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# On the contact homology of the first exotic contact form of J. Gonzalo and F. Varela

*Alla Memoria di Giovanni Prodi*

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**Abstract** This paper contains a detailed study of the behavior of the first exotic contact structure of J. Gonzalo and F. Varela on $S^3$ along a remarkable vector field $v$ in its kernel found by Martino (Adv Nonlinear Stud, 2011). All contact forms are assumed to verify the condition that reads as follows: $\mathrm{d}(\theta\alpha)(v, .)$ is a contact form with the same orientation than $\alpha$. $\alpha$ is the first exotic contact form of Gonzalo and Varela (Third Schnepfenried Geometry Conference, vol 1, Asterisque no 107–108, pp 163–168. Société Mathématique de France, Paris, 1983). We also prove in this paper that the contact homology (via dual Legendrian curves) is non-zero for a sequence of indexes tending to infinity for the contact forms $\theta\alpha$ of the first exotic contact structure of J. Gonzalo and F. Varela on $S^3$, under the assumption that they can be connected to the first contact *form* of this contact structure through a path along which a special pseudo-gradient which we build in this paper is assumed to verify a Fredholm condition (see the Sect. 1 and Bahri in Morse relations and Fredholm deformations of $v$-convex contact forms, 2014 for the definition of this notion). We do not know whether this assumption is verified for the pseudo-gradient which we use here.

**الملخص**

تحتوي هذه الورقة على دراسة مفصلة لسلوك بناء الاتصال الغريب الأول لـِ ج. جونزالو وَ ف. ڤاريلا على $S^3$ على طول حقل متجهي مميز $v$ في نواته والذي تم اكتشافه من قبل ڤ. مارتينو[19]. نفترض أن جميع أشكال الاتصال تحقق الشرط التالي: $d(\theta\alpha)(v,.)$ هو شكل اتصال بنفس توجه $\alpha$ حيث $\alpha$ هو شكل الاتصال الغريب الأول لـِ ج. جونزالو وَ ف. ڤاريلا [15]. نثبت أيضاً في هذه الورقة أن هومولوجية الاتصال (بواسطة منحنيات لجندر الثَّنْوِيَّة) ليست صفرية لمتتابعة من المؤشرات تميل إلى مالانهاية لأشكال الاتصال $\theta\alpha$ لبناء الاتصال الغريب الأول لـِ. جونزالو وَ ف. ڤاريلا على $S^3$، بافتراض أنه يمكن توصيلها بشكل الاتصال الأول لبناء الاتصال هذا من خلال مسار يحقق على طوله شبه – تدرج خاص يتم تكوينه في هذه الورقة شرط فريدهولم (انظر المقدمة وَ [8] لتعريف هذا المفهوم). لا نعرف فيما إذا كان هذا الفرض متحققا لشبه – التدرج الذي نستخدمه هنا.

**Mathematics Subject Classification** 37J45 · 37J55 · 53D10 · 55N99 · 58E10

## 1 Introduction

The standard contact structure $\alpha_0$ on $S^3$ is, following the terminology introduced by Eliashberg [14], "tight". $\alpha_0$ has an explicit formula: $(x_2\mathrm{d}x_1 - x_1\mathrm{d}x_2) + (x_4\mathrm{d}x_3 - x_3\mathrm{d}x_4)$ and one can easily find a vector field $v_0$ in its kernel (in fact, a family of vector fields) that defines a Hopf fibration. The orbits of such a vector field $v_0$ are

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all closed and the dynamics of $\alpha_0$ is periodic and well understood. $v_0$ can also be used to complete Legendre duality, see below, for convex contact forms of this contact structure.

$(\alpha_0, v_0)$ with these nice features is, therefore, an (nearly) explicit example where computations can be carried and a precise idea of the dynamics of the contact structure can be understood using $v_0$. Also the periodic orbits of the related Reeb vector fields have been studied and understood in some detail.

For over-twisted contact forms, few explicit examples are available and the understanding of the dynamics of such contact structures and their Reeb vector fields is, therefore, less advanced.

Explicit examples of over-twisted contact structures are known, e.g. the family of exotic, pairwise non-isomorphic contact structures on $S^3$ provided by Gonzalo and Varela [15]. However, their geometry has not been studied completely. A definite progress has become possible in the past few years after the discovery by Martino [19] of an explicit vector field $v$ in the kernel of the first contact form $\alpha_1$ of J. Gonzalo and F. Varela that commutes with a vector field $X_0$ defining a standard $S^1$-action on $S^3$.

For future reference, $\alpha_1$ reads as

$$\alpha_1 = -\cos\left(\frac{\pi}{4} + \pi r_2\right)(x_2 dx_1 - x_1 dx_2) - \sin\left(\frac{\pi}{4} + \pi r_2\right)(x_4 dx_3 - x_3 dx_4)$$

$S^3$ is the set $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$; $r_1 = x_1^2 + x_2^2$, $r_2 = x_3^2 + x_4^2$.

The vector field $v$ found by Martino [19] has nice properties: its orbits are not closed, but they are integrable in two variables $(a, p)$, see below. In addition, Legendre transform can be performed for $\alpha_1$ along $v$, that is

$(A)$ the form $\beta = d\alpha_1(v, .)$ is a contact form with the same orientation than $\alpha_1$.

It follows that there is a non-empty class of contact forms $\theta\alpha_1, \theta : S^3 \longrightarrow \mathcal{R} \setminus \{0\}, C^2$, such that $d(\theta\alpha(v, .)$ is also a contact form with the same orientation than $\alpha_1$.

Because $v$ commutes with the vector field $X_0$ and because its orbits are almost explicit, whereas $\ker\alpha_1$ is tangent to these orbits, the dynamics of $\ker\alpha_1$ and of $\alpha_1$ along this vector field $v$ can be very well understood.

Because the condition $(A)$ is verified, the variational problem corresponding to the action functional $\int_0^1 \alpha_1(\dot{x})dt$ on the space $C_\beta = \{x \in H^1(S^1, S^3), \theta\alpha(\dot{x}) = c \gneq 0, \beta(\dot{x}) = 0\}$ ($c$ is not prescribed) is well defined.

We can study its features and study the Morse relations in this variational problem; also we can compute the homology that we have defined in [2,3], see also the more recent extension in [8]; the issue of invariance of this homology under deformation is distinct from this computation, see [8] and other related works.

Let us describe now the results that we prove in some more detail:

We recall [1,2] that the *coincidence points* for $(\ker\alpha_1, v)$ of any given point $x_0$ of $S^3$ are the points $x_{\bar{s}}$ on the $v$-orbit $x_s$ through $x_0$ such that $\ker\alpha$ is mapped onto itself in the $v$-transport between $x_0$ and $x_{\bar{s}}$. These coincidence points can be understood for $(\alpha_1, v)$ and, therefore, the Fredholm violation for the variational problem $(J, C_\beta)$ can be described very precisely (see Propositions 8.2, 8.3, Sect. 8.4 of this paper). This Fredholm violation has strong consequences on the variational problem $(J, C_\beta)$, which we recall here:

### 1.1 The Fredholm violation for the variational problem $(J, C_\beta)$;

$J$ on $C_\beta$ does not verify the Fredholm assumption. This can be seen easily from the formula for its differential:

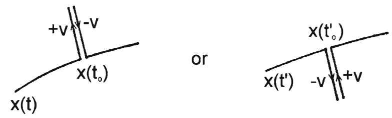
$$\partial J(x).z = -\int_0^1 b\eta, \quad z = \lambda\xi + \mu v + \eta w \in T_x C_{\beta_1}, \ \dot{x} = a\xi + bv$$

with $\lambda, \mu, \eta \in H^1(S^1, \mathcal{R})$ and verify the conditions, see [1,2]:

$$\overline{\dot{\lambda + \bar{\mu}\eta}} = b\eta + C, \quad \dot{\eta} = \mu a - \lambda b$$

$\bar{\mu}$ above is $\alpha_1(w)$, where $w$ is the contact vector field of $\beta_1$.

The violation of the Fredholm assumption has the following consequence: given a curve $x$ of $C_{\beta_1}$, we can add to this curve a back and forth or forth and back run along $v$ at a time $t_0$. The value of $J$ is not changed and even if the new derived curves are not in $C_\beta$ anymore, they are "almost" in this space. Cutting details, once this "Dirac mass" along $v$ is inserted along the curve, it can be "opened" up at its top or at its bottom depending on the cases and a small piece of $\xi$-orbit can be inserted:




If the "Dirac mass" is chosen with the appropriate length and location, $J(x_\epsilon)$ can be made smaller than $J(x)$.

The phenomenon is subtle because if the "Dirac mass" is small in size, $J(x_\epsilon)$ is always more than $J(x)$; but this changes with a larger "Dirac mass".

## 1.2 Fredholm violation and intersection operator restricted to periodic orbits;

The simple phenomenon described above has drastic implications on the corresponding variational problem. The main (negative) consequence can be described as follows:

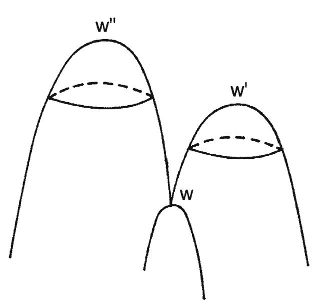
Every periodic orbit $w_m$ of index $m$ has a companion "shadow critical point at infinity" $(\delta + w_m)^\infty$ that "cancels its effect topologically". This follows from the fact that the lack of Fredholm properties of the linearized operator implies that no Morse lemma is available in the vicinity of a periodic orbit.

This phenomenon is described in [1–4]; see [2, pp 151–178] in particular.

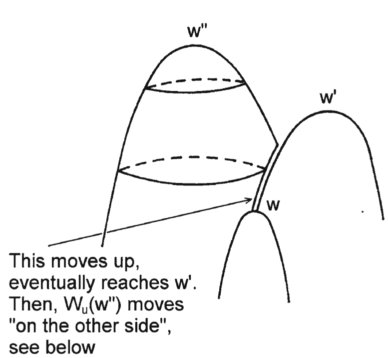
The "companion" $(\delta + w_m)^\infty$ to the periodic orbit $w_m$ is of index $(m+1)$ and $J((\delta + w_m)^\infty) = J(w_m)$, that is, the periodic orbit $w_m$ and its shadow $(\delta + w_m)^\infty$ are at the same level and cancel each other topologically (a byproduct of the fact that there is no Morse Lemma).

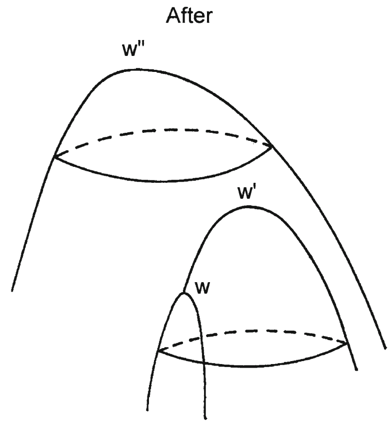
In terms of Morse Theory, this has a fundamental consequence: given a periodic orbit $w_{m+1}$ of index $(m+1)$ such that $J(w_{m+1})$ is larger than $J(w_m)$ and such that no other critical point (at infinity) of $J$ has a critical value in $(J(w_m), J(w_{m+1})$, the intersection number $i(w_{m+1}, w_m)$ is not defined intrinsically: it depends on the choice of a pseudo-gradient. Indeed, if we "swipe" $W_u(w_{m+1})$ across $W_u((\delta + w_m)^\infty)$ along a change of pseudo-gradient vector fields, see the drawing below, then the intersection number $i(w_{m+1}, w_m)$ changes and decreases or increases by 1:

Before




In between






This change can be accomplished whereas the initial and final pseudo-gradients have very similar properties: no increase along decreasing flow-lines of the number of zeros of the $v$-component $b$ of the tangent vector $\dot{x}$ to the curve under deformation, control on $\int_0^1 |b|\mathrm{d}t$ along these decreasing flow-lines, which all end up at periodic orbits or critical points at infinity (up to "Dirac masses" in $v$, that is back and forth or forth and back runs along $v$-orbits, see the Fredholm violation above for example).

Therefore, the intersection operator restricted to periodic orbits $\partial_{\text{per}}$ *depends on the pseudo-gradient* and the existence of an invariant attached to the contact structure itself appears to be difficult to establish.

### 1.3 "Symplectic" deformations:

We thus will say, as in [8]—which uses a different method to prove Theorems 1.1, 1.2 and 1.3 below—that a deformation of pseudo-gradients for $J$ having all the properties listed above(or $J_t$ if we also deform the contact form $\alpha_1$ into $\theta_t\alpha$, under $(A)_t$) is "symplectic" or "Fredholm" if, along this deformation and for every periodic orbit $w^t_{m+1}$ and every periodic orbit $w^t_m$—as long as they do not degenerate—the unstable manifold $W_u(w^t_{m+1})$ is never tangent to the stable manifold $W_s((\delta + w^t_m)^\infty)$; that is $W_u((\delta + w^t_m)^\infty) \nsubseteq \overline{\underset{t\in[0,1]}{\cup} W_u(w^t_{m+1})}$, see [8, section 4, Definition 1] for the precise description of $W_u((\delta + w^t_m)^\infty)$.

With this definition and these "symplectic deformations", the very basic problem of definition of intersection numbers between periodic orbits described above is overcome.

### 1.4 Value of the homology:

On the other hand, the Fredholm violation described above (Propositions 8.2 and 8.3, Sect. 8.4) can be used in a different way in the case of $\theta\alpha_1$, to prove that the flow-lines originating at a periodic orbit $w_m$ of index $m$ can be made to "bypass" any critical point at infinity of $(J, C_\beta)$. This is established throughout Sect. 8, see also Sect. 15 for additional remarks. This "bypassing" occurs while preserving the description of the unstable manifold of $w_m$ with $m*$s, see [3, Proposition 1, p469], or $m$ families of $\pm v$-jumps (in each family, the $\pm v$-jumps follow each other—no overlapping of families—and they all have the same orientation).

Assuming that the decreasing deformation thereby defined is "Fredholm" or "symplectic" as described above, that is that no tangency occurs over the flow-lines of this pseudo-gradient between the unstable manifold of a periodic orbit of index $m$ with a "shadow critical point at infinity" $(\delta + x_{m-1})^\infty$ arising because of the Fredholm violation ($x_{m-1}$ is a periodic orbit of index $(m-1)$), we derive the following results:

**Theorem 1.1** *There exists a sequence $k_n$ tending to $\infty$ such that the homology group of order $2k_n - 1$ (see [2,3,7]) for $\alpha$ has at least two generators.*

This result implies the following:

**Theorem 1.2** *Let $\theta$ be a function on $S^3$ valued into $\mathcal{R}$. If $|\theta - 1|_{C^2}$ is small enough, then the Reeb vector field $\xi_\theta$ of $\theta\alpha$ has two distinct geometric closed orbits.*

We then have

**Theorem 1.3** *Let $\theta$ be a function on $S^3$ valued into $\mathcal{R}^+$ such that $\beta_{1\theta} = \mathrm{d}(\theta\alpha)(v, .)$ is a contact form with the same orientation than $\alpha$. Assume that there exists a uniform positive constant $C$ such that, for every $s \in [0, 1]$, the periodic orbits $x = x(t)$ of the Reeb vector field $\xi_s$ of $\alpha_s = \frac{1}{1-s+\frac{s}{\theta}}\alpha$ of Morse index $k$ satisfy the estimate*[1]*:*

$$\int_0^1 \alpha_s(\dot{x})\,\mathrm{d}t \le C(k+1).$$

*Assume in addition that the above deformation from $\alpha$ to $\theta\alpha$ is "Fredholm" or "symplectic". Then, the Reeb vector field $\xi_\theta$ of $\theta\alpha$ has at least two distinct geometric closed orbits.*

The "symplectic" requirement on the deformation is discussed in [6] for other pseudo-gradients than the one defined here.

The proof of the above-stated results is based on a detailed study of the properties of a vector field discovered by Martino [19] in $\ker\alpha_1$. These results read as follows:

### 1.5 The vector field $v$ of Martino [19] and the behavior of $\alpha_1$ along $v$:

Martino provides in [19] a vector field $v_1$ in the nucleus of the first exotic contact form of Gonzalo and Varela [15] $\alpha$ such that $\mathrm{d}\alpha(v_1, .) = \beta_1$ is a contact form with the same orientation than $\alpha$. Over the process of publication, V. Martino found another such vector field $v$, much simpler and with nicer properties than $v_1$. This is the one that we provide in Sect. 2 and that we use throughout this paper.

A striking feature with this $v$ is that the computations are almost explicit. Indeed, there is a vector field on $S^3$, denoted $X_0$, the orbits of which are all closed:

$$X_0 = x_2\partial_{x_1} - x_1\partial_{x_2} + x_4\partial_{x_3} - x_3\partial_{x_4}$$

such that

$$[X_0, \xi] = 0, \quad [X_0, v] = 0$$

Thus, $X_0$ commutes with $\xi$ and with $v$. It, therefore, defines an $S^1$-action on $C_\beta$ and the action functional $J(x) = \int_0^1 \alpha(\dot{x})$ is invariant under this action.

The set of periodic orbits is, therefore, made of tori, $X_0$-circles of periodic orbits of $\xi$. These tori $T_t$ are defined by the equation $T_t = \{(x_1, x_2, x_3, x_4) \in S^3, r_2 = x_3^2 + x_4^2 = t + \frac{1}{2}\}$ for a countable set of values $t \in [\frac{-1}{2}, \frac{1}{2}]$.

These periodic orbits of $\xi$ are studied in Sect. 7 of this paper:

Three integers are associated with each periodic orbit on a torus $T_t$: $k$, which is its multiplicity, $p$, which is the number of counter-clockwise rotations that the associated simple periodic orbit completes in the $(x_3, x_4)$-plane and $q$, which is the number of counter-clockwise rotations that the same associated simple periodic orbit completes in the $(x_1, x_2)$-plane.

Introducing the functions of $r_2 = t + \frac{1}{2}$ $(r_1 = 1 - r_2)$:

$$A = \cos\left(\frac{\pi}{4} + \pi r_2\right), \quad B = \sin\left(\frac{\pi}{4} + \pi r_2\right)$$
$$\tilde{A} = A + \pi r_1 B, \quad \tilde{B} = B + \pi r_2 A$$

the critical value of the periodic orbit reads:

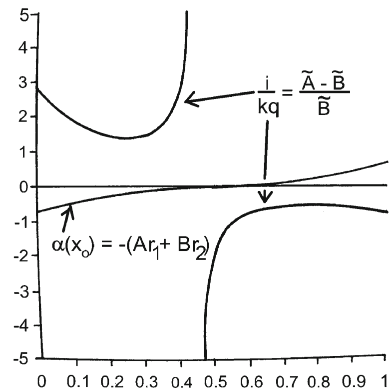
$$c = i \times \mathrm{cv}(t) = i \times \frac{\pi(A\tilde{B}r_1 + B\tilde{A}r_2)}{\tilde{A} - \tilde{B}}$$

[1] See Sect. 9.2 for the very limited use of this assumption. This assumption has weaker partial forms, see also 9.2. The same result is established in [8] without the use of this condition.

with $i = 2k(p-q)$ representing the Morse index of the periodic orbit. In addition,

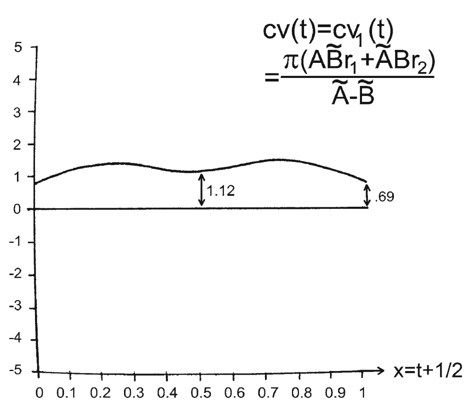
$$\frac{p}{q} = \frac{\tilde{A}}{\tilde{B}}$$

The model is almost explicit and the functions of $r_2 = x = t + \frac{1}{2}$ defined by $\alpha(X_0) = -Ar_1 - Br_2$, $\text{cv}(t) = \text{cv}_1(x)$ and $\frac{i}{kq} = \frac{\tilde{A}-\tilde{B}}{\tilde{B}}$ may be viewed explicitly. Their "graphed" behavior can be checked by a direct, rigorous study.




Observe that $kq$ represents the number of times that the periodic orbit, with multiplicity $k$, rotates counter-clockwise in the $(x_1, x_2)$-plane. We can, therefore, replace the notation $\frac{i}{kq}$ by $\frac{i}{q}$, where $q$ designates now $kq$.

The graph of $\text{cv}(t)$ is




Once the behavior of the periodic orbits of the Reeb vector field of $\alpha_1$, the related critical values and the Morse indexes have been understood, we turn to another property of the (semi)-flow associated with the variational problem $(J, C_\beta)$. This property is a new property, which is different from the fundamental property of decrease of the number of zeros of the $v$-component of the tangent vector $\dot{x}$ of the curve under deformation that we have discussed extensively in our earlier works, see [2–4].

It is related to the linking of two curves in $C_\beta$ subject to this (semi)-flow and it reads as follows:

## 1.6 Linking numbers, flow at infinity:

Tangent vectors $z$ to $S^3$ or to a more general contact manifold $(M^3, \alpha)$, with the datum of a $v$ as above, can be written as a linear combination $z = \lambda\xi + \mu v + \eta w$. $w$ is the contact vector field of $\beta = d\alpha(v, .)$

($\beta \wedge d\beta = \alpha \wedge d\alpha$). Tangent vectors to the curves $x$ of $C_\beta(\dot{x} = a\xi + bv)$ also read $z = \lambda\xi + \mu v + \eta w$, only that $\lambda, \mu, \eta$ are now $H^1$-functions valued into $\mathcal{R}$ satisfying appropriate ODEs, see [2–4]. A tangent vector field to $C_\beta$ may be viewed, in some generalized sense-a smoothing effect is required-as the datum, at every curve $x$ of $C_\beta$ of $(\eta, C)$. $\eta$ here is an $H^1$ or $L^2$-function, $C$ is a constant of integration.

A pseudo-gradient for our variational problem, that is for $\int_0^1 \alpha(\dot{x})$ on $C_\beta$ is given by $\eta = b$. This vector field has the striking property, see Lemma 6.1, Sect. 6 for an idea of the proof, that *the linking number of any two curves* $x_1(s), x_2(s)$*, subject to the semi-flow that it generates, does not decrease* (does not increase with the reverse orientation) as the time $s$ of this semi-flow increases.[2] For that very same reason, this semi-flow must have a large,uncontrolled set of blow-up curves (not in $C_\beta$, but in a natural completion of $C_\beta$, $C_\beta^+ = \{x \in H^1(S^1, M), \dot{x} = a\xi + bv, a \geq 0\}$). This is why we ruled it out for the variations of $J(x) = \int_0^1 \alpha_x(\dot{x})$ and we replaced it by the semi-flow of the $H^{-1}$-vector field of [4]: the information that it provided at the blow-up time was too poor, whereas, with the vector field $Z$ of [4], the behavior at the blow-up time is very well understood, the curves are in the $\cup\Gamma_{2k}$s; these are the spaces of curves made of $k$-pieces of $\xi$-orbits alternating with $k$-pieces of $\pm v$-orbits.

Our thinking led us to believe, for years, that this more precise information had a downside, namely that this non-decreasing property of the linking had to be given up for the semi-flow $Z$ of [4]. This turns out, see Lemma 6.1, Sect. 6, to be partly wrong: Under $Z$, the linking numbers of the curve $x(s)$, subject to $\frac{\partial x}{\partial s} = Z(x)$, with any periodic orbit of $\xi$ never decreases. Accordingly, if a semi-flow-line of $Z$ flows from a periodic orbit $PO_1$ of $\xi$ to another periodic orbit $PO_2$ of $\xi$, then for any periodic orbit of $\xi$ $PO_3$, we have

$$\text{link}(PO_2, PO_3) \geq \text{link}(PO_1, PO_3)$$

This provides a very strong information about the flow-lines of $Z$.

However, the story does not stop here because the semi-flow $Z$ is only "half" of the global flow. The other "half" is the flow at infinity $Z_\infty$, that is, it is the flow in the space $\cup\Gamma_{2k}$s. This second "half" might decrease the linking number so that the property

$$\text{link}(PO_2, PO_3) \geq \text{link}(PO_1, PO_3)$$

fails for the global flow. This sounds hopeless and, to understand what can be done, the need for concrete examples, over which these phenomena can be read, becomes compelling.

Whereas the framework of $(S^3, \alpha_0)$, $\alpha_0$ being the standard contact form of $S^3$ is too explicit and symmetric (the contact homology of [2,3] for $\alpha_0$ can be seen to carry at least one generator for each odd index larger then or equal to 3)[3], the framework of $(S^3, \alpha_1)$ has less symmetries and is more complicated. After $(\ker\alpha_0, S^3)$, this could be the next nearly explicit example to explore whether some properties do or do not hold.

Further understanding of the behavior of the linking of a curve of $C_\beta$ subject to a suitable decreasing (semi)-flow for $J$ with a periodic orbit of a Reeb vector field of the contact form is completed in [7].

### 1.7 Conclusion:

We conclude this introduction with two observations.

We first observe that the finding by Martino [19] of this $v$, with respect to which Legendre transform for $\alpha_1$ can be performed and the related geometry that one can explicitly study, see, e.g. the present paper, should be useful: the first exotic contact form/structure of Gonzalo and Varela [15], equipped with this $v$ is a new example, which provides a framework that is different from the framework of the standard contact form $\alpha_0$ on $S^3$ and the vector fields defining a Hopf fibration in its kernel.

On the other hand, the tools developed in the present paper are a direct continuation of the earlier work completed in [1–4] (see also [5,9,10]). The framework of the variational problem $(J, C_\beta)$ is a collaboration

[2] S. Angenent pointed out this property to me during a visit, some 15 years ago, to Madison, Wisconsin.

[3] Usually, the cycles derived from variations carry an index which coincides with the geometric dimension of their "support" or "carrier". This equality has to be abandoned here, because the cycles of the contact homology of [2,3] might have boundaries at infinity or $S^1$-invariant boundaries. Equality between dimension of "support" and index may be recovered if a "modding-out" by these additional objects at infinity is carried out.

with D. Bennequin (see [13] for a beautiful theorem proved by D. Bennequin on exotic contact forms. This work motivated Eliashberg [14] for the introduction of the notion of “tight” and “over-twisted” contact structure). Whereas these techniques have been useful in the study of some Partial Differential Equations, see [11,12], the full contribution of these techniques to the study of the invariants that can be attached to a contact structure is still open. The Weinstein conjecture has been formulated by Weinstein [24], after the work of Rabinowitz [22]. Solutions to this conjecture have been claimed, provided, through the work of Hofer [16], Hutchings [17], [18] and Taubes[23]. We provide in [8] a proof of this conjecture on $S^3$, with an understanding of the general Morse relations for the action functional on $c_\beta$.

We proceed now with the proof of the statements and claims described above.

## 2 Basic definitions and identities

Gonzalo and Varela [15] have given explicit formulae for an infinite family of contact forms/structures over $S^3$ that cannot be identified by a diffeomorphism of $S^3$. This family reads

$$\alpha_n = -\cos\left(\frac{\pi}{4} + n\pi r_2\right)(x_2 dx_1 - x_1 dx_2) - \sin\left(\frac{\pi}{4} + n\pi r_2\right)(x_4 dx_3 - x_3 dx_4)$$

$n$ is above a non-negative integer. $\alpha_0$ is the standard contact form of $S^3$. $S^3$ is the set $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$; $r_1 = x_1^2 + x_2^2$, $r_2 = x_3^2 + x_4^2$.

We study in what follows $\alpha_1 = \alpha$. The kernel of $\alpha$ is spanned by the two (singular when $r_1$ or $r_2$ are zero) vector fields:

$X = \sqrt{2}(\frac{B}{r_1}(x_2\partial_{x_1} - x_1\partial_{x_2}) - \frac{A}{r_2}(x_4\partial_{x_3} - x_3\partial_{x_4}))$

$Y = \frac{1}{r_1}(x_1\partial_{x_1} + x_2\partial_{x_2}) - \frac{1}{r_2}(x_3\partial_{x_3} + x_4\partial_{x_4})$

Let

$A = \cos(\frac{\pi}{4} + \pi r_2)$, $B = \sin(\frac{\pi}{4} + \pi r_2)$

$\tilde{A} = A + \pi r_1 B$, $\tilde{B} = B + \pi r_2 A$

Let

$\zeta = -(\tilde{B}(x_2\partial_{x_1} - x_1\partial_{x_2}) + \tilde{A}(x_4\partial_{x_3} - x_3\partial_{x_4}))$

The Reeb vector field of $\alpha$ is

$$\xi = \frac{\zeta}{\alpha(\zeta)}$$

Let

$$a = x_1x_3 + x_2x_4$$
$$b = x_1x_4 - x_2x_3$$

Martino [19] has found a non-singular vector field $v_1$ such that

**Proposition 2.1** (Martino [19]) *$\beta_1 = d\alpha(v_1, .)$ is a contact form with the same orientation than $\alpha$.*

The vector field $v_1$ given by Martino [19] is a bit complicated. But, Martino [19] also found later another, simpler vector field $v$ for which Proposition 2.1 holds, with the same proof (basically) than $v_1$. This vector field, which is due again to V. Martino, reads with the above notations:

$$v = aY + bX$$

The notation is a bit unfortunate for us. Indeed, let $\beta = d\alpha(v, .)$. We have denoted throughout our work $\dot{x} = a\xi + bv$ the tangent vector to the curves $x$ of the variational space $C_\beta = \{x \in H^1(S^1, S^3); \beta(\dot{x}) = 0; \alpha(\dot{x}) = C\}$. $C$ is a positive, not prescribed constant.

Having pointed out this duplication in the notations, we will nevertheless denote the components of $v$ on $X$ and $Y$, $a$ and $b$, respectively. No confusion allowed, we will denote $a$ and $b$ ($a$ now is a positive constant, $b$ is an $L^1(S^1, \mathcal{R})$-function) the components of the tangent vector $\dot{x} = a\xi + bv$ to a curve $x$ of $C_\beta$.

We first have

**Lemma 2.2** *$v$ is $C^\infty$.*

*Proof of Lemma 2.2* We write $v$ near, e.g. $r_1 = 0$. $v$ reads

$$aY + bX = (x_1x_3 + x_2x_4)\left(\frac{1}{r_1}(x_1\partial_{x_1} + x_2\partial_{x_2}) - \frac{1}{r_2}(x_3\partial_{x_3} + x_4\partial_{x_4})\right)$$
$$+(x_1x_4 - x_2x_3)\sqrt{2}\left(\frac{B}{r_1}(x_2\partial_{x_1} - x_1\partial_{x_2}) - \frac{A}{r_2}(x_4\partial_{x_3} - x_3\partial_{x_4})\right)$$

The components on $\partial_{x_3}, \partial_{x_4}$ are fine. Let us consider the component on $\partial_{x_1}$. It reads

$$(x_1x_3 + x_2x_4)\frac{1}{r_1}x_1 + (x_1x_4 - x_2x_3)\sqrt{2}\frac{B}{r_1}x_2$$

which rereads

$$x_3\frac{1}{r_1}(x_1^2 - \sqrt{2}Bx_2^2) + x_4x_2x_1\frac{1}{r_1}(1 + \sqrt{2}B)$$

Observe that $1 + \sqrt{2}B = 1 + \sqrt{2}\sin(\frac{\pi}{4} + \pi r_2) = 1 - \cos(\pi r_1) + \sin(\pi r_1)$
Clearly $\frac{1}{r_1}(1 + \sqrt{2}B)$ is $C^\infty$. The claim follows. □

The following identities are very basic and easy to prove. They are used throughout this paper:

**Proposition 2.3** (i) $Y.r_1 = 2, Y.r_2 = -2, Y.(\frac{\pi}{4} + \pi r_2) = -2\pi, Y.A = 2\pi B, Y.B = -2\pi A$
(ii) $Y.\tilde{A} = 2\pi(2B - \pi r_1 A), Y.\tilde{B} = -2\pi(2A - \pi r_2 B)$
(iii) $\zeta.a = -(\tilde{A} - \tilde{B})b, \zeta.b = (\tilde{A} - \tilde{B})a$
(iv) $[\zeta, X] = 0, [\zeta, Y] = Y.\tilde{B}(x_2\partial_{x_1} - x_1\partial_{x_2}) + Y.\tilde{A}(x_4\partial_{x_3} - x_3\partial_{x_4})$
(v) $[\zeta, v] = \zeta.aY + \zeta.bX + a[\zeta, Y]$
(vi) $Y.a = \frac{r_2 - r_1}{r_2 r_1}a, Y.b = \frac{r_2 - r_1}{r_1 r_2}b$
(vii) $X.a = -\sqrt{2}b\frac{Ar_1 + Br_2}{r_1 r_2}, X.b = \sqrt{2}a\frac{Ar_1 + Br_2}{r_1 r_2}$
(viii) $v.r_1 = 2a, v.r_2 = -2a$
(ix) $v.a = \frac{(r_2 - r_1)a^2}{r_1 r_2} - \sqrt{2}b^2\frac{Ar_1 + Br_2}{r_1 r_2}$.

Let

$$X_0 = x_2\partial_{x_1} - x_1\partial_{x_2} + x_4\partial_{x_3} - x_3\partial_{x_4}$$

$X_0$ is a vector field that commutes with $\zeta, \xi, X, Y$ and $v$. Indeed, simple computations verify the following:

**Lemma 2.4** (i) $[X_0, X] = 0, [X_0, Y] = 0, [X_0, \zeta] = 0, [X_0, \xi] = 0$
(ii) $X_0.a = X_0.b = 0$. *Therefore*, $[X_0, v] = 0$.

The commutation of $X_0$ with $\xi$, $v$, $X$ and $Y$ is a key feature of this framework. Using this feature, the computations about the one-parameter groups of $\xi$, $v$ etc, which are usually quite involved, simplify and the related phenomena can be read in an easy way. This is an important new example in Contact Form Geometry.
In the next section, we understand the dynamics of ker$\alpha$ along $v$ ($v$ commutes with $X_0$):

## 3 The dynamics of $v$

The evolution equations describing the dynamics of $v$ read in the coordinates $(x_1, x_2, x_3, x_4)$ of $S^3$:

$$\dot{x}_1 = \frac{ax_1}{r_1} + \frac{\sqrt{2}bBx_2}{r_1}$$
$$\dot{x}_2 = \frac{ax_2}{r_1} - \frac{\sqrt{2}bBx_1}{r_1}$$
$$\dot{x}_3 = -\frac{ax_3}{r_2} - \frac{\sqrt{2}bAx_4}{r_2}$$
$$\dot{x}_4 = -\frac{ax_4}{r_2} + \frac{\sqrt{2}bAx_3}{r_2}$$

With $x = r_2$, $v.b = (aY + bX).b = ab(\frac{2x-1}{x(1-x)} + \sqrt{2}\frac{A(1-x)+Bx}{x(1-x)})$. Thus,

$\frac{v.b}{b} = -\frac{v.x}{2}(\frac{2x-1+\sqrt{2}(A(1-x)+Bx)}{x(1-x)}) = v.xf(x)$

$f(x) = \frac{-1}{2}\frac{2x-1+\sqrt{2}(A(1-x)+Bx)}{x(1-x)} = \frac{-1}{2}\frac{2x-1+\sqrt{2}g(x)}{x(1-x)}$

$b = b_0 e^{\int_{y_0}^{y} f(x)\mathrm{d}x}$

Since $a^2 + b^2 = y(1-y)$, with $y = r_2$,

$$a^2 = y(1-y) - b_0^2 e^{2\int_{y_0}^{y} f(x)\mathrm{d}x}$$

We consider a trajectory along which $b$ is never zero (which is equivalent to $b_0$ not zero).

### 3.1 The evolution equations of $v$ in the $(a, y)$-variables and their behavior:

The evolution equations along $v$ in the $(a, y)$-variables read

$$v.y = -2a,$$
$$v.a = \frac{2y-1}{y(1-y)}a^2 - \sqrt{2}\frac{g(y)}{y(1-y)}b^2$$

This rewrites

$$v.y = -2a$$
$$v.a = -2f(y)a^2 - \sqrt{2}g(y)$$

or

$$v.y = p,\ v.p = \frac{f(y)}{4}p^2 + \frac{\sqrt{2}}{2}g(y)$$

the function $-g(y)$ is negative for $y \in [0, \frac{1}{2}]$ and it is positive for $y \in [\frac{1}{2}, 1]$.

**Lemma 3.1** *Assume that $y_0 \neq 0, 1$. Then, either $a \equiv 0$ and $y \equiv \frac{1}{2}$ or $y$ is an oscillating periodic function between two values $y_{\min}$ and $y_{\max} = 1 - y_{\min}$. If the period is $T$, then $(y - \frac{1}{2})$ is $\frac{T}{2}$ anti-periodic.*

*Proof of Lemma 3.1* $-f$ is positive in $[\frac{1}{2}, 1]$ and negative in $[0, \frac{1}{2}]$. It remains in absolute value bounded away from zero outside of any $(\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$. The same result holds for $g$.

Assume that $y$ remains larger than or equal to $\frac{1}{2} + \epsilon$ for $t \leq t_0$. Then, $p$ has to tend to zero on a sequence of times $t_n$ tending to $\infty$. $(v.p)(t_n)$ is negative, bounded away from zero.

