.e., $h = u_1 \ldots u_n$ with each $u_i \in U$. We then have $g = u_1 \ldots u_n u$, i.e., $g \in U^{n+1} \subseteq W$. Hence, W is $\mathcal{G}$-closed. Since G is $\mathcal{G}$-connected and W is $\mathcal{G}$-open and $\mathcal{G}$-closed, we must have $W = G$. This means that G is generated by U. $\square$

Theorem 2.10 *Let K be a discrete invariant subgroup of a $\mathcal{G}$-connected $\mathcal{G}$-topological group G. If for any $\mathcal{G}$-open neighborhood U of x in G there exists a $\mathcal{G}$-open symmetric neighborhood V of e in G such that $VxV \subset U$, then every element of K commutes with every element of G, i.e., K is contained in the center of the group G.*

Proof Assume that the subgroup K is non-trivial. Take an arbitrary element $x \in K$ distinct from the identity e of G. Since the group is discrete, we can find a $\mathcal{G}$-open neighborhood U of x in G such that $U \cap K = \{x\}$. It follows from the $\mathcal{G}$-continuity of the multiplication in G and the obvious equality $exe = x$ that there exists a $\mathcal{G}$-open symmetric neighborhood V of e in G such that $VxV \subset U$. Let $y \in V$ be arbitrary. Since K is an invariant subgroup of G, we have that $yxy^{-1} \in K$. It is also clear that $yxy^{-1} \in VxV^{-1} = VxV \subset U$. Therefore, $yxy^{-1} \in U \cup K = \{x\}$, i.e., $yxy^{-1} = x$. This implies that $yx = xy$ for each $y \in V$.

Since the group G is $\mathcal{G}$-connected, Lemma 2.8 implies that the sets V^n, with $n \in \mathbb{N}$, cover the group G. Therefore, every element $g \in G$ can be written in the form $g = y_1 \ldots y_n$, where $y_1, \ldots, y_n \in V$ and $n \in \mathbb{N}$. Since x commutes with every element of V, we have

$$gx = y_1 \cdots y_n x = y_1 \cdots x y_n = \cdots = y_1 x \cdots y_n = x y_1 \cdots y_n = xg.$$

We have proved that the element $x \in K$ is in the center of the group G. Since x is an arbitrary element of K, we conclude that the center of G contains K. $\square$

By Theorem 2.9, we have the following result.

Theorem 2.11 *If H is a $\mathcal{G}$-dense subgroup of a $\mathcal{G}$-connected $\mathcal{G}$-topological group, then every $\mathcal{G}$-neighborhood U of the identity element in H algebraically generates the group H.*

Definition 2.12 A space X is called $\mathcal{G}$-**resolvable** if there exists $\mathcal{G}$-dense disjoint subsets A and B of X.

Let G be a $\mathcal{G}$-topological group. Since G is homogeneous and the union of resolvable spaces is again resolvable then we have the following results.

Theorem 2.13 (i) *If a subgroup H of a $\mathcal{G}$-topological group G is $\mathcal{G}$-resolvable, then so is G.*
(ii) *If a $\mathcal{G}$-topological group G contains a proper $\mathcal{G}$-dense subgroup, then G is $\mathcal{G}$-resolvable.*
(iii) *If a $\mathcal{G}$-topological group G contains a non-$\mathcal{G}$-closed subgroup, then G is $\mathcal{G}$-resolvable.*

Proof (i) We reach the aim by generalized subspace topology.
(ii) It is coming from the first result **(i)** since closure of a generalized topological subgroup is subgroup.
(iii) It is coming from **(i)** and **(iii)**. $\square$

Theorem 2.14 *Let $f : G \to H$ be a $\mathcal{G}$-continuous mapping of $\mathcal{G}$-topological spaces. If G is $\mathcal{G}$-compact and H is $\mathcal{G}$-ultra Hausdorff and $\mathcal{G}$-normal, then f is $\mathcal{G}$-closed.* $\square$

Proof Let K be a $\mathcal{G}$-closed set in G. Since G is $\mathcal{G}$-compact then K is $\mathcal{G}$-compact. So, by $\mathcal{G}$-continuity of f, $f(K)$ is $\mathcal{G}$-compact in H. By assumption, H is ultra Hausdorff and normal, so $f(K)$ is $\mathcal{G}$-closed. $\square$

Definition 2.15 Let X and Y be $\mathcal{G}$-topological spaces. A $\mathcal{G}$-continuous onto mapping $f : X \to Y$ is called **identification map** if f is $\mathcal{G}$-open or $\mathcal{G}$-closed.

Theorem 2.16 *Let $f : G \to H$ be a $\mathcal{G}$-continuous onto homomorphism of $\mathcal{G}$-topological groups. If G is $\mathcal{G}$-compact and H is $\mathcal{G}$-ultra Hausdorff and normal, then f is $\mathcal{G}$-open.*

Proof By Theorem 2.14, the mapping f is $\mathcal{G}$-closed, and hence it is quotient. Let K be the kernel of f. If U is $\mathcal{G}$-open in G, then $f^{-1}(f(U)) = KU$ is $\mathcal{G}$-open in G, by Theorem 2.9 of [8]. Since f is quotient, it follows that the image $f(U)$ is $\mathcal{G}$-open in H. Therefore, f is a $\mathcal{G}$-open mapping.   □

Lemma 2.17 *Suppose that $f : X \to Y$ is a $\mathcal{G}$-open $\mathcal{G}$-continuous mapping of a space X onto a space Y, $x \in X$, $B \subset Y$, and $f(x) \in Cl_{\mathcal{G}}(B)$ where $Cl_{\mathcal{G}}(B)$ is generalized closure of B. Then, $x \in f^{-1}(Cl_{\mathcal{G}}(B))$.*

Proof Take $y = f(x)$, and let O be a $\mathcal{G}$-open neighborhood of x. Then, $f(O)$ is a $\mathcal{G}$-open neighborhood of y. Therefore, $f(O) \cap B \neq \emptyset$ and, hence, $O \cap f^{-1}(B) \neq \emptyset$. It follows that $x \in Cl_{\mathcal{G}}(f^{-1}(B))$. Equality is evident.   □

Theorem 2.18 *Let H be a $\mathcal{G}$-closed subgroup of a $\mathcal{G}$-topological group G. If the spaces H and G/H are $\mathcal{G}$-separable, then the space G is also $\mathcal{G}$-separable.*

Proof Let π be the natural homomorphism of G onto the quotient space G/H. Since G/H is $\mathcal{G}$-separable, we can fix a $\mathcal{G}$-dense countable subset B of G/H. Since H is $\mathcal{G}$-separable and every coset xH is $\mathcal{G}$-homeomorphic to H, we can fix a $\mathcal{G}$-dense countable subset M_y of $\pi^{-1}(y)$, for each $y \in B$. Put $M = \bigcup\{M_y : y \in B\}$. Then, M is a countable subset of G and M is $\mathcal{G}$-dense in $\pi^{-1}(B)$. Since π is a $\mathcal{G}$-open mapping of G onto G/H, it follows from Lemma 2.17 that $Cl_{\mathcal{G}}(\pi^{-1}(B)) = G$. Hence, M is $\mathcal{G}$-dense in G and G is $\mathcal{G}$-separable.   □

Theorem 2.19 *Let H be a $\mathcal{G}$-closed invariant subgroup of a $\mathcal{G}$-topological group G. If H and G/H are $\mathcal{G}$-connected, then so is G.*

Proof Suppose that H and G/H are $\mathcal{G}$-connected and $f : G \to \{0, 1\}$ be an arbitrary $\mathcal{G}$-continuous map. We have to show that f is constant. The restriction of f to H must be constant and since each coset gH is $\mathcal{G}$-connected, f must be constant on gH as well taking value $f(g)$. Thus, we have a well-defined map $\tilde{f} : G/H \to \{0, 1\}$ such that $\tilde{f} \circ \pi = f$. By the fundamental property of quotient spaces, it follows that $\tilde{f}$ is $\mathcal{G}$-continuous and so must be constant since G/H is $\mathcal{G}$-connected. Hence, f is also constant and we conclude that G is $\mathcal{G}$-connected.   □

3 $\mathcal{G}$-topological group actions on $\mathcal{G}$-topological spaces

In this section, we will introduce $\mathcal{G}$-topological Group Actions on $\mathcal{G}$-topological spaces and we want to improve some results from topological group action theory.

Definition 3.1 If G is a $\mathcal{G}$-topological group and X is a $\mathcal{G}$-topological space, then an **action** of G on X is a map $G \times X \to X$, with the image of (g, x) being denoted by $g(x)$, such that $(gh)(x) = g(h(x))$ and $e(x) = x$.