Thus, either $p$ is negative and remains negative all the time, bounded away from zero; $y$ cannot remain then larger than $\frac{1}{2} + \epsilon$.

Or $p(t_{n_0})$ is positive; $v.p(t_{n_0})$ being negative, $p$ remains small thereafter and has to cross zero.

$y$ cannot remain larger than $\frac{1}{2} + \epsilon$ for all times. It cannot as well remain smaller than $\frac{1}{2} - \epsilon$ for all times.

Observe now that if $(p, y)$ solves the above system of equations, then $(-p, y)$ solves the system of equations with $v$ replaced by $-v$. Thus, if $p(t_0) = 0$ and $(p, y)(t)$ is the solution for $t \gneq t_0$, with $(p(t_0) = 0, y(t_0) = y_0)$, then the solution for $t \lneq t_0$ is $(-p(2t_0 - t), y(2t_0 - t))$. The anti-periodicity follows.

Last, $y$ cannot remain constant equal to $\frac{1}{2}$, unless $p = -2a \equiv 0$.

After increasing from $y_{\min}$ to $y_{\max}$, $y$ decreases from $y_{\max}$ to $y_{\min}$.

Observe now that the function

$$e^{-2\int_{y_0}^{y} f(x)\mathrm{d}x}$$

has a derivative equal to

$$e^{-2\int_{y_0}^{y} f(x)\mathrm{d}x}(2y - 1 + \sqrt{2}g(y) + 1 - 2y) = \sqrt{2}e^{-2\int_{y_0}^{y} f(x)\mathrm{d}x}g(y)$$

It is, therefore, monotone on each $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Setting $y_0 = y_{\max}$, we have $b_0 = b_{\max}$. We want to solve

$$y(1-y)\mathrm{e}^{-2\int_{y_0}^{y} f(x)\mathrm{d}x} = b_{\max}^2 = y_{\max}(1-y_{\max})$$

Because $\int_{\bar{y}}^{1-\bar{y}} f(x)\mathrm{d}x = 0$ ($f$ is odd around $\frac{1}{2}$), $1 - y_{\max}$ is a solution in $[0, \frac{1}{2}]$. By monotonicity, it is the only solution and $y_{\min} = 1 - y_{\max}$. □

### 3.2 The rotation over a full "cycle" of a piece of $v$-orbit in the $(a, y)$-variables

We rewrite now, using the fact that $a = \frac{v.r_1}{2} = -\frac{v.r_2}{2}$, the evolution of equations for $v$ stated above. We find

$$\left(\frac{x_1}{\sqrt{r_1}}\right)^{\cdot} = \frac{\sqrt{2}bB}{r_1}\frac{x_2}{\sqrt{r_1}}$$
$$\left(\frac{x_2}{\sqrt{r_1}}\right)^{\cdot} = -\frac{\sqrt{2}bB}{r_1}\frac{x_1}{\sqrt{r_1}}$$
$$\left(\frac{x_3}{\sqrt{r_2}}\right)^{\cdot} = -\frac{\sqrt{2}bA}{r_2}\frac{x_4}{\sqrt{r_2}}$$
$$\left(\frac{x_4}{\sqrt{r_2}}\right)^{\cdot} = \frac{\sqrt{2}bA}{r_2}\frac{x_3}{\sqrt{r_2}}$$

so that, with $z = \frac{x_1+ix_2}{\sqrt{r_1}}$, $z_1 = \frac{x_3+ix_4}{\sqrt{r_2}}$,

$$z = \mathrm{e}^{i\tilde{\phi}}z_0, \quad z_1 = \mathrm{e}^{i\tilde{\psi}}z_{1,0},$$

with

$$\tilde{\phi} = -\int_0^t \frac{\sqrt{2}bB}{r_1}\,\mathrm{d}\tau, \quad \tilde{\psi} = \int_0^t \frac{\sqrt{2}bA}{r_2}\,\mathrm{d}\tau$$

Observe that $\mathrm{d}z(Y) = \mathrm{d}z_1(Y) = 0$; $Y$ is in the kernel of $\alpha$ and is always transverse to $v$ along its flow-lines, provided $b_0$; thus $b$ is non-zero along such a flow-line.

Let now $x$ be a point on the torus $T_0$ defined by the equation $r_2 = \frac{1}{2}$. We claim that

**Lemma 3.2** *Over a full cycle, that is from $x$ to the next intersection point of the $v$-orbit through $x$ with $T_0$, the total rotation of each of $z$ and $z_1$ is*

$$-2\sqrt{2k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{\cos\left(\frac{\pi}{4}+\pi y\right)}{y\sqrt{k(y)-k(\bar{y})}}\,\mathrm{d}y - 2\tan^{-1}\left(\sqrt{\frac{1}{4k(\bar{y})-1}}\right)$$

*Proof of Lemma 3.2* Let $T^+(x)$ be the first time on the positive $v$-orbit through $x$ (assume that, e.g. $a$ is positive at $x$) such that $a$ becomes zero. Let $T^-(x)$ be the first time on the negative $v$-orbit through $x$ such that $a$ becomes zero. From $T^-(x)$ to $T^+(x)$, the total rotation of the unit vector $z_1$ is

$$\bar{\psi} = \int_{T^-(x)}^{T^+(x)} \frac{\sqrt{2}bA}{r_2}\,\mathrm{d}t$$

$b$ is equal to

$$b = b_0\mathrm{e}^{\int_{y_0}^{y} f(z)\mathrm{d}z}$$

Setting $y_0$ at $\bar{y}$, that is at the maximum value of $r_2 = y$, and assuming that $b_0$ is positive (we will derive the other case by symmetry), we find

$$\begin{aligned} b &= \sqrt{\bar{y}(1-\bar{y})}\mathrm{e}^{\int_{\bar{y}}^{y} f(z)\mathrm{d}z} \\ a &= \sqrt{y(1-y) - \bar{y}(1-\bar{y})\mathrm{e}^{2\int_{\bar{y}}^{y} f(z)\mathrm{d}z}} \\ &= \mathrm{e}^{\int_{\bar{y}}^{y} f(z)\mathrm{d}z}\sqrt{y(1-y)\mathrm{e}^{-2\int_{\bar{y}}^{y} f(z)\mathrm{d}z} - \bar{y}(1-\bar{y})} \end{aligned}$$

Setting $k(y)$ to be

$$k(y) = y(1-y)\mathrm{e}^{-2\int_{\frac{1}{2}}^{y} f(z)\mathrm{d}z}$$

we find

$$\begin{aligned} b &= \sqrt{k(\bar{y})}\mathrm{e}^{\int_{\frac{1}{2}}^{y} f(z)\mathrm{d}z} \\ a &= \mathrm{e}^{\int_{\frac{1}{2}}^{y} f(z)\mathrm{d}z}\sqrt{k(y) - k(\bar{y})} \end{aligned}$$

Therefore,

$$\bar{\psi} = \frac{-1}{2}\sqrt{2k(\bar{y})}\int_{1-\bar{y}}^{\bar{y}} \frac{\cos\left(\frac{\pi}{4} + \pi y\right)}{y\sqrt{k(y) - k(\bar{y})}}\,\mathrm{d}y$$

Going over a full cycle, we have to double $\bar{\psi}$. we can view this as

$$-\sqrt{2k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{\cos(\frac{\pi}{4} + \pi y)}{y\sqrt{k(y) - k(\bar{y})}}\,\mathrm{d}y$$

to which we add

$$-\sqrt{2k(\bar{y})}\int_{1-\bar{y}}^{\frac{1}{2}} \frac{\cos\left(\frac{\pi}{4} + \pi y\right)}{y\sqrt{k(y) - k(\bar{y})}}\,\mathrm{d}y$$

Changing $y$ into $1-y$ in this last integral, we find ($k(y) = k(1-y)$):

$$\begin{aligned} &-\sqrt{2k(\bar{y})}\int_{\bar{y}}^{\frac{1}{2}} \frac{\sin\left(\frac{\pi}{4} + \pi y\right)}{(1-y)\sqrt{k(y) - k(\bar{y})}}\,\mathrm{d}y \\ &= \sqrt{2k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{\sin\left(\frac{\pi}{4} + \pi y\right)}{(1-y)\sqrt{k(y) - k(\bar{y})}}\,\mathrm{d}y \end{aligned}$$

Thus, our total rotation is

$$-2\sqrt{2k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{\cos\left(\frac{\pi}{4} + \pi y\right)}{y\sqrt{k(y) - k(\bar{y})}}\,\mathrm{d}y + \sqrt{2k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{y\sin\left(\frac{\pi}{4} + \pi y\right) + (1-y)\cos\left(\frac{\pi}{4} + \pi y\right)}{y(1-y)\sqrt{k(y) - k(\bar{y})}}\,\mathrm{d}y$$

Observe that $k'(y) = \sqrt{2}g(y)\mathrm{e}^{-2\int_{\frac{1}{2}}^{\bar{y}} f(x)\mathrm{d}x}$. Thus,

$$\sqrt{2k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{g(y)}{y(1-y)\sqrt{k(y)-k(\bar{y})}}\,\mathrm{d}y = \sqrt{k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{k'(y)}{k(y)\sqrt{k(y)-k(\bar{y})}}\,\mathrm{d}y$$

$$= \sqrt{k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{k'(y)}{((\sqrt{k(y)-k(\bar{y})})^2+k(\bar{y}))\sqrt{k(y)-k(\bar{y})}}\,\mathrm{d}y$$

$$= 2\sqrt{k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{\mathrm{d}(\sqrt{k(y)-k(\bar{y})})}{((\sqrt{k(y)-k(\bar{y})})^2+k(\bar{y}))}\,\mathrm{d}y$$

$$= 2\int_{\frac{1}{2}}^{\bar{y}} \frac{\mathrm{d}\left(\sqrt{\frac{k(y)}{k(\bar{y})}-1}\right)}{\left(\left(\sqrt{\frac{k(y)}{k(\bar{y})}-1}\right)^2+1\right)}\,\mathrm{d}y$$

$$= -2\tan^{-1}\left(\sqrt{\frac{1}{4k(\bar{y})-1}}\right)$$

Our total rotation is, therefore,

$$-2\sqrt{2k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{\cos\left(\frac{\pi}{4}+\pi y\right)}{y\sqrt{k(y)-k(\bar{y})}}\,\mathrm{d}y - 2\tan^{-1}\left(\sqrt{\frac{1}{4k(\bar{y})-1}}\right)$$

□

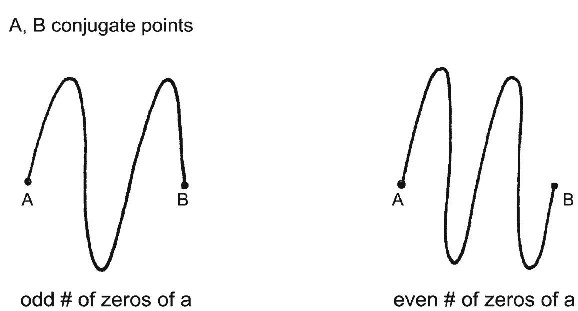
## 4 Conjugate points and characteristic surface

Conjugate points are points $x_0$, $x_1$ on the same $v$-orbit such that the form $\alpha$ is transported onto itself in the $v$-transport map between $x_0$ and $x_1$. Accordingly, ker$\alpha$ completes between the two points a number $k$ of full revolutions.

Because $X_0$ commutes to $v$, the value of $\alpha(X_0)$ must be the same at two points that are conjugate. $\alpha(X_0)$ is on the other hand a function of $r_2$ (or equivalently of $r_1$) only. Its behavior as a function of $r_2$ is described in (F0) (see the Sect. 1).

It is clearly a monotone function of $r_2$. Therefore, conjugate points must live on the same torus $T_t = \{x; r_2(x) = t + \frac{1}{2}\}$.

Accordingly, we can consider the number of *inside* zeros of the $Y$-component $a$ of $v = aY + bX$ along the piece of $\pm v$-orbit connecting the conjugate points. We will distinguish the case when this number is odd and the case when this number is even when $t \neq 0$.




F1

$T_0$ is a part of the characteristic hyper-surface ((i) of Lemma 4.2 below). We, therefore, will consider below conjugate points that are not on $T_0$.

4.1 Conjugate points with $k = 1$ and an even number of inside zeros of $a$ along the $\pm v$-jump separating them:

We study in the beginning of this section the conjugate points and the characteristic hyper-surface corresponding to exactly one full rotation ($k = 1$) of $\ker\alpha$ along $v$ and such that the number of zeros of $a$ separating these two conjugate points is even. We observe

**Lemma 4.1** *If $k = 1$ and the number of zeros of $a$ is even, then this number is exactly 2.*

*Proof of Lemma 4.1* $X_0$ is $v$-transported. It is in $\ker\alpha$ on the torus $T_0$. Therefore, if the number of inside zeros is 4 or more, the rotation of $\ker\alpha$ along the piece of $v$-orbit connecting the two conjugate points is too large. If it is less than 2, it is too little. The conclusion follows. □

$T_t$ is as above the torus of $S^3$ defined by the equation

$$\left\{r_2 = x_3^2 + x_4^2 = \frac{1}{2} + t, t \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\}$$

This torus degenerates into a circle for $t = \pm\frac{1}{2}$.

Given a torus $T_t$, let $\psi$ denote the map generated by the one-parameter group of $v$ from $T_t$ to $T_t$. Since $v = aY + bX$, with $Y$ transverse to $T_t$ ($t \neq \pm\frac{1}{2}$), this map is well defined and differentiable at any point $z$ of $T_t$ with $a(z) \neq 0$.

**Lemma 4.2** (i) *$T_0$ is a characteristic surface.*

(ii) *If a point $z$ belonging to a torus $T_t$ with $t$ non-zero and such that $b(z) \neq 0$, is on a characteristic surface, then $\mathrm{d}\psi_z$ is equal to* Id*; $z$ is then on a $v$-flow-line through a point $z_0$ on $T_0$ and $\mathrm{d}\psi_{z_0}$ is also equal to* Id.

*Proof of Lemma 4.2* Because the differential equation defined by $v$ on $(a, b)$ is integrable, the values of $a$ and of $b$ at $z_0$ are equal to their values at $\psi(z_0)$. This holds also true for $z$ and $\psi(z)$. In fact, to each point on the $v$-orbit from $z_0$ to $z$, we can associate a corresponding point on the $v$-orbit from $\psi(z_0)$ to $\psi(z)$.

This implies that the $v$-orbit from $\psi(z_0)$ to $\psi(z)$ is derived through the flow of $X_0$ from the $v$-orbit from $z_0$ to $z$, with a constant time along this flow $s_0$.

The following identity always holds:

$$\mathrm{d}\psi(X_0) = X_0$$

Indeed, because $X_0$ and $v$ commute, $\mathrm{d}\psi(X_0) = X_0 + \mu v$. Because $\psi$ is a map from a subset of the torus into itself, $\mu$ is zero whenever $a$ is non-zero. By density, $\mu$ is zero whenever $\psi$ is well defined and differentiable.

Assume that $z$ and $z'$ are conjugate. Because $X_0$ and $v$ commute and because $\alpha_z(X_0)$ is a monotone function of $r_2(z)$, $z$ and $z'$ must belong to the same torus $T_t$. Thus, $z'$ is then $\psi(z)$.

Assume in a first step that $a(z)$ is non-zero. The case when $a(z) = 0$, $b(z) \neq 0$ is studied below. If $a(z)$ is non-zero, then

$$\mathrm{d}\psi(X) = \theta X$$

Indeed, since $z$ and $\psi(z)$ are conjugate and since $a(z)$ is non-zero, $d\psi(X)$ reads as $\theta X + \mu v$. On the other hand, $\mathrm{d}\psi(X)$ is tangent to $T_t$ and this, when $a(z) = a(\psi(z))$ is non-zero, can only happen if $\mu$ is zero.

We use now the identity

$$a(\psi(z)) = a(z)$$

This identity implies that

$$\mathrm{d}a_{\psi(z)}(\mathrm{d}\psi(X)) = \theta \mathrm{d}a_{\psi(z)}(X) = \mathrm{d}a_z(X).$$

Since $b(z)$, $b(\psi(z))$ are not zero and since $z$, $\psi(z)$ are not in $T_0$, $\mathrm{d}a_z(X) = \mathrm{d}a_{\psi(z)}$ are non-zero and $\theta$ must be equal to 1.

Thus,

$$\mathrm{d}\psi(X_0) = X_0, \quad \mathrm{d}\psi(X) = X$$

We also claim, and this is less obvious, that

$$\mathrm{d}\psi_{z_0}(\zeta) = \zeta$$

Let $\Gamma$ be the differential of the $v$-one parameter group map from $z_0$ to $z$, that is from $T_0$ to $T_t$; this differential is taken at $z_0$.

Let $g$ be the $X_0$-one parameter group map from $[z_0, z]$ to $[\psi(z_0), \psi(z)]$. Let $\Gamma_1$ be the differential of the $v$-one parameter group map from $\psi(z_0)$ to $\psi(z)$.

Then,

$$\Gamma_1 = \mathrm{d}g \circ \Gamma \circ \mathrm{d}g^{-1}$$

and

$$\mathrm{d}\psi_{z_0} = \mathrm{d}g \circ \Gamma^{-1} \circ \mathrm{d}g^{-1} \circ \mathrm{d}\psi_z \circ \Gamma$$

We know that $d\psi_z(X) = X$, so that

$$\mathrm{d}g \circ \Gamma^{-1} \circ \mathrm{d}g^{-1} \circ \mathrm{d}\psi_z \circ \Gamma(\Gamma^{-1}(X)) = \mathrm{d}g \circ \Gamma^{-1} \circ \mathrm{d}g^{-1}(X) = \mathrm{d}g \circ \Gamma^{-1}(X)$$

($\mathrm{d}g(X) = X$ because $X$ and $X_0$ commute).

Therefore,

$$\mathrm{d}\psi_{z_0}(\Gamma^{-1}(X)) = \mathrm{d}g \circ \Gamma^{-1}(X)$$

$\Gamma^{-1}(X)$ reads

$$\Gamma^{-1}(X) = a_1 X_0 + b_1 \zeta, b_1 \neq 0$$

It follows that

$$\mathrm{d}\psi_{z_0}(a_1 X_0 + b_1 \zeta) = \mathrm{d}g(a_1 X_0 + b_1 \zeta) = a_1 X_0 + b_1 \zeta$$

Since $\mathrm{d}g(X_0) = X_0$, $\mathrm{d}\psi_{z_0}(\zeta) = \zeta$ as claimed. □

To see that $T_0$ is also a characteristic hyper-surface, we observe that, whenever it is well defined and differentiable, the map $\psi$ from $T_0$ to $T_0$ maps $X_0$ to $X_0$ as noted above, $X_0$ is in $\ker\alpha$ along $T_0$. $v$ is of course mapped onto itself through its own one-parameter group, so that the one-parameter group of $v$ maps $\ker\alpha$ into $\ker\alpha$ from $z$ of $T_0$ into $\psi(z)$ of $T_0$.

We evaluate $\mathrm{d}\psi(\zeta) = a_1 X_0 + b_1 \zeta$.

We know that

$$\begin{aligned} \mathrm{d}a_{\psi(z)}(\mathrm{d}\psi(\zeta)) &= a_1 \mathrm{d}a(X_0) + b_1 \mathrm{d}a(\zeta) = -b_1 b(\tilde{A} - \tilde{B}) \\ &= \mathrm{d}a_z(\zeta) = -b(\tilde{A} - \tilde{B}) \end{aligned}$$

Since $b$ is non-zero, $b_1 = 1$ and

$$\alpha(\mathrm{d}\psi(\zeta)) = \alpha(\zeta)$$

The one-parameter group of $v$ maps $T_0 - \{b = 0\}$ into itself and the map $\psi$ maps $\alpha$ onto $\alpha$. $T_0$ is a characteristic hyper-surface.

To complete the proof of Lemma 4.2, we need now to study the case $b = 0$, also $a = 0, b \neq 0$.

We have

**Lemma 4.3** *Let $z$ and $\psi(z)$ be two consecutive points on the same torus $T_t$ such that $b$ is zero at $z$ (hence along all the flow-line from $z$ to $\psi(z)$). Then $z$ and $\psi(z)$ are conjugate points if and only if* $\mathrm{d}\psi_z = \mathrm{Id}$. *This is equivalent to the statement that* $\mathrm{d}\psi_{z_0} = \mathrm{Id}$, *where $z_0$ is the nearest point on $T_0$ on the same $v$-flow-line.*

*Proof of Lemma* 4.3 The evolution differential equations of $v$ show that the angles $\tilde{\phi}$ and $\tilde{\psi}$ defined above are identically zero when $b = 0$.

At each crossing of, e.g. $r_1 = 0$, $(x_1, x_2)$ changes into $(-x_1, -x_2)$, whereas $(x_3, x_4)$ remains unchanged. This can be seen as follows: the evolution equations for $v$ are continuously differentiable in the variables $a, r_1$ throughout $r_1 = 0$ because the function $f$ extends into a differentiable function through $x = 0, x = 1$. If the $v$-orbit crosses $r_1 = 0$ at $s = s_0$, then in the variables $a, y$, we have $a(s+s_0) = -a(s_0-s), r_1(s+s_0) = r_1(s_0-s)$. This implies that $b(s + s_0) = b(s - s_0) = 0$.

Coming back to the equations in the $x_i$-variables, we see that $x_i(s + s_0) = x_i(s_0 - s), i = 3, i = 4$. These equalities, coupled with $b(s_0 + s) = b(s_0 - s) = 0, a(s_0 + s) = a(s_0 - s)$ imply that $x_i(s_0 + s) = -x_i(s_0 - s), i = 1, i = 2$ as claimed.[4]

After two crossings of $r_1 = 0$ (or $r_2 = 0$), $a$ comes back to its initial values. All these $v$-orbits are closed.

We will see below that $\psi$ is well defined and differentiable on the tori $T_t, t \neq \pm\frac{1}{2}$.

Considering $z$ a point on a characteristic hyper-surface, with $b(z) = 0, r_1(z) \neq 0, r_2(z) \neq 0$, $z$ not in $T_0$, we write

$$\mathrm{d}\psi(\zeta) = a_1 X_0 + b_1 \zeta$$

We then have $(\mathrm{d}b(X_0) = 0)$:

$$\mathrm{d}b(\mathrm{d}\psi(\zeta)) = \mathrm{d}b(\zeta) = a(\tilde{A} - \tilde{B}) = b_1 \mathrm{d}b(\zeta)$$

Thus $b_1 = 1$ and $\alpha(\mathrm{d}\psi(\zeta)) = a_1\alpha(X_0) + \alpha(\zeta)$.

Equality implies that $a_1 = 0$. It follows that $d\psi(\zeta) = \zeta$. We also know that $\mathrm{d}\psi(X_0) = X_0$ $(a \neq 0)$. Thus, $\mathrm{d}\psi_z = \mathrm{Id}$ and arguing as above $\mathrm{d}\psi_{z_0} = \mathrm{Id}$. □

**Lemma 4.4** *Along* $b = 0$, $\mathrm{d}\psi_{z_0}$ *is never equal to* Id.

*Proof of Lemma* 4.4 We know that the total rotation is, when $b$ is positive,

$$R(\bar{y}) = -2\sqrt{2k(\bar{y})}\int_{\frac{1}{2}}^{\bar{y}} \frac{\cos\left(\frac{\pi}{4} + \pi y\right)}{y\sqrt{k(y) - k(\bar{y})}}\,\mathrm{d}y - 2\tan^{-1}\left(\sqrt{\frac{1}{4k(\bar{y}) - 1}}\right)$$

We will discuss later the change of sign that occurs when $b$ is negative. Restricting for the time being $b$ to remain positive, we observe that the above formula extends by continuity to $b = 0$.

Indeed, at $b = 0$, as crossing occurs of e.g $r_1 = 0$, the vector $\frac{x_1+ix_2}{|x_1+ix_2|}$ undergoes a change of direction equal to $\pi$. This can be seen in the two-dimensional frame provided by $x_3 + ix_4$ and $x_4 - ix_3$. $b$ remains unchanged equal to zero whereas $a$ changes from $a$ to $-a$. Accordingly, $\frac{x_1+ix_2}{|x_1+ix_2|}$ does not follow a continuous process; its orientation is reversed through the crossing.

When $b$ is not zero, this change occurs continuously. $b$ does not change sign, but $a$, which satisfies $a^2 = y(1 - y) - b_0^2 e^{2\int_{y_0}^{y} f(x)\mathrm{d}x}$, decreases to zero ($a = 0$ when $y = \bar{y}$)and changes sign. As $b_0$ tends to zero, $\bar{y}$ tends, e.g. to 1, $x_1 + ix_2$ tends to zero, but the normalized $\frac{x_1+ix_2}{|x_1+ix_2|}$ converges over each half a cycle because the rotation of this vector over this half-cycle is monotone near its edges (on a uniform—w.r.t to the values of $(b, a)$, with $b$ close to zero—neighborhood of these edges) and has a total value equal to $\frac{R(\bar{y})}{2}$. $\frac{R(\bar{y})}{2}$ is finite and has a finite limit when $\bar{y}$ tends to 1. The limiting position of $\frac{x_1+ix_2}{|x_1+ix_2|}$ over each half-cycle, as $r_1$ tends to zero with $b \neq 0$ ($b$ is close to zero here), is defined by the equations $a = 0, b \neq 0$. However, this vector rotates over a small interval of $r_1$-values from a vector having $b$ small, $\frac{a}{b}$ large, to a vector having $a = 0, b \neq 0$ (the signs of the components do not change). This is a $\frac{\pi}{2}$-rotation in the $(x_3+ix_4, -x_4+ix_3)$ corresponding frame. This $\frac{\pi}{2}$-rotation (multiplied by 2) is already accounted for in the formula for $R(\bar{y})$ and yields when $b = 0$ the $\pi$-rotation that $\frac{x_1+ix_2}{|x_1+ix_2|}$ undergoes over a full cycle. The claim about the continuous extension of $R(\bar{y})$ to

[4] We have proved in Sect. 2, Lemma 2.2 that $v$ is $C^\infty$. However, the argument stated above, slightly modified, implies the existence and continuity of the one-parameter group of $v$ through $r_1 = 0$ or $r_2 = 0$. At each occurrence, $r_1$ and $a, b$ are uniquely defined along the flow-line. Half of the variables satisfy then a non-singular evolution equation. They extend, therefore, in a unique, continuous way. The other half follows using the $(a, b)$ equations it satisfies.

$\bar{y}=1,0$ follows. Over this extension, the meaning of $R(\bar{y})$, that is that it represents the total rotation of, e.g $\frac{x_1+ix_2}{|x_1+ix_2|}$ (we could also use $\frac{x_3+ix_4}{|x_3+ix_4|}$ instead), reversal of orientation included, is unchanged.

Let us now consider the rotation $R(\bar{y})$ when $b$ is negative. The value of $k(\bar{y})$ can be easily recognized to be

$$k(\bar{y})=\left|b\left(\frac{1}{2}\right)\right|=|b_0|$$

Therefore, the general formula for $R(\bar{y})$ that will work whether $b$ is positive or negative reads

$$R(\bar{y})=-2\sqrt{2}b_0\int_{\frac{1}{2}}^{\bar{y}}\frac{\cos\left(\frac{\pi}{4}+\pi y\right)}{y\sqrt{k(y)-k(\bar{y})}}\,\mathrm{d}y-2\frac{b_0}{\sqrt{k(\bar{y})}}\tan^{-1}\left(\sqrt{\frac{1}{4k(\bar{y})-1}}\right)$$

Setting $u=\sqrt{\frac{4k(\bar{y})}{1-4k(\bar{y})}}$, the above formula can be manipulated into

$$R(\bar{y})=-2\sqrt{2}b_0\int_{\frac{1}{2}}^{\bar{y}}\frac{\cos\left(\frac{\pi}{4}+\pi y\right)}{y\sqrt{k(y)-k(\bar{y})}}\,\mathrm{d}y-\frac{b_0}{|b_0|}\pi+\frac{4b_0}{\sqrt{1-4k(\bar{y})u}}\tan^{-1}u$$

The discontinuity at the crossing of $b=b_0=0$ of the formula for $R(\bar{y})$ is clear; but it is a $2\pi$-discontinuity and $R(\bar{y})$ is an angle. We can, therefore, replace it, for $\bar{y}$ close to 0 and to 1 by

$$R_1(\bar{y})=-2\sqrt{2}b_0\int_{\frac{1}{2}}^{\bar{y}}\frac{\cos\left(\frac{\pi}{4}+\pi y\right)}{y\sqrt{k(y)-k(\bar{y})}}\,\mathrm{d}y+\frac{4b_0}{\sqrt{1-4k(\bar{y})u}}\tan^{-1}u$$

Observe that $\bar{y}$ is a differentiable function of the point $z_0$ on the same $v$-flow-line in $T_0$. Indeed, $k(\bar{y})=b_0^2$ and $k'(y)=\sqrt{2}g(y)$ is non-zero near $\bar{y}=0,1$.

$\frac{4b_0}{\sqrt{1-4k(\bar{y})u}}\tan^{-1}u$ contains $\frac{\tan^{-1}u}{u}$ which is a function of $u^2$. $u^2$ is a differentiable function of $z_0$ again. Therefore, $\frac{4b_0}{\sqrt{1-4k(\bar{y})u}}\tan^{-1}u$ is a differentiable function of $z_0$. Its differential if $b_0$ is zero is clearly $4\mathrm{d}b_0(.)$.

We, therefore, consider the first term in $R_1(\bar{y})$:

$$-2\sqrt{2}b_0\int_{\frac{1}{2}}^{\bar{y}}\frac{\cos\left(\frac{\pi}{4}+\pi y\right)}{y\sqrt{k(y)-k(\bar{y})}}\,\mathrm{d}y$$

We manipulate it to show that it is continuously differentiable at $b_0=0$. We rewrite it as

$$2\sqrt{2}b_0\int_{\frac{1}{2}}^{\bar{y}}\frac{\cos\left(\frac{\pi}{4}+\pi y\right)}{y\sqrt{\bar{y}-y}}w(y,\bar{y})\,\mathrm{d}y$$

$w(y,\bar{y})$ is $C^\infty$ and negative+ near $\bar{y}=1$.

Denoting $\gamma(y,\bar{y})$ the expression $\frac{\cos(\frac{\pi}{4}+\pi y)w(y,\bar{y})}{y}$ and integrating by parts, we rewrite

$$-2\sqrt{2}b_0\left[\left(2\sqrt{\bar{y}-\frac{1}{2}}\right)\gamma\left(\frac{1}{2},\bar{y}\right)+2\int_{\frac{1}{2}}^{\bar{y}}\sqrt{\bar{y}-y}\frac{\partial\gamma}{\partial y}\,\mathrm{d}y\right]$$

Under this form, we clearly see that the expression is differentiable. At $b_0 = 0$, its differential reads

$$2\sqrt{2}\mathrm{d}b_0(.)\int\limits_{\frac{1}{2}}^{1}\frac{\cos\left(\frac{\pi}{4}+\pi y\right)}{y\sqrt{1-y}}w(y,1)\,\mathrm{d}y$$

The differential of $R_1$ is, therefore,

$$-\mathrm{d}b_0(.)\left(4+\int\limits_{\frac{1}{2}}^{1}\frac{\cos\left(\frac{\pi}{4}+\pi y\right)}{y\sqrt{1-y}}w(y,1)\,\mathrm{d}y\right)$$

$\cos(\frac{\pi}{4}+\pi y)$ is negative in $[\frac{1}{2},1]$. $w(y,1)$ is negative also. The differential of $R_1$ is, therefore, non-zero and Lemma 4.4 follows for all points $z$ on the $v$-orbits with $b=0$ such that the map $z \to z_0$ is differentiable.

A special argument needs to be made for the points on the circles $r_1 = 0$, $r_1 = 1$ since the map generated by the one-parameter group of $v$ from these points to the corresponding points $z_0$ on $T_0$ is not differentiable in an obvious way. These points are also similar to the points having $a = 0$, the maxima and minima of $y$ on a $v$-orbit, when $b$ is non-zero.

We have not yet proved the conclusion of Lemma 4.4 for these points. Let us proceed with this now and see how the argument can be adjusted to cover also the case of the points on the circles $r_1 = 0$, $r_1 = 1$.

Assuming that $b \neq 0$ on a $v$-orbit, we consider a point $z$ on this $v$-orbit where $a = 0$, and we then introduce the next such point $\psi(z)$, with $r_2(\psi(z)) = r_2(z)$. We assume that $(z, \psi(z))$ is a conjugate pair.

We claim that $\mathrm{d}\psi_z(Y)$ is then $Y$. Indeed, $\psi$ is generated by the one-parameter group of $v$ (properly parametrized). Since $z$ and $\psi(z)$ are conjugate,

$$\mathrm{d}\psi_z(Y) = \theta Y + \theta_1 X.$$

On the other hand, $a(\psi(x)) = a(x) = 0$. Span$\{Y, X_0\}$ is the space tangent to $\{a = 0\}$. Thus, $\mathrm{d}\psi_z(Y)$ is in Span$\{Y, X_0\}$. If $z$ is not in $T_0$—which we can assume—$\theta_1$ must be zero.

On the other hand,

$$r_2(\psi(x)) = r_2(x)$$

This implies that $\theta = 1$, that is $\mathrm{d}\psi_z(Y) = Y$

Let $\gamma$ be the $v$-generated map from $T_0$ into $\{x; a(x) = 0; r_2(x) \gtrsim \frac{1}{2}\}$. $\gamma$ is well defined and it is differentiable near $z_0$ if $v$ is not tangent to $T_0$ at $z_0$. Indeed, let $T(z_0)$ be the time along $v$ from $z_0$ to $z$.

$$\begin{aligned}T(z_0) &= c\int\limits_{\frac{1}{2}}^{\bar{y}}\frac{\mathrm{d}y}{\sqrt{k(y)-k(\bar{y})}}\\ &= c\int\limits_{\frac{1}{2}}^{\bar{y}}\frac{\mathrm{d}y}{\sqrt{\bar{y}-y}}w(y,\bar{y})\end{aligned}$$

$w(y, \bar{y})$ is a smooth function for $\bar{y} \neq \frac{1}{2}$.

Integrating per parts,

$$T(z_0) = 2c\sqrt{\bar{y}-\frac{1}{2}}\,w\left(\frac{1}{2},\bar{y}\right) + 2c\int\limits_{\frac{1}{2}}^{\bar{y}}\sqrt{\bar{y}-y}\frac{\partial w}{\partial y}\,\mathrm{d}y$$

Under this form, we see that $T$ depends in a differentiable way on $\bar{y}$ ($C^0$). Furthermore, $a^2(z_0) + b^2(z_0) = \bar{y}(1-\bar{y})$.

Thus, for $\bar{y} \gneqq \frac{1}{2}$, $\bar{y}$ is a differentiable function of $z_0$. $\gamma$ is, therefore, a continuously differentiable function of $z_0$ and it also has a differentiable inverse.

Let

$$w = \mathrm{d}\gamma_z^{-1}(Y) \in T_{z_0}T$$

Let $\theta_s$ be the one-parameter group of $X_0$. $z_1$ and $z_0$ have the same $a$ and the same $b$. They are, therefore, on the same $X_0$-orbit; $z_1 = \theta_{s_0}(z_0)$.

$w$ at $z_0$ reads $w = a_1\zeta + b_1 X_0$, $a_1 \neq 0$.

We have ($X_0$ commutes to $v$):

$$\theta_{s_0}(z_0) = z_1; \mathrm{d}\theta_{s_0}(w) = a_1\zeta + b_1 X_0$$

Observe that $\theta_{s_0}(z) = \psi(z)$, $\mathrm{d}\theta_{s_0,z}(Y) = Y(\psi(z)))$.

Using the commutation relation between $X_0$ and $v$, we then have

$$\gamma(z_1) = \gamma \circ \theta_{s_0}(z_0) = \theta_{\tilde{s_0}} \circ \gamma(z_0)$$

$\tilde{s_0}$ is the time along $X_0$ from $z$ to $\psi(z)$. We observed earlier that

$$s_0(z_0) = \tilde{s_0}(z)$$

Thus,

$$\mathrm{d}\gamma_{z_1}(\mathrm{d}\theta_{s_0}(w) + \mathrm{d}s_0(w)X_0) = \mathrm{d}\theta_{\tilde{s_0}}(\mathrm{d}\gamma_{z_0}(w)) + \mathrm{d}\tilde{s_0}(\mathrm{d}\gamma_{z_0})X_0$$

We know that $\mathrm{d}s_0(w) = \mathrm{d}\tilde{s_0}(\mathrm{d}\gamma_{z_0})$, $\mathrm{d}\gamma_{z_1}(X_0) = X_0$.

$X_0$ is pure rotation and $X_0.a = 0$. Thus, we find

$$\mathrm{d}\gamma_{z_1}(\mathrm{d}\theta_{s_0}(w) = \mathrm{d}\theta_{\tilde{s_0}}(\mathrm{d}\gamma_{z_0}(w)) = \mathrm{d}\theta_{\tilde{s_0}}(Y) = Y$$

Considering then $z_1 = \tilde{\psi}(z_0) = \gamma^{-1} \circ \psi \circ \gamma(z_0)$, we find that

$$\delta z_1 = \mathrm{d}\tilde{\psi}(\delta z_0) = \mathrm{d}\gamma^{-1} \circ \mathrm{d}\psi \circ \mathrm{d}\gamma(\delta z_0)$$

With $\delta z_0 = w$:

$$\delta z_1 = \mathrm{d}\gamma^{-1}(Y) = w$$

and the claim follows since $\mathrm{d}\tilde{\psi}(X_0) = X_0$, $\mathrm{d}\tilde{\psi}(v) = v$.