For a point $x \in X$, the set $G(x) = \{gx : g \in G\}$ is called the **orbit** of x.
If this map is $\mathcal{G}$-continuous, then the action is said to be $\mathcal{G}$-**continuous**. The space X, with a given $\mathcal{G}$-continuous action of G on X, is called G-**space**.

Proposition 3.2 *Every $\mathcal{G}$-continuous action $\theta : G \times X \to X$ of a $\mathcal{G}$-topological group G on a space X is a $\mathcal{G}$-open mapping.*

Proof It is sufficient to verify that the images under θ of the elements of some base for $G \times X$ are $\mathcal{G}$-open in X. Let $O = U \times V \subset G \times X$, where U and V are $\mathcal{G}$-open in G and X, respectively. Then, $\theta(O) = \bigcup_{g \in G} \theta_g(V)$ is $\mathcal{G}$-open in X since every θ_g is a $\mathcal{G}$-homeomorphism of X onto itself. Since the $\mathcal{G}$-open sets $U \times V$ form a base for $G \times X$, the mapping θ is $\mathcal{G}$-open.   □

Proposition 3.3 *The $\mathcal{G}$-continuity of an action $\theta : G \times X \to X$ of a $\mathcal{G}$-topological group G with identity e on a space X is equivalent to the $\mathcal{G}$-continuity of θ at the points of the set $\{e\} \times X \subset G \times X$.*

Proof Let $g \in G$ and $x \in X$ be arbitrary and U be a neighborhood of gx in X. Since θ_h is a homeomorphism of X for each $h \in G$, the set $V = \theta_{g^{-1}}(U)$ is a neighborhood of x in X. By the $\mathcal{G}$-continuity of θ at (e, x), we can find a neighborhood O of e in G and a neighborhood W of x in X such that $hy \in V$ for all $h \in O$ and $y \in W$. Clearly, if $h \in O$ and $y \in W$, then $(gh)(y) = g(hy) \in gV = \theta_g(V) = U$. Thus, $ky \in U$, for all $k \in gO$ and all $y \in W$, where $O' = gO$ is a neighborhood of g in G. Hence, the action θ is $\mathcal{G}$-continuous. □

Here are some examples of $\mathcal{G}$-continuous actions of $\mathcal{G}$-topological groups.

Example 3.4 Any $\mathcal{G}$-topological group G acts on itself by left translations, i.e., $\theta(x, y) = xy$ for all $x, y \in G$. The $\mathcal{G}$-continuity of this action follows from the $\mathcal{G}$-continuity of the multiplication in G.

Example 3.5 Let G be a topological group, H a $\mathcal{G}$-closed subgroup of G, and let G/H be the corresponding left coset space. The action ϕ of G on G/H, defined by the rule $\phi(g, xH) = gxH$, is $\mathcal{G}$-continuous. Indeed, take any $y_0 \in G/H$, and fix a $\mathcal{G}$-open neighborhood O of y_0 in G/H. Choose $x_0 \in G$ such that $\pi(x_0) = y_0$, where $\pi : G \to G/H$ is the $\mathcal{G}$-quotient mapping. There exists $\mathcal{G}$-open neighborhoods U and V of the identity e in G such that $\pi(Ux_0) \subset O$ and $V^2 \subset U$. Clearly, $W = \pi(Vx_0)$ is $\mathcal{G}$-open in G/H and $y_0 \in W$. By the choice of U and V, if $g \in V$ and $y \in W$, then $\phi(g, y) \in O$. Indeed, take $x_1 \in Vx_0$ with $\pi(x_1) = y$. Then, $y = x_1H$ and $\phi(g, y) = gx_1H \in VVx_0H \subset \pi(Ux_0) \subset O$. It follows that ϕ is $\mathcal{G}$-continuous at $(e, y_0) \in G \times G/H$; hence, ϕ is $\mathcal{G}$-continuous, by Proposition 3.3.