We conclude the proofs of Lemmas 4.3 and 4.4 with the case $b = 0$, $a = 0$. Let us consider the case, e.g. $r_1 = 0$. $v$ at such a point is $x_3\partial_{x_1} + x_4\partial_{x_2}$. If $\ker\alpha$ is mapped onto itself from $z$ on $r_1 = 0$ onto $\psi(z)$ again on $r_1 = 0$, then, since $\ker\alpha = \mathrm{Span}\{\partial_{x_1}, \partial_{x_2}\}$, we know that

$$\mathrm{d}\psi_z(-x_4\partial_{x_1} + x_3\partial_{x_2}) = A_1\partial_{\tilde{x_1}} + B_1\partial_{\tilde{x_2}}$$

We know that $r_2$, $a$ and $b$ are unchanged under $\psi$.

Since $\mathrm{d}a(-x_4\partial_{x_1} + x_3\partial_{x_2}) = 0$ and $\mathrm{d}b(-x_4\partial_{x_1} + x_3\partial_{x_2}) = -x_4^2 - x_3^2 = -r_2$,

$$\mathrm{d}a(A_1\partial_{\tilde{x_1}} + B_1\partial_{\tilde{x_2}}) = A_1\tilde{x_3} + B_1\tilde{x_4} = 0$$
$$\mathrm{d}b(A_1\partial_{\tilde{x_1}} + B_1\partial_{\tilde{x_2}}) = A_1\tilde{x_4} - B_1\tilde{x_3} = -\tilde{x_3}^2 - \tilde{x_4}^2$$

Thus,

$$A_1 = -\tilde{x_4}, \quad B_1 = \tilde{x_3}$$

and

$$\mathrm{d}\psi_z(-x_4\partial_{x_1} + x_3\partial_{x_2}) = -\tilde{x_4}\partial_{\tilde{x_1}} + \tilde{x_3}\partial_{\tilde{x_2}}$$

The previous argument extends then verbatim. □

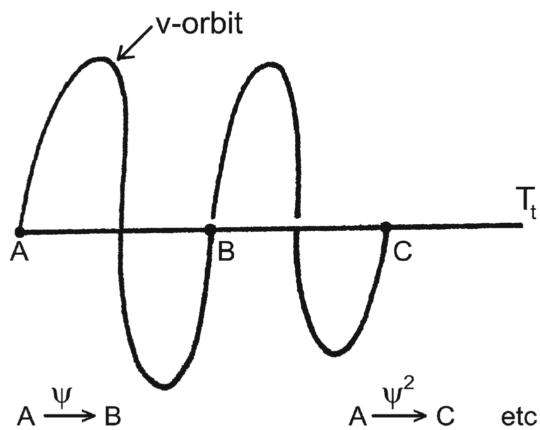
4.2 The case when the number of inside zeros of $a$ is even $\gneq 2$

$\mathrm{d}\psi_n$ denotes in what follows the differential of the composition $\psi^n$ of $\psi$ with itself $n$-times.

When there are more than 2, but an even number $m = 4, 6, 8$, etc. of zeros of $a$ separating the conjugate points, the conclusion that $\mathrm{d}\psi_{\frac{m}{2}}$ is the Id remains unchanged. We can add

**Lemma 4.5** *Let* $m \neq 0$. $d\psi_m = \mathrm{Id} \Rightarrow d\psi = \mathrm{Id}$ *so that there are no additional conjugate points for* $m = 4, 6, 8$ *with respect to* $m = 2$.

Proof of Lemma 4.5



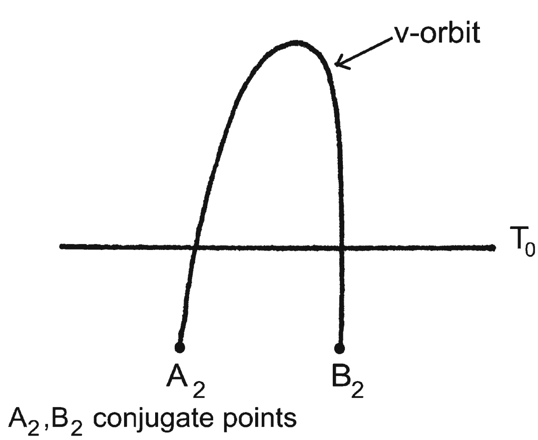

F2

If $\mathrm{d}\psi_m (m \neq 0)$ is equal to Id anywhere on a $v$-orbit, it is then equal to Id at any other point of the same $v$-orbit. This is derived as in Lemma 4.2. $\psi_m$, from $T_0$ to $T_0$, is derived from $\psi$ by iteration. Considering such a point on $T_0$ and setting $\mathrm{d}\psi(\xi) = \xi + \mu X$, we find that $\mathrm{d}\psi_m = \xi + m\mu X$ ($X = X_0$ on $T_0$), so that $\mathrm{d}\psi_m = \mathrm{Id}$ implies that $\mu = 0$ as claimed. □

We have studied above conjugate points separated by an even number of zeros of $a$. This case reduced, as we have shown above, to the case of exactly one inside crossing.

4.3 The case of conjugate points separated by exactly one zero of $a$:

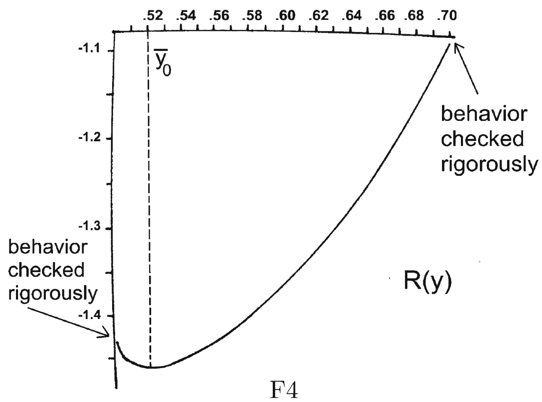
We study now conjugate points that are separated by exactly one zero of $a$:




F3

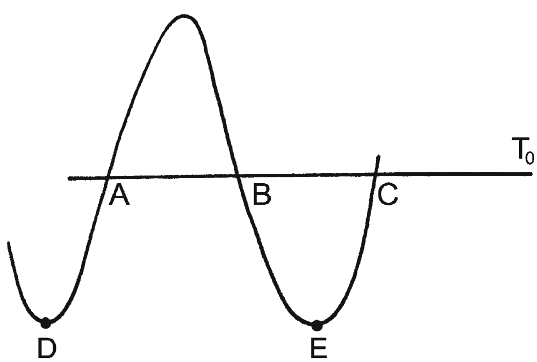
The more general case of conjugate points separated by an odd, non-zero number of zeros of $a$ is not studied here; however, the general behavior can de understood from our arguments given below.

Let $\bar{y}_0$ be the critical value of $R(\bar{y})$, $\bar{y}_0 \gneq \frac{1}{2}$, corresponding to conjugate points as in Lemmas 4.3 and 4.4 above, with $k = 1$ ($2\pi$-rotation, two zeros of $a$ separating the conjugate points[5]).




F4

Let us consider a flow-line of $v$ crossing $T_0$ at three consecutive points $A, B, C$. Assume that, e.g. $a$ is negative at $A$. Let $D$ be the first point on this $v$-orbit *before* $A$ such that $a$ is zero at $D$ and let $E$ be the first point *after* $B$ such that $a$ is zero at $E$.




F5

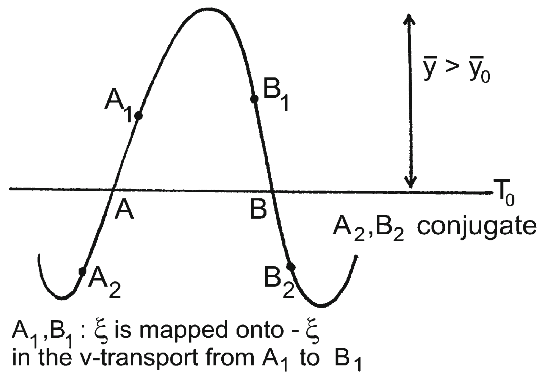
Let $\theta$ be the map from $T_0$ to $T_0$ that assigns to $A$ the point $B$ and let $\psi$ be the map that assigns to the point $A$ the point $C$.

We then claim that

**Lemma 4.6** *Assume that $\bar{y} \gneq \bar{y}_0$ on this $v$-flow-line. Then,*

(i) *There exists $\delta \gneq 0$ such that* $\mathrm{d}\theta(\xi) = -\xi - \frac{\delta^2}{2}X$, $\mathrm{d}\psi(\xi) = \xi - \delta^2 X$
(ii) *There are two points $A_1$, $B_1$ on the same torus $T_t$ in $r_2 \gneq r_1$ that are on the portion of $v$-orbit from $A$ to $B$ such that $\beta = \mathrm{d}\alpha(v, .)$ is mapped onto $-\beta$ and $\xi$ is mapped onto $-\xi$ in the $v$-transport from $A_1$ to $B_1$.*
(iii) *There are two points $A_2$, $B_2$ on the same torus $T_t$ in $r_1 \gneq r_2$, $A_2$ on the portion of $v$-orbit from $v$-orbit from $D$ to $A$ and $B_2$ on the portion of $v$-orbit from $B$ to $E$ that are conjugate.*

[5] Would there be more critical points of $R(y)$ than the single one indicated by the graph below, our proofs would only be slightly modified. The modifications are minor: the torus $T^1$ that we will choose below should have $r_2(T^1) \gneq$ largest critical value of $\bar{y}$.




*F6*

*Proof of Lemma* 4.6 The map $\psi$ reads from $T_0$ to $T_0$:

$$\psi(z_0, z_1) = (\mathrm{e}^{iR(\bar{y})}z_0, \mathrm{e}^{iR(\bar{y})}z_1) = (z, z_1)$$

We know that $R(\bar{y})$ has a unique critical point $\bar{y}_0$ (see footnote 5, above) in $r_2 \gtrapprox r_1$. Thus, we can prove (i) at $\bar{y}$ close or equal to 1, the result will follow at any other $\bar{y} \gtrapprox \bar{y}_0$.

Taking $\bar{y}$ close, not equal to 1,

$$\mathrm{d}\psi(\xi) = \xi + i\frac{\partial R}{\partial \xi}(z, z_1)$$

At $\bar{y} = 1$, we have computed the differential of $R$ in Lemma 4.4. We have found that this differential is

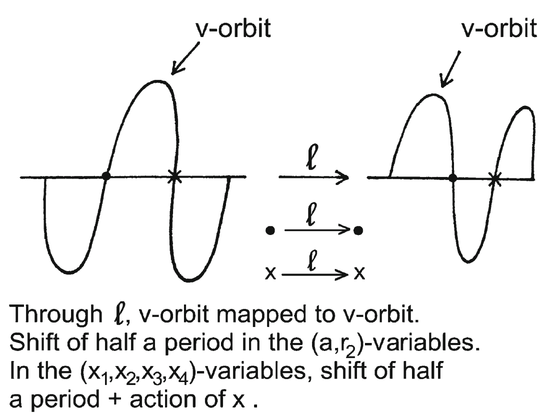
$$-\mathrm{d}b_0(.)\left(4 + \int_{\frac{1}{2}}^{1} \frac{\cos\left(\frac{\pi}{4} + \pi y\right)}{y\sqrt{1-y}} w(y, 1)\,\mathrm{d}y\right)$$

Observe that $i(z, z_1)$ reads $-X_0$ and observe that $\mathrm{d}b_0(\xi) = \frac{(\tilde{A}-\tilde{B})a}{\alpha(\zeta)}$ is negative at $A$ since we are assuming that the $v$-orbit starting from $A$ enters into $r_2 \gtrapprox r_1$. Thus $i\frac{\partial R}{\partial \xi}(z, z_1)$ reads as $-\delta^2 X = -\delta^2 X_0$ and the claim about $\mathrm{d}\psi$ in (i) follows.

Assume now that $\mathrm{d}\theta(\xi) = -\xi + \nu v$. We want to compute $\psi$ using $\theta$. Let $\ell$ be the transformation of $S^3$ defined by

$$\ell : (x_1, x_2, x_3, x_4) \to (x_3, x_4, -x_1, -x_2)$$

We prove in Lemma 5.1 below that $\ell(v\text{-orbit}) = v$-orbit. The new $v$-orbit derived under $\ell$ has the same $\bar{y}$, etc., but it is shifted by half a period in the $(a, b, r_1)$-variables.



We can, therefore, find a map $\Gamma$, defined with the use of the one-parameter group of $X_0$ such that $\Gamma \circ \ell$ maps our $v$-orbit onto itself, with a shift forward of exactly half a period. Then, the map from $A$ to $B$ as well as the map from $B$ to $C$ reads

$$\theta = (\Gamma \circ \ell), \quad \theta_1 = \Gamma \circ \ell$$

and $\psi$ reads

$$\psi = \theta_1 \circ \theta$$

Observe that because $\Gamma$ is generated by $X_0$, we can write in a $(\xi, v, X_0)$-frame that

$$\mathrm{d}\Gamma = \mathrm{Id} + \mathrm{d}c(.)X_0$$

$c$ is a suitable differentiable function and $\mathrm{d}c(X_0) = 0$.

We have to distinguish the two $\mathrm{d}\Gamma$s, the one involved in $\mathrm{d}\theta$ from the one involved in $\mathrm{d}\theta_1$. $\mathrm{d}c$ for the first one is denoted $\mathrm{d}c_-$, whereas $\mathrm{d}c$ for the second one is denoted $\mathrm{d}c_+$. Since $\mathrm{d}\theta$ maps $\xi$ on $-\xi + \nu X_0$, we know that $\mathrm{d}c_+(\xi) = \mathrm{d}c_-(-\xi)$. We know that $\mathrm{d}\ell(X_0) = X_0$, $\mathrm{d}\ell(\xi) = -\xi$. Therefore,

$$\mathrm{d}\psi = \mathrm{d}\theta_1 \circ \mathrm{d}\theta = (\mathrm{Id} + \mathrm{d}c_+(.)X_0) \circ \mathrm{d}\ell \circ (\mathrm{Id} + \mathrm{d}c_-(.)X_0) \circ \mathrm{d}\ell$$

and $\mathrm{d}\psi(\xi)$ is, therefore,

$$\begin{aligned}
&(\mathrm{Id} + \mathrm{d}c_+(.)X_0) \circ \mathrm{d}\ell(-\xi - \mathrm{d}c_-(\xi)X_0) \\
&\quad = (\mathrm{Id} + \mathrm{d}c_+(.)X_0)(\xi - \mathrm{d}c_-(\xi)X_0) \\
&\quad = \xi - \mathrm{d}c_-(\xi)X_0 + \mathrm{d}c_+(\xi)X_0 = \xi - 2\mathrm{d}c_-(\xi)X_0
\end{aligned}$$

(i) follows.

We now establish (ii) and (iii).

In order to prove (ii), that is the existence of $A_1$, $B_1$, we simply prove that $\xi$ turns, if $\bar{y} \gneq \bar{y}_0$, more than $2\pi$ from $A$ to $B$.

For this, we take $-\xi - \frac{\delta^2}{2}X = -\xi - \frac{\delta^2}{2}X_0$ at $B$ and we transport it "backwards" along the $v$-orbit. we prove that the component of the transported vector on $X_0$ increases. If $\delta^2$ is small enough (if $\bar{y} \gneq \bar{y}_0$ is close enough to $\bar{y}_0$), the existence of a point $B'$ on the portion of $v$-orbit between $A$ and $B$ such that $\xi$ maps onto $-\lambda^2\xi$ in the $v$-transport between $A$ and $B'$ follows and, by a continuity and connectedness argument, (ii) follows.

The backwards transport equations along $v$ in the $(\xi, w = -[\xi, v] + \bar{\mu}\xi, v)$-frame [2,4] ($w$ is the Reeb vector field of $\beta$, $\eta$ is the component of the transported vector on $w$, $\lambda$ on $\xi$) yield

$$\dot{\eta} = \lambda, \quad \dot{\overline{\lambda + \bar{\mu}\eta}} = -\eta$$

Transporting backwards $-\xi$, we find that $\eta = -\epsilon + O(\epsilon^2)$, $\lambda + \bar{\mu}\eta = -1 + O(\epsilon^2)$ for $\epsilon \gneq 0$ small. Observe that, near $B$, $a$ is positive, so that $\mathrm{d}\alpha(v, X_0) = v.\alpha(X_0) \lneq 0$. Since $\mathrm{d}\alpha(v, [\xi, v]) = -1$, $[\xi, v]$ has a positive component on $X_0$ in the $(\xi, v, X_0)$ frame. In fact, the backwards transported vector $z$ reads transversally to $v$:

$$-\xi + \epsilon[\xi, v] + O(\epsilon^2) = (-1 + O(\epsilon))\xi + C\epsilon X_0 + O(\epsilon^2)$$

with $C \gneq 0$. The claim follows.

We move now to prove (iii). The reference figure is F5, with two additional points: $A'$ on the portion of $v$-orbit between $D$ and $A$, close to $A$; and $C'$ on the portion of $v$-orbit between $C$ and $E$, close to $C$. $A'$ and $C'$ are on the same torus $T_t$.

We claim that ker$\alpha$ has turned more than $2\pi$ between $A'$ and $C'$ and that this establishes (iii), that is, this establishes the existence of $A_2$, $B_2$. Indeed, let then $C''$ be the point, very near $C$ if $A'$ is very near $A$, such that ker$\alpha$ has turned $2\pi$ between $A'$ and $C''$. $C''$ is on a torus $T_{t'}$ *below* $T_t$ (that is $r_2(C'') \lneq r_2(A')$).

On the other hand $X_0$ is $v$-transported; it is in Span$\{\xi, v\}$ ($a = 0$) at $D$ and $E$. Therefore, $\xi$ is $v$-transported parallel to itself between $D$ and $E$. In fact, $\xi$ is $v$-transported onto itself between $D$ and $E$ since $(a, b)$ is mapped onto itself after a "period". Differentiating, we find the collinearity coefficient to be 1.

Since $\xi$ is mapped onto itself between $D$ and $E$, ker$\alpha$ cannot be mapped onto itself by the $v$-transport map between $D$ and $E$; otherwise, using a continuity argument, $Y$ would map into Span$\{X, Y\}$. Using then the fact that $(a, b)$ is mapped onto itself after one period, differentiating with the use of the formulae of Sect. 2 (we can assume that $b$ is not zero on this flow-line and derive the general result by continuity from the result under this restriction), we find that $Y$ maps on $Y$. The differential of the $v$-transport map would then be Id, which is impossible since $\bar{y} \gtrsim \bar{y}_0$.

Using a continuity argument, the existence of $(A_2, B_2)$ follows.

In order to see that ker$\alpha$ has turned more than $2\pi$ between $A'$ and $C'$, we consider a vector $X' = X_0 - \alpha_{A'}(X_0)\xi$ in ker$\alpha$ at $A'$. We $v$-transport it to $A$. Let $\epsilon$ be the time along $v$ from $A'$ to $A$.

Using the transport equations of $\xi$ ($X_0$ is $v$-transported), see above, we find that the $v$-transport of $X'$ at $A$ reads transversally to $v$:

$$\begin{aligned} &X_0 + \alpha_{A'}(X_0)[C\epsilon X_0 - \xi + O(\epsilon^2)] \\ &\quad = (1+\alpha_{A'}(X_0)C\epsilon)X_0 - \alpha_{A'}(X_0)[\xi + O(\epsilon^2)] \\ &\quad = (1+\alpha_{A'}(X_0)C\epsilon)\left[X_0 - \frac{\alpha_{A'}(X_0)[\xi + O(\epsilon^2)]}{1+\alpha_{A'}(X_0)C\epsilon}\right] \\ &\quad = (1+\alpha_{A'}(X_0)C\epsilon)F \end{aligned}$$

We $v$-transport $F$ from $A$ to $C$. The image is easy to find using (i). It is

$$\begin{aligned} &X_0 - \frac{\alpha_{A'}(X_0)[\xi - \delta^2 X_0 + O(\epsilon^2)]}{1+\alpha_{A'}(X_0)C\epsilon} \\ &= \left(1 + \frac{\alpha_{A'}(X_0)\delta^2}{1+\alpha_{A'}(X_0)C\epsilon}\right)\left(X_0 - \frac{\alpha_{A'}(X_0)[\xi + O(\epsilon^2)]}{(1+\alpha_{A'}(X_0)C\epsilon)\left(1 + \frac{\alpha_{A'}(X_0)\delta^2}{1+\alpha_{A'}(X_0)C\epsilon}\right)}\right) \\ &= \left(1 + \frac{\alpha_{A'}(X_0)\delta^2}{1+\alpha_{A'}(X_0)C\epsilon}\right)G \end{aligned}$$

Observe now that $\alpha_{A'}(X_0) = \alpha_{C'}(X_0)$, both are negative and that $\epsilon'$, the time along $v$ from $C'$ to $C$ is $\epsilon$ (both are positive).

Therefore, the component of $G$ along $\xi$ is strictly larger than the component of $F$ along $\xi$ for $\epsilon$ small. $G$ can be viewed as the transport of a vector parallel to $X$, $X'$ as above, starting from a point $C''$ preceding (close to) $C$ on the $v$-orbit. Comparing the components of $F$ and $G$, we see that $C''$ must come from a torus $T_{t''}$ *below* the torus $T_{t'}$ of $A'$, $C'$. □

## 5 The symmetry of the $v$-orbits and the symmetry of the vectors transported along them

$\alpha$ reads

$$-\frac{\sqrt{2}}{2}[\cos(\pi y)(x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4) + \sin(\pi y)(x_4 dx_3 - x_3 dx_4 - x_2 dx_1 + x_1 dx_2)$$

Let us consider the transformation of $S^3$, as above, defined by

$$\ell : (x_1, x_2, x_3, x_4) \to (x_3, x_4, -x_1, -x_2)$$

We claim that

**Lemma 5.1** *$\ell^*(\alpha) = -\alpha$, $d\ell(v) = v$, $d\ell(\zeta) = -\zeta$, $d\ell([\xi, v]) = -[\xi, v]$ (with $\xi = \frac{\zeta}{\alpha(\zeta)}$, the Reeb vector field of $\alpha$).*

We defer for the time being the proof of Lemma 5.1 and we move to understand the behavior of $v$-transported vectors along pieces of $\pm v$-orbit:

Let us consider a piece $x_s$ of $\pm v$-orbit from $z_0$ in $T_0$, crossing again, after $z_0$, $T_0$ at $z_1$ (next intersection point) and again, after $z_1$, $T_0$ at $z_2$. Let $[0, s_1]$ be the time interval of $[z_0, z_1]$ and $[0, s_2]$ be the time interval of $[z_1, z_2]$.

We can define a map from $[0, s_1]$ into $[0, s_2]$ as follows:

**Definition 5.2** Given $s \in [0, s_1]$, $\tau(s) \in [0, s_2]$ is the unique time such that if $x_s$ belongs to the torus $T_t$, then $x_{\tau(s)}$ belongs to $T_{1-t}$ and $a$ at $x_{\tau(s)}$ is equal to $-a$ at $x_s$.

$\tau(s)$ is recognized below to be equal to $s + s_1$.

Let also $z_1', z_2'$ be two consecutive points on this piece of $\pm v$-orbit belonging to the same torus $T_t$. Assuming for example that $T_t$ lies in $r_2 \gneq r_1$, the piece of $\pm v$-orbit comes from $T_0$, intersects $T_t$ at $z_1'$ one first time, then "moves deeper" in $r_2 \gneq r_1$ before coming back to intersect $T_t$ at $z_2'$. Let $s_1'$ be the time along this piece of $\pm v$-orbit for $z_1'$, $s_2'$ be the time for $z_2'$.

Let $z(s)$ be a $v$-transported vector along $x_s$. Let $\eta(s)$ be the $[\xi, v]$-component of $z(s)$, $\eta_0(s)$ be this component when $z(s) = X_0$.

We then claim that

**Lemma 5.3** (i) $\eta_0(s_1') = -\eta_0(s_2')$
(ii) *There is a constant $\mu$ which might depend on the transported vector $z$, but does not depend on $s$ once $z = z(s)$ is given such that $\eta(\tau(s)) = -\eta(s) + \mu\eta_0(\tau(s))$.*

*Proof of Lemmas 5.1 and 5.3* In order the see that $d\ell(v) = v$, we take $v$ under the form $aY + bX$. We observe $a(\ell(x)) = -a(x)$, $b(\ell(x)) = b(x)$. We then observe that $\ell$ exchanges $r_1$ and $r_2$ and transforms $x_1\partial x_1 + x_2\partial x_2$ into $x_3\partial x_3 + x_4\partial x_4$ and vice-versa. Thus, $d\ell(Y) = -Y$. On the other hand, $x_2\partial x_1 - x_1\partial x_2$ transforms into $x_4\partial x_3 - x_3\partial x_4$ and vice-versa, whereas $\sin(\frac{\pi}{4} + \pi(1-y)) = -\cos(\frac{\pi}{4} + \pi y)$ and $\cos(\frac{\pi}{4} + \pi(1-y)) = -\sin(\frac{\pi}{4} + \pi y)$, so that $d\ell(X) = X$. It follows that $d\ell(aY + bX) = aY + bX$; $d\ell(v) = v$, as claimed.

Using the formula for $\alpha$, we derive easily that $\ell^*\alpha = -\alpha$: Indeed,

$$\cos(\pi(1-y)) = -\cos(\pi y), \sin(\pi(1-y)) = \sin(\pi y),$$

whereas

$$\ell^*(x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4) = x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4$$

and

$$\ell^*(x_4 dx_3 - x_3 dx_4 - x_2 dx_1 + x_1 dx_2) = x_2 dx_1 - x_1 dx_2 - x_4 dx_3 + x_3 dx_4$$

Thus, $d\ell(\xi) = -\xi$. Combined with $d\ell(v) = v$, this yields

$$d\ell([\xi, v]) = -[\xi, v]$$

Finally, $\ell$ changes $A$ into $-B$ and $B$ into $-A$. It also changes $r_2$ into $r_1$ and vice-versa. Thus, $\tilde{A}$ is changed into $-\tilde{B}$ and $\tilde{B}$ is changed into $-\tilde{A}$. Thus, $\zeta$ turns into $-\zeta$, as claimed. This concludes the proof of Lemma 5.1.

For (i) of Lemma 5.3, we compare the projections at $z_1'$ and at $z_2'$ of $[\xi, v]$ on the tangent space to the torus $T_t$, parallel to $v$.

We have $[\xi, v] = [\xi, aY + bX] = \xi.aY + \xi.bX + a[\xi, Y] = \xi.a\frac{1}{a}v + \xi.bX + a[\xi, Y] - \xi.a\frac{b}{a}X$. Observe now that $\xi.a = -\frac{(\tilde{A}-\tilde{B})b}{\alpha(\zeta)}$ so that $\xi.a\frac{b}{a} = -\frac{(\tilde{A}-\tilde{B})b^2}{a\alpha(\zeta)}$ and observe that $\xi.b = \frac{(\tilde{A}-\tilde{B})a}{\alpha(\zeta)}$.

Since $b(z_1') = b(z_2')$ and $a(z_1') = -a(z_2')$, the projection of $[\xi, v]$ at $z_2'$ reads $-a_1\xi - b_1 X$ if the projection of $[\xi, v]$ at $z_1'$ reads $a_1\xi + b_1 X$. $a_1$ is zero since $[\xi, v]$, hence its projection (which is completed parallel to $v$, $v$ is in $\ker\alpha$), is in $\ker\alpha$. Thus, the first projection reads $b_1 X$, whereas the second one reads $-b_1 X$. (i) follows readily.

For (ii), we observe that, by construction and by symmetry (for $b$: $a^2 + b^2 = y(1-y)$), $a(x_s) = -a(x_{\tau(s)})$, $b(x_s) = b(x_{\tau(s)})$. Therefore,

$$a(\ell(x_s)) = a(x_{\tau(s)}), \quad b(\ell(x_s)) = b(x_{\tau(s)})$$

It follows that $\ell(x_s)$ and $x_{\tau(s)}$ are on the same $X_0$-orbit. Since $X_0$ and $v$ commute, the time required to go from $\ell(x_s)$ to $x_{\tau(s)}$ along $X_0$ does not depend on $s$. If $\theta_t$ is the one-parameter group of $X_0$, we thus have

$$x_{\tau(s)} = \theta_{s_0}(\ell(x(s))(*)$$

Observe that the above formula implies, since $d\ell(v) = v$, $\ell(z_0) = z_1$ and since $X_0$ and $v$ commute, that $\tau(s) = s + s_1$ as claimed above.

$s_0$ in $(*)$ does not depend on the time $s$ of the $v$-orbit. However, $s_0$ depends on the $v$-orbit. When this $v$-orbit changes, $s_0$ changes as a differentiable function of the $v$-orbit, a function that is a constant on each of these $v$-orbits.

Since we are assuming that the vector $z = z(s)$ is $v$-transported, we can differentiate the above formula along $z$. Observe that $\mathrm{d}x_{\tau(s)}(v) = v$. Therefore, the map

$$g : x_s \to x_{\tau(s)}$$

commutes to the one-parameter group of $v$. Denoting $\phi_s$ this one-parameter group of $v$,

$$\mathrm{d}g \circ d\phi_s = \mathrm{d}\phi_s \circ \mathrm{d}g$$

On the other hand, because $z(s)$ is $v$-transported and because $g$ is generated by the one-parameter group of $v$, with $z_1 = \phi_{s_1}(z_0)$,

$$z(s_1) = \mathrm{d}g(z(0)) + \gamma(v)$$

The introduction of the additional component along $v$ follows from the fact that $z(0)$ need not be tangent to the tori $T_t$. It also follows from the fact that the time along $v$ does not identify with $r_1(s)$ or $r_2(s)$. It can be recognized, if we view $g$ as $x_{s+s_1}$, as the contribution of $\mathrm{d}s_1(.)v$.

Using then

$$\mathrm{d}g \circ \mathrm{d}\phi_s = \mathrm{d}\phi_s \circ \mathrm{d}g$$

or using in a direct proof and in a simpler way the fact that $g(x_s) = x_{s+s_1}$, we derive that

$$\mathrm{d}g(z(s)) = z(\tau(s)) + \gamma v$$

Differentiating $(*)$ and using the above identity, we derive

$$\mathrm{d}g(z(s)) = z(\tau(s)) + \gamma v = \mathrm{d}\theta_{s_0}(\mathrm{d}\ell(z(s))) + \mathrm{d}s_0(z)X_0 + \gamma v$$

Observe now that $X_0$ commutes to $v, \xi, [\xi, v]$. Thus, $\mathrm{d}\theta_{s_0}([\xi, v]) = [\xi, v]$ and (ii) of Lemma 5.3 follows using Lemma 5.1 and the fact that $\mathrm{d}s_0(z)$ is constant since $z$ is $v$-transported along a piece of $\pm v$-orbit. Lemma 5.3 is thereby established. □

## 6 Linking

The flow of [2,4] is made of two distinct parts. The first part is generated by a semi-flow $\frac{\partial x}{\partial s} = Z(x)$, where $x$ is a curve of the space of Legendrian "dual" curves $C_\beta$. The second part is the "flow at infinity", on $\cup\Gamma_{2k}$. We will find this dichotomy here again.

$Z(x)$ can be represented along the curve $x$ with the use of coordinate functions $\lambda, \mu, \eta$ using the moving frame defined by $\xi, v, w$ ($w = [\xi, v] + \bar{\mu}\xi$ is the Reeb vector field of $\beta$).

$$Z(x) = \lambda\xi + \mu v + \eta w$$

$\dot{x}$ reads as $a\xi + bv$ and $Z(x)$ enjoys the property that

$$b\eta \geq 0$$

This property implies the following:

**Lemma 6.1** *Let $x(s, .)$ be a solution of $\frac{\partial x}{\partial s} = Z(x)$. The linking number of the curve $x(s, .)$ with a given periodic orbit of $\xi$ $O$ never decreases along the flow-lines of $Z$.*

There is another pseudo-gradient for the functional defined by $J(x) = \int_0^1 \alpha_x(\dot{x})\,\mathrm{d}t$ on the space $C_\beta$. This (semi)-flow is defined by the differential equation $\frac{\partial x}{\partial s} = Z_0(x)$ and the $[\xi, v]$-component of $Z_0$ reads as $b$. $b$ is the $v$-component of $\dot{x}$. For $Z_0$, the linking number of any two families of solutions $x_1(s, .)$, $x_2(s.)$ of $\frac{\partial x}{\partial s} = Z_0(x)$ never decreases when $s$ increases. One might ask why we do not use $Z_0$ instead of $Z$. The reason is that the curves under the evolution equation defined by $Z_0$ blow up "too often" and their behavior at the blow-up time is not characterized by simple models. We were thinking that the choice of $Z$ in lieu of $Z_0$ involved some "trade" along which the behavior at the blow-up time was improved, whereas the linking property has to be lost. It turns that we do not have to entirely give up this linking property when we choose $Z$ instead of $Z_0$, as indicated by Lemma 6.1 above.

*Proof of Lemma* 6.1 Assume that $x_1$ is a periodic orbit bounding a surface $c$, $\partial c = x_1$. We orient $c$ near its boundary by $\dot{x}_1$ (parallel to $\xi$) and the outgoing normal. This gives an orientation of $c$.

Assume that a flow-line of our differential equation "touches" $x_1$ at some time $s_0$. Let us then denote $x_2 = x(s_0, .)$. We may assume that $\dot{x}_1$ and $\dot{x}_2$ are independent at the "touching" time $t_0$ and that $(\dot{x}_1, \dot{x}_2, \frac{\partial x_2}{\partial s})(t_0)$ is an independent frame. We then compute ($a_1$ and $a_2$ are the $\xi$-components of $\dot{x}_1, \dot{x}_2$, respectively. $b_2$ is the $v$-component of $\dot{x}_2$):

$$\alpha \wedge \mathrm{d}\alpha \left(\dot{x}_1, \frac{\partial x_2}{\partial s}, \dot{x}_2\right)(t_0) = \frac{a_1}{a_2}(s_0)(b_2\eta)(s_0, t_0)\alpha \wedge \mathrm{d}\alpha(\xi, w, v)$$

This is negative because $b_2\eta$ is positive.

Thus, if $x(s, .)$ "enters" $c$ over $s_0$, $\frac{\partial x_2}{\partial s}(t_0)$ defines the negative normal. The linking number increases. If, on the other hand, $x(s, .)$ "leaves" $c$ over $s_0$, $\frac{\partial x_2}{\partial s}(t_0)$ defines the positive normal and the linking number increases again. □

We now establish a formula for the linking of two periodic orbits. This formula reads as follows:

**Lemma 6.2** *Let $y_1$ and $y_2$ be two periodic orbits such that $r_2(y_1) \gneqq r_2(y_2)$. Assume that the total algebraic rotation (clockwise, in the $(x_3, x_4)$-frame) over $y_1$ of the vector $(x_3, x_4)$ is $q_1$ and that the total algebraic rotation of $(x_1, x_2)$ over $y_2$ (counter-clockwise, in the $(x_1, x_2)$-frame) is $p_2$. Then, the linking number of $(y_1, y_2)$, link$(y_1, y_2)$is equal to $q_1 p_2$.*

*Proof of Lemma* 6.2 It is enough to establish the formula for $y_2$ defined by $r_2 = 0$ and $y_1$ defined by $r_1 = 0$. We can take for $c$ the disk $D = \{(\sqrt{1 - r_2}, 0, x_3, x_4), r_2 \leq 1\}$. The boundary of $D$ is $y_1$. We orient it along $+\xi$, which reads $\dot{x}_3 = \frac{\sqrt{2}}{\alpha(\zeta)}x_4$, $\dot{x}_4 = -\frac{\sqrt{2}}{\alpha(\zeta)}x_3$ (clockwise in the $(x_3, x_4)$-frame). The orientation of $D$ along its boundary is given by $(\xi, -e_1)$ ($e_1$ is the first vector in the canonical basis of $\mathcal{R}^4$): $D$ can be considered as the image of $D_0 = \{(x_3, x_4), x_3^2 + x_4^2 \leq 1\}$ through the map:

$$s : (x_3, x_4) \to (\sqrt{1 - r_2}, 0, x_3, x_4)$$

The boundary of $D_0$ is then oriented by pull-back clockwise. This orientation of the boundary with the outgoing normal as a second vector (which maps on $-e_1$ through the map $s$ above) yields the positive orientation in the $(x_3, x_4)$-plane. This yields at $(0, 0)$ the positive frame $(\partial_{x_3}, \partial_{x_4})$. At $(x_3 = 0, x_4 = 0)$ in $D$, $y_2$ is oriented along $+\xi$ which is parallel to $(x_2 = 0)$ $\partial_{x_2}$ so that the linking number is given by $\mathrm{sign}(\alpha \wedge \mathrm{d}\alpha(\partial_{x_3}, \partial_{x_4}, \partial_{x_2})) = \mathrm{sign}(\alpha(\partial_{x_2})\mathrm{d}\alpha(\partial_{x_3}, \partial_{x_4})) = 1$. The claim follows. □

## 7 Periodic orbits, critical values, index, cycles that are not boundaries

Let $h = \alpha(\zeta) = A\tilde{B}r_1 + B\tilde{A}r_2$.

The equations governing the dynamics of $\xi$ are

$$\dot{x}_1 = -\frac{\tilde{B}}{h}x_2, \quad \dot{x}_2 = \frac{\tilde{B}}{h}x_1$$

$$\dot{x}_3 = -\frac{\tilde{A}}{h}x_4, \quad \dot{x}_4 = \frac{\tilde{A}}{h}x_3$$

In order to define a periodic orbit, we need to have, with $p, q$ integers:

$$\frac{\tilde{A}}{\tilde{B}} = \frac{p}{q}$$

The minimal period of this periodic orbit is then

$$\tau = \frac{2\pi hp}{\tilde{A}} = \frac{2\pi hq}{\tilde{B}}$$

If this periodic orbit is iterated $k$-times, then it runs $kq$-times over the full circle in the $(x_1, x_2)$-variables and $kp$-times over the full circle in the $(x_3, x_4)$-variables.

The action of this periodic orbit(with multiplicity $k$) is

$$c = k \int_0^\tau \alpha_x(\xi)\,\mathrm{d}t = k\frac{2\pi hp}{\tilde{A}}$$
$$= 2\pi k\left(A\tilde{B}r_1\frac{p}{\tilde{A}} + B\tilde{A}r_2\frac{q}{\tilde{B}}\right)$$
$$= 2\pi k(Aqr_1 + Bpr_2)$$

Thus,

$$c = 2\pi kq(Ar_1 + Br_2) + 2\pi Br_2k(p-q)$$

The total index, including the $X_0$-degeneracy, is $i = 2|p-q|$ since the rotation of $v$ with respect to $X_0$ is at most $2|p-q|\pi$ along the periodic orbit.[6]

Observe now that $p$ has the sign of $\tilde{A}$ and that $q$ has the sign of $\tilde{B}$. Since $\frac{\tilde{A}}{\tilde{B}} = \frac{p}{q}$ and since $\tilde{A} \gneq \tilde{B}$, $p \gneq q$ and $i = 2k(p-q)$. Calculating,

$$i = 2k(p-q) = 2k\left(\frac{\tilde{A}}{\tilde{B}} - 1\right)q$$

Thus,

$$2kq = \frac{\tilde{B}i}{\tilde{A}-\tilde{B}}$$

and

$$c = \pi\frac{\tilde{B}i}{\tilde{A}-\tilde{B}}(Ar_1 + Br_2) + \pi Br_2 i = \frac{i\pi(A\tilde{B}r_1 + B\tilde{A}r_2)}{\tilde{A}-\tilde{B}}$$

We denote in the sequel

$$\mathrm{cv}(t) = \frac{\pi(A\tilde{B}r_1 + B\tilde{A}r_2)}{\tilde{A}-\tilde{B}}$$

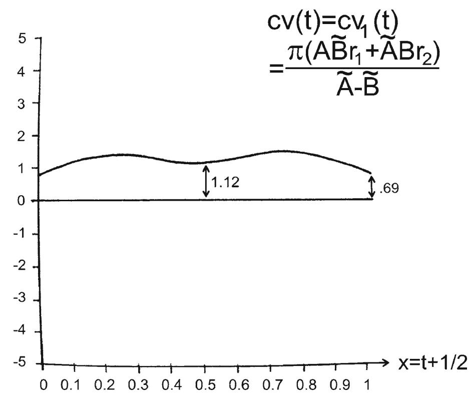
the coefficient of $i$ in $c$ for the torus $T_t$.

The behavior of $\mathrm{cv}(t)$ as a function of $t$ has been described in Sect. 1:

Next, we have the following useful observation:

**Lemma 7.1** *Periodic orbits appear in critical circles* (*circles generated by the action of* $X_0$) *of even index* $2k$. *If a critical circle of index* $2k$ *dominates a based periodic orbit of index* $2k-1$, *then the intersection number is zero.*

[6] See [2,4] in order to understand how the rotation of $v$ and the index of a periodic orbit are linked; this can be understood using the linearized operator $-(\ddot{\eta} + \eta\tau)$ of the functional $J = \int_0^1 \alpha_x(\dot{x})\,\mathrm{d}t$ on the space $C_\beta$. A sketch of the argument would go as follows: eigenvalues of this operator, taken with periodic or with $H_0^1$-boundary conditions, have a variational characterization (min–max type, on symmetric sets of $\mathcal{Z}_2$-genus equal to $i$ for the $i$th-eigenvalue)that implies that the $j$th-eigenvalue of the $H_0^1$-linearized problem taken at any based point is larger than or equal to the $j$th-eigenvalue of the periodic orbit problem. Because $v$ rotates $2|p-q|\pi$ with respect to $X_0$, starting from a point where they are parallel, the $(2k-1)$-first eigenvalues under both sets of boundary conditions are negative. Would the $2k$th-eigenvalue be positive for a based periodic orbit,that is for some $H_0^1$-problem (with a prescribed base point) then it would be positive for the periodic orbit problem whereas we know it to be zero. In this way, one can see how to prove that the periodic orbit index is $2k$ whereas the $H_0^1$-one is $2k-1$.




*Proof of Lemma* 7.1 If the based periodic orbit is in the circle, the result is immediate. Otherwise, by $X_0$-invariance, the critical circle of index $2k$ would have to dominate another critical circle of the same index. This can be ruled out using arguments of transversality: one considers the $X_0$-invariant unstable manifold of the dominating circle and a section to $X_0$ in the stable manifold (degeneracy included) of the dominated one. A dimension argument rules out the existence of an intersection between these two sets (transversality holds, see [2] p 53[7]). □

**Corollary 7.2** *All the cycles of the homology corresponding to odd dimensional periodic orbits* (*i.e corresponding to "based" periodic orbits*) *that are not images through the intersection operator of iterates of the two simple periodic orbits* $O_0$ *and* $O_1$ *, corresponding respectively to* $r_1 = 0$ *and* $r_2 = 0$[8], *of* [2,7] *are not boundaries.*

To end this section, we compute, using the integers $(p, q)$ associated with a periodic orbit, the linking number of two periodic orbits $y_1, y_2$. Coming back to the equations governing the dynamics of $\xi$, we see that, if $\tilde{A}_1$ is negative, $y_1$ runs clockwise in the $(x_3, x_4)$-plane. Similarly, if $\tilde{B}_2$ is positive, $y_2$ runs counter-clockwise in the $(x_1, x_2)$-plane. With respect to the notations of last section,we observe that the roles of $(p, q)$ are reversed. $kq$ is now the algebraic rotation in the $(x_1, x_2)$-plane and $kp$ is the algebraic rotation in the $(x_3, x_4)$-plane. $\tilde{A}$ and $p$ have the same sign, whereas $\tilde{B}$ and $q$ have the same sign.

Thus, if $(\bar{p}_i, \bar{q}_i)$ characterize $y_i$, we find that

**Lemma 7.3** *Assume that* $r_2(y_1) \gtrsim r_2(y_2))$. *Then,* $link(y_1, y_2) = -\bar{p}_1\bar{q}_2$

*Proof of Lemma* 7.3 The clockwise algebraic rotation of $y_1$ in the $(x_3, x_4)$-plane is $-\bar{p}_1$ and the counter-clockwise algebraic rotation of $y_2$ in the $(x_1, x_2)$-plane is $\bar{q}_2$. The result follows then from Lemma 6.2. □

[7] We have established in [2] that transversality held for the flow of [4], with preservation of the property that the number of zeros of the $v$-component of $\dot{x}$ did not increase along decreasing flow-lines. The argument "at infinity" is based on the following simple remark: considering a non-characteristic $\xi$-piece of a curve at infinity, we denote $E_m$ an eigenspace of the $H_0^1$-linearized operator $-(\ddot{\eta} + \eta\tau)$ corresponding to increasing eigenvalues from $\lambda_1$ to $\lambda_m$. $\phi_1, \ldots, \phi_m$ are the eigenfunctions. $E_{m+1}^+$ and $E_{m+1}^-$ are then the corresponding "half"-eigenspaces built by convex-combination of functions of $E_m$ with $\mathbb{R}^+\phi_{m+1}$ or $\mathbb{R}^-\phi_{m+1}$, respectively. Let $\phi$ be a smooth function of $H_0^1(0, 1)$. Let $\tilde{\phi}$ be then the function derived from $\phi$ on $(-\infty, \infty)$ after extending $\phi$ to $(-\infty, 0)$ with the function identically $+1$ if the incoming $\pm v$-piece is along $+v$, $-1$ otherwise and after extending $\phi$ to $(0, \infty)$ in the same way. Let $i(\phi)$ be the minimal number of zeros of a regularization of $\tilde{\phi}$ and let for $E \subset H_0^1$, $i(E)$ be equal to $\text{Sup}\{i(\phi), \phi \in E\}$. Then $i(E_m) = \inf(i(E_{m+1}^+), i(E_{m+1}^-))$.

[8] The definition of the homology in [2,7] contains contributions of the critical points at infinity. In *Compactness* [3], it was established that most of the critical points at infinity did not interfere with this homology. The result was stronger for even indexes than for odd indexes. We actually, after a few observations, strengthen in Sect. 15.1, Theorem 1' of [3] and establish that compactness in the sense of Theorem 1.1 of [3] holds for large enough even indexes. However, this compactness is about ruling out "genuine" critical points at infinity. There is another type of "critical points at infinity", see [2], Chapter (IV)2., after p161 , that are actually periodic orbits carrying in addition one or two (at most two, see Lemma 7.3, page 161 and see pp 176–178 of [2]) back and forth runs along $v$. These objects are also discussed further in the next section, Sect. 8 of this paper, below, when we address the violation of the Fredholm condition by the variational problem that we are studying. These additional objects leave our conclusions unchanged: our cycles are built with genuine periodic orbits. The shift in the index associated with these new cycles is 2, the action of $X_0$ is identical; Corollary 7.2 applies to these new objects. It is worth observing here though that, because these additional objects are usual periodic orbits, with at most two (it could be a finite, bounded number, the conclusion would be the same)back and forth run along $v$, the energy estimates provided above, also the conclusions about the flow, about $L + L^*$ (see below, after Lemma 7.3 for the defintion of $L$ and $L^*$) carry on to these "modified" periodic orbits, unchanged.

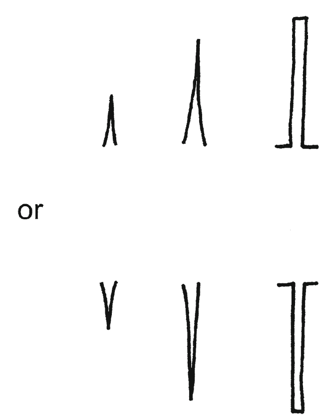
In what follows, $T^1$ will designate a torus $T_{t_0}$ among the tori $T_t$, $t \gneq 0$ which is made of periodic orbits of the Reeb vector-field $\xi$. $T_{-t_o}$ will be the torus $T^{1*}$. Considering then a curve at infinity $\tilde{x}$, that is a curve in $\cup\Gamma_{2s}$, we compute the linking numbers of $\tilde{x}$ with each of the periodic orbits of $\xi$ in $T^1$. We then average these linking numbers over the circle of periodic orbits of $\xi$ spanned by $X_0$. This average is denoted $L$. $L^*$ is the corresponding average with $T^{1*}$.

## 8 (Non)-Fredholm aspects of the problem

We now turn to the Fredholm aspects of this problem.

### 8.1 Ruling out critical points at infinity:

Given a configuration bearing already “large” $\pm v$-jumps in it (this configuration might either be supported by a critical point at infinity that includes large $\pm v$-jumps or/and this configuration might also include developed or fully developed, even “opened” “Dirac” masses in $\pm v$; a “Dirac mass” here is in fact a rather large back and forth or forth and back run along $v$, with possibly a (small) $\xi$-piece inserted in between, see [2, pp 28–30]),




FA3

we need to prove that we can build over this configuration “Dirac” masses in a continuous way, that is, to introduce negative or positive “Dirac” masses so as to decrease $J$ below the critical level at infinity corresponding to the critical point (at infinity) that supports this configuration.

We start with an important observation, namely that given two “Dirac masses” of this configuration, all the $H_0^1$-unstable directions between these two “Dirac masses” are involved in the process. We need to explain what we mean by this statement:

When we try to understand the flow-lines that are reaching this configuration and neighboring ones, we are led in a natural way to consider the possibility that such “Dirac masses” could develop along some or part of the $H_0^1$-unstable directions and not along others: this would depend on what unstable directions would support these “Dirac masses” as we approach this configuration.

Fortunately, it turns out that we need to consider all the $H_0^1$-unstable directions related to a $\xi$-piece defined by two basic $\pm v$-jumps of the configuration *together*.

In order to see this, we need to “break” the degeneracy related to the fact that introducing a back and forth or forth and back jump along $v$ does not change the value of $J$.

In order to break this degeneracy, we, therefore, consider the space $\cup\Gamma_{2s}$ and we modify the functional $J_\infty$ on this stratified space; we replace it by

$$J_\infty + \epsilon\Sigma(\delta s_i)^4$$

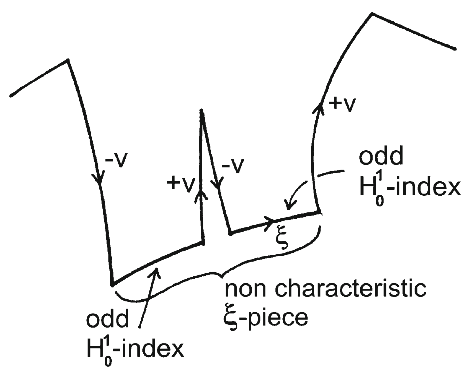
The sum runs over the (finite) range of $\pm v$-jumps of the curve $x$.

This functional is differentiable on this stratified space and the $H_0^1$-unstable manifold is not changed by the introduction of the additional term $\epsilon\,\Sigma(\delta s_i)^4$. However, the degeneracy is now broken.

The unstable manifold of the new critical points on $\cup\Gamma_{2s}$ that we have introduced with this modified functional contains all the $H_0^1$-unstable manifold for $J_\infty$ (this $H_0^1$-unstable manifold is defined for $J_\infty$ at any curve of $\cup\Gamma_{2s}$, whether critical or non-critical). Indeed the term $\epsilon\,\Sigma(\delta s_i)^4$ is of higher order on this $H_0^1$-unstable manifold. The contributions of the small $\pm v$-jumps that are used to represent this $H_0^1$-unstable manifold, see [3, pp 554–560], is quadratic, maybe cubic for one small $\pm v$-jump if the related $\xi$-piece is characteristic.

As $\epsilon$ goes to zero and disappears, we find again the functional $J_\infty$ and its variations. Thus, the critical points of the modified functional will behave as the underlying critical points (at infinity) that support them, with the addition of some "Dirac masses". The whole $H_0^1$-unstable manifold between these "Dirac masses" is thereby involved in the process, as claimed above.

Considering such $H_0^1$-unstable manifolds over a given configuration, we assume in a first step, for the sake of simplicity, that the "Dirac masses" are all "positive", that is that they are all made of a run along $+v$ followed by a run along $-v$. We also assume in this first step that the $\xi$-piece which we are considering runs between a basic run along $-v$ to a basic run along $+v$ and is non-characteristic:




FA4

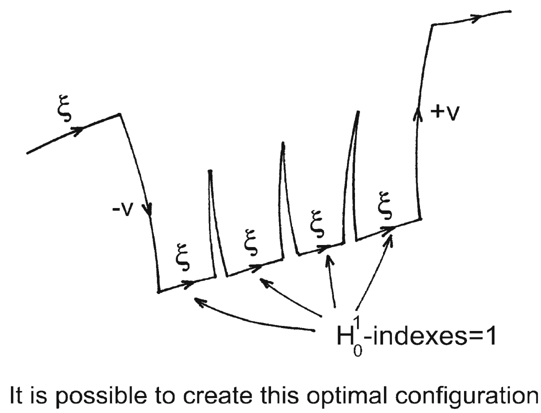
Under such circumstances, we claim, and the argument has been made in [2, pp 154–156], that on this $\xi$-piece, if it is of $H_0^1$-index 2 or higher, it is possible to introduce a "positive Dirac mass" somewhere along the $\xi$-piece (and its $H_0^1$-unstable variations) without increasing the maximal number of zeros of $v$. We have also established in [2, pp 163–168], that there are in fact preferred positions for these positive "Dirac masses" where we can introduce them without increasing the maximal number of zeros of $b$ until the configuration is "saturated".

Once the configuration is "saturated", the $H_0^1$-index of the $\xi$-sub-pieces formed by two consecutive "Dirac masses" is 1; this is embedded in the definition of the "optimal positions" where these "Dirac masses" can be inserted. This is discussed more below.

We can evolve from this "saturated" configuration to another one that is clearly unique: we consider the left (initial) negative $-v$-jump of the basic $\xi$-piece that we are considering. There is a positive "Dirac mass" on this $\xi$-piece that immediately follows this initial $-v$-jump. We "push away" from this initial $-v$-jump this "Dirac mass" until the $\xi$-piece lying between them supports a $v$-rotation along $\xi$ a bit less than $2\pi$. In this way, the $H_0^1$-index of this $\xi$-piece that they define stays 1 or increases to 1, being close to become 2, but not yet there. We then go, if "there is room", to the next "Dirac mass" (they are all positive by assumption) and we complete the same operation between the first and second "Dirac mass" and so forth. The maximal number of zeros of $b$ never increases over such configurations because the $H_0^1$-index of each sub-$\xi$-piece remains less than or equal to 1 (it is equal to 1 after deformation), whereas the $H_0^1$-index to the right never increases (and this implies that the maximal number of zeros of $b$ does not increase).

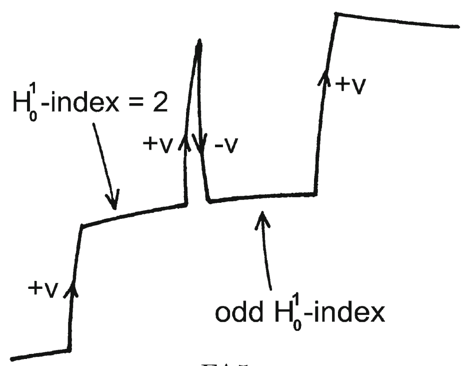
Using the following drawing, one can check that the two processes are in fact the same: we reach in this way an optimal configuration. The $H_0^1$-index between two consecutive positive "Dirac masses" is then always 1.

Configurations over which we choose to introduce negative "Dirac masses" rather than positive ones also behave in the same way if the initial $v$-jump of the basic $\xi$-piece that we consider is positive and the final one is negative.




FA4'

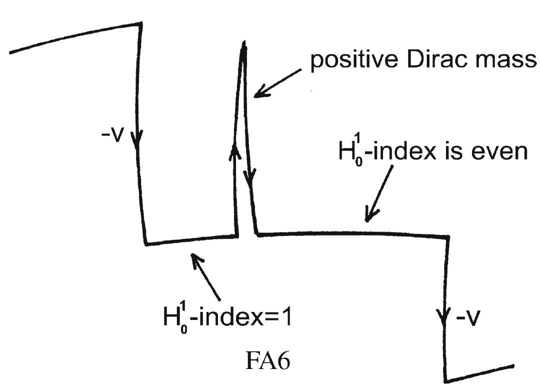
There are other possibilities: we could introduce, e.g. positive "Dirac masses" between an initial and a final $v$-jump that are both running along $+v$. The preferred position for the first positive "Dirac mass" is then such that the $H_0^1$-index of the left $\xi$-sub-piece that it defines is 2, whereas the $H_0^1$-index of the right $\xi$-sub-piece that it defines is odd:




FA5

Such a position exists (the argument is very similar to the one used in [2, pp 162–168]) and the maximal number of zeros of $b$ does not increase after this "Dirac mass" is introduced. Thereafter, on the right $\xi$-piece, we proceed as above.

In case the initial and final $v$-jumps are along $-v$, we introduce the first "Dirac mass" at a position such that the $H_0^1$-index to the left is 1 and the right $H_0^1$-index is even. We iterate then the process on the right $\xi$-piece (which runs also between two negative $v$-jumps):




FA6

If the initial $v$-jump is positive and the final one is negative, the $H_0^1$-indices are even to the right and to the left. If the initial $v$-jump is negative and the final one is positive, they are both odd.

There are similar rules if the additional "Dirac masses" are negative rather than positive.

Finally, and this is an important observation, between two consecutive "Dirac masses" that are both positive and both negative, since the $H_0^1$-index is 1, we can introduce a "Dirac mass" of the other type (negative if the Dirac masses were positive or positive if they were negative) at a position such that the right and left $H_0^1$-indices are zero. Again, the maximal number of zeros of $b$ will not increase.

The continuity of this whole process, as the basic configurations evolve, is straightforward, except under two circumstances: one of the $\xi$-sub-pieces defined above could change $H_0^1$-index. We would then have to consider what happens as one of these sub-$\xi$-pieces becomes characteristic. It could also happen that a basic configuration, as these basic configurations evolve, "loses" one of its "Dirac masses" (positive or negative).

Considering then our set of configurations as a parametrized set, the parameters being the position of the various $\pm v$-jumps and "Dirac masses" and how large they are, etc., we find a stratified space. On each stratum, the number and behavior of the various $\pm v$-jumps are given; none of these $\pm v$-jumps reduces to zero.

Each stratum has the other sub-strata as singular subsets. If we remove these substrata, then the $\pm v$-jumps of the configurations deform continuously, without ever canceling or changing orientation. Using the arguments above, we can add other "Dirac masses", positive first, then negative if we please. The whole process is continuous, this follows from the uniqueness of these optimal configurations, unless a basic $\xi$-piece becomes characteristic.

We prove below that the process of introducing additional, positive, then negative "Dirac masses" can be made in a continuous way over the configurations where there are characteristic $\xi$-pieces.

Let us for the moment address the issue of extension of this process, in a continuous way to sub-strata of a given stratum. In order to analyze this process well, let us consider the simpler case where there is exactly one sub-stratum. The process is well-defined on the sub-stratum and on the complement of the sub-stratum in the stratum. We need to glue the two processes. In order to do this, we "thicken" the sub-stratum, that is we think of it as having some thickness. We can use this thickness to transform the continuous process on the complement of the sub-stratum into the continuous process on the substratum. Because both processes are based onto optimal configurations that can be reached through the introduction of positive "Dirac masses" (additional ones, if some are already there and there are "holes" to be filled) and, then, once these "Dirac masses" are introduced in suitable positions, adjustments if needed (the distribution of positions for these "Dirac masses" is unique up to small deformations), addition of negative masses etc, they "fit" into each other and they can be glued. Again, the main issue is the fact that these processes need to be better understood when some $\xi$-piece(s) becomes characteristic.

We address this issue now:

As a $\xi$-basic piece of orbit, part of a critical point (at infinity) becomes characteristic and its $H_0^1$-index grows from $i_0^j$ to $i_0^j + 1$, the maximal number of zeros of $b$ on this piece of orbit can either remain unchanged, equal to $i_0^j + 1$, or it can change from $i_0^j$ to $i_0^j + 2$.

An example of the first occurrence is the case of a $\xi$-piece between an initial negative $-v$-jump and a final positive $v$-jump, with $i_0^j$ even, becoming equal to the next odd integer. There is then no additional positive "Dirac mass" to introduce (we forget here about the negative "Dirac masses", we introduce them—they are not needed—after if we wish; the processes will have been already made into continuous processes). The process deforms continuously.

On the other hand, if $i_0^j$ is odd and increases to the next, larger, even integer, an additional positive "Dirac mass" should be introduced all over the transition. On one side (once the index is larger), it will fit into an optimal configuration. On the other side, it will introduce two more zeros on the $\xi$-sub-piece.

It follows that the deformation/creation of new "Dirac masses" is not an obvious process , especially if several consecutive characteristic $\xi$-pieces are involved.

This occurs, for example, with a characteristic $\xi$-piece of a critical point at infinity. Then, the complement of a family of characteristic sub-pieces can be made only of characteristic sub-pieces. Whatever we gain on one side might be lost on the other one. Considering, e.g. one characteristic $\xi$-piece of a critical point at infinity, broken into two sub-pieces over a stratum, we encounter intermediate configurations over which the first of these $\xi$-sub-pieces is characteristic and hence the second one as well. Their respective $H_0^1$-index increases, e.g. for the first one from $i_0^1$ to $i_0^1 + 1$, whereas for the second one it decreases from $i_0^2$ to $i_0^2 - 1$. The maximal number of $b$ on the first one can either increase from $i_0^1$ to $i_0^1 + 2$ or stay unchanged, equal to $i_0^1 + 1$. A similar phenomenon can occur for the second one, on the decreasing side.

If on both sub-pieces the maximal number of zeros of $b$ remains unchanged, the continuity of the associated "saturated" configurations follows immediately.

If it changes on both sides, we may see that the loss of two zeros on one side is compensated by a gain of two zeros on the other side. Over the process, an additional "positive Dirac mass" can be tracked as it "transits" from one $\xi$-sub-piece to the neighboring one.

Finally, if the change is asymmetrical, then the maximal number of zeros, e.g. drops by 2 at the transition. Over the configurations where the number of zeros is less by 2, we can freely introduce an additional "positive Dirac mass", without going above the prescribed maximum in the number of zeros of $b$. Continuity follows at the transition; once the transition is over, we can revert gradually to the "saturated" configurations as above. The claim follows for the characteristic pieces of the critical pints at infinity.

## 8.2 Extending (but not ruling out) the above arguments to the periodic orbits:

We now move to study the violation of the Fredholm assumption for the case of the periodic orbits. Their contribution to the Morse complex has been studied in [2, pp 174–189]. These results are recalled and used in Sect. 9, below.

We introduce here some more precisions to the arguments of [9] to make them more transparent. The arguments follow closely Sect. 8.1, above:

For elliptic periodic orbits, given a configuration of "Dirac masses" on it, of various sizes (never all totally vanishing), there is always a $\xi$-interval among those defined by these "Dirac masses" that is not characteristic. Therefore, the argument developed for critical points at infinity works for periodic orbits.

For hyperbolic periodic orbits, these arguments also work if the locations of the "positive or negative Dirac masses" is not, for all of them, precisely at the nodes, see [3, pp 471–475], and see the construction of the unstable manifold of a periodic orbit (Proposition 1 of [3]); then, again, some $\xi$-interval is not characteristic.

If all the "Dirac masses" are located at the nodes, we run into the difficulty that all the $\xi$-intervals are characteristic and we have to decide whether we should, for each of these intervals, consider the strict or the full $H_0^1$-index to count the maximal number of zeros on the flow-lines through this configuration.

The total count should not exceed a certain bound $2k$. This could leave room to allow for the full $H_0^1$-index on some $\xi$-intervals, whereas, on the other ones, only the strict index would be taken. The $\xi$-intervals of the first type and the $\xi$-intervals of the second type are not a priori prescribed. The parameterization derived from the location, size, and type of the "basic Dirac masses" of the configuration does not suffice anymore when there are characteristic $\xi$-pieces; the specification of the type of indexes used to define the cycles is a required additional information.

It follows that the "optimal distributions" of "Dirac masses" are not only multiple, as is the case as soon as several characteristic pieces are involved. They could also carry some topology as the additional "Dirac masses" corresponding to the full $H_0^1$-indexes might travel around the periodic orbit, thereby building some $S^1$-topology, contrary to what happens for characteristic pieces of critical points at infinity and contrary to what happens to elliptic periodic orbits.

However, we can associate with each basic configuration along the periodic orbit an optimal distribution corresponding to all the strict indexes.

In order to "see" this optimal distribution, we perturb a bit one of the basic points where the basic "Dirac masses" defining the configuration are located, so that the total rotation, if it were $k\pi$, becomes a bit less than $k\pi$, equal to $k\pi - \epsilon$. It is indeed possible to "move" one basic point a bit and achieve any rotation close to $k\pi$ (this follows from some "simple" argument involving the stable and unstable directions, transversally to $\xi$, near a node and the related rotation of $v$ along a full turn of the Poincare-return map. The $v$-rotation, if the orbit is hyperbolic of index $2p+2$, is in $((2p+1)\pi, (2p+2)\pi)$ for an interval defined by two consecutive nodes and it is in $((2p+2)\pi, (2p+3)\pi)$ for the next interval of the same type).

Once the rotation is $k\pi - \epsilon$, we can move each other basic point of the configuration so that all the $\xi$-intervals are a bit shorter than the associated characteristic intervals. The optimal distribution of "Dirac masses" on this "shortened" configuration extends to the original one, providing a distribution of "Dirac masses" corresponding to the strict indexes. It is not difficult to "translate" a bit the "Dirac masses" so that they all end up located at the nodes where the small $\pm v$-jumps representing the $H_0^1$-unstable manifolds of the various $\xi$-intervals are also located. This shift or translation is costless if the "Dirac masses" are not open. If they are open, then $J_\infty$ can be decreased by opening them more; translation to the nodes and decrease can be combined into one decreasing deformation. In order to prepare for this process, we can redistribute, using the techniques of [2], pp 82–102, the $v$-rotation along the periodic orbit so that all the nodes are located on a small piece of the periodic orbit, where points are either in $A^+$, or in $A^-$, or in $A^+ \cup A^-$(see below for the definition of $A^+$ and $A^-$).

Thus a "strict optimal" configuration of "Dirac masses" exists and is unique. Any other allowed configuration contains this configuration and contains additional "Dirac masses" to account for some full indexes.

Would the $2k$th-eigenvalue be positive for a based periodic orbit, that is for some $H_0^1$-problem, then it would be positive for the periodic orbit problem, whereas we know it to be zero.

As deformation occurs, we might find two of them, rather than one, in the vicinity of a "basic Dirac mass" of the configuration. "Vicinity" indicates here that the $v$-rotation separating them is less than $\frac{3\pi}{4}$. All these points are close to each other on the $\xi$-orbit as explained above.

The "basic Dirac masses" can be assumed to be all "essential" in that we can decrease $J_\infty$ by "opening them".

Assume for example that we have a basic "positive Dirac mass" and that a distribution that it supports involves some full $H_0^1$-index on a $\xi$-piece that has this basic "Dirac mass" as an edge. We could then have, on this $\xi$-piece, very close if not immediately preceding and following this edge, e.g. immediately preceding, a "positive Dirac mass" followed by a "negative Dirac mass".

Opening the basic "positive Dirac mass", we can "close and absorb" the "negative one" and then "coalesce into it", see [2, pp 159–162], the preceding "positive Dirac mass". In all, we end up with a single "open positive Dirac mass" and we cancel the "preceding positive and negative Dirac masses".

We thus can continuously deform and contract all allowed distributions into the "strict" optimal one, with maybe some additional opening of the basic "Dirac masses". As some basic "Dirac masses" cancel over the configuration space, we can glue these optimal distributions, even after some of their "Dirac masses", these and others, have been opened a bit to account for the various glueings/deformations/cancellations.

The previous arguments thereby extend to all periodic orbits. Some additional special argument must be made for the case when the strict $H_0^1$-index is zero for the characteristic $\xi$ pieces with reverse edge orientations. The argument, as an additional combination of "positive and negative Dirac masses" travel, is an adaptation of the previous argument.

8.3 Extension of the compactness arguments of [3] to the new cycles related to the violation of the Fredholm assumption:

For our homology to work, for the compactness results to hold, we need to be able to track along our cycles, along our topological classes, the location of the $\pm v$-jumps under evolution, of the $*$s to use the terminology that we have introduced in [3]. We need to define $*$s also for the new topological classes derived from the violation of the Fredholm hypothesis. These topological classes have been defined and described in [2, Chapter IV.2, pp 174–189]. We check now that they fit in the framework of this homology: these topological classes are defined by the addition of the two chains: a chain $c_1$ spanned by two "Dirac masses" having the same "orientation", e.g. "positive", separated by an $H_0^1$-index equal to 1, located at two points $a, a+\eta$, combined with the $H_0^1$-unstable manifolds in $H_0^1[a, a+\eta]$ and in $H_0^1[a+\eta, a+1]$; the "height" of each "Dirac mass" is a parameter that changes from 0 (when the "Dirac mass" is not there) to 1 (when it is fully expanded). Another chain $c_2$ is added to $c_1$. Over $c_1$, there is only one "Dirac mass", with the same orientation than the previous one; this "Dirac mass" is located at a time $t$ that travels from $a$ to $a+\eta$. It is combined over $c_2$ with the $H_0^1$-unstable manifolds of $H_0^1[t, t+1]$. We refer to [2, pp 178–181], for more details.

We introduce, over $c_1$, two $*$s for each "Dirac mass", one positive and one negative, and an additional $*$ for each $H_0^1$-index in $H_0^1[a, a+\eta]$. The $v$-rotation in the $\xi$-transport from $a$ to $a+\eta$ is close enough (less than) to $2\pi$ so that the dimension of the unstable manifold in $H_0^1[a, a+\eta]$ is $2p+1$ if the periodic orbit is of index $2p+3$ and is $2p$ if it is of index $2p+2$. There is, of course, an additional, "internal", small $\pm v$-jump in $[a, a+\eta]$ related to the additional $H_0^1$-index in $[a, a+\eta]$. This should yield an additional $*$, but we may view it as a companion of one of the $*$s that are associated with the second (negative) $v$-jump of the "Dirac mass" at $a$, or the first (positive) $v$-jump of the "Dirac mass" at $a+\eta$ (we build two companions, one to the $*$ to the left, the other one to the $*$ to the right and we scale and locate them appropriately in $[a, a+\eta]$.

We, therefore, derive in all $2p+5*$s if the $H_0^1$-index of the periodic orbit is $2p+3$, $2p+4*$s if it is $2p+2$. This number of $*$s is coherent with the dimension of the additional cycles associated with a periodic orbit, see [8, pp 177–178], as a consequence of the violation of the Fredholm assumption.

We have to check that their definition extends to $c_2$, so that they are globally defined over the topological class $c_1+c_2$.

It suffices for this purpose to create two $*$s equal to 0 at a location $t'$ such that the $v$-rotation in the transport along $\xi$ over the interval $[t, t']$ is larger than $\pi$ and less than $2\pi$ and the $v$-rotation in $[t', t+1]$ be larger than $(2p+1)\pi$ and less than $(2p+2)\pi$ if the index of the periodic orbit is $2p+3$, whereas this $v$-rotation in $[t', t+1]$ should be larger than $2p\pi$ and less than $(2p+1)\pi$ if the index of the periodic orbit is $2p+2$. This can be done, because the total $v$-rotation on a time 1-interval changes as the initial point changes; but it changes by an amount less than $\pi$: we can lose or gain at most one node of the associated function $\eta$, see [2, pp 174–176], over this initial time change. This is a consequence of the Sturm–Liouville properties of the linearized operator associated with the second derivative.

The definition of the $*$s, in the right amount is, therefore, possible and the number of sign changes is coherent with the dimension of the cycles. The results that we have established for the unstable manifolds of the periodic orbits, therefore, extend to this framework. The only difference resides in the fact that some of these $*$s have companions from the onset, whereas for the $*$s of the unstable manifolds of the periodic orbits, there were no companions as they were given birth to near the periodic orbits.

### 8.4 Violation of the Fredholm condition: the example of the first exotic contact structure of J. Gonzalo and F. Varela; also a general theorem about its impact on the homology:

Let

$$A^+ = \{x_0 \in M, \alpha_{x_0}(D\Phi_s(\xi(x_{-s})) = \alpha_{x_0}(D\Phi_s(\xi(D\Phi_{-s}(x_0))) \gneqq 1\} \text{ for some } s \lneqq 0\}$$

and

$$A^- = \{x_0 \in M, \alpha_{x_0}(D\Phi_s(\xi(x_{-s})) = \alpha_{x_0}(D\Phi_s(\xi(D\Phi_{-s}(x_0))) \gneqq 1\} \text{ for some } s \gneqq 0\}$$

**Proposition 8.1** $M \setminus (\{y = \bar{y}_0\} \cup T_0) \subset A^+ \cup A^-$.

We then have

**Proposition 8.2** *Let $\lambda\alpha$, $\lambda : \mathbb{R}^3 \longrightarrow \mathbb{R}^+ \setminus \{0\}$ being a smooth function, be a contact form belonging to the same contact structure than $\alpha$.*

(i) *Generically on $\lambda$ in $C^\infty(M, \mathbb{R}^+ \setminus \{0\})$, $M \setminus (\{y = \bar{y}_0\} \cup T_0) \subset A^+_\lambda \cup A^-_\lambda$, where $A^+_\lambda$ and $A^-_\lambda$ are defined just as $A^+$ and $A^-$ are, with $\lambda\alpha$ in lieu of $\alpha$. If $R(\bar{y}_0)$ is irrational, then the set $\{\bar{y} = \bar{y}_0\}$ is also contained in $A^+_\lambda \cup A^-_\lambda$.*

(ii) *Furthermore, under the same assumption, all points $z$ in $T_0$ such that $R(z)$ is irrational are in $A^+_\lambda \cup A^-_\lambda$*

We will then derive from Proposition 8.2 the following key result:

**Proposition 8.3** *Let $M$ be a three-dimensional manifold.*
*Assume that either*

(i) *There are a finite number of curves in $M$, $C_1, \ldots, C_p$ and a hyper-surface $T_0$ such that $M \setminus (C_1 \cup C_2 \cdots \cup C_p \cup T_0) \subset A^+_\lambda \cup A^-_\lambda$ and these curves do not intersect the critical points at infinity of the contact form $\lambda\alpha$ and the periodic orbits of its Reeb vector field. Furthermore, these critical points at infinity intersect $T_0$ at points that are in $A^+_\lambda \cup A^-_\lambda$.*
*or*

(ii) $A^+_\lambda = A^-_\lambda$
*or*

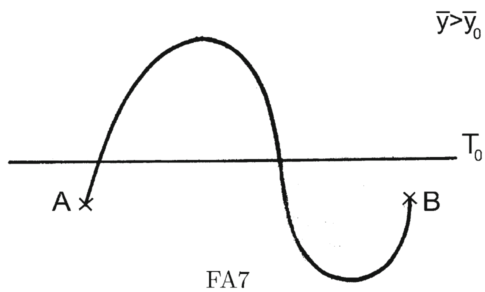
(iii) *A $\xi$-piece of orbit of a critical point at infinity that crosses the union of the hyper-surfaces of conjugate points of $M$ for the contact form $\alpha_1 = \lambda\alpha$ at least twice has an $H^1_0$-index equal to 1 or more.*
*Under any of the conditions described above, the critical points at infinity for which $i_0 + \gamma$[9] is bounded (independently of the Morse index of the critical point at infinity) do not interfere with the homology for a large enough index.*

[9] The arguments of Proposition 8.3 extend to $i_0 + \gamma$ large. We will address this case in Proposition 9.1, below.