Suppose that a $\mathcal{G}$-topological group G acts continuously on a space X and that $Y = X/G$ is the corresponding orbit set. Let Y carry the quotient $\mathcal{G}$-topology generated by the orbital projection $\pi : X \to X/G$ (a set $U \subset Y$ is $\mathcal{G}$-open in Y if and only if the preimage $\pi^{-1}(U)$ is $\mathcal{G}$-open in X). The $\mathcal{G}$-topological space X/G so obtained is called the orbit space or the orbit space of the G-space X. The orbital projection is always a $\mathcal{G}$-open mapping:

Proposition 3.6 *If $\theta : G \times X \to X$ is a $\mathcal{G}$-continuous action of a topological group G on a space X, then the orbital projection $\pi : X \to X/G$ is $\mathcal{G}$-open.*

Proof For a $\mathcal{G}$-open set $U \subset X$, consider the set $\pi^{-1}\pi(U) = GU$. Every left translation θ_g is a homeomorphism of X onto itself, so the set $GU = \bigcup_{g \in G} \theta_g(U)$ is $\mathcal{G}$-open in X. Since π is a $\mathcal{G}$-quotient mapping, $\pi(U)$ is $\mathcal{G}$-open in Y. $\qquad\square$

Theorem 3.7 *If a $\mathcal{G}$-compact $\mathcal{G}$-topological group H acts continuously on an ultra-$\mathcal{G}$-Hausdorff space X, then the orbital projection $\pi : X \to X/H$ is a $\mathcal{G}$-open and $\mathcal{G}$-perfect mapping.*

Proof Let $Y = X/H$. If $y \in Y$, choose $x \in X$ such that $\pi(x) = y$ and note that $\pi^{-1}(y) = Hx$ is the orbit of x in X. Since the mapping of H onto Hx assigning to every $g \in H$, the point $gx \in X$ is $\mathcal{G}$-continuous, the image Hx of the $\mathcal{G}$-compact group H is also $\mathcal{G}$-compact. Hence, all fibers of π are $\mathcal{G}$-compact.

To verify that the mapping π is $\mathcal{G}$-closed, let $y \in Y$ and $x \in X$ be as above, and let O be a $\mathcal{G}$-open set in X containing $\pi^{-1}(y) = Hx$. Since the action of H on X is $\mathcal{G}$-continuous, we can find, for every $g \in H$, $\mathcal{G}$-open neighborhoods $U_g \ni g$ and $V_g \ni x$ in H and X, respectively, such that $U_g V_g \subset O$. By the $\mathcal{G}$-compactness of H and of the orbit Hx, there exists a finite set $F \subset H$ such that $H = \bigcup_{g \in F} U_g$ and $Hx \subset \bigcup_{g \in F} gV_g$. Then, $V = \bigcap_{g \in F} V_g$ is a $\mathcal{G}$-open neighborhood of x in X, and we claim that $HV \subset O$. Indeed, if $h \in H$ and $z \in V$, then $h \in U_g$, for some $g \in F$, so that $hz \in U_g V \subset U_g V_g \subset O$. Thus, $W = \pi(V)$ is a $\mathcal{G}$-open neighborhood of y in Y, and we have $\pi^{-1}\pi(V) = HV \subset O$. Hence, the mapping π is $\mathcal{G}$-closed. Finally, π is $\mathcal{G}$-open, by Proposition 3.6. $\qquad\square$

Definition 3.8 Let X and Y be G-spaces with $\mathcal{G}$-continuous actions $\theta_X : G \times X \to X$ and $\theta_Y : G \times Y \to Y$. A $\mathcal{G}$-continuous mapping $f : X \to Y$ is called G**-equivariant** if $\theta_Y(g, f(x)) = f(\theta_X(g, x))$, i.e., $gf(x) = f(gx)$, for all $g \in G$ and all $x \in X$. Clearly, f is G-equivariant if and only if the diagram below commutes,

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\ \theta_X\ } & X \\
\Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle f} \\
G \times Y & \xrightarrow{\ \theta_Y\ } & Y
\end{array}
$$

where $F = id_G \times f$ is the product of the identity mapping id_G of G and the mapping f.

Example 3.9 Let H be a $\mathcal{G}$-closed subgroup of a $\mathcal{G}$-topological group G, and $Y = G/H$ be the left coset space. Denote by θ_G the action of G on itself by left translations, and by θ_Y the natural $\mathcal{G}$-continuous action of G on

Y. Then, the quotient mapping $\pi : G \to G/H$ defined by $\pi(x) = xH$ for each $x \in G$ is equivariant. Indeed, the equality $g(\pi(x)) = gxH = \pi(gx)$ holds for all $g, x \in G$. Equivalently, the diagram is commutative,

$$
\begin{array}{ccc}
G \times G & \xrightarrow{\theta_G} & G \\
\downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \pi} \\
G \times Y & \xrightarrow{\theta_Y} & Y
\end{array}
$$

where $\Pi = id_G \times \pi$.