*Proof of Proposition 8.1* Observe that the function $\Gamma(s) = \alpha_{x_0}(D\Phi_{-s}(\xi(x_s)))$ achieves its extrema when $\alpha_{x_0}(D\Phi_{-s}([\xi, v]) = 0$, i.e. at the coincidence points of $x_0$. Since ker$\alpha$ turns well along $v$, we can derive whether a given point is in $A^+$ or $A^-$ from the behavior of the function $\Gamma(s)$ at the various coincidence points of $x_0$.

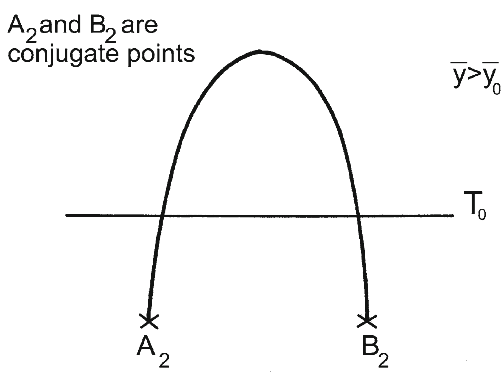
Assume first that we are considering a flow-line of $v$ such that $\bar{y} \gneq \bar{y}_0$.

Let us consider a point $A$ in $r_1 \gneq r_2$ very close to the torus $T_0$ and let us consider its first (positive, rotation of ker$\alpha = 2\pi$) coincidence point $B$. We know (Lemma 4.6) that, if $A$ is close enough to $T_0$, then $B$ satisfies $r_1(B) \gneq r_1(A)$. Furthermore, if $A$ is close enough to $T_0$, then all the negative iterates of $A$ under the coincidence map (the map that assigns to a given point its next coincidence point, see [7, p 25, Definition 1]) are closer and closer to $T_0$ (a consequence of Lemma 4.6 again). It follows that such $A$s and $B$s are in $A^-$ and it follows as well that $A$ and $B$ are in $A^-_\lambda$ for every function $\lambda$ valued into $\mathbb{R} \setminus \{0\}$.



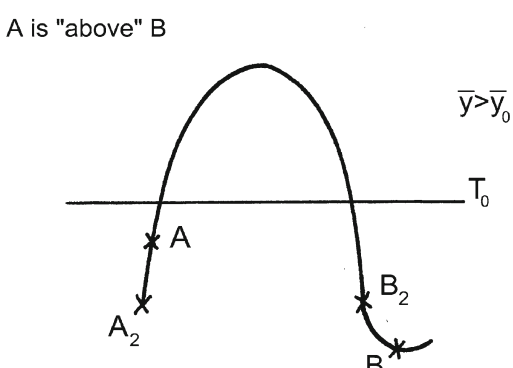
FA7

We now move $B$ backwards along the $v$-flow-line. $A$ moves also backwards along the $v$-flow-line. As long as $B$ does not cross $T_0$, neither does $A$. Also, we know that since $\bar{y} \gneq \bar{y}_0$ there are two conjugate points $A_2$, $B_2$, with a single zero of $a$ separating them (Lemma 4.6):



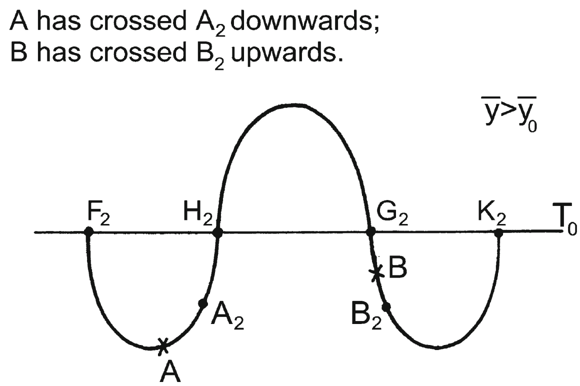
FA8

As long as $B$ has not reached, over its backwards movement, $B_2$, $A$ remains "above" $A_2$, that is $r_1(A) \lneq r_1(A_2)$ and $r_1(B) \gneq r_1(A_2)$.



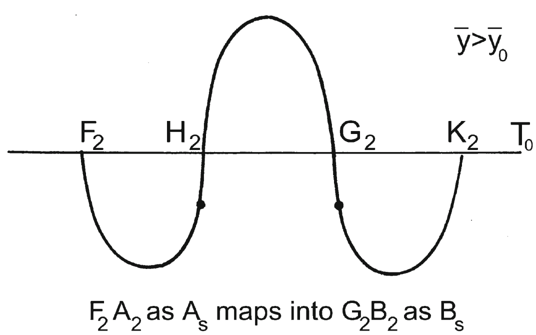
FA9

so that the interval $[A_2, H_2]$ of the $v$-flow-line is mapped through the coincidence map on the interval $[B_2, K_2]$ of the same flow-line:




FA10

Now $B$ "crosses over" $B_2$ and "enters" into $[G_2, B_2]$, so that, by strict monotonicity of the rotation of ker$\alpha$ along $v$, $A$ "enters" into the interval $[F_2, A_2]$. When $B$ reaches $G_2$, $A$ reaches $F_2$; there are, by Lemma 4.6, no further conjugate points with $2\pi$-rotation over the span of these intervals and, therefore, $r_1(B) \lneq r_1(A)$ now. The $A$-interval $[F_2, A_2]$ is mapped through the coincidence map on the interval $[G_2, B_2]$:



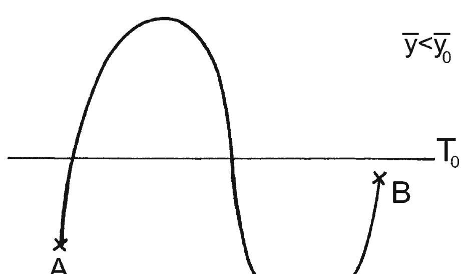

FA11

There is periodicity of the pattern and, therefore, now that $B$ is in $[G_2, B_2]$, thinking of it as an $A$, it is *inside* an interval $[F'_2, A'_2]$ of its own and its image $B'$ through the positive coincidence map is, therefore, "closer" to $T_0$ than $B$ is, that is, $r_1(B') \gneq r_1(B)$. We thus conclude that $A$ and $B$, the original $A$ and $B$ are in $A^+$, whereas they were before in $A^-$. When $B$ is $B_2$, it is, as an $A$, in an $[F_2, A_2]$-interval; therefore $B_2$ and $A_2$ are in $A^+$. By symmetry, they are also in $A^-$.

This concludes the study of the case $\bar{y} \gneq \bar{y}_0$.

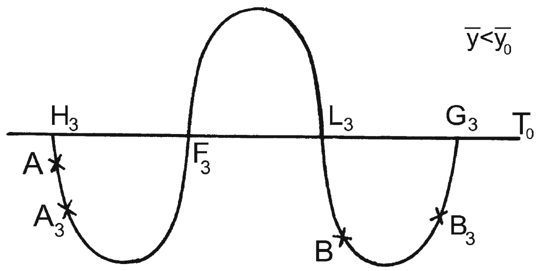
We now move to study the case when $\bar{y} \lneq \bar{y}_0$.

Using the proof of Lemma 4.6 and the fact that $\bar{y}_0$ is the unique critical point of the function $R$, we can conclude that, now, with $A$ very close to $T_0$ and $a(A)$ negative, the image $B$ through the positive coincidence map is even closer to $T_0$. Such $A$s and $B$s are in $A^+$.



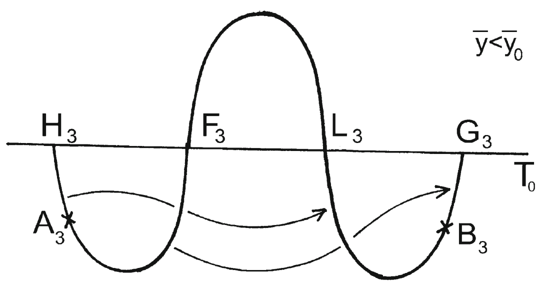

FA12

As above, starting from such a configuration, we move $B$ "backwards". $A$ also then moves backwards. Two zeros of $a$ are separating $A$ and $B$; there are no conjugate points. $B$ stays "above" $A$; eventually, $A$ reaches $D$ and $B$ is still "above" $E$. Thereafter, $A$ moves "up", whereas $B$ continues to move down towards $E$. Their tori have to cross. This implies the existence of two conjugate points $A_3$, $B_3$, with $k = 1$, that is a rotation of $2\pi$, but separated by three zeros of $a$:




FA13

As $B$ reaches $B_3$ from "above", $A$ reaches $A_3$ from "below". They cross over. Now $A$ is "above $B$". Over this process, we derive that the $A$-interval $[F_3, A_3]$ is mapped on the $B$-interval $[G_3, B_3]$, whereas the $A$-interval $[A_3, H_3]$ is mapped onto the $B$-interval $[B_3, L_3]$:




FA14

Iterating, we find that in the former case, $A$ and $B$ are in $A^-$, whereas in the latter one they are in $A^+$. In case $A$ is $A_3$ and $B$ is $B_3$, both $A$ and $B$ are in both $A^+$ and $A^-$.

The proof of Proposition 8.1 is thereby complete. □

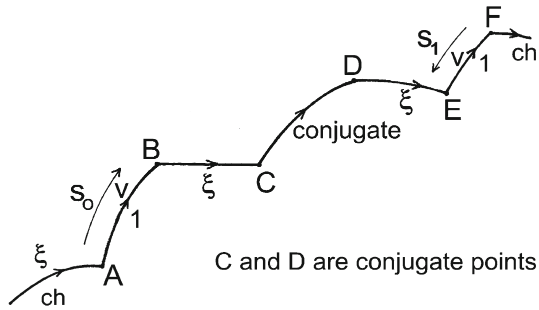
*Proof of Proposition* 8.2 Taking another contact form $\lambda\alpha$ of the same contact structure than the first exotic contact form of J. Gonzalo–F. Varela, we observe that outside of the characteristic hyper-surface defined by $\bar{y} = \bar{y}_0$ and outside of the curves defined by the equation $a = 0$ on the torus $T_0$, all points of $M$ are in $A^+ \cup A^-$ for $\lambda\alpha$. Indeed, by Proposition 8.1 and its proof, it is clear that, given such a point, the standard contact form $\alpha$ will be mapped under the iterated coincidence map to $\epsilon^{-1}\alpha$, with $\epsilon$ as small as we wish. This implies the claim of the proposition, with $\lambda\alpha$ in lieu of $\alpha$ as well.

For the points of the characteristic hyper-surface $\bar{y} = \bar{y}_0$, would we assume, an assumption that requires verification, that the rotation number $R(\bar{y}_0)$ is an irrational number, then, the iterates of the coincidence map for $\lambda\alpha$ on this hyper-surface will all belong to the same circle on a given torus and they will form a dense subset of this circle. Unless $\lambda$ is constant on such a circle, which is not generic, this will not happen. This holds also true for the points $z$ of $T_0$ with irrational rotation number $R(z)$.

Finally, we can further analyze the transport equation on the two circles defined by $a = 0$ on $T_0$. But generically on $\lambda$, its periodic orbits and critical points at infinity will not intersect these curves. Therefore, Proposition 8.2 holds. □

*Proof of Proposition* 8.3 We remove the index $\lambda$ for the sake of simplicity.

Let us consider for simplicity a characteristic $\xi$-piece, followed by a non-degenerate $\xi$-piece of $H_0^1$-index $i_0^j = 0$, with $\gamma_j = 0$, followed by another non-degenerate $\xi$-piece, again with $H_0^1$-index 0 and same orientation of its initial and final $\pm v$-jumps ($\gamma_j = 0$), followed then by a characteristic $\xi$-piece:




$C$ and $D$ are then conjugate points. $s_0$ and $s_1$ are the lengths of the $\pm v$-jumps from $A$ to $B$ and from $E$ to $F$, respectively. Since they have the same orientation, we will assume, without loss of generality, that they are both positive.

We then know that

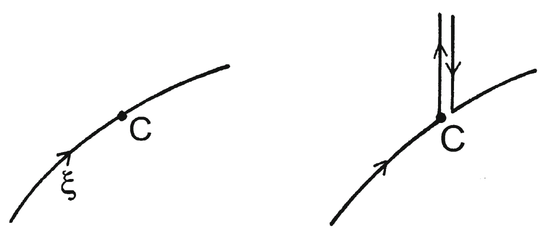
$$\alpha(D\Phi_{s_0}(\xi(A))) = 1, \quad \alpha(D\Phi_{-s_1}(\xi(F))) = 1$$

If $\xi$, $v$-transported from any point of $[A, B]$ to $B$, as well as from any point of $[E, F]$ to $E$, satisfies $\alpha_{B \text{ or } E}(D\Phi_s(\xi)) \gneqq 1$, then this critical point at infinity is false and we can discard it. Thus $\alpha_{B \text{ or } E}(D\Phi_s(\xi)) \leq 1$ on these intervals. It follows then from the identities $\alpha(D\Phi_{s_0}(\xi(A))) = 1$, $\alpha(D\Phi_{-s_1}(\xi(F))) = 1$ that $B \in A^-$ and $A \in A^+$.

It might happen that although $B$ is in $A^-$, for a form $\lambda\alpha$, it is "ultimately", that is under iterations of the coincidence maps, in $A^+ = (A^+_\lambda)$. This might happen as well, with $A^+$ in lieu of $A^-$ for $E$.

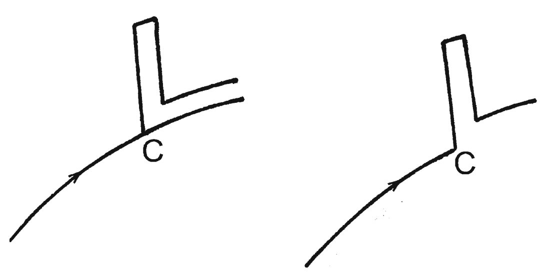
However, whatever happens, the following observations hold:

First, given a point $C$ of $M$ and a piece of $\xi$-orbit through $M$:



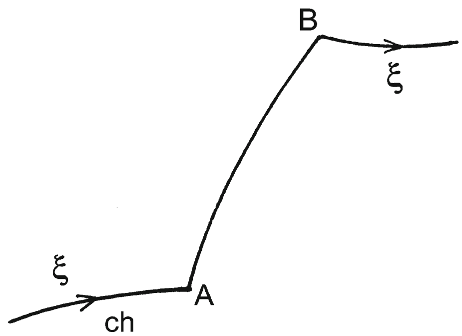

FA16

of length $a$, it is possible, because $C$ is in $A^+$ or $C$ is in $A^-$, e.g. $C$ is in $A^+$, to introduce a "Dirac mass" at $C$, that is a back and forth or forth and back run along $v$ at $C$, to grow it to an appropriate "height" and then to "open it" and insert in it a small $\xi$-piece of length $\epsilon$, so that the total action $\int_0^1 \alpha_x(\dot{x})\, dt$ decreases:



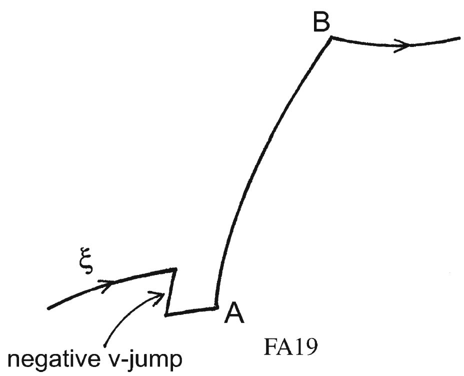

FA17

Secondly, starting from $A$:



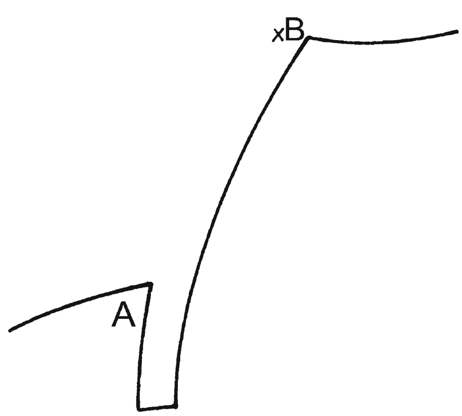

FA18

we consider a negative small $v$-jump along the characteristic piece near $A$. This negative small $v$-jump pushes our cycle "down". This negative small $v$-jump is now "exiting" the characteristic $\xi$-piece through $A$:



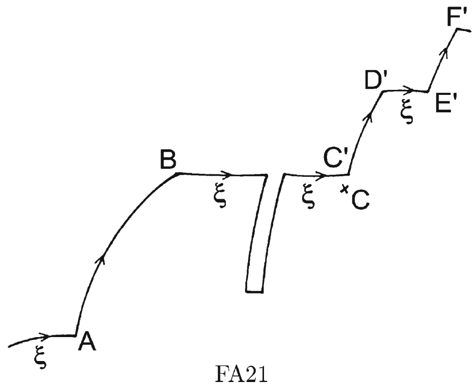

FA19

Over this exit, we can use the fact that $B$ is in $A^{-}$ and continue this decreasing process by extending the negative $v$-jump into a "negative Dirac mass" that decreases the total action:




FA20

This "decreasing process", with the use of "negative Dirac masses" continues as the base point for this "Dirac mass" travels now along the first non-degenerate $\xi$-piece:



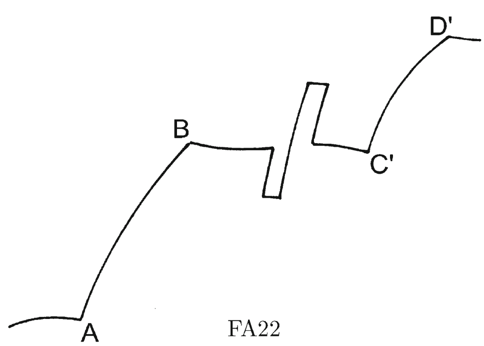

FA21

It might happen though that this ability to decrease the action over the curves which follows from the fact that the points in a portion of these curves are in $A^-$, disappears somewhere along the trip from $B$ to $C$. Using Propositions 8.1 and 8.2, it would then be replaced by the ability to decrease the action through the use of a "positive Dirac mass", since the points must be either in $A^-$ or in $A^+$.

Here is where our third observation comes into play:

Up to this point in this decreasing process, we can think of this "negative Dirac mass" as an extension of the negative $v$-jump on the first characteristic piece and the count of the maximal number of zeros of $b$ does not increase with these "Dirac masses" with respect to the configurations with the negative $v$-jump inside the characteristic piece. In fact, we can think of a process over which, as this negative $v$-jump is still on this characteristic piece, we would "accelerate" its process of leaving the characteristic piece and try to make it "reach" the next characteristic $\xi$-piece under the form of "Dirac masses", without increase in the maximal number of zeros of $b$.

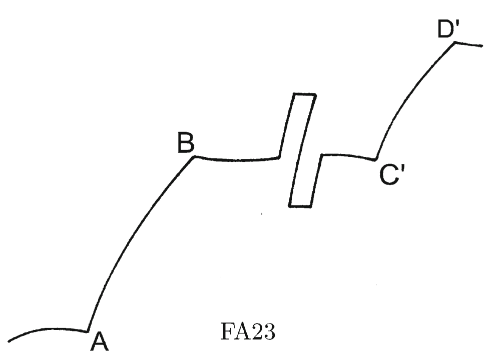
However, if we are forced to reverse the "orientation of the Dirac masses" and to now change and use "positive ones", whereas we were using "negative ones", then, we might face two occurrences. The first one is as follows:




FA22

The outcome is that we have introduced two additional sign-changes here.

However, if we introduce the "positive Dirac mass" before the "negative one", we find



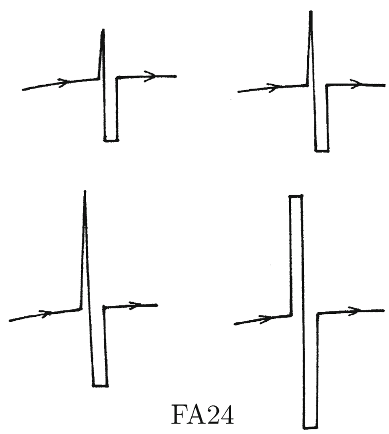

FA23

and the count of the number of zeros is now the right count.

Starting from $B$, we can continue with the "negative Dirac mass" all along the non-degenerate $\xi$-piece and we can in fact continue across $CD$ onto the next non-degenerate $\xi$-piece, then to the next characteristic $\xi$-piece where our "negative Dirac mass" can return to its form of negative small $v$-jump (see the proof of Proposition 21 in [3], the metamorphosis is identical). Thus this decreasing process continues, unhindered, unless we encounter a point $\bar{B}$ that ceases to be in $A^-$.

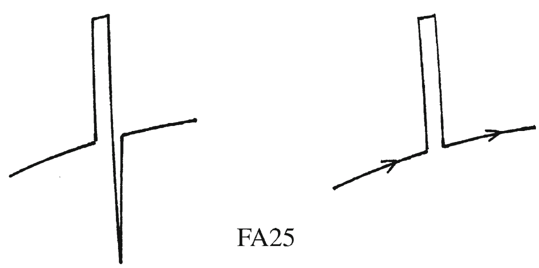
However, then, this point is, by Propositions 8.1 and 8.2, in $A^+$ and, in fact, a whole open neighborhood of this point is in $A^+$.

We can then use the second construction above to switch the "orientations" of the "Dirac masses" across $\bar{B}$. We create, as the base point reaches $\bar{B}$ from the "left", a "positive Dirac mass" to the left of the already existing "negative Dirac mass", over a gradual process:



FA24

Then, we "close" the "negative Dirac mass", also over a gradual process.



FA25

and we can then continue with the "positive Dirac mass".

This is in fact a natural switch to complete near the jump $CD$, because the point $C$ might be in $A^-$, but it is certainly in $A^+$. Near $C$, we can switch from a "negative Dirac mass" to a "positive Dirac mass" as described above.

As we observed earlier, if through the analysis of the "ultimate behavior" (that is, as $s$ tends $\pm\infty$, on the $v$-orbit through a point), we find that $B$ "ultimately" in $A^+$ rather than in $A^-$, we can switch from the "special negative Dirac mass" that we would have defined at $B$ to a "positive Dirac mass" through the process described above, without increase in the maximal number of zeros of $b$ over the cycle.

This establishes Proposition 8.3 under (i). (ii) is a variant of (i), only that now, we might encounter configurations over which there are points that are neither in $A^+$ nor in $A^-$. However, near each characteristic $\xi$-piece, the points are in one of them. Therefore, if over a given interval, inserted into a sequence of non-degenerate $\xi$-pieces, the points are neither in $A^-$ nor in $A^+$, then there are points on each side of the interval that are in these sets.

Over each of these intervals, the $H_0^1$-index being zero and the points being neither in $A^+$ nor in $A^-$, every $v$-jump that does not correspond to a basic $v$-jump of the critical point at infinity can be brought to be zero over a decreasing process. We thus, because $A^- = A^+$, can order our decreasing "Dirac masses" over these intervals, one before and one after, in a way that does not increase the maximal number of zeros of $b$ and use this combination as a small $v$-jump or $*$, now equal to zero, travels over these intermediate intervals. The combination decreases the action and defines a transition between the "Dirac masses" on each side of the interval. This establishes Proposition 8.3 under (ii).

Proposition 8.3 under (iii) will follow from arguments that we introduce below, when we discuss the case when $i_0^f + \gamma$ is very large, in the proof of Proposition 9.1, to get rid of the critical point at infinity under this assumption.

The proof of Proposition 8.3 is thereby complete. □

Proposition 8.2 (and its proof) has the following consequence:

### 8.5 The Palais–Smale condition and the flow at infinity for a generic form $\lambda\alpha$ of the contact structure of J. Gonzalo and F. Varela:

We have studied in [9] the Palais–Smale condition on the flow at infinity under some conditions on $v$. These conditions are not satisfied by the vector field $v$ of Martino [19], but the proof can be extended.

However, there is a simpler way to derive from Proposition 8.2 the fact that the flow at infinity, the flow on the $\cup\Gamma_{2s}$, see [2] and [4], does verify the Palais–Smale condition for a generic $\lambda\alpha$. Indeed, we know, from the proof of Proposition 8.2, that, as we start from some point of $M$, if this point does not verify $\bar{y} = \bar{y}_0$, then the iterated coincidence map will drive the images of this point to $T_0$ either under positive, or under negative iterations. We know better: we know that, if on a $v$-orbit $\bar{y} \neq \bar{y}_0$, there must be two conjugate points that are as $(A_2, B_2)$ (F.A7) or as $(A_3, B_3)$ (F.A12). $A_i$ and $B_i$ are conjugate, one is in $A_\lambda^+$, whereas the other one is in $A_\lambda^-$. Therefore, each of them is in the intersection of the two sets. Using an argument of connectedness, we derive that $S^3 \setminus (T_0 \cup \{\bar{y} = \bar{y}_0\})$ is in $A_\lambda^+ \cap A_\lambda^-$.

Using a general position argument, we may assume that the end-points of the $\xi$-pieces of every critical point at infinity are not in $T_0 \cup \{\bar{y} = \bar{y}_0\}$. This implies that if a $\pm v$-jump of a critical point at infinity is very large, the critical point at infinity is false because the conditions of Proposition 21 of [2], p112, are violated. A bound on the size of the $\pm v$-jumps, as well as a sizable decrease in the value of the functional at infinity $J_\infty$ as soon as a $\pm v$-jump is "too large", implies the verification of the Palais–Smale condition for a related suitable pseudo-gradient flow for $J_\infty$. This pseudo-gradient will introduce companions to the existing large $\pm v$-jumps. The decrease, with each companion, is "sizable" and this implies the verification of the Palais–Smale condition.

However, this condition, using the results of the present paper, is also verified *without* the introduction of companions. The proof is more involved. This is established in [8, section 9, Appendix 2].

## 9 Computation of homology for the first exotic contact form of J. Gonzalo and F. Varela

We complete now the last three steps to compute the homology for the first exotic contact *form* of Gonzalo and Varela [15]:

### 9.1 Arrows and conclusion:

With the use of the Fredholm violation, see Sects. 8.1, 8.2, 8.3 and Sects. 15.1, 15.2, below, we have built a flow such that all flow-lines originating at periodic orbits avoid critical points at infinity (including those having "Dirac masses" in them).

In particular, dominations $w_m - (\delta + w_{m-2})^\infty$, with $w_j$ denoting a periodic orbit of index $j$, do not exist. Again, as pointed out in the Sect. 1, $(\delta + w_j)^\infty$ stands for the periodic orbit $w_j$ with a "Dirac mass" built on it.

Manipulation of these flow-lines, with the insertion of additional "Dirac masses" along $(\delta + w_{m-2})^\infty$, as in Sects. 8.1 and 8.2, is allowed: we are considering $(\delta + w_{m-2})^\infty$ vis a vis $w_m$, not $(\delta + w_{m-1})^\infty$. These flow-lines are not covered by the assumption that the deformation is "symplectic" or "Fredholm".

It follows that we have $\partial_{\text{per}} \circ \partial_{\text{per}} = 0$ and a homology related to periodic orbits can be defined.

Let us compute this homology for the contact form $\alpha$. We will prove in Sect. 10 that this homology is invariant under "Fredholm" deformation of the contact form.

Let $A_{2k}$ be the free group generated by the periodic orbits of $\xi$ of index $2k-2$ and let $A_{2k-1}$ be the free group generated by the periodic orbits of index $2k-1$.

The Fredholm assumption is (strongly) violated only outside $T_0 \cup \{\bar{y} = \bar{y}_0\}$. However, a generical perturbation of $\alpha$ into $\lambda\alpha$ transforms the set of conjugate points for $\lambda\alpha$ in $\bar{y} = \bar{y}_0$ into a discrete set of curves. We may assume, using again general position, that no periodic orbit or critical point at infinity intersects these curves. In addition, for $\lambda$ generical, periodic orbits and critical points at infinity will cross $T_0$ at points where $R(z)$ is irrational.

Thus, under general position, the intersection operator $\partial_i$ (from $A_i$ to $A_{i-1}$), will verify

$$\partial_i \circ \partial_{i-1} = 0$$

There might be a contribution to the $A_i$ coming from the iterates of the two simple periodic orbits $O_0$ and $O_1$ corresponding to $r_1 = 0$ and $r_2 = 0$, respectively.

Let $H_i$ be the subset of $A_i$ made of periodic orbits of index $i$ that are not iterates of $O_0$ and of $O_1$.

Observe now that the map induced by the restriction of the intersection operator $\partial$, composed with the projection on $H_{2k-1}$:

$$\partial_{2k/2k-1} : H_{2k} \longrightarrow H_{2k-1}$$

is zero. Indeed, by Corollary 7.2 above , this map is zero for the first contact *form* of J. Gonzalo and F. Varela.

Let us now estimate the contribution of the iterates of $O_0$ and of $O_1$: we claim that the $v$-rotation on the simple orbits corresponding to $r_1 = 0$ or $r_2 = 0$ is at least $7\pi$. It follows that the index of the iterate of order $\bar{p}$, $i_{\bar{p}}$, is at least $7\bar{p}$:

The proof of this claim is provided with some further studies of the linking numbers with periodic orbits in [7]. We provide the argument here for the sake of completeness of this paper:

Let us first observe that $O_0$ is elliptic. This follows from the computation of the linearized operator at $O_0$, of the quantity $\tau$ in particular, see [1, p 2], [4, p 21], involved in the formula of the linearized operator $\ddot{\eta} + \eta\tau$. We skip the details of this computation here.

We now consider, in order to establish our claim, the neighboring periodic orbits to, e.g. $O_0$ that are not iterates of $O_0$. These neighboring periodic orbits have associated numbers $(p, q)$ that tend both to $-\infty$ as $r_1$ tends to zero: the ratio $\frac{\tilde{A}}{\tilde{B}}$ is irrational at $r_1 = 0$.

$p$ is the number of counter-clockwise rotations in the "surviving" $(x_3, x_4)$-plane. We thus may consider our neighboring periodic orbits as made of $p$ distinct pieces of nearly closed $\xi_0$-pieces of orbits. Each of this distinct piece converges to the periodic orbit $O_0$ as $r_1$ tends to zero.

We consider some base point $x_0$ on $O_0$. We pick up $v$ at $x_0$, equal, therefore, to $v(x_0)$ and we $\xi_0$-transport it around the periodic orbit $O_0$ over $p$-revolutions. This transported vector is denoted $u = u(s)$, where $s$ is the running parameter over the periodic orbit $O_0$, based at $x_0$ and iterated an infinite number of times. Over each of these $p\xi$-pieces, $u(s)$ will coincide with $v$ a certain number of times. This number of times can be $n$ or $n-1$, where $n$ is the $H_0^1$-index of $O_0$, with no base point assigned, that is, starting from any point of $O_0$, $v$ turns more than $n\pi$ and less than $(n+1)\pi$ over $O_0$.

On the approaching $\xi_0$-orbits, we can take a base point close to $x_0$ and define a $\xi_0$ transported vector $\hat{u}(s)$, equal to $v$ at the base point. Using continuity, $v$ will coincide, on each of the $p$-pieces of $\xi$-orbit with $\hat{u}(s)$ at most $n$-times. It follows that on the whole approaching $\xi_0$ periodic orbit, $v$ will coincide with the transported vector $\hat{u}(s)$ at most $pn$-times. The index of this periodic orbit is then less than or equal to $pn+1$, since it is less than or equal to $pn$ under the constraint that the variation of the curve is along $v$ at the base point.

Thus, the ratio of the index $i_p$ to $p$ is less than or equal to $\frac{pn+1}{p}$. It limit-sup, as $p$ tends to infinity is, therefore, less than or equal to $n$. The ratio $\frac{i_p}{p}$ is equal to $\frac{-2(\tilde{A}-\tilde{B})}{\tilde{A}}$ at the periodic orbit. This ratio is $2\pi$ at $O_0$. It follows that $n$ is strictly larger than 6. The claim follows.

Next, we claim that:

Recalling that $H_{2k}$ is the set of periodic orbits of index $2k$ of $\xi_0$ that are not iterates of $O_0$ and $O_1$, we define $n_k$ to be its cardinal. Then, $n_{k-1} + 4 \geq n_k \geq n_{k-1} + 2$ as $k$ tends to infinity.

Indeed, for $r_2 \geq \frac{1}{2}$, we consider the ratio of the index $i$ to the number $q$. This ratio is equal to $\frac{-2(\tilde{A}-\tilde{B})}{\tilde{A}}$. The minimum $m$ of this function on $[\frac{1}{2}, 1]$ is strictly larger than 1 and strictly less than 2.

It follows that if $\frac{i}{q+1} \lnsim m \leq \frac{i}{q}$, then $\frac{i+2}{q+3} \lnsim \frac{i}{q+1} \lnsim m \leq \frac{i}{q} \leq \frac{i+2}{q+1}$ for $p$ or $q$ large enough. There is at least one more periodic orbit in $H_{i+2}$ with respect to $H_i$ in the $r_2$-interval $[\frac{1}{2}, 1]$, maybe 2. The claim follows using the symmetry between $r_1$ and $r_2$.

We are now ready to prove Theorems 1.1, 1.2 and 1.3:

We consider the periodic orbits of prescribed index $i$. This set has been denoted above $A_i$. $A_i$ is made of two subsets. To see this, we first consider the odd index $(2k-1)$. Then $A_{2k-1}$ is made of the periodic orbits of index $(2k-1)$ with $0 \lneq r_1 \lneq 1$ and of the iterates of the elliptic orbits $O_0$ and $O_1$ (corresponding to $r_1 = 0$ and $r_2 = 0$, respectively). The set of periodic orbits of index $(2k-1)$ with $0 \lneq r_1 \lneq 1$ is in one to one correspondence with the set $H_{2k}$ of periodic orbits of index $2k$ introduced earlier. The iterates of $O_0$ and $O_1$ have a strictly increasing index since the $v$-rotation on each of them is larger than $3\pi$, so that their index is at least 7, see our first claim above. Therefore, there are either two iterates contributing to the index $i$ or none. Their contribution at each index $i$ is denoted $K_i$.

Thus, $C_{2k-1}$ is made of $H_{2k-1}$ that has as many elements as $H_{2k}$ and of $K_{2k-1}$, that is empty or has two elements which are iterates of $O_0$ and $O_1$.

The same conclusion applies to $A_{2k}$.

By Corollary 7.2, the intersection operator from $H_{2k}$ to $H_{2k-1}$ is zero. Furthermore, by the first claim above, there must be an infinite numbers of intervals of iterations $[p_m, p_m+5]$ where the $K_j = \emptyset$ for $j \in [p_m, p_m+5]$. Considering an odd index $(2l-1)$ in this interval, such that $2l$ and $(2l-2)$ are also in this interval, Theorems 1.1, 1.2 and 1.3 follow now from our second claim above ($H_{2k-1}$ has at least two more elements than $H_{2k-2}$) and from the fact that the intersection operator from $A_{2l} = H_{2l}$ into $A_{2l-1} = H_{2l-1}$ is zero.

It follows that the homology is non-zero for the standard first exotic contact form of J. Gonzalo and F. Varela $\alpha$, for large enough indexes. This homology can be seen, due to the symmetry $(x_1, x_2) \longrightarrow (x_3, x_4)$ that $(\alpha_1, v)$ exhibits, to have at least two generators at the indexes when it is non-zero.

### 9.2 Carrying rotation "around a critical point at infinity"; completion of the proofs of Theorems 1.1, 1.2 and 1.3:

We turn now to the critical points at infinity. We ruled out, by means of general and theoretical tools, in [3] and using also our observations above about the Fredholm aspects of this problem, see also our "final observations" in Sect. 15, below, that they would interfere with the homology that we defined, for most of them.

Some are left though; they are as follows: none of their $\xi$-pieces, degenerate or non-degenerate, can be of large $H_0^1$-index, because it would follow then that the Fredholm condition is violated and the topological classes can be "moved" below the critical level defined by these critical points at infinity. Two characteristic $\xi$-pieces of non-zero strict $H_0^1$-index cannot follow each other; this follows from [3, Proposition 21, p 518]. Furthermore, non-degenerate $\xi$-pieces cannot be of $H_0^1$-index 2 or more. Otherwise, we can, arguing as in [2, Proposition 15 and Lemma 11, pp 81–102], modify the number of zeros of $b$ on the unstable manifold of the associated cycle.

Let also $n$ be the number of characteristic $\xi$-pieces of such a critical point at infinity of non-zero strict $H_0^1$-index. A critical point at infinity supports several cycles of different dimensions if it has characteristic pieces, see [3, p 518 and 532]. Let $\ell$ be the number of full $H_0^1$-half-unstable manifolds of characteristic $\xi$-pieces used in a combined way to define the cycle associated with this critical point at infinity that is interfering with the homology.

Using [3, pp 507–508, also Theorem 1 p 478], we must have[10]:

$$\ell \geq n - 2$$

Also, denoting $i_0$ the strict $H_0^1$-index of this critical point at infinity and denoting $\gamma$, as in [2] and as in [3], the number of non-degenerate $\xi$-pieces (each of $H_0^1$-index $i_0^j$, $j = 1, \ldots, s$, contributing $i_0^j + 1$ to the total number of zeros of $b$, see [2, p 78] and [3, p 469]), we have ([2, pp 138–139], [3, pp 513–516]):

[10] Theorem 1.1 and Theorem 1.1' (see Sect. 15, below about a line that has been omitted in [3] from the statement of this theorem) are based on Hypothesis 2(B) of [3]. This assumption can be seen to hold, because, on the one hand, one can use "decreasing normals", see [3, p 482, also Appendix 3], on small $\pm v$-jumps and their families in order to bring them to be "comparable". Also the one-parameter group of $\xi$ $\psi_s$ near the $\xi$-pieces of the critical points at infinity can be brought to satisfy a uniform estimate of the type $||d\psi_s|| \leq 1$. If there are a large number of characteristic $\xi$-pieces with a bounded $H_0^1$-index, one of them will be such the related decreasing normal will then have the smallest norm among all these characteristic $\xi$-pieces; more precisely, "moving" some families away from some nodal zones, see [3, p 502], will increase the functional $J_\infty$ by the smallest amount (when compared to the same "movement" on the other, similar characteristic $\xi$-pieces that are in a large number). We can use this characteristic $\xi$-piece to complete the operation of "moving away", p 502 of [3], whereas we can decrease in size, again as in [3, p 502], the (small)-$\pm v$-jumps of the other characteristic $\xi$-pieces. The rate of decrease, i.e. the effect of the normal, is larger on these other characteristic $\xi$-pieces. This allows to complete the modifications that we need, so that Theorems 1.1 and 1.1' hold, without this assumption.

$$i_0 + \gamma + 2\ell = 2k$$
$$i_0 + i_\infty + \ell = 2k(+1)$$

$i_\infty$ is the index at infinity of the critical point at infinity [2, pp 109–126]. $2k$ or $(2k+1)$ is the index of the cycle that we are considering.

Since the index of each $\xi$-piece is a priori bounded, there must be, as $k$ tends to $\infty$, a large number of $\xi$-pieces. Let us first assume that these critical points at infinity are simple. We claim that there must be a fixed positive constant $c$ such that

**Proposition 9.1**

$$\ell \geq ck$$

*as k tends to* $\infty$.

*Proof of Proposition* 9.1 Indeed, if the estimate of Proposition 9.1 does not hold, then $i_0 + \gamma$ is comparable to $2k$. More specifically, because the $H_0^1$-index of each characteristic $\xi$-piece is a priori bounded, $i_0^f + \gamma$, where $i_0^f$ is the $H_0^1$-index of the non-characteristic $\xi$-pieces, is comparable to $2k$. Therefore, $\frac{i_0^f+\gamma}{\ell}$ tends to $\infty$ with $k$. It follows that there is a sequence of consecutive non-characteristic $\xi$-pieces such their the sum of the $i_0^j + \gamma_j$ over all these consecutive $\xi$-pieces tends to $\infty$.

Because these $\xi$-pieces are non-characteristic, the $\pm v$-jumps connecting them are $\pm v$-jumps between conjugate points. We may assume that $i_0^j$ is at most 1 on any of these $\xi$-pieces; otherwise, the argument used in [2, Proposition 15], allows us to change the maximal number of zeros of $b$ on the unstable manifold of our critical point at infinity, thereby canceling its interference with the homology. The argument has been made in [2, pp 81–102]; but we will repeat it at the end of this proof, for the sake of completeness.

Since the sum of $i_0^j + \gamma_j$ over all the family that we are considering is large, we may assume that either $\gamma_j$ is non-zero or $i_0^j$ is 1 on a large number of these $\xi$-pieces.

Picking up on of these $\xi$-pieces, we start, following the procedure of [2, p 96, Lemma 11], to "pile up" $v$-rotation from other neighboring $\xi$-pieces on this $\xi$-piece. This is a process through which some $v$-rotation in a $\xi$-transported frame is "transferred" from a $\xi$-piece to another one, without change in the global transport map of the critical point at infinity. Since there are many neighboring $\xi$-pieces, there is a large amount of rotation available to be transferred from these $\xi$-pieces to this $\xi$-piece, without changing the $H_0^1$-index of these neighboring $\xi$-pieces: if the $H_0^1$-index is $i_0^j$, the $v$-rotation is $i_0^j\pi + \alpha_j, \alpha_j \gneqq 0$. We can show that $\alpha_j$ is bounded away from zero. The claim follows.

It is unclear though that this large amount of rotation remains large when transported on the $\xi$-piece that we have singled out because the partial transport maps, from one $\xi$-piece to the other $\xi$-piece, enter into the argument. The key observation here states that, if the rotation is small, close to zero, when transported from one $\xi$-piece to the other one, it is then "large", larger than a fixed positive constant $\theta_0$, provided that the determinant of the partial transport map from one $\xi$-piece to the other one is positive. This is established below (Proposition 9.2).

The assumption about the determinant of the partial transport maps, that is the fact that we can take it to be positive on a large number of these non-degenerate $\xi$-pieces is easy to verify.

It follows that the $H_0^1$-index is 1 on both of these $\xi$-pieces; we can create, iterating this transport of rotation process, an additional rotation of $\pi$ on one of them, thereby crossing the index 2. If $\gamma_j$ is initially zero on both $\xi$-pieces, we are done. Otherwise, we might need to add more rotation. This is done as follows: we may assume that we have completed this process on a large number of distinct $\xi$-pieces, which, therefore, now carry a $v$-rotation equal to $2\pi = \alpha_j, \alpha_j$ positive, bounded away from zero.

We now repeat the argument used above. Either "piling up" a portion of the $\alpha_j$-$v$-rotation from these $\xi$-pieces on one of them that we would have singled out, we find that the $v$-rotation, once transported, is larger than $\pi$. We are done. Otherwise, there is a $\xi$-piece such that the transport of the $v$-rotation from this $\xi$-piece to the one that we have singled out is very small. Then, completing the transport from the latter to the former, we conclude. □

It follows that $\ell$ must be larger than $ck$, where $c$ is a given positive constant, independent of $k$. Using the estimate introduced in Theorem 1.3 on the action of the periodic orbits, we find that, if such a critical point at infinity interferes with the homology, then it must be that the number of non-degenerate $\xi$-pieces separating two consecutive characteristic $\xi$-pieces is a priori bounded by a given integer $n_0$, independent of $k$ *in average*.

We now can assume that we have a large number of sequences of $n_0$ or less non-degenerate $\xi$-pieces separating two consecutive characteristic $\xi$-pieces. We also can assume, without loss of generality, that these $\xi$-pieces are all of $H_0^1$-index 0 with $\gamma_j = 0$, that is with their ingoing and outgoing $v$-jumps having the same orientation.

Assuming now that the size of the $\pm v$-jumps between them is bounded independently of $k$, or that the $v$-transport maps between the hyper-surfaces of conjugate points where the edges of these consecutive non-degenerate $\xi$-pieces sit are bounded, we prove that we can transfer $v$-rotation so that all of a sequence becomes a sequence of characteristic $\xi$-pieces of strict $H_0^1$-index 0. Using then Proposition 21, p 518, of [3], we derive that the cycle that we are considering is a boundary, without increase in the number of zeros.

Let us observe here that, within the framework of the, e.g. first exotic contact structure of J. Gonzalo and F. Varela, the assumption that the $v$-transport maps associated with the $\pm v$-jumps of genuine critical points at infinity (in particular, critical points that are not "false" in the sense of [2, Proposition 21, p112]) are bounded holds true: this claim has been established above (Propositions 8.1 and 8.2 and their proofs, also Sect. 8.4).

In order to conclude this set of observations about the behavior of the critical points at infinity that might interfere with our homology, we now consider two non-degenerate $\xi$-pieces and we use the technique of [2, pp 81–102], to transport $\xi$-rotation from one $\xi$-piece to the other one. The following technical result was used in our arguments above; we prove it now:

**Proposition 9.2** *Assume that the rotation derived by transport from one non-degenerate $\xi$-piece to the other one, is small, close to zero. Then the rotation derived by transport from the latter non-degenerate $\xi$-piece to the former one is larger than a fixed positive constant $\theta_0$, provided that the determinant of the partial transport map from one $\xi$-piece to the other one is positive.*

*Proof* We consider the matrix of transport, from one $\xi$-piece to the other one. These transport matrices are the ones defined in [2, pp 78–79 and pp 134–136]; typically, they map a point from a characteristic hyper-surface of conjugate points into a conjugate point, but there are more general forms of these maps, see, e.g. [2, pp 134–136]. This transport matrix is adjusted using the one-parameter group of $\xi$ so that its differential maps $\ker\alpha$ at the end-point of a $\xi$-piece into $\ker\alpha$ at the starting point of the target $\xi$-piece.

Let $P$ be the related matrix. When we transport a $v$-amount of rotation equal to $\theta$ from the first $\xi$-piece to the second one, we need to consider on the second $\xi$-piece, see [2, pp 82–96], the matrix $PR_\theta P^{-1}$. Indeed, if we want that the Poincare return map of the critical point at infinity does not change, $\xi$ needs to be modified in the vicinity of the target $\xi$-piece so that the portion of the transport map related to this target $\xi$-piece changes from Id to $PR_\theta P^{-1}$.

We know how to complete such a transformation, see [2, pp 81–87],[11]if $P\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$. $P$ does not necessarily satisfy this.

However, we observe that for every rotation matrix $R_\gamma$ and every non-zero constant $c$,

$$P(cR_\gamma)R_\theta(cR_\gamma)^{-1}P^{-1} = PR_\theta P^{-1}$$

We can, therefore, replace $P$ by $P(cR_\gamma)$. Completing an appropriate transformation of this type, we may assume from the onset in our arguments that $P$ has the more special form used in [2, pp 81–85] ($P$ has positive determinant by choice of the $\xi$-pieces):

$$P = \begin{pmatrix}1 & \delta\\ 0 & \gamma^2\end{pmatrix}$$

We compare the rotation that $v = \begin{pmatrix}1\\0\end{pmatrix}$ completes under $PR_\theta P^{-1}$ to the rotation that the same $v$ completes under $P^{-1}R_\theta P$. $\theta$ is any given positive number. The only constraint on $\theta$ is that $\theta$ should be less than $\alpha_j$, if we are removing rotation from the $j$th-non-degenerate $\xi$-piece to transport it to another non-degenerate $\xi$-piece. In this way, $\beta$ remains a contact form with the same orientation than $\alpha$.

[11] There is no need to worry about preserving, through the modification, the fact that $\beta \wedge d\beta$ is a contact form with the same orientation than $\alpha$: we are adding rotation on this $\xi$-piece and we are subtracting it from the first one in an amount less than the total $v$-rotation in the $\xi$-transport that it supports.

Writing explicitly the two matrices, namely $PR_\theta P^{-1}$ and $P^{-1}R_\theta P$, comparing their action on $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$, using the fact that we may assume that $\alpha_j$ is bounded away from zero (this claim requires a proof. When the number of non-degenerate $\xi$-pieces is large, the proof is straightforward for a large number of them for a generic contact form of the contact structure), we derive the result.

Considering now critical points at infinity as above, but that are iterates of order $m$, we observe that the arguments above extend with little modification if $m = o(k)$.

On the other hand, the value of the action functional on these critical points at infinity is $O(k)$ because they are assumed to interfere with the homology. We are assuming that condition (H) holds; that is that the critical values for the contact form that we are considering in the first exotic contact structure of J.Gonzalo and F.Varela for the index $2k(+1)$ are $Ck$ as $k$ tends to $\infty$. $C$ is a given, a priori bounded, constant. This assumption is verified by the first exotic contact form $\alpha$ of J. Gonzalo and F. Varela and by $C^1$-perturbations of this contact form.

This implies that $m = O(k)$.[12]

Once we know that $m$ and $k$ are of the same order as $k$ tends to $\infty$, we derive that the simple critical point at infinity associated with this iterated critical point at infinity must have a bounded action as well as a bounded number of $\xi$-pieces (this follows from estimates on the action; it can also be partially derived—using bounds on $\ell + i_0 + \gamma$—as above, using the arguments of Propositions 9.1 and 9.2). It follows, using again the arguments of 8.4, that these critical points at infinity must be in a compact set and, therefore, they do not substantially modify the value of the homology groups since their dimension for odd indexes $2k + 1$ tends to $\infty$ with $k$. Our claims follow, that is the proofs of Theorems 1.1, 1.2 and 1.3 are complete. The case when the function $\lambda$ is generic implies the general case, without this assumption, by a simple perturbation argument. The action of the periodic orbits of Morse index $k$ is a priori bounded under the assumptions of Theorem 1.3.

□

## 10 Discussion of the invariance of the intersection operator ∂ under deformations of the contact form

As we complete deformations of the contact form over the same contact structure, under the assumptions that underline our approach, we would like to derive that the homology that we define does not change.

In the classical "compact" variational theory, the invariance of the homology associated with the intersection operator comes "for free". The intersection operator itself is greatly affected by isotopies in the definition of the pseudo-gradient flow for the variational problem. However, the final algebraic calculation ignores these transitional changes. The result is independent of the flow and is the same even if we continuously deform compact variational problems.

This conclusion is less obvious in our framework, because we are considering only part of the intersection operator. It is natural to think that the critical points at infinity can interfere with this part of the intersection operator that we use to compute this homology along isotopies of flows and deformation of contact forms.

If a critical point at infinity interferes with our special flow-lines, then it might be of several types: it might be that, along isotopies, a critical point at infinity of index $m$ is dominated by a chain of our homology of dimension $m$ also, whereas this critical point at infinity also dominates another chain of our homology of dimension $m-1$; or that it dominates a chain (of our homology again) of dimension $m$, whereas it is dominated by a chain (of our homology) of dimension $m + 1$.

Chains of our homology are unstable manifolds of periodic orbits.

It follows that the critical points at infinity interfering with our homology along isotopies are subject exactly to the same conditions as the ones that interfered with the "triangles" in our variational problem, that is the ones that interfered with the verification of the relation $\partial \circ \partial = 0$ in our homology.

The arguments of "Compactness" in [3] and the use of the Fredholm violation allow us to define deformations that "bypass" the critical points at infinity, see Sects. 8.1, 8.2, 8.3 above and Sects. 15.1, 15.2 below.

[12] This estimate could also be derived from the knowledge of the fact that, after a given, a priori bounded, number of iterations, the Morse index of the critical point at infinity is to the least 4; we could then infer that the index of the iterates of order $m$ is of the order of $m$ for $m$ large. We could even allow that such a property holds once a compact set of simple critical points at infinity is excluded: the results about the value of the homology are not modified by a finite set of critical points at infinity and their iterates. Observe that, by the arguments of Sect. 8.4, the $\pm v$-jumps of these critical points at infinity cannot be too large; otherwise, these critical points at infinity are "false" [2, p 112] and they do not interfere with the homology.

We are assuming in addition that, as we "bypass" these critical points at infinity, no tangency occurs between $W_u(x_m)$ and $W_s((\delta + x_{m-1})^\infty)$, that is that our deformation is "Fredholm", see the Sect. 1. $x_s$ stands here for a periodic orbit of index $s$, $(\delta + x_s)^\infty$ stands for the shadow critical point at infinity built with the addition of a "Dirac mass" along the periodic orbit $x_s$. We do not know whether this "Fredholm" condition is verified.

We are left with cancelations of periodic orbits.

Since the flow-lines originating at periodic orbits avoid critical points at infinity, they involve only periodic orbits.

That the related homology does not change along the deformation hinges then upon the fact that the cancelations of periodic orbits happen "normally". Again, the "Fredholm" or the "symplectic" assumption that we are requiring on the deformation implies that these cancelations occur "normally" or "classically", that is as in the usual, compact, Morse Theory, between periodic orbits,

We are left with the possibility that a periodic orbit could cancel with a critical point at infinity. Following [2, pp 103–107], this cannot happen if the periodic orbit does not degenerate. The cancelation then takes place between two periodic orbits of indexes $m$ and $(m-1)$. It is, however, true that critical points at infinity come to collapse with the degenerating periodic orbits.

These critical points at infinity must have all their $\xi$-pieces characteristic: this follows from the fact that their $\pm v$-jumps are small.

Using the results of the Appendix, section 16, below, they cannot be in $\Gamma_2$. We can then introduce a companion to one of their $\pm v$-jumps—there are at least two of them—and "bypass" them. Since there is more than one $\pm v$-jump, we can introduce this companion and still spare a $*$ along the deformation as a single $\pm v$-jump: we have a choice. This will be useful in Sect. 15.1. The flow then ignores these false critical points at infinity.

Invariance of the homology through "Fredholm" deformations follows.

Another approach to this difficult question of invariance of the $\partial$-operator through deformation of contact forms is studied in [6].

## 11 Flow-lines at infinity, estimates on the number of $\pm v$-jumps, linking numbers and flow-lines at infinity

11.1 Flow-lines at infinity, estimates on the number of $\pm v$-jumps:

The estimate (ii) in Lemma 3 of [13] reads

$$\frac{\partial}{\partial s}\left(\int (|b| - \nu)^+\right) \leq -\frac{C(a+1)}{\nu}\frac{\partial a}{\partial s}(**)$$

This estimate is concerned with the first part of the flow, that is with the semi-flow $\frac{\partial x}{\partial s} = Z(x)$. It is not concerned with the flow at infinity. Along the flow-lines of the flow of [13], additional $\pm v$-jumps are created with the first part of the flow, not with the flow at infinity. The flow at infinity does not create in general new $\pm v$-jumps in the curves it deforms, except under two occurrences that we will discuss below.

Let us first estimate, considering a (semi)-flow-line (combining the two portions of the flow, $Z(x)$ and also the flow at infinity) that starts at a periodic orbit of index $i$ on the torus $T_{\bar{t}}$ and that ends at a periodic orbit of index $(i-2)$ on the torus $T_{\underline{t}}$, the number of $\pm v$-jumps that can be created by the flow of $Z(x)$. The indexes contain the degeneracies. We are studying here flow-lines connecting based periodic orbits of index $(i-1)$ with circles of periodic orbits of index $(i-2)$.

$(**)$ implies that at the blow-up time of $\frac{\partial x}{\partial s} = Z(x)$, if any, the following estimate holds:

$$c_0 N \leq C[i^2(\mathrm{cv}(\bar{t}) - \mathrm{cv}(\underline{t})) + C_1]$$

$N$ designates here the number of "large" ($\leq c_0$) $\pm v$-jumps of the curve at the blow-up time. $N$ designates by extension the maximal number of pieces of $\pm v$-orbits, of length $\ell$, $\frac{c_0}{2} \leq \ell \leq c_0$, that we can create with the $\pm v$-jumps of the curve.

It follows that, if $i$ is large and if $\mathrm{cv}(\bar{t}) - \inf \mathrm{cv}(t)$ is small ($\leq \delta'$), then

$$N \leq \delta i^2 (***)$$

with $\delta$ small, $\delta$ tends to zero with $\delta'$ if $i$ tends to $\infty$. Ideally, we would like to replace $i^2$ by $i$. As we will see, the flow at infinity will obey an estimate of the type of $(***)$, but with $i$ in lieu of $i^2$. This is a much better estimate that would allow us, using the tool provided by the linking number, to study flow-lines connecting periodic orbits. We are not able to establish this better estimate.

We prove however below that the flow at infinity satisfies this estimate.

We also discuss throughout this section the behavior of $L$, $L^*$, $L + L^*$ along the flow-lines at infinity. $L$ and $L^*$ have been defined at the end of Sect. 7 above. Their definition involves the choice of tori of perodic orbits of $\xi T^1$ and $T^{1*}$. This will be completed in two steps:

In a first step (11.1 and 11.2), we assume that none of the $\xi$-pieces of the curves under consideration is on $T^1$ and $T^{1*}$. The case when one of the nearly $\xi$-pieces is close to $T^1$ and $T^{1*}$ is discussed later in a second step, in Sect. 11.4

The reason to study also $L + L^*$ independently of $L$ and $L^*$ is that, if we consider a periodic orbit of $\xi$ $PO_2$ such that $r_2(PO_2) \lesssim r_2(T^1)$, then $L + L^*(PO_1) = -p_1(p_2 - q_2)$ where $(p_1, q_1)$ are the integers associated with $(PO_1)$ in Sect. 7 and $(p_2, q_2)$ are associated, with $(PO_2)$. This also reads $-p_1 \times \frac{i_2}{2}$. $i_2$ is the Morse index of $PO_2$; thus, $L + L^*$ carries a direct relationship with the Morse index, a key notion in Variational Theory.

## 11.2 Flow-lines at infinity, estimates on the number of $\pm v$-jumps, non-decreasing property of $L + L^*$

The flow at infinity is built through a sequence of Lemmata, ranging from Lemma 11.1 below until Lemma 11.8 of Sect. 11.2. $\nu$ is in Lemma 11.1 below a small positive constant, $\nu \ll c_0$; it is the constant used in the definition of the semi-flow of [10].

**Lemma 11.1** *Let $\bar{x}$ be a curve of the $\Gamma_{2k}$s. Assume that none of the $\xi$-pieces of $\bar{x}$ is neither on $T^1$ and $T^{1*}$.*

- (i) *If $\bar{x}$ has two non-characteristic $\xi$-pieces belonging to two different tori $T_t$, $T_{t'}$, the flow at infinity can be defined so that it decreases $J_\infty$ and does not decrease $L$ and $L^*$.*
- (ii) *If every $\pm v$-jump of $\bar{x}$ that intersects $T^1$ and $T^{1*}$ an equal number of times and vice-versa, the flow at infinity at this curve decreases $J_\infty$ and does not decrease $L + L^*$.*
- (iii) *Assuming that a curve of the $\Gamma_{2k}$s has two $\xi$-pieces, one of them to the least being a characteristic $\xi$-piece, on two different tori $T_t$, $T_{t'}$, the flow at infinity can be defined so that it decreases $J_\infty$ and does not decrease $L$ and $L^*$. Furthermore, on all the curves having two $\xi$-pieces (characteristic or non-characteristic) on two different tori $T_t$ and $T_{t'}$, with $|t - t'| \geq \sqrt{\nu}$, the following differential inequality holds:*

$$\frac{\partial \int_0^1 (|b| - \nu)^+}{\partial s} \leq -\frac{C}{\nu} \frac{\partial \left( \int_0^1 \alpha_x(\dot{x})\, dt \right)}{\partial s}$$

*$C$ is above a uniform constant. This flow can be extended to an appropriate neighborhood of these curves in $C_\beta^+$ with the same differential inequality.*

This implies the following corollary: let $T_{t_0}$ be the torus $T^1$ ($t_0 \leq \frac{1}{2}$).

**Corollary 11.2** *Under the same assumptions than Lemma 11.1, the flow at infinity can be defined so that its rest points have their $\xi$-pieces on tori $T_t$ with $t \in [t_0 - \nu, 1 - t_0 + \nu]$. More precisely, one of its $\xi$-pieces lies on a $T_t$ with $t \in [t_0, 1 - t_0]$ and all the other $\xi$-pieces lie on tori $T_s$ with $|s - t| \leq \nu$. Furthermore, $(***)$ holds for the combination of $Z$ and this flow at infinity and $L$ and $L^*$ do not decrease along the flow-lines of this combination.*

*Proof of Lemma* 11.1 (i) of Lemma 11.1 follows from the construction of a decreasing deformation using the vector field $X_0$. This vector field commutes to $\xi$ and to $v$ and, therefore, can be transported between the two non-characteristic $\xi$-pieces of the rest point that do not belong to the same torus. This transport takes place between the end $x^+$ of the first $\xi$-piece (one is before the other one) and the beginning $x^-$ of the second one. $\alpha_{x^+}(X_0)$ is not equal to $\alpha_{x^-}(X_0)$. A tangent vector at infinity to our rest point, without increase in the number of $\pm v$-jumps, can then be defined using the $\pm v$-jumps at the other end of each of these $\xi$-pieces, see [2,4], perhaps the best reference is [3, section 4.2] (the construction of normals; however, there is an increase in the number of $\pm v$-jumps with normals, but the argument adapts, without increase for non-characteristic

$\xi$-pieces): we consider $v$ at one edge, $\xi$transport it at the other edge, creating thereby some $[\xi, v]$-component. We then scale $v$ at the edge so that the component over $[\xi, v]$ created in this way matches that of $X_0$. We can then match the $\xi$ and $v$-components of $X_0$ by including in our variation changes $\delta a_i$ in the lengths of the $\xi$-pieces and changes $\delta s_i$ in the length of the $\pm v$-jumps abutting at $x^{\pm}$. We derive a tangent vector $z$ and $\partial J_\infty.z = \alpha_{x^+}(X_0) - \alpha_{x^-}(X_0) \neq 0$. Since $L$ and $L^*$ are derived by averaging the linking with the periodic orbits of $T^1$, $T^1_*$ which are derived one from the other through the circle action of $X_0$, $L$ and $L^*$ do not change under a $\pm z$-variation (assuming the $\xi$-pieces are not on $T^1$, $T^1_*$). (i) follows.

(ii) follows from Lemma 5.3.

The proof of the first part of (iii) is very close to the proof of (ii), only that the construction of $z$ cannot be the same: the $\xi$-pieces can be characteristic and , therefore, they might not offer the freedom to create a $[\xi, v]$-component at their other edge by taking $v$ at one edge. Such a $v$ must come from a new $\pm v$-jump located inside the characteristic piece. A $z$ is defined in this way. Again, $L$ and $L^*$ do not change, whereas $\delta J_\infty.z = \alpha_{x^+}(X_0) - \alpha_{x^-}(X_0)$ is larger than or equal to $c_3|t-t'|$. $c_3$ is a uniform, appropriate positive constant. The construction of $z$ can be completed in a neighborhood of the rest curves in $\cup\Gamma_{2s}$. For the second part of (iii) and the differential inequality, we observe that, under all circumstances, we can decide to locate the small $\pm v$-jumps that we use to "compensate" after $\xi$-transport the $-[\xi, v]$-component of $X_0$ at a given edge at a distance, measured in terms of $v$-rotation along $\xi$, less than $\frac{\pi}{4}$. The flow introduces additional $\pm v$-jumps. But it also changes the size of the existing $\pm v$-jumps of the curve along which $X_0$ is $v$-transported. The $\xi$-transport of a vector reads on the $\eta$, $\mu$-component, see [2,3, p 468]:

$$\dot{\mu} + \eta\tau = 0, \dot{\eta} = \mu$$

The construction of this flow is completed by convex-combination of vector fields $z_i$. For each $z_i$, e.g. $z_{i_0}$, the "compensation" process is completed on exactly two $\xi$-pieces and uses for each of them one edge. The first one is denoted $A$ and the second one $B$. Let $A_1$ and $B_1$ be the two points inside the $\xi$-pieces at which the additional small $\pm v$-jumps are created. Using the above transport equations, we find

$$|\mu(A_1) - \mu(A)| + |\mu(B_1) - \mu(B)| \leq C[|\eta(A)| + |\eta(B)|$$

$\mu(A)$ and $\mu(B)$ are, on the other hand, the $v$-components of $\pm X_0$ at $A$ and $B$, respectively; they are $O(1)$, they have the same sign because the rotation is $\leq \frac{\pi}{4}$. $\eta(A)$, $\eta(B)$ are of course $O(1)$.

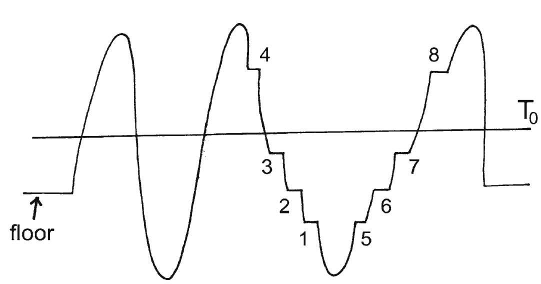
Along a piece of $v$-orbit of length at least $\sqrt{\nu}$, there is a time interval of length $\geq \frac{1}{C'}\sqrt{\nu}$, $C'$ might be large, over which the $\eta$-component of $X_0$ is itself $\geq \frac{1}{C'}\sqrt{\nu}$. It follows that

$$z_{i_0}.\int_0^1 |b| \leq (|\mu(A_1) - \mu(A)| + |\mu(B_1) - \mu(B)| + |\mu(A)| + |\mu(B)|) = O(1) \tag{1}$$

The differential inequality follows for $z_{i_0}$ and, by convex combination, it follows for the flow. □

**Definition 11.3** We denote, given a "rest point" of the flow at infinity, in what follows by the expression "the floor" a value of $t$ which we choose and such that all the $\xi$-pieces of this "rest point", after the use of Corollary 11.2 above, are on tori $T_s$, with $|t - s| \leq \nu$.

We then have




F8

The nearly $\pm v$-jump of F8 above has to go "over" the $\bar{y}$ of the first characteristic hyper-surface; otherwise, it does not hinder the deformation: we are assuming that the torus $T^1$ with respect to which the linking $L + L^*$ is computed (it is computed with respect to $T^1$ and $T^{1*}$) is "above" this $\bar{y}$.

In Lemma 11.4 below, we assume again that none of the $\xi$-pieces involved in the deformation is on $T^1$ or on $T^{1*}$, see Sect. 11.4 below to include this case in our proofs.

**Lemma 11.4** (i) *For any index $j \geq 1$, there is a $J_\infty$-decreasing, $L$ and $L^*$-not decreasing tangent vector $z_j$ that we can build using the small $\xi$-piece numbered $j$. Along $z_j$, when $j = 1$ or $j = 5$, these small $\xi$-pieces move "up" towards the "floor" or decrease in size until they disappear . When $j = 3$ or $j = 7$, $j = 4$, $j = 8$, they move "down" until they reach the "floor", or they decrease in size until they disappear. Assuming that the limit $\pm v$-jump has not $a = 0$ all along itself, all these $\xi$-pieces stay small.*

(ii) *Assume that the torus $T^1$ is "above" the level of the characteristic hyper-surface $\Sigma^1$. For $j = 2$ and $j = 6$, if the $\xi$-pieces at these locations* (*on the "floor"*) *are small enough, not on $T_0$ and once all the other $\xi$-pieces in the other locations* (*that is, not "on the floor"*) *have been canceled or are tiny, there is a $J_\infty$-increasing, $L + L^*$-not decreasing tangent vector $z$.*

**Observation 11.5** *If along a $\pm v$-jump, a is identically zero, then this $\pm v$-jump must be on $T_0$. Accordingly, the "floor" of the "rest point" for the flow at infinity is within the "distance" $v$ from $T_0$. The conclusion of* (i), *that the $\xi$-pieces stay small if they start small, should be understood in the sense that, given a lower-bound for* $\text{Sup}|a|$ *on the limit $\pm v$-jump, these intermediate small $\xi$-pieces need to be "small" enough to start with, in relation to this lower-bound. As* $\text{Sup}|a|$ *tends to zero, the "floor" is very close to $T_0$ and the intermediate $\xi$-pieces are already close to $T_0$. Lemma 11.4 holds then, without further modification or use of a flow at infinity, for these curves.*

**Observation 11.6** *Lemma 11.4 is established under the assumption that the $\xi$-pieces of the curves do not cross $T^1$ or $T^{1*}$. Clearly, this assumption might be satisfied on the initial curve and violated after the use of the deformation defined in Lemma 11.4. We establish below, in Sect. 11.4, that* (i) *of Lemma 11.4 holds after this assumption is removed. This is related to the fact that the flow for* (i) *is built with the use of $X_0$ along the $\pm v$-jumps.*

*Proof of Lemma 11.4* When $j = 1$ or when $j = 5$, we take $-X_0$ at the starting point of the small $\xi$-piece and we transport it along the curve (made of $\pm v$ and of $\xi$-pieces) to the end-point of the $\xi$-piece having $j = 0$. At this end-point, the vector which we derive through this transport is equal to $-X_0$ since $X_0$ commutes with $\xi$ and $v$. We then "compensate" as usual the vectors $-X_0$ defined at these edges, using the $\pm v$-jumps found at the other edges of each $\xi$-piece. We have taken care above of the characteristic $\xi$-pieces: each of them, after the introduction of one or two additional $\pm v$-jumps (two if the characteristic $\xi$-piece is "very long") yields a decrease in $J_\infty$ of an amount equal to $\delta_0$ to the least. The total number of new $\pm v$-jumps introduced in this way is $o(i)$ ($i$is the Morse index) because the flow-lines which we consider include a decrease of $J_\infty$ equal to $o(i)$ to the most.

With this "compensation", we have defined a tangent vector $z_j$. Calculating, we find

$$\partial J_\infty.z_j = \alpha(X_0)_t - \alpha(X_0)_h \lesssim 0$$

$\alpha(X_0)_t$ is the value of $\alpha(X_0)$ at the "tail" or starting point of the small $\xi$-piece located at $j = 1$ or $j = 5$, $\alpha(X_))_h$ is the value of $\alpha(X_0)$ at the "head" or end-point of the $\xi$-piece having $j = 0$. These notations will be used below as well.

Observe that, as we compensate the $[\xi, v]$-component of $-X_0$ at the beginning of the $\xi$-piece having $j = 1$ or $j = 5$, we take a vector $\mu_0 v$ at the end point of this $\xi$-piece and we transport it back, seeking to "compensate" the $[\xi, v]$-component of $-x_0$, along this $\xi$-piece to its starting point. The transport equations (see, e.g. [9, p 468]) yield

$$\dot{\eta} = -\mu$$

Therefore,

$$\text{sgn}(\eta(\text{starting point})) = \text{sgn}(-\mu_0)$$

On the other hand,

$$\begin{aligned} d\alpha(v, [\xi, v]) &= -1 \\ d\alpha(v, -X_0) &= -v.\alpha(X_0) \end{aligned}$$

Assuming that the $\pm v$-jump is oriented, e.g. along $+v$, we find when $j = 1$ that $v.\alpha(X_0)$ is negative. $-X_0$ has, therefore, a negative component along $[\xi, v]$. We thus have to take $\mu_0$ to be negative, clearly large because the $\xi$-piece is so small unless $v.\alpha(X_0)$ is close to zero. It follows that, with $j = 1$, this small $\xi$-piece can only move up. Because $\alpha(-X_0)_t$ is positive, this $\xi$-piece "shortens" at a rate that depends only on $\alpha(X_0)_t - \alpha(X_0)_h \lnapprox 0$. Thus, either this small $\xi$-piece reaches the "floor" or it disappears. The argument adjusts, with identical conclusions, for $j = 5$.

When $j = 3, 4, 7$ or when $j = 8$, we take $X_0$ instead of $-X_0$. We build a $z_j$ and we find that

$$\partial J_\infty.z_j = \alpha(X_0)_h - \alpha(X_0)_t \lnapprox 0$$

Now, the $\xi$-pieces move down. When $j = 4$ or $j = 8$, they "shorten". When $j = 3$ or when $j = 7$, the $\xi$-piece expands, but moves down very fast ($|\mu_0|$ is large: the local piece of curve is essentially a $\pm v$-jump; therefore, $v.\alpha(X_0)$ (or $a$) is "far from zero" in this region; this follows from the assumption, stated in (i) that the limit $\pm v$-jump has not $a = 0$ all along itself). Putting these two steps of the argument together, we find that all these $\xi$-pieces eventually end up on the "floor". In the case where $a$ is zero on the limit $\pm v$-jump, these intermediate small $\xi$-pieces are, "from inception", close to the "floor" since they are close to $T_0$. The statement about $L$ and $L^*$ follows from the fact that this portion of the flow is built using $X_0$ along the $\pm v$-jumps. We will partially remove in Sect. 11.4 the assumption about the $\xi$-pieces of the curves not crossing $T^1$ and $T^{1*}$.

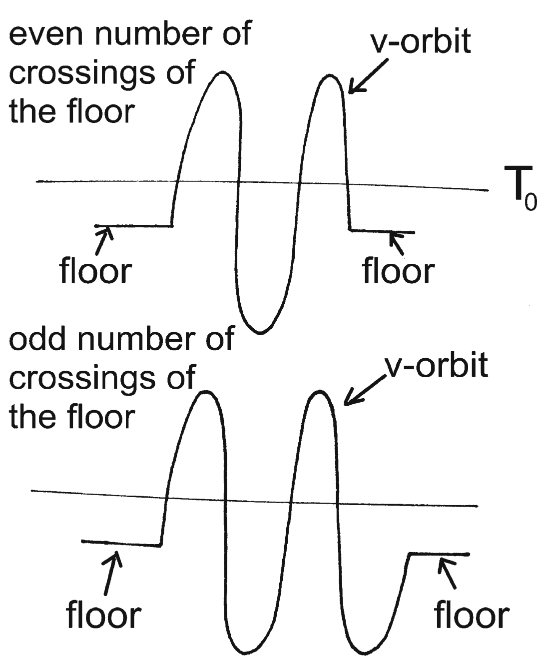
The proof of (i) is thereby complete.