Let $\eta = \{X_i : i \in I\}$ be a family of G-spaces. Then, the product space $X = \prod_{i \in I} X_i$, if X is ultra-$\mathcal{G}$-Hausdorff, is a G-space. To define an action of G on X, take any $g \in G$ and any $x = (x_i)_{i \in I} \in X$, and put $gx = (gx_i)_{i \in I}$. Thus, G acts on X coordinatewise. The following result guarantees the $\mathcal{G}$-continuity of this action.

Proposition 3.10 *The coordinatewise action of G on the product $X = \prod_{i \in I} X_i$ of G-spaces is $\mathcal{G}$-continuous, i.e., X is a G-space, if X is ultra-$\mathcal{G}$-Hausdorff.*

Proof By Proposition 3.3, it suffices to verify the continuity of the action of G on X at the neutral element $e \in G$. Let $x = (x_i)_{i \in I} \in X$ be an arbitrary point and $O \subset X$ a neighborhood of gx in X. Since canonical open sets form a base of X, we can assume that $O = \prod_{i \in I} O_i$, where each O_i is a $\mathcal{G}$-open neighborhood of x_i in X_i and the set $F = \{i \in I : O_i \neq X_i\}$ is finite. Since all factors are G-spaces, we can choose, for every $i \in F$, $\mathcal{G}$-open neighborhoods $V_i \ni e$ and $V_i \ni x_i$ in G and X_i, respectively, such that $U_i V_i \subset O_i$. Put $U = \bigcup_{i \in F} V_i$ and $W = \prod_{i \in I} W_i$, where $W_i = V_i$ if $i \in F$ and $W_i = X_i$ otherwise. It follows immediately from the definition of the sets U and W that $UW \subset O$. Therefore, the action of G on X is $\mathcal{G}$-continuous.

Theorem 3.11 *Let G act $\mathcal{G}$-continuously on X and suppose that both G and X/G are $\mathcal{G}$-connected, then X is $\mathcal{G}$-connected.*

Proof Suppose X is the union of two disjoint nonempty $\mathcal{G}$-open subsets U and V. Now $\pi(U)$ and $\pi(V)$ are $\mathcal{G}$-open in X/G. Since X/G is $\mathcal{G}$-connected, $\pi(U)$ and $\pi(V)$ cannot be disjoint. If $\pi(x) \in \pi(U) \cup \pi(V)$, then both $U \cup O(x)$ and $V \cup O(x)$ are nonempty, where $O(x)$ is the orbit of x. It means $O(x)$ is a disjoint union of two nonempty $\mathcal{G}$-open sets. But $O(x)$ is the image of G under the $\mathcal{G}$-continuous function $f : G \to X$ defined by $f(g) = g(x)$. $O(x)$ is therefore $\mathcal{G}$-connected, and thus a contradiction. □

Then, we can give the relationship between group actions and separation axiom in the following.

Theorem 3.12 *If X is $\mathcal{G}$-compact topological group and G a $\mathcal{G}$-closed subgroup acting on X by left translation, then X/G is $\mathcal{G}$-regular, and so X/G is $\mathcal{G}$-Hausdorff.*

Proof Since G is a $\mathcal{G}$-closed subgroup and the left translation map $L_x : X \to X$ is a $\mathcal{G}$-homeomorphism then $\pi^{-1}\pi(x) = xG = L_x(G)$ is $\mathcal{G}$-closed. Thus, every point $\pi(x)$ of X/G is $\mathcal{G}$-closed, and it follows that X/G is $\mathcal{G}$-T_1 space.

Now we will show that for a $\mathcal{G}$-closed subset F of X/G and a point $p \notin F$, there are $\mathcal{G}$-open sets U, V satisfying $p \in U$, $F \subset V$, $U \cap V = \emptyset$. Since X acts transitively on X/G, we may suppose that p is element of the class $eG = G$ of the identity element e. Since F is $\mathcal{G}$-closed, there exists a $\mathcal{G}$-open set U_0 such that $F \cap U_0 = \emptyset$ and $p \in U_0$. From the continuity of group action of X, there is a $\mathcal{G}$-open set W such that $e \in W$ and $W^{-1}W \subset \pi^{-1}(U_0)$. The set $W\pi^{-1}(F) = \bigcup_{x \in \pi^{-1}(F)} Wx$ is $\mathcal{G}$-open. Since π is $\mathcal{G}$-open map, both the sets $U = \pi(W)$ and $V = \pi(W\pi^{-1}(F))$ are $\mathcal{G}$-open and such that $p \in U$ and $F \subset V$. Last, we will show that $U \cap V = \emptyset$ by contradiction. So assume that there exists $y \in U \cap V$. Then, there exists $x_1, x_2 \in W$ and $x \in \pi^{-1}(F)$ such that $y = \pi(x_2) = \pi(x_1x)$. Thus, we have $g \in G$ such that $x_2g = x_1x$, from which we deduce that $\pi(xg^{-1}) \in F \cap U_0 = \emptyset$ from $xg^{-1} = x_1^{-1}x_2 \in W^{-1}W \subset \pi^{-1}(U_0)$. So, we get $U \cap V = \emptyset$. □