We move now to prove (ii). All the intermediate small $\xi$-pieces are assumed to be on "the floor", that is they are all on the same torus, the torus where the $\xi$-pieces with $j = 0$ or $j = 9$ are. The other ones have been canceled through the process described in (i), or brought to be so small that they do not interfere with our arguments.

Let us assume for simplicity that we have a single intermediate small $\xi$-piece, e.g. with $j = 2$. There are two possible cases: either the $\pm v$-jump changes orientation across this $\xi$-piece or it does not.

Let us assume, for the sake of simplicity, in a first step, that it does not.

Between the end point of the "initial" $\xi$-piece and the starting point of the "final" $j = 9$ $\xi$-piece, the (nearly) $\pm v$-jump has a number of "oscillations", of "crossings" of the "floor".




F9

If the number of inside crossings (ends not counted) of the "floor" by this $\pm v$-jump were odd and if there were no intermediate small $\xi$-pieces, we could infer from Lemma 5.3(i) and (ii) that any transported vector along this $\pm v$-jump does not change $L + L^*$. "Compensating" at the edge pieces as above, we would define a tangent vector $z$. We can take for example $z$ to be $\xi$ at the end point of the "initial" $\xi$-piece. $z$ or $-z$ can be used. If $\partial J_\infty.z$ is non-zero, we find a suitable decreasing pseudo-gradient. If, on the other hand, $\partial J_\infty.z$ is zero, then $\alpha$ is mapped onto itself in the $v$-transport between the end point of the "initial" $\xi$-piece and the starting point of the "final" $\xi$-piece. $\xi$ is mapped onto $\xi + \mu X$. Using the identities for $X.a$ in Proposition 2.3 above, $\mu$ is zero. Arguing as in Lemma 4.2, we find that the end-point of the "initial" $\xi$-piece is on a hyper-surface of conjugate points, with $k = 1$ (as we start, we know through the use of arguments such as the ones used in the proof of lemma that it lies on a hyper-surface of conjugate points after a number of full oscillations. Coming back to the nearest intersections of the $\pm v$-jump with $T_0$, arguing as in the proof of Lemma 4.2, we find that the differential of the $v$-transport map between these extreme points on $T_0$ must be Id. Using then the properties of $X_0$, we can see that the differential of the $v$-transport map for a *single* oscillation must be Id).

However, the oscillations of the $\pm v$-jump must cross $T^1$ and $T^{1*}$ is, by choice, "above" the level of the hyper-surface $\Sigma^1$ of conjugate points with $k = 1$ (after a full oscillation).

We thus see that, if this intermediate small $\xi$-piece were not there, we could build in this case a pseudo-gradient $z$ that would decrease $J$ and not decrease $L + L^*$. When this intermediate small $\xi$-piece is present, we need to modify slightly the argument. We consider $z$ as being "split" $z$ into two pieces, one "before" a small $\xi$-piece reduced to zero and the other one "after" the same $\xi$-piece reduced to zero.

For extremely small (now not necessarily zero) $\xi$-pieces roughly at the same location for very close $\pm v$-jumps, the splitting extends. The first part of the extended $z$ first one is denoted $z^-$, and the second one $z^+$. We scale $z^+$ with a constant $c$ very close to 1 so that $L + L^*$ does not change under the combination of $z^-$ and $z^+$. We also "compensate" $z^-$ and $z^+$ across the small $\xi$-piece. we derive a tangent vector $z'$. It is clear that $\partial J.z' \lneq 0$, whereas $L + L^*$ does not decrease under $z'$. (ii) follows in this case.

If we are now in the case where the number of inside crossings of the "floor" by the (nearly) $\pm v$-jump between the end-point of the "initial" $\xi$-piece and the starting point of the "final" $\xi$-piece is even, we then simply observe that between the intermediate small $\xi$-piece on the "floor" and one of these edges, the number of "crossings" of the "floor" (boundary crossings not included) is again even, so that the above argument extends.

The last case involves a change of orientation of the $\pm v$-jump across the small $\xi$-piece. There are again "initial" and "terminal" edges of the next (preceding and, respectively, following) $\xi$-pieces. If the number of inside crossings on either side is even, we can use the above arguments. If it is odd, then their difference is even. If this difference is non-zero (even), the argument for (i), slightly modified as above, repeats. If it is zero, we are facing a back and forth run along $v$, adding up basically to zero if it were not for the small intermediate $\xi$-piece. If there any direction that can be built, which is $J_\infty$-decreasing and $L + L^*$-not decreasing, we can use it and cancel along it this intermediate small $\xi$-piece and the back and forth run along $v$. □

## 11.3 The non-decreasing property of $L$ and $L^*$ through the flow at infinity:

The basic idea in order to find cycles of the homology of [2] that are not boundaries is to study the flow-lines connecting periodic orbits. We would like to prove that these flow-lines do not exist ideally. The tool for this is the "averaged" linking number $L$ and $L^*$ and its behavior along the decreasing flow-lines of a suitable pseudo-gradient of $J$.

We observed above (Lemma 6.1) that the linking number with a given periodic orbit of $\xi$ did not increase under the first part of our flow.

We then singled out a torus $T^1$. We considered a "curve at infinity" $\bar{x}$ and we computed the linking number of $\bar{x}$ with the "tori $T^1$ and $T^{1*}$, $L$ and $L^*$". $L$ and $L^*$ are the integrals along the an orbit of $X_0$ of the linking numbers of $\bar{x}$ with the periodic orbits of $\xi$ on $T^1$ and $T^{1*}$ respectively.

We proved (Lemma 11.1(ii) and Corollary 11.2) that if each $\pm v$-jump of $\bar{x}$ that intersected $T^1$ intersected also $T^{1*}$, then the flow at infinity could be defined near $\bar{x}$ so that $L + L^*$ did not decrease. However, we did not establish that $L$ and $L^*$, taken each one apart, did not decrease. This is what we establish now:

We assume now that we are given a curve $\bar{x}$ of $\Gamma_{2s}$, with no $\xi$-piece on $T^1$ or on $T^{1*}$, that crosses the torus $T^1$ with one of its $\pm v$-jumps (maybe more) at a point $\bar{x}(t_0)$ and at another point $z(t_1)$, both not end-points of this $\pm v$-jump. We assume that we are given along $\bar{x}$ a "tangent vector" $z$ that might be in the tangent space to $\Gamma_{2s}$ at $\bar{x}$, but might also be an "entering normal" that creates additional small $\pm v$-jumps along the $\xi$-pieces of $\bar{x}$ and therefore "pushes" $\bar{x}$ "inside" a $\Gamma_{2s+2}$ or another $\Gamma_{2s'}$, with $s' \gneq s$. We also assume (for simplicity, this assumption can easily be removed after splitting a general variation $z$ into a sum of $z_i$ satisfying this assumption) that "the support of $z$", that is the range of values of the time parameter $t$ over which $z$ is non-zero, is restricted to the $\pm v$-jump that we singled out and to its two neighboring $\xi$-pieces (the preceding and the following ones). We then claim that

**Lemma 11.7** *$L$ and $L^*$ do not change under $z$.*

*Proof of Lemma 11.7* In order to compute $L$, we choose a two dimensional chain $c$ that has one of the periodic orbits $O_1$ of $\xi$ on $T^1$ as its boundary. Let $c_\tau$ be the result of the action of the one parameter group generated by $X_0$ on $c$ during the time $\tau$. $c_{2\pi}$ is, therefore, $c$, $c_\tau^*$, $c*$ are the symmetric chains. In order to compute $L$ and $L^*$, we integrate the intersection number $\ell(\tau)$ of $c_\tau$ and of $c_\tau^*$ with the curve $\bar{x}$ or the curve "$\bar{x} + \epsilon z$" from 0 to $2\pi$ with respect to the measure $d\tau$.

We first "break" the $\pm v$-jump into two pieces separating $z(t_0)$ and $z(t_1)$. We also break the $\pm v$-jump, if it also crosses $T^{1*}$, into sub-pieces with only one point of intersection with either $T^1$ or $T^{1*}$. Each of them is treated separately, but the argument is the same for all of them. Considering the first piece and its displacement along the tangent vector $z$, we assume first that $z(t_0)$ is parallel to $\xi$. We then denote $z$ as $z_1$. Since $\ell(\tau)$ is the intersection number of $c_\tau$ with "$\bar{x} + \epsilon z_1$" and since the boundary of $c_\tau$ is a periodic orbit of $\xi$ ($\xi$ and $X_0$ commute), $\ell(\tau)$ does not change with $\epsilon$ and, therefore, the differential of $L$, hence of $L$ under $z_1$, $z_1.L$ is zero in this case.

Similarly, if $z$, denoted $z_2$ now, is equal to $CX_0$ along the $\pm v$-jump of $\bar{x}$, $C$ being a constant, $z_2.L$ is zero. Indeed, since the $\xi$-pieces that precede and follow the $\pm v$-jump are on other tori than $T^1$, any change in the intersection number $L$ would be only due to a change of the intersection number of the $\pm v$-jump, under $z_2$, with the $c_\tau$s, after integrating. However, this $\pm v$-jump evolves under the action of the one-parameter group of $X_0$, whereas $\cup c_\tau$ is obtained through the action of this very same one-parameter group on $c$. It follows that the contribution of this $\pm v$-jump to the linking number $\ell(\tau)$ might change, but the value derived after integration does not. $z_2$ can be taken here to be equal to $CX_0$ along this $\pm v$-jump, but the tangent vector might also include a change of length of the $\pm v$-jump, that is the argument works if we replace the assumption on $z_2$ to be equal to $CX_0$ along this $\pm v$-jump by the assumption that $z_2$ is equal to $CX_0 + \lambda v$, where $\lambda$ is an arbitrary smooth function equal to zero in a neighborhood of $t_0$. The conclusion is the same: $z_2.L$ is zero.

We claim now that any "tangent vector" $z$ to $\bar{x}$, subject to the restrictions of Lemma 11.7, can be written as the addition $z_1 + z_2$, $z_1$ and $z_2$, respectively, as above.

Indeed, we first write $z(t_0)$, which we may assume to be tangent to $T_1$ as $A_1\xi + B_1X_0$. The values of $z_1$ and $z_2$ along the $\pm v$-jump are then derived by $v$-transport. Considering then a $\xi$-piece that precedes or follows, we can, if this $\xi$-piece is not characteristic, "compensate" the $[\xi, v]$-components of $z_1$ and $z_2$ at the edges of the $\pm v$-jump by transport along $\xi$ of $\delta v$, with a suitable value of $\delta$. $\delta v$ is taken at the other edge of the $\xi$-piece. If, on the other hand, one or both of these $\xi$-pieces is characteristic, $z$ is then built by introducing an additional (small) $\pm v$-jump somewhere on this $\xi$-piece in order to be able to "compensate" as above its $[\xi, v]$-component at the corresponding edge of the $\pm v$-jump. The location of this additional (small) $\pm v$-jump can then also be used to create each of $z_1$ and $z_2$. The proof of Lemma 11.7 is thereby complete. Of course, any variation at infinity $z$ can be decomposed in smaller pieces that behave as $z$ of Lemma 11.7 above.

The argument above extends to include the process of introduction of a small $\xi$-piece along a $\pm v$-jump (typically, what is completed to "bypass" a false critical point at infinity: see pp 111–112, Proposition 21 of [2]). □

The above argument breaks down if the $\pm v$-jump is tangent to $T^1$ or $T^{1*}$ (if it is tangent to one of them, then, any point of intersection with either of them is a point of tangency). Indeed, then, $z(t_0)$ cannot always be taken tangent to the torus $T^1$ or $T^{1*}$. However, the flow of Lemma 11.1, Corollary 11.2 and Lemma 11.4 is equal to $X_0$, transversally to the $\pm v$-jump. For this flow, $L$ and $L^*$ do not change.

We need to use another flow and, therefore, we need Lemma 11.7 only once the curves have been driven by the flow of Lemma 11.1, Corollary 11.2 and Lemma 11.4 to be nearby a "rest-point" of Definition 11.3. In such a neighborhood, the tangent decreasing directions are taken transverse to $X_0$ along the $\pm v$-jumps, because the use of $X_0$ brings very little decrease for $J_\infty$. Assuming that a $\pm v$-jump of a curve at infinity $x$ close to a "rest-point" is almost tangent to, e.g. $T^1$, that is, $a$ is small at $x(t_0) \in T^1$, a decreasing direction $z$ will along the $\pm v$-jump be transverse to $\text{Span}(X_0, v)$. Taking a neighborhood of this rest-point that has a point of tangency with $T^1$ along its $\pm v$-jump, we find that any decreasing direction $z$, if it exists—it could not exist if the $\pm v$-jump takes place between conjugate points—has at the rest-point a non-zero component on $Y$ near the point of tangency. Convex-combining this $z$, which is defined in a small neighborhood of these special rest-points, as long as there are not critical points at infinity , with the remainder of the flow, we find a global flow. This global flow has a non-zero component on $Y$ at a point of tangency along a $\pm v$-jump for which a direction a decrease exists. For the other $\pm v$-jumps, for which this direction of decrease do not exist, namely those taking place between conjugate points, we do not care about not increasing the number of zeros of $b$ because we need only to find cycles; therefore, we want to know what the intersection numbers are for one given flow; for the other ones, these numbers might be different.

We now study the case of curves having some of their $\xi$-pieces on $T^1$ and $T^{1*}$.

### 11.4 Taking care of curves having one or more $\xi$-pieces on $T^1$ or $T^{1*}$:

If a curve is close to a curve at infinity, that is, close to a curve in the $\Gamma_{2s}$s that has one or more $\xi$-pieces on $T^1$ or on $T^{1*}$, then it becomes more difficult to define a $J/J_\infty$-decreasing deformation that does not decrease $L$ and $L^*$. The reason for this is that the $\xi$-piece on, e.g. $T^1$ could, under the flow, stay on $T^1$ and move along $T^1$, changing the linking numbers with a continuum of periodic orbits of $T^1$. However,

**Lemma 11.8** (i) *The flow of Lemma 11.1, Corollary 11.2 and under the flow of Lemma 11.4*(i) *can be defined so that, if a curve $\bar{x}$ is in the $\Gamma_{2s}$s, $L$ and $L^*$ at $\bar{x}$ do not change along the deformation i.e.:*

$$\frac{\partial L}{\partial s}(\bar{x}) = 0;\ \frac{\partial L^*}{\partial s}(\bar{x}) = 0$$

(ii) *The rest points for this flow at infinity are the curves having all their $\xi$-pieces on $T^1$ (respectively, on $T^{1*}$) running between points where $a = 0$ and points that belong to some hyper-surface of conjugate points (with an arbitrary number of full revolutions of ker$\alpha$ along the $\pm v$-jump in between).*

(iii) *If $r_2(T^1)$ is close enough to 1, there are no conjugate points on $T^1$ and on $T^{1*}$. Therefore, under this assumption, the $\xi$-pieces of the rest points run between points having $a = 0$.*

(iv) *If now $x$ is not a curve at infinity, but if it is only a curve close to a curve at infinity, then the flow $z = \frac{\partial}{\partial s}$ can be defined so that, the following inequality holds:*

$$z.L(x) \geq -\epsilon \partial J(x).z;\ z.L^*(x) \geq -\epsilon \partial J(x).z$$

*$\epsilon$ is above a specified small positive parameter.*

*Proof of Lemma 11.8* (i) follows from the fact that $\bar{x}$ "touches" along its $\xi$-pieces that are on $T^1$ and $T^{1*}$ only a finite set of the periodic orbits of $T^1$ or $T^{1*}$. Given one of these $\xi$-pieces that is, e.g. on $T^1$, we can build the flow so that it will either move this $\xi$-piece "above" $T^1$ or "below" $T^1$, or it will not move this $\xi$-piece at all, only make it shorter or longer. In this last case, the claim is obvious. If the $\xi$-piece moves up and if we denote the initial time $s = 0$, then for $s \gneq 0$, $s$ small, the only intersection of the curve $x(s)$ derived by evolution along the flow with the torus $T^1$ will be either the intersection of the two $\pm v$-jumps with $T^1$. These $\pm v$-jumps under our flow, "far" from their edges, can be considered to be displaced along $X_0$, whereas $L$ and $L^*$ are computed by integrating over the action of $X_0$. Therefore, $L$ and $L^*$ will not change under this flow for $s \gneq 0$ small. If, on the other hand, the $\xi$-piece moves down, then there is no further intersection with $T^1$. The conclusion is the same. If we add to this the fact that $L$ and $L^*$is obviously a continuous function of $s$, the claim follows.

To make the construction of the flow more precise, we are given, according to the framework, $X_0$ along a $\pm v$-jump connecting $\xi$-pieces that are not on the same torus $T_t$ and we build the deformation $z$ using this datum. In order to build $z$, we choose a point "inside" each of the $\xi$-pieces (would they be characteristic or not, the construction ignores this feature) involved in the construction. This point should be located less than $\frac{\pi}{4}$-$v$-rotation inside each $\xi$-piece (starting from the edge). Accordingly, the transport of $v$ from this point to this edge reads $a_1[\xi, v] + b_1 v$, with $a_1$ non-zero. We then choose $\gamma$ so that $X_0 + \gamma v \in \text{Span}(\xi, a_1[\xi, v] + b_1 v)$ at each edge (with the corresponding values of $a_1, b_1$). Clearly, if $X_0$ at the edge has a component on $[\xi, v]$, then the (scaled) $\pm v$-jump that we introduce in order to build $v$ is non-zero. The $\xi$-piece of the curve under $z$ will move up or down, but will leave the torus. On the other hand, if $X_0$ has a zero component on $[\xi, v]$ at this edge, then this (scaled) $\pm v$-jump will be zero, $X_0 + \gamma v$ will be collinear to $\xi$. The $\xi$-piece will only under $z$ become shorter or larger. $z$ is of course derived by convex-combinations of local pieces, but the local pieces all use the same $X_0$; the $X_0$ at the edge and the orientation of the (scaled) $\pm v$-jumps that are introduced are all the same. The conclusion follows for (i) for the flow of Lemmas 11.1 and 11.4(i).

The rest points of this flow have their $\xi$-pieces on the same torus $T_t$. But if $T_t$ is different from $T^1$ and $T^{1*}$, we can use Lemma 11.7 and continue our deformation.

If now we are considering a rest point on, e.g. $T^1$, we observe that the above construction, with exactly the same conclusion, can be completed along a $\pm v$-jump connecting two points of $T^1$ that are neither conjugate points, nor have $a = 0$. Indeed, we can consider $\xi$ at one edge of the $\pm v$-jump and transport it to the other edge. If $a$ is non-zero at one edge, it is non-zero at the other one, so that $X_0$ does not belong to Span$(\xi, v)$. $X_0$ has, therefore, a non-zero component on $[\xi, v]$ at this second edge. We then repeat the construction completed above for a tangent vector $z$ that decreases $J_\infty$. Because $z$ only makes the $\xi$-piece on one side of the $\pm v$-jump shorter or longer without changing it, whereas, on the other side, $z$ moves this $\xi$-piece up or down according solely to the $[\xi, v]$-component of $X_0$, the conclusion about $z.L$ is unchanged. (i) holds for this extended flow and the rest points must have their $\xi$-pieces on $T^1$ (respectively, $T^{1*}$); these $\xi$-pieces run between points that either have $a = 0$ or are conjugate points. (ii) follows.

Assume now that $r_2(T^{1*})$ is close to 0. Let $A'_2$ be a point on $T^{1*}$, with $a(A'_2) \neq 0$. Assume that $a(A'_2) \lesssim 0$ or that $v.r_2 \gtrsim 0$. Let us consider the $v$-orbit starting at this point. The reference figures are F3, F4, F5, and F6. We use Lemmas 4.4, 4.5, and 4.6 above. Let us consider the first point $D$ on the negative $v$-orbit through $A'_2$ where $a = 0$ and let us consider the *second* point $E$ on the positive $v$-orbit where $a = 0$. By Lemma 4.6 (the assumption of this lemma is satisfied since $r_2(T^{1*})$ is small), there are two points $A_2$ and $B_2$ on the piece of $v$-orbit between $D$ and $E$ that are conjugate. Using Lemma 4.4, these points are not close to $D$ and $E$. Therefore, $A_2$ and $B_2$ are on the positive $v$-orbit starting at $A'_2$. Considering then the point $B'_2$ on this positive $v$-orbit such that ker$\alpha$ has turned $2\pi$ between $A'_2$ and $B'_2$, we find that $B'2$ is *before* $B_2$ on this piece of $v$-orbit and *after* the point $B$ (Figure F6). Thus $r_2(A'_2) \lesssim r_2(B'_2) \lesssim \frac{1}{2}$.

Observe that $a(B'_2)$ is now positive. The next point $C'_2$ such that ker$\alpha$ has turned $2\pi$ between $B'_2$ and $C'_2$, $4\pi$ between $A'_2$ and $C'_2$ must be on the positive $v$-orbit through $B'_2$, after two crossings of $T_0$: in between two crossings, the rotation of ker$\alpha$ is $\pi$. Thus we must have two crossings on this piece of $v$-orbit and only two. Furthermore, because $\bar{y}$, the maximum of $r_2$ on this $\pm v$-jump is so large, it is larger than $\bar{y}_0$ and this implies that (see Lemma 4.6 and its proof) $r_2(B'_2) \lesssim r_2(C'_2) \lesssim \frac{1}{2}$. We also know that $a$ at $C'_2$ is again positive. The argument for $B'_2$ repeats with $C'_2$. $r_2$ keeps increasing, remaining $\lesssim \frac{1}{2}$, $A_2$ has no conjugate points.

If $a(A'_2)$ is positive instead of being negative, $A'_2$ behaves as $B'_2$ does; but the conclusion is unchanged: $A'_2$ has no conjugate points. (iii) follows.

If now $x$ is not a curve at infinity, but if it is only a curve close to a curve at infinity, then we can specify how close it is by requiring, in addition to the fact that $x$ is close in the appropriate sense to the $\Gamma_{2s}$s, that under the $z = \frac{\partial}{\partial s}$-flow defined in a neighborhood of the $\Gamma_{2s}$s the following inequality holds:

$$z.L(x) \geq -\epsilon \partial J(x).z;\ z.L^*(x) \geq -\epsilon \partial J(x).z$$

$\epsilon$ is above a specified small positive parameter.

In this way, $L$, $L^*$ might decrease, but very little under the deformation and our arguments go through, virtually unchanged. □

## 12 The $H^1_0$-semi-flow

We consider now the $H^1_0$-flow of [2,4]; it is a key piece in all the arguments used to deform the curves of $C_\beta$ onto the set formed by the union of the unstable manifolds of the periodic orbits with the unstable manifolds of the critical points at infinity in $\cup\Gamma_{2k}$, see [2,4]. We describe in what follows its definition and properties in detail.

This flow requires the choice of a differentiable family of points that are the starting and ending points of the nearly $\xi$-pieces; equivalently they are the ending and starting points of the nearly $\pm v$-pieces.

Let us first consider a nearly $\xi$-piece, defined between two points $x_i^-, x_i^+$. We consider the two $v$-orbits through these two points. Assume that the piece of curve between these two points has $\dot{x} = a\xi + bv$ , with the running time $t$ in [0, 1] for the sake of simplicity. Assume that $b \in H^1_0(0, 1)$ on this interval and consider the differential evolution equation:

$$\frac{\partial b}{\partial s} = \frac{\ddot{b} + a^2 b\tau}{a} + \frac{\overline{(\int_0^t b^2 - t\int_0^1 b^2)b - \overline{\mu}\dot{b}^2}^{\,\cdot}}{a} - b^2\bar{\mu}_\xi, \quad b \in H^1_0(0, 1)$$

If we denote $b_{|[0,1]}$, $b_i$, then we can introduce the "tangent vector":

$$Z(x) = \left(\int_0^t b_i^2 - t\int_0^1 b_i^2\right)\xi + \frac{\dot{b}_i + \left(\int_0^t b_i^2 - t\int_0^1 b_i^2 - \bar{\mu} b_i\right) b_i}{a} v - b_i[\xi, v]$$

This "tangent" vector is defined only on the portion of curve between $x_i^-$ and $x_i^+$. At time 0 and at time 1, $Z(x)$ is parallel to $v$, so that $x_i^{\pm}$ move along the $v$-orbits that they (respectively) define.

We have established in [2, pp 39–49] and [4, pp 123–134] existence, continuity, etc. for this flow. We refine here the results of [2] and [4]. We consider the operator

$$A = -(\ddot{\eta} + a\eta\tau)$$

under $H_0^1(0,1)$ boundary conditions. The nearly piece of $\xi$-orbit between $x_i^-$ and $x_i^+$ is close to a $\xi$-orbit. Assuming for simplicity (we will discuss the more general case later) that it is "far" from being characteristic, this $\xi$-piece can be identified as the unique piece of $\xi$-orbit $\varkappa$ connecting the two $v$-orbits through $x_i^-$ and $x_i^+$. It has a Morse index, which we denote $i_0^i$. Accordingly the operator $A$ defined above, under its boundary conditions, has the same index.

Let $A_0$ be this operator for the $\varkappa$-piece of $\xi$-orbit and let $E_+ \oplus E_-$ be the related decomposition on positive and negative eigenspaces for the $L^2$-scalar product. $b_i$ can then be decomposed into $b_i^+ + b_i^-$. The following differential inequalities satisfied by $b_i^+, b_i^-$ are not difficult to establish (one uses in particular the equivalence of norms in $E^-$ which is finite dimensional):

$$\frac{\partial \int b_i^{+2}}{\partial s} \le -c_0 \int \dot{b}_i^{+2} + o\left(\int b_i^{-2}\right)$$

$$c_2 \int b_i^{-2} + o\left(\int b_i^{+2}\right) \le \frac{\partial \int b_i^{-2}}{\partial s} \le c_1 \int b_i^{-2} + o\left(\int b_i^{+2}\right)$$

A more difficult estimate reads as follows: We write the evolution equation of $b_i$, $\lambda + \bar{\mu}\eta$ is $(\int_0^t b_i^2 - t\int_0^1 b_i^2) b_i$. We have

$$\frac{\partial b_i}{\partial s} = \frac{\ddot{b}_i + a^2 b_i \tau}{a} + \frac{\dot{\overline{\left(\int_0^t b_i^2 - t\int_0^1 b_i^2\right) b_i - \overline{\mu} b_i^2}}}{a} - b_i^2 \bar{\mu}_\xi$$

We project onto the space $E^+$ of $A_0$. We find

$$\frac{\partial b_i^+}{\partial s} = \frac{\ddot{b}_i^+ + a_0^2 b_i^+ \tau}{a_0} + (o(b_i) - o(b_i)^-) + \left(\frac{\dot{\overline{\left(\int_0^t b_i^2 - t\int_0^1 b_i^2\right) b_i - \overline{\mu} b_i^2}}}{a} - b_i^2 \bar{\mu}_\xi\right) - \left(\frac{\dot{\overline{\left(\int_0^t b_i^2 - t\int_0^1 b_i^2\right) b_i - \overline{\mu} b_i^2}}}{a} - b_i^2 \bar{\mu}_\xi\right)^-$$

We multiply the above equation by $-\ddot{b}_i^+ = -\ddot{b}_i + \ddot{b}_i^-$ and we integrate between 0 and 1.

The most difficult term is

$$\int_0^1 (\bar{\mu}\dot{b}_i^2)\ddot{b}_i$$

This gives rise to terms that are $O(\int_0^1[(b_i^2 + |b_i^3|)|\ddot{b}_i|])$ and $O(\int_0^1[|b_i||\dot{b}_i||\ddot{b}_i|])$. The term $\int_0^1(\frac{\overline{(\int_0^t b_i^2 - t\int_0^1 b_i^2)b_i}^{\cdot}}{a})\ddot{b}_i$ requires the addition to the above terms of $O(\int_0^1 b_i^2 \times \int_0^1[(|b_i| + |\dot{b}_i|)|\ddot{b}_i|])$.

Observe that

$$\int_0^1 b_i^2|\ddot{b}_i| \le \left(\int_0^1 \ddot{b}_i^2\right)^{\frac{1}{2}} \times \left(\int_0^1 \dot{b}_i^2\right)^{\frac{1}{2}} \times \left(\int_0^1 b_i^2\right)^{\frac{1}{2}} = o\left(\int_0^1(\ddot{b}_i^2 + \dot{b}_i^2)\right)$$

Also

$$\int_0^1[|b_i||\dot{b}_i||\ddot{b}_i|] + \int_0^1 b_i^2 \times \int_0^1[(|b_i| + |\dot{b}_i|)|\ddot{b}_i|] = o\left(\int_0^1(\ddot{b}_i^2 + \dot{b}_i^2)\right)$$

We are left with $\int_0^1 |b_i^3||\ddot{b}_i|$. For this we use the Nash inequality [20], Nirenberg [21], in dimension 1:

$$\int_0^1 b_i^6 \le C\left(\int_0^1 b_i^2\right)^2 \int_0^1 \dot{b}_i^2$$

This yields

$$\int_0^1 |b_i|^3|\ddot{b}_i| \le C_1\left(\int_0^1 \ddot{b}_i^2\right)^{\frac{1}{2}} \times \int_0^1 b_i^2 \times \left(\int_0^1 \dot{b}_i^2\right)^{\frac{1}{2}} = o\left(\int_0^1(\ddot{b}_i^2 + \dot{b}_i^2)\right)$$

All terms containing a projection $u^-$ onto $E^-$ are in fact combination of a finite set of functions that span $E^-$ with suitable coefficients. With an $L^2$-orthonormal basis $f_1, \ldots, f_s$, the coefficients are $\int_0^1 uf_j$. The time-derivatives in all these terms can be switched, after integration by parts, to be taken on $u^-$; therefore, they are in fact taken on the $f_j$ and $u^-, \dot{u^-}, \ddot{u^-}$ are all $L^\infty$ bounded by $\mathrm{Sup}(\int_0^1 uf_j)$. One can also consider $(\dot{u})^-, (\ddot{u})^-, (\dddot{u})^-$. The component of these terms on $f_j$ are equal to as $-\int_0^1 u\dot{f}_j, \int_0^1 \ddot{f}_j, \int_0^1 u\dddot{f}_j$. There are, therefore, bounded by $|u|_{L^1}$.

In addition,

$$\int u^-\ddot{b}_i^+ = \int u^- A_0(b_i^+) - \int u^- a_0^2 b_i^+\tau = -\int u^- a_0^2 b_i^+\tau$$

This implies that

$$\int_0^1\left[\left(\frac{\overline{\left(\int_0^t b_i^2 - t\int_0^1 b_i^2\right)b_i - \overline{\mu}b_i^2}^{\cdot}}{a} - b_i^2\bar{\mu}_\xi\right)^- + o(b_i)^-\right]\ddot{b}_i^+ = o\left(\int_0^1 b_i^2\right)$$

On the other hand, using our observations above and the fact that all norms are equivalent on $E^-$,

$$\int_0^1\left(\frac{\overline{\left(\int_0^t b_i^2 - t\int_0^1 b_i^2\right)b_i - \overline{\mu}b_i^2}^{\cdot}}{a} - b_i^2\bar{\mu}_\xi\right)\ddot{b}_i^- = o\left(\int_0^1 b_i^2\right)$$

Using the fact that $b_i$ is in $H_0^1$ and the equivalence of norms in $E^-$, we also have

$$\int_0^1 (a_0\tau b_i^+ + o(b_i))(\ddot{b}_i - \ddot{b}_i^-) = o\left(\int_0^1 \ddot{b}_i^2\right) + O\left(\int_0^1 \dot{b}_i^{+2}\right) + o\left(\int_0^1 b_i^{-2}\right)$$

We thus find, using the equivalence of norms on $E^-$

$$\frac{\partial \int_0^1 \dot{b}_i^{+2}}{\partial s} \leq -C_2 \int_0^1 \ddot{b}_i^{+2} + O\left(\int_0^1 \dot{b}_i^{+2}\right) + o\left(\int_0^1 b_i^{-2}\right)$$

We recall that we also have

$$\frac{\partial \int b_i^{+2}}{\partial s} \leq -c_0 \int \dot{b}_i^{+2} + o\left(\int b_i^{-2}\right)$$

Combining both equations with the use of a suitable constant $C \geq \frac{2}{c_0}$ (we will use this inequality later), we find

$$\frac{\partial \left(\int (\dot{b}_i^{+2} + C b_i^{+2})\right)}{\partial s} \leq -C_3 \left(\int \ddot{b}_i^{+2} + \int \dot{b}_i^{+2}\right) + o\left(\int b_i^{-2}\right),$$

whereas

$$c_2 \int b_i^{-2} + o\left(\int b_i^{+2}\right) \leq \frac{\partial \int b_i^{-2}}{\partial s}$$
$$\leq c_1 \int b_i^{-2} + o\left(\int b_i^{+2}\right)$$

We use in the sequel these three differential inequalities. $\delta_1 \ll \delta_0$ are two positive constants. $|b_i(0)|_{L^2}^2$ is assumed to be $\leq \delta_1$. We first claim

**Lemma 12.1** *There exists a positive constant $C_1$, independent of $\delta_0, \delta_1$, such that,*

- (i) *if $\int_0^1 b_i^{+2}(0) \leq C_1 \int_0^1 b_i^{-2}(0)$, then for any later time $s$ and as long as both quantities are small (measured with a fixed positive constant $\epsilon_0$), $\int_0^1 b_i^{+2}(s) \leq 2C_1 \int_0^1 b_i^{-2}(s)$. $\int_0^1 b_i^{-2}(s)$ increases then exponentially, whereas there exists a constant $C_2$, independent of $\delta_0, \delta_1$, such that $\int_0^1 \dot{b}_i^{+2}(s) \leq o(\int_0^1 b_i^{-2}(s)) + C_2 \int_0^1 b_i^{+2}(s-1)$ for $s \geq 1$ as long as $\int_0^1 b_i^{+2}(s)$ and $\int_0^1 b_i^{-2}(s)$ are small (measured as above). Finally, on each $[s-1, s]$, there exists then some time $s'$ such that $\int \ddot{b}_i^{+2}(s') \leq o(\int_0^1 b_i^{-2}(s)) + C_2 \int_0^1 b_i^{+2}(s-2)$ for $s \geq 2$.*
- (ii) *If $\int_0^1 b_i^2(0) \leq \delta_1$ and if $\int_0^1 b_i^{+2}(0) \geq \frac{C_1}{2} \int_0^1 b_i^{-2}(0)$, then $(\int \dot{b}_i^{+2} + C \int b_i^{+2})(s)$ decreases exponentially or faster as long as $\int_0^1 b_i^{+2}(s) \geq C_1 \int_0^1 b_i^{-2}(s)$ and, with a suitable positive constant c, the following estimate holds: $\int_0^1 b_i^2(s) \leq \delta_1 \mathrm{e}^{-cs}(1 + \frac{2}{C_1})$. In particular, either for some time $s_0$ (which we assume then to be the first time for which this inequality holds), $\int_0^1 b_i^{+2}(s_0) \leq C_1 \int_0^1 b_i^{-2}(s_0)$, with $\int_0^1 b_i^2(s_0) \leq \delta_1 \mathrm{e}^{-cs_0}(1 + \frac{2}{C_1})$ :* (i) *then applies, or $\int_0^1 b_i^2(s)$ tends to zero and the flow-line never exits a neighborhood of the rest points through the boundary defined by the inequality $\int_0^1 b_i^2 \leq \delta_0$.*

*Proof of Lemma* 12.1 (ii) follows readily from the two first inequalities among the three displayed above.

We thus prove (i). Assuming $\int_0^1 b_i^{+2}(0) \geq \frac{C_1}{2} \int_0^1 b_i^{-2}(0)$, then we derive from the third inequality above that $\int_0^1 b_i^{-2}(s)$ increases for $s$ small. As long as $\int_0^1 \dot{b}_i^{+2}(s)$ is not $o(\int_0^1 b_i^{-2})(s)$, $\int_0^1 b_i^{+2}(s)$ decreases by the first inequality and, therefore, $\int_0^1 b_i^{+2}(s) \leq 2C_1 \int_0^1 b_i^{-2}(s)$ holds. However, if $\int_0^1 \dot{b}_i^{+2}(s)$ is $o(\int_0^1 b_i^{-2})(s)$, then $b_i$

being in $H_0^1$, this inequality holds without further argument; $\int_0^1 b_i^{-2}(s)$ goes on as an increasing function of $s$ and the assumptions of (i) at time zero are satisfied at all further time $s$ as long as $\int_0^1 b_i^2(s)$ is small.

The third inequality implies then that $\int_0^1 \dot{b_i}^{-2}$ increases exponentially. Starting from $\delta_1$, with $\delta_1 \ll \delta_0$, it takes a long time to reach the level $\frac{2\delta_0}{3}$. Let us consider the time interval $[s, s+1]$, $s \geq 0$.