To give the last important result, we need the following definitions.

Definition 3.13 Let G act on the generalized topological space X. Then, for a point x of X the set

$$G_x = \{g \in G : gx = x\} \quad (or \quad G_x = \{g \in G : xg = x\})$$

is a subgroup of G and it is called **stabilizer of x in G**.

We should note that G_x is $\mathcal{G}$-closed since the singleton set $\{x\}$ is $\mathcal{G}$-closed.

Definition 3.14 Let G act on the generalized topological space X. Then, for a point x of X, we define a map

$$\mu_x : G \to X$$

by $\mu_x(g) = gx$ (or $\mu_x(g) = xg$).

By continuity of action μ_x is $\mathcal{G}$-continuous. Obviously, we have the following facts.

(i) μ_x is surjective iff G acts transitively on X.
(ii) $\bigcap_{x \in X} G_x = \{e\}$ iff G acts effectively on X.

Now we are ready to prove the following result.

Theorem 3.15 *If G is $\mathcal{G}$-compact, X ultra $\mathcal{G}$-Hausdorff, and if G acts transitively on X, then X is homeomorphic to the orbit space G/G_x for any $x \in X$.*

Proof Let X be a homogeneous $\mathcal{G}$-space of G. First, we claim that μ_x induces a bijection $h_x : G/G_x \to X$ such that $\mu_x = h_x \circ \pi_x$, where $\pi_x : G \to G/G_x$ is orbital projection. We have $\pi(g_1) = \pi(g_2)$ iff $g_1^{-1}g_2 \in G_x$ iff $g_1^{-1}g_2 x = x$ iff $\mu(g_1) = \mu(g_2)$. The equality $h_x(\pi_x(g)) = \mu_x(g)$ determines the injection h_x. On the other hand, from the first fact above μ_x is surjective implies h_x is surjective. Since U is $\mathcal{G}$-open in X, $\mu_x^{-1}(U) = \pi_x^{-1}(h_x^{-1}(U))$ is $\mathcal{G}$-open in G and hence, $h_x^{-1}(U)$ is $\mathcal{G}$-open. Thus, h_x is $\mathcal{G}$-continuous bijection.

Next, we claim that for $g \in G$, $x \in X$ and $y = gx$, the diagram

$$
\begin{array}{ccc}
G/G_x & \xrightarrow{\ h_x\ } & X \\
\downarrow{\scriptstyle A_g} & & \downarrow{\scriptstyle L_g} \\
G/G_y & \xrightarrow{\ h_y\ } & X
\end{array}
$$

is commutative, where $G_y = gG_xg^{-1}$ and A_g is a homeomorphism given by $A_g(g'G_x) = (gg'g^{-1})G_y$. Indeed, for $g_1 \in G$, $L_g(h_x(\pi_x(g_1))) = L_g(\mu_x(g_1)) = gg_1x = gg_1g^{-1}y = h_y(A_g(\pi_y(g_1)))$. It implies that the diagram is commutative. Clearly, we have $G_y = A_gG_x = gG_xg^{-1}$.

Last if μ_x(or h_x) is $\mathcal{G}$-open map then h_x is a homeomorphism. By hypothesis, μ_x is $\mathcal{G}$-open map. So, if O is $\mathcal{G}$-open set in G/G_x then $\pi_x^{-1}(O)$ is $\mathcal{G}$-open (by Theorem 3.7) and hence, $h_x(O) = \mu_x(\pi_x^{-1}(O))$ is $\mathcal{G}$-open. It means that h_x is $\mathcal{G}$-open map, which implies that h_x is a homeomorphism. $\qquad\square$

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