Either for some $s_1 \in [s, s+1]$, $\int_0^1 \dot{b_i}^{+2}(s_1)$ is $o(\int_0^1 b_i^{-2})(s_1) = o(\int_0^1 b_i^{-2})(s+1)$ or the first inequality implies that

$$\frac{\partial \int b_i^{+2}}{\partial s} \leq -\frac{c_0}{2} \int \dot{b_i}^{+2}, \quad x \in [s, s+1]$$

Integrating between $s$ and $s+1$, we find

$$\int b_i^{+2}(s+1) + \frac{c_0}{2} \int_s^{s+1} \int_0^1 \dot{b_i}^{+2}(s,t)\, \mathrm{d}t\, \mathrm{d}x \leq \int b_i^{+2}(s)$$

It follows that, for some time $s_1 \in [s, s+1]$:

$$\int_0^1 \dot{b}_i^{+2}(s_1, t)\, \mathrm{d}t \leq \frac{2}{c_0} \int b_i^{+2}(s)$$

Over both cases, we can claim the existence of $s_1 \in [s, s+1]$ such that

$$\int_0^1 \dot{b}_i^{+2}(s_1, t)\, \mathrm{d}t \leq \frac{2}{c_0} \int b_i^{+2}(s) + o\left(\int_0^1 b_i^{-2}\right)(s+1)$$

Integrating then the second inequality between $s_1$ and $s+1$ and using the increasing property of $\int_0^1 b_i^{-2}(s)$, we find

$$\int (\dot{b}_i^{+2} + C b_i^{+2})(s+1) \leq o\left(\int_0^1 b_i^{-2}\right)(s+1) + \int (\dot{b}_i^{+2} + C b_i^{+2})(s_1)$$

Using then the above inequality, we derive that

$$\int_0^1 \dot{b}_i^{+2}(s+1) \leq o\left(\int_0^1 b_i^{-2}(s+1)\right) + C_2 \int_0^1 b_i^{+2}(s)$$

for $s \geq 0$, or the inequality of (i) at time $s$ for $s \geq 1$ as stated. Having established this inequality, we integrate the second inequality over the interval $[s-1, s]$, with $s \geq 2$. (ii) follows. □

We also have

**Lemma 12.2** *Under the conditions of* (i) *of Lemma* 12.1, *the following estimate holds for a suitable fixed positive constant* $c$ :

$$\int_0^1 b_i^{+2}(s) \leq \int_0^1 b_i^{-2}(s)(o(1) + C_1 \mathrm{e}^{-cs})$$

*Proof of Lemma* 12.2 We consider the first of the three inequalities. At any time $s$, either $\int_0^1 b_i^{+2}(s) = o(\int_0^1 b_i^{-2}(s))$ or $\int_0^1 b_i^{+2}(s)$ is exponentially decreasing. Since $\int_0^1 b_i^{-2}$ increases, the claim follows then from the assumption in (i) at the initial time. □

Combining the conclusions of Lemmas 12.1 and 12.2, we see that we can assume, on a flow-line that exits a neighborhood of the rest points at infinity, that from the time $\underline{s}$, which we set for convenience to be 0, at which $\int b_i^2 = \sqrt{\delta_1} \ll \delta_0$, to the exit time $\bar{s}$ at which $\int b_i^2 = \delta_0$, we do have:

$$\int_0^1 b_i^{+2} = o\left(\int_0^1 b_i^{-2}(s)\right)$$

We then claim:

**Lemma 12.3** *At the exit time, the function $b_i$ has at most $(i_0^i - 1)$ interior zeros.*

*Proof of Lemma* 12.3 Once $b_i$ has $(i_0^i - 1)$ or less at some time $s$, the non-increasing property of the number of zeros of $b_i$, a feature of the differential equation that it verifies, implies the result for later times.

On the other hand, (i) of Lemma 11.8 combined with Lemma 12.1 implies that on each $[s-1, s]$, $s$ large enough, there is a time $s'$ such that

$$|b_i^+|_{C^1} = o(|b_i^-|_{L^2})$$

Consider a non-zero function $c_i^-$ of $E^-$. Assume now that there exists a fixed positive constant $c_3$ and another small fixed positive constant $\rho$ such that, near each value $t_0$ such that $c_i^-(t_0) = o(|c_i^-|_{L^2}$, the following estimate holds:

$$(***) \quad |c_i^{-'}(t)| \geq c_3 |c_i^-|_{L^2}, t \in [t_0 - \rho, t_0 + \rho]$$

It then follows that if we add to $c_i^-$ a function $c_i^+$ satisfying

$$|c_i^+|_{C^1} = o(|c_i^-|_{L^2}),$$

then the addition $c_i$ of both functions has not more zeros than $c_i^-$, that is $(i_0^i - 1)$-zeros at most. This follows from a simple application of the mean value theorem to the function $c_i$: its zeros must be very close to values $t_0$ of the parameter $t$ for which $c_i^-(t_0) = o(|c_i^-|_{L^2}$. The assumption on $c_i^{-'}(t)$ in a uniform neighborhood of $t_0$ allows then to reach the stated conclusion.

$o(|c_i^-|_{L^2})$, $o(|b_i^-|_{L^2})$ are here $\leq \delta' |c_i^-|_{L^2}$, $\delta' |b_i^-|_{L^2}$, where $\delta'$ is as small as we please, whereas $c_3$, $\rho$ are fixed constants, albeit small.

Lemma 12.3 then follows from the claim that $b_i^-(s)$ will satisfy the condition on $c_i^-$ for some $s \in [0, \bar{s}]$.

To see why this claim holds, we come back to the evolution equation satisfied by $b_i$. $f_1, \ldots, f_p$ is an orthonormal basis of $E^-$. $E^-$ is the negative eigenspace of the operator $-(\ddot{\eta} + a_0^2 \eta \tau_0)$ under Dirichlet boundary conditions. We may assume that $f_1, \ldots, f_p$ are its normalized eigenfunctions. Let $w_1(s), \ldots, w_p(s)$ be the components of $b_i^-(s)$ along $f_1, \ldots, f_p$. Multiplying the evolution equation by $f_j$, integrating between 0 and 1, integrating by parts (all $f_j$ are $C^\infty$), and using the fact that $\int_0^1 b_i^{+2} = 0(\int_0^1 b_i^{-2}(s)) = o(\Sigma |w_i|^2)$, we find that

$$\frac{\partial w_j}{\partial s} = -\mu_j w_j + o(\Sigma |w_i|)$$

The $-\mu_j$s are the negative eigenvalues of the operator $\frac{\ddot{\eta} + a_0^2 \eta \tau_0}{a_0}$. If we remove $o(\Sigma |w_i|)$, this rereads

$$\frac{\partial b_i^-}{\partial s} = -\frac{A_0}{a_0} b_i^-$$

$u_i = \frac{b_i^-}{|b_i^-|_{L^2}}$ then satisfies

$$(1) \quad \frac{\partial u_i}{\partial s} = -\frac{A_0}{a_0} u_i - \lambda u_i,$$

where $\lambda$ is a constant in time, that varies with s, derived from the fact that $|u_i|_{L^2} = 1$. The actual evolution differential equation on $u_i = \frac{b_i^-}{|b_i^-|_{L^2}}$ reads:

$$(1)' \quad \frac{\partial u_i}{\partial s} = -\frac{A_0}{a_0}u_i - \lambda_1 u_i + o(1)$$

$\lambda_1$ behaves as $\lambda$ does.

Considering (1), we recognize that its rest points are the $f_j$s. We, therefore, claim that either $u_i(0)$ is in a small neighborhood of one of the $f_j$, $j = 1, \dots, p$, so small that $(***)$ is satisfied at $u_i(0)$, or $u_i(0)$ is not in none of these neighborhoods. Then, $u_i(s)$ has to enter such a neighborhood before some a priori bounded (perhaps large, but a priori bounded) time $s_0$. The same property holds then for $(1)'$ if $o(1)$ is small enough. We then take $\delta_1$ so small with respect to $\delta_0$ that $\bar{s} \geq s_0$. The conclusion then follows. □

We next establish the following lemma, Lemma 12.4, that holds also under periodic boundary conditions, completely unchanged:

**Lemma 12.4** *Let $x(s) = x(s,t)$ be the solution of the differential equation corresponding to the $H_0^1$-flow $\frac{\partial x}{\partial s} = Z(x)$ and let $b(s,t)$ be the $v$-component of $\frac{\partial x}{\partial t}$ between two of the $v$-verticals of $x(s)$. $b(s,t)$ is the solution of the evolution partial differential equation:*

$$\frac{\partial b}{\partial s} = \frac{\ddot{b} + a^2 b\tau}{a} + \frac{\overline{\left(\int_0^t b^2 - t\int_0^1 b^2\right) b - \overline{\mu} b_i^2}^{\,\cdot}}{a} - b^2\bar{\mu}_\xi, \quad b \in H_0^1(0,1)$$

*Let $T$ be the blow-up time for this equation. There exists a positive constant $c_5$ such that if $\overline{\lim} \int_0^1 |b(s,t)|\mathrm{d}t \leq c_5$ as $s$ tends to $T^-$, then $T = \infty$, $b(s,t) \in H^1(0,1)$ exists for all time $s$. In addition, $\int_0^1 (b(s,t)^2 + \dot{b}(s,t)^2)\mathrm{d}t$ tends to zero as $s$ tends to $\infty$, $\int_0^\infty \int_0^1 |\ddot{b}|^2 \mathrm{d}t\mathrm{d}s \lneq \infty$ and the end-points $x(s,0)$ and $x(s,1)$ of the piece of curve between the two $v$-verticals of the curve $x$ converge as $s$ tends to $\infty$.*

**Corollary 12.5** *Consider the same evolution equation then in Lemma 12.4. Assume that $\int_0^1 \alpha_x(\dot{x}) \leq a_0$ and assume that $\int_0^1 |b(0,t)|\mathrm{d}t \leq \frac{c_5}{2}$. There exists a positive constant $c_6$, depending only on $c_5$ and $a_0$, such that, if the blow-up time $T$ is finite, then $\int_0^1 \alpha_x(\dot{x})\mathrm{d}t(T^-) \leq \int_0^1 \alpha(\dot{x})\mathrm{d}t(0) - c_6$.*

*Proof of Lemma 12.3* We are changing under $Z(x)$ portions of a given curve of $C_\beta^+$, which has a set of $v$-verticals under the $H_0^1$-flow. This flow does not change the $v$-verticals, but tries to evolve the portions of curves connecting them to pieces of $\xi$-orbits. $b(s,t)$ designates, therefore, the $v$-component of the time-derivative of $x$ between two of these verticals. We take the evolution equation satisfied by $b$, multiply it by $b$ and integrate between 0 and 1. We find with suitable constants $c$ and $C$:

$$\frac{\partial \int_0^1 b^2}{\partial s} + c\int_0^1 \dot{b}^2 \leq C\left(\int_0^1 b^4 + \int_0^1 b^2\right)$$

Using the Nash inequality for $n = 1$ [20,21], we bound $\int_0^1 b^4$ by $C(\int_0^1 |b|)^2 \times \int_0^1 \dot{b}^2$. Taking $c_5$ small enough, we can absorb this term in $c\int_0^1 \dot{b}^2$. Keeping the same notations for the sake of simplicity, we find

$$\frac{\partial \int_0^1 b^2}{\partial s} + c\int_0^1 \dot{b}^2 \leq C\int_0^1 b^2$$

The basic equation is $\frac{\partial x}{\partial s} = Z(x)$ and it implies that $\frac{\partial \int_0^1 \alpha_x(\dot{x})}{\partial s} \leq -\int_0^1 b^2$ (we do not have equality because we might have several distinct nearly $\xi$-pieces and we might be using the $H_0^1$-flow on each of them). We thus know that, whatever the blow-up time $T$ might be, $\int_0^T \int_0^1 b^2 \leq \infty$. Integrating, this implies that

$$\int_0^1 b^2\,\mathrm{d}t(T^-) + \int_0^T\int_0^1 \dot{b}^2\,\mathrm{d}t\,\mathrm{d}s \leq \int_0^1 b^2\,\mathrm{d}t(0) + C\int_0^T\int_0^1 b^2\,\mathrm{d}t\,\mathrm{d}s$$

In fact, $\int_0^1 b^2(s)$ is bounded independently of $s \in [0, T]$.

We now multiply the evolution equation on $b$ by $-\ddot{b}$. We observe that, with $C_1$ a large constant,

$$\int_0^1 |b|^3|\ddot{b}| \, \mathrm{d}t \leq C_1 \int_0^1 b^6 + \frac{1}{C_1}\int_0^1 \ddot{b}^2 \leq C_1' \left(\int_0^1 b^2\right)^2 \times \int \dot{b}^2 + \frac{1}{C_1}\int_0^1 \ddot{b}^2$$

Since $\int_0^1 b^2$ is bounded, we derive that

$$\int_0^1 |b|^3|\ddot{b}| \, \mathrm{d}t \leq C_2 \times \int \dot{b}^2 + \frac{1}{C_1}\int_0^1 \ddot{b}^2$$

We also observe that

$$\int_0^1 \bar{\mu}\dot{b}^2\ddot{b} \, \mathrm{d}t \leq O\left(\int_0^1 (b^2 + |b^3|)|\ddot{b}| \, \mathrm{d}t\right) + 2\int_0^1 b\bar{\mu}\dot{b}\ddot{b}\mathrm{d}t$$

$\int_0^1 |b^3||\ddot{b}|\mathrm{d}t$ has been estimated above. On the other hand, using again [20,21],

$$\int_0^2 b^2|\ddot{b}| \, \mathrm{d}t \leq \left(\int_0^1 b^4\right)^{\frac{1}{2}} \times \left(\int_0^1 |\ddot{b}|^2\right)^{\frac{1}{2}} \leq C_2 \int_0^1 \dot{b}^2 + \frac{1}{C_1}\int_0^1 |\ddot{b}|^2$$

We are left with $\int_0^1 b\bar{\mu}\dot{b}\ddot{b}\mathrm{d}t$. This is upper-bounded by $C_1 \int_0^1 b^2\dot{b}^2 + \frac{1}{C_1}\ddot{b}^2$. Integrating by parts in $\int_0^1 b^2\dot{b}^2$, we find that this rereads $\frac{1}{3}\int_0^1 b^3\ddot{b}$. This has been already estimated. Finally,

$$\begin{aligned}\int_0^1 \ddot{b}\frac{d(b[\int_0^1 b^2 - t\int_0^1 b^2])}{\mathrm{d}t} &= O\left(\int_0^1 |b|^3|\ddot{b}|\right) + \int_0^1 \ddot{b}\dot{b}\left(\int_0^1 b^2 - t\int_0^1 b^2\right) \mathrm{d}t \\ &= O\left(\int_0^1 |b|^3|\ddot{b}|\right) + O\left(\int_0^1 \dot{b}^2 b^2\right)\end{aligned}$$

All these terms have been estimated above. Taking $C_1$ large enough, we derive, with suitable constants $c$ and $C$:

$$\frac{\partial \int_0^1 \dot{b}^2}{\partial s} + c\int_0^1 \ddot{b}^2 \leq C\int_0^1 \dot{b}^2$$

Combining this with the estimate on $\frac{\partial \int_0^1 b^2}{\partial s} + c\int_0^1 \dot{b}^2$ above, we derive with the use of another large constant $C'$:

$$\frac{\partial\left(\int_0^1 \dot{b}^2 + C'\int_0^1 b^2\right)}{\partial s} + c\int_0^1 \ddot{b}^2 \leq C_3 \int_0^2 b^2$$

We derive from this inequality that $T = \infty$. Since $\int_0^\infty \int_0^1 b^2 \mathrm{d}t\mathrm{d}s$ is then finite, we derive also that $\int_0^\infty \int_0^1 (\dot{b}^2 + \ddot{b}^2)\mathrm{d}s\mathrm{d}t \leq \infty$. It follows that $\int_0^1 b^2 + \int_0^1 \dot{b}^2$ must tend to zero. The piece of curve, which carries a finite amount of "energy" ($\int_0^1 \alpha_x(\dot{x})\mathrm{d}t$ is decreasing, positive), must converge to a piece of $\xi$-orbit connecting the two preassigned verticals. Because these form an isolated set and because $\int_0^1 b^2$ tends to zero $x(s, 0$ and $x(s, 1)$ must converge as $s$ tends to $\infty$. Lemma 12.4 is thereby established. □

*Proof of Corollary* 12.5 We can prove for this evolution equation that, for almost every $\nu \gtrsim 0$, the following estimate holds, see [10, p 33 and Appendix 4]:

$$\frac{\partial \int_0^1 (|b|-\nu)^+}{\partial s} \le \frac{C}{\nu}\int_0^1 b^2 \le -\frac{C}{\nu}\frac{\partial\left(\int_0^1 \alpha_x(\dot{x})\,\mathrm{d}t\right)}{\partial s}$$

Taking $\nu \le \frac{c_5}{6}$ and using the above inequality, we derive after integration Corollary 12.5. □

## 13 Deforming a "nearly" $\pm v$-jump to "infinity" without decrease of $L$ and of $L^*$

Let us now consider a nearly $\pm v$-jump. It might contain some back and forth nearly runs along $v$ if $b$ has zeros. Let us assume, in a first step, that $b$ has no zero along this nearly $\pm v$-jump and let us assume that it is, e.g. a $+v$-jump.

We are given a constant $C_0$. This constant will depend on the geometry of the contact form $\alpha$ along $v$. We divide the nearly $v$-jump into sub-pieces of length (counted along $v$) $\ell$ between $\frac{C_0}{2}$ and $C_0$.

We consider one of these sub-pieces, between its two extremal points $y_i^-$ and $y_i^+$. $b$ on this sub-piece, see [2], is very close in the $L^1$-topology to a very large constant $|b|_\infty$. The time $t$ spanned between these two extremal points is, therefore, very small.

Let us consider the two $v$-orbits, through $y_i^-$ and through $y_i^+$. The sub-piece of curve that we are considering is "small" (depending on the value of $C_0$) and runs from one $v$-orbit to the other one. On each $v$-orbit, a point is "above" another one if the piece of $v$-orbit between them is along $+v$. It is "below" if the piece of $v$-orbit between them is along $-v$. We claim

**Lemma 13.1** (i) *There is a unique "small" piece of orbit of $\xi$ running from a point $z_i^-$ on the first $v$-orbit "above" $y_i^-$ to a point $z_i^+$ on the second $v$-orbit "below" $y_i^+$.*

(ii) *Under a $J_\infty$-decreasing, $L+L^*$-not decreasing deformation, this sub-piece of curve will converge to the curve made of the $v$-orbit from $y_i^-$ to $z_i^-$ combined with the piece of $\xi$-orbit from $z_i^-$ to $z_i^+$, followed by the piece of $v$-orbit from $z_i^+$ to $y_i^+$.*

(iii) *Combining this with the flow that uses $X_0$ in lemma, we can define a $J_\infty$-decreasing, $L+L^*$-not decreasing deformation that retracts by deformation all these curves onto curves having nearly $\xi$-pieces close to a given "floor", alternating with $\pm v$-jumps going from a torus $T_t$ close to the torus of the "floor" to another torus $T_{t'}$ also close to the "floor".*

*Proof of Lemma* 13.1 We first prove (i). We consider a small section $\sigma$ to $v$ at $y_i^-$. The $v$-orbit through $y_i^+$ intersects $\sigma$ at a point $u_i$. We may assume that $\xi$ is tangent to $\sigma$ and we may consider coordinates of $\sigma$ where $\xi$ is constant. Let $w_0$ be a vector field in $\sigma$ independent of $\xi$. We may assume that $w_0$ and $\xi$ commute; therefore, we may assume that they are both constant.

We pull back to $\sigma$, using the one-parameter group $\gamma_s$ of $v$, the sub-piece of curve. We find a curve in $\sigma$ running from $y_i^-$ to $u_i$. Let $s(t)$ be the time required along $-v$ for the pull-back. $x(t)$, $t \in [0,1]$ denotes the sub-piece of curve. Let us denote $x_s$ the $v$-orbit through $y_i^-$. $s$ will be running from 0 to $s_0 = s(1)$. The tangent vector to the curve after pull-back is $d\gamma_{-s(t)}(\xi(x(t))$. Because the sub-piece of curve is a nearly $v$-piece, we can write

$$\mathrm{d}\gamma_{-s(t)}(\xi(x(t)) = \mathrm{d}\gamma_{-s(t)}(\xi(x_{s(t)})) + o(1) = a_1(t)(\xi + c_1(t)w_0)$$

$a_i(t)$ is close to 1 because $C_0$ is small. We can re-parameterize the curve so that the component of the tangent vector on $\xi$ is now 1:

$$\xi + c_2(t)w_0,\ t \in [0,\epsilon]$$

On the other hand, $\xi$ can be seen ker$\beta$ transversally to $v$ and $\beta$ is a contact form with $v$ in its kernel. Therefore, if $C_0$ is small enough and if the frame $(\xi, v, w_0)$ has the proper orientation (otherwise, change $w_0$ into $-w_0$)

$$\mathrm{d}\gamma_{-s}(\xi(x_s)) = a(s)(\xi + c(s)w_0)$$

with $a(s)$ positive, close to a constant, and $c(s)$ an increasing function of $s$, for $s$ small in $[0, s_0]$. This follows from the monotone rotation of ker$\beta$, that is of $\xi$, in a $v$-transported frame. Replacing $\xi$ by the re-scaled $\frac{\xi(x_s)}{a(s)}$, we find that the pull-back vector is directed by $\xi + c(s)w_0$. If instead of the vector $\frac{\xi(x_s)}{a(s)}$ at $x_s$, we consider a small piece of curve tangent to $\lambda\xi$, starting at $x_s$, during the time $\epsilon$, we find after pull-back a piece of curve on $\sigma$ tangent to $\xi + c(s,t)w_0$, $t \in [0, \epsilon]$ (the choice $\lambda$ is embedded in the way the tangent vector reads after pull-back: the component of this vector on $\xi$ is identically 1). The function of $t$ defined by $c(s,t) - c(s)$ is $O(\epsilon)$, uniformly for $s \in [0, s_0]$, in the $C^1$-sense to the least.

We claim that, under our assumptions, there is a positive constant $\delta$ which depends only on $C_0$ such that, if $\epsilon$ is small enough:

$$\delta c(s_0) \le \frac{\int_0^\epsilon c_2(t)\,\mathrm{d}t}{\epsilon} \le (1-\delta)c(s_0)$$

Indeed, as $\epsilon$ tends to zero, this estimate reduces to a "limiting" estimate along a piece of $v$-orbit through $y_i^-$ of length $s_0$. $s_0$ is of the same order than $C_0$. The function $c(s)$ defined above is strictly monotone increasing, with a derivative bounded away from zero.The estimate follows.

Observe that we also have, after the same arguments

$$\delta_1 C_0 \le c(s_0) \le \delta_2 C_0$$

$\delta_1, \delta_2$ are again here positive constants that depend only on $C_0$.

The function $\theta(s,\epsilon) = \frac{\int_0^\epsilon c(s,t)\mathrm{d}t}{\epsilon}$ is a monotone increasing function of $s$ (following the strict monotonicity of $c(s)$, that is, the positivity of its derivative). It is equal, uniformly for $\epsilon$ small, to $O(\epsilon)$ for $s = 0$ and it is equal to $c(s_0) + O(\epsilon)$ for $s = s_0$, with $s_0$ of the same order than the fixed constant $C_0$, whereas $\epsilon$ is as small as we please. The equation

$$\theta(s,\epsilon) = \frac{\int_0^\epsilon c_2(t)\,\mathrm{d}t}{\epsilon}$$

has, therefore, a unique solution $\bar{s}$ and, using the above estimates on $\frac{\int_0^\epsilon c_2(t)\,\mathrm{d}t}{\epsilon}$, we can assert the existence of a small positive constant $\delta_3$ that depends only on $C_0$ such that

$$\delta_3 C_0 \le \bar{s} \le (1-\delta_3)C_0$$

(i) then follows.

Let us solve, under Dirichlet boundary conditions for $\eta$ on the sub-piece, the following linear differential equation in $\eta$ on the interval $[0, \epsilon]$:

$$\frac{\ddot{\eta} + a^2\eta\tau - \overline{\dot{(\bar{\mu}b\eta\,)}}}{a} + \frac{\overline{\dot{\left(\left(\int_0^t b\eta - t\int_0^1 b\eta\right)b\right)}}}{a} - b\eta\bar{\mu}_\xi = -b$$

□

We claim that

**Lemma 13.2** *Assume that b is positive and that* $\frac{C_0}{2} \le \epsilon|b|_\infty \le C_0$. *Then, the solution* $\eta$ *satisfies* $b\eta \ge 0$.

*Proof of Lemma* 13.2 Assume that $\eta$ is negative somewhere on $[0, \epsilon]$. Up to a change of notations, we might as well assume that $\eta$ is negative all over this interval. Multiplying the equation by $\eta$ and integrating on this interval, we find

$$-\int_0^\epsilon \dot{\eta}^2 + \int_0^\epsilon b\eta + o(|b|_\infty^2)\int_0^\epsilon \eta^2 + O\left(|b|_\infty \int_0^\epsilon |\eta\dot{\eta}|\right) = 0$$

Since $\eta$ is in $H_0^1(0,\epsilon)$, this implies

$$-\frac{1}{2}\int_0^\epsilon \dot{\eta}^2 + \int_0^\epsilon b\eta + o(|b|_\infty^2)\int_0^\epsilon \eta^2 \ge 0$$

We know that $\frac{C_0}{2} \le \epsilon|b|_\infty \le C_0$. The conclusion follows. □

## 14 The deformation argument and the choice of the family of tori $T^1$

We have built in [3, Proposition 1], a model for the unstable manifold of a periodic orbit. This model is achieved near the periodic orbit in $\Gamma_{2m}$ if the index of the periodic orbit is $m$. Accordingly, starting from a periodic orbit of odd index $2k-1$ (the periodic orbits here are degenerate; however, the arguments of Proposition 1 of [3] extend to the present framework for the based periodic orbits, of odd index, transverse to the degeneracy), the unstable manifold of such a periodic orbit is achieved with the use of $(2k-1)$ trackable $\pm v$-jumps.

Using this local model, we flow our curve down, using the flow $Z_\infty$ defined on $\cup\Gamma_{2k}$ in Lemma 11.1, Corollary 11.2, Lemma 11.4(i), Lemmas 11.7 and 11.8. The curves under deformation either "move down" past a given reference level $\underline{c}$ for the functional, or they can enter a neighborhood of periodic orbits of $\xi$, maybe with additional back and forth runs along $v$ added to them (see [2], Chapter (IV)2. after p 161); this has been discussed above in Sect. 8.3 and we ignore it here. Or, a last possibility, they can enter a neighborhood of the rest points defined in (i) of Lemma 11.8. We reach in this way a neighborhood, as small as we please, of these "rest points". These "rest points" could include genuine critical points at infinity; but since we are studying flow-lines connecting periodic orbits of consecutive Morse indexes, they are ruled out on such flow-lines. For the other "rest points", a decreasing flow can always be defined at infinity; but this flow might decrease $L(T^1)$ or $L^* = L(T^{1*})$. Using Lemma 11.8(ii) and (iii), $T_t$ must be one of the $T^1$s or $T^{1*}$s: for a "rest point" having its $\xi$-pieces on any other torus, there is a decreasing deformation at infinity, unless this rest point is a critical point at infinity. Genuine critical points at infinity do not interfere with flow-lines connecting periodic orbits of consecutive indexes. For the "false ones", that is for those admitting a decreasing "normal", Lemma 11.7 extends as we already pointed out above.

Furthermore, by (iii), if we choose $r_2(T^1)$ large enough, these rest points are curves having all their $\xi$-pieces on $T^1$ or $T^{1*}$ and the edges of these $\xi$-pieces all verify $a = 0$.

We then claim that

**Proposition 14.1** *Given a non-empty open interval $(a_1, b_1)$ contained in $[0, 1]$ and a positive real A, we can choose $r_2(T^1)$ in this interval so that $T^1$ is a torus of periodic orbits. Furthermore, for any rest point $x_\infty$ of the flow at infinity having its $\xi$-pieces on $T^1$ running between points verifying $a = 0$, $J_\infty(T^1) \geq A$.*

*Proof of Proposition 14.1* Values $x$ in $[0, 1]$ such that $\frac{\tilde{A}}{\tilde{B}}(x)$ is rational are dense. We can also choose $x$ in $[a_1, b_1]$ so that the rational $\frac{p_1}{q_1} = \frac{\tilde{A}}{\tilde{B}}(x)$ has a very large numerator and denominator. It follows that $p_1 - q_1$ is very large. Therefore, the action of the periodic orbits of $T^1$ is very large. We can choose $x$ so that $p_1$ and $q_1$ are odd.

Considering then $x_\infty$, we claim that, generically, $x_\infty$ can be "re-normalized" into a periodic orbit on $T^1$; in particular $J_\infty(x_\infty)$ is equal to the value of the action functional on a periodic orbit of $T^1$. This implies Proposition 14.1.

In order to prove this claim, we observe that the time along $\zeta$ from a point where $a = 0$ to another point where $a = 0$ is an integer multiple of $\frac{\pi}{\tilde{A}-\tilde{B}}$. On the other hand, along a $\pm v$-jump connecting two such points on the same torus $T^1$, the rotation along $X_0$ is given by Lemma 3.2. Let us call $\theta(T^1) = \theta(r_2(T^1)) - \bar{\theta}$ this rotation. For $x_\infty$ to be a close curve, we need that there are two integers $n, s, n \gtrneqq 0$ so that

$$k\frac{\pi\tilde{A}}{\tilde{A}-\tilde{B}}(r_2(T^1)) + n\bar{\theta} = k\frac{\pi p_1}{p_1 - q_1} + n\bar{\theta}$$

and

$$k\frac{\pi\tilde{B}}{\tilde{A}-\tilde{B}}(r_2(T^1)) + n\bar{\theta} = k\frac{\pi q_1}{p_1 - q_1} + n\bar{\theta}$$

are both multiples of $2\pi$.

This implies that $\frac{\bar{\theta}}{\pi}$ is a rational, which we can rule out generically. Proposition 14.1 follows. □

## 15 Final observations

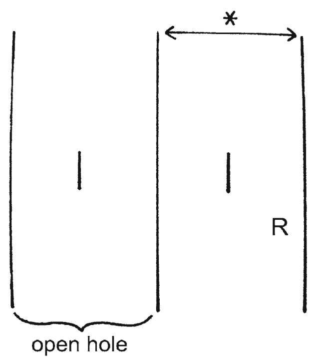
15.1 Observations about the arguments of compactness in [9]:

When a $*$ around which a Hole Flow of the "old" type, see [3, pp 484–485], becomes a family, we have to switch and center this Hole Flow around another $*$.

In order to complete this switch, we use the New Hole Flow, see [3, pp 560–561], nearby and over the configurations when a given $*$ changes nature.

Over such configurations, we study the existence of "open holes", see [3, pp 563–564] in particular. If there is no such "open hole", then the configuration is "saturated" and, if the configuration cannot be decreased through the new Hole Flow and the introduction of companions to existing $*$s[13], the count in the number of zeros of $b$ completed in [3, p 567] gives us the freedom of two zeros. We use this freedom to complete the switch.

On the other hand, if there is an "open hole", then the $*$ which is immediately to the right of this open hole is "steady"; it is also alone in its new nodal zone:




FA1

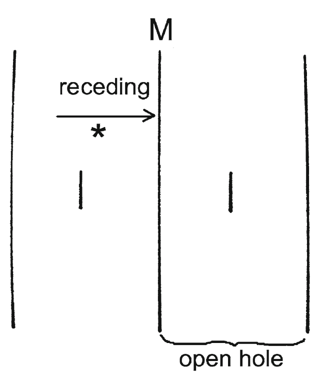
We can then complete a decrease of $J$ by introducing a companion of $*$ in $R$. This companion may be viewed in *two* ways: it may be viewed as part of the new Hole Flow on $*$ and it may also be viewed as part of the "old" Hole Flow on $*$.

Thus, over the switch, this can be used, whether this $*$ is or is not the $*$ around which we are to center now our ("old") Hole Flow, as a support flow for the switch.

This argument breaks down in one single case, that is it breaks down in the case when the open hole is occurring in the last new nodal zone to the right, so that the $*$ immediately to its right defines the $*$ of the right boundary of the (e.g. characteristic) $\xi$-piece.

Indeed, under such a circumstance, there is no $R$ and, therefore, there is no such companion. We then have to introduce a companion to the $*$ defining the right boundary *inside* the open hole, that is *in front* of the $*$ of the boundary, in the last new nodal zone to the right.

Again, this can be used as a support flow for the switch. However, let us imagine that an "open hole" closes because the $*$ in front of it, immediately to its left, recedes on the new nodal zone $M$:




FA2

[13] Companions are introduced in the new Hole Flow when a "steady" $*$ is on or very close to a "new nodal line", see [3, p 561], and the two new nodal zones around this new nodal line do not contain other $*$s. Then, for each such occurrence, we can count that there is an independent condition or constraint, namely that a $*$ is precisely on the new nodal line or very close. This provides *independently* of the flow with companions that can be built around such $*$s a family of conditions, one for every such occurrence. Turning now to the remaining $*$s, either there is an "open hole" and a decrease without the use of companions follows or there is none and $i_0^j$ conditions are satisfied on this $\xi$-piece. This is why the "old hole flows" can be convex-combined, across the dividing lines along which $*$s change nature and become families, to define a global flow, as long as these $*$s, see further remarks about this below, is a single $\pm v$-jump.

If no other open hole opens, the configuration is to be saturated. *Before* $*$ reaches $M$, we reach the freedom of two zeros and the switch can be completed.

If another open hole opens, we can use the flow attached to this *new open hole* and it can be convex-combined with the flow related to the previous open hole because both are defined by companions *behind* (immediately to the right) of the two open holes; since these two open holes are distinct, the new nodal lines behind them do not overlap and the convex-combination is possible; *unless* the previous open hole was related to the last new nodal zone to the right and the new open hole is related to the new nodal zone that is next to the left of this last new nodal zone.

We then would have an overlap, with companions of opposite orientations, in the use of the last new nodal zone to the right.

We can instead introduce a companion to this receding $*$, over such an occurrence, in the new nodal zone next to the last one (immediately to its left). This would be compatible with the switch *unless* the receding $*$ is the $*$ around which the "old" hole flow is to be centered. Then, the use of companions to this $*$, to the right and to the left of it (to the left as it is receding, to the right once it has receded), combined with the use of the hole flow, leads to two additional zeros of $b$, thereby violating the constraint on this number of zeros.

We, therefore, have to avoid to center our "old" hole flow around $*$s defining the right boundary of the $\xi$-piece.

We have confronted already such an issue when we discussed the configurations over which $*$s were exiting a $\xi$-piece and we had to switch to other $*$s that were more to the "middle", inside the $\xi$-piece. In order to avoid centering our flow around an exiting $*$, we used the fact that exiting $*$s yielded repetitions in the sign distribution and we used the fact that the $\xi$-piece supported $(i_0^j - 1)*$s ($i_0^j$ is the $H_0^1$-index of the $\xi$-piece). These combined facts allowed us to complete the switch.

This very same procedure allows us now to overcome the present issue in the switching process.[14]

If, on the other hand, all $*$s on a given characteristic $\xi$-piece are families, except for the one neighboring the right boundary of this $\xi$-piece, then we can, over the configurations where this is happening, switch the orientation of the "open holes" of the new Hole Flow, viewing them starting from the left edge rather than from the right edge. These two uses of the new Hole Flow can be convex-combined over the transition lines, using the additional companions that were not introduced because they involved $\pm v$-jumps involving old Hole Flows around which the switch is to be completed.

### 15.2 Completing the switching process between two $*$s that are on the same characteristic piece without the use of a second $\xi$-piece:

The switching process in [3] involved also the use of another characteristic piece; the two $*$s involved in the switch could belong to the same characteristic $\xi$-piece, but we needed the use of another $*$, from another characteristic $\xi$-piece, in order to complete the switch, see below for more precisions.

We want to improve this argument here and allow that the two $*$s involved in the switch belong to the same characteristic piece.

Then, an additional issue over this switching process needs to be overcome, namely that the two "old" hole flows, see [3, p 484], for the definition of the "old" hole flow, centered over these two $*$s might be "incompatible"; this means that if we use them simultaneously, the number of zeros of $b$ might increase beyond the prescribed upper-bound.

It turns out that, using the violation of the Fredholm assumption, we can complete such a switch without increasing the number of zeros of $b$ beyond the prescribed upper-bound. Let us recall here the main steps in the argument of [3] and indicate how to modify them to allow for this more general framework:

In the last pages, pp 560–568 of Compactness, [3], we developed a deformation argument based on the definition of "new nodal zones" and on the definition of a "new Hole flow".

[14] Using the present observations, Theorem 1' of [3, p 568], can be improved: the assumption that the dominated critical point at infinity $y_{m-1}^\infty$ has more than one characteristic piece of large $H_0^1$-index can be replaced by the weaker assumption that $y_{m-1}^\infty$ has at least one characteristic $\xi$-piece with again a large $H_0^1$-index. The alternative assumption in Theorem 1' of [3] must be read: "or if the number of characteristic $\xi$-pieces of $y_{m-1}^\infty$ which are separated by non-degenerate $\xi$-pieces that are either of $H_0^1$-index $\geq 1$ or have zero $H_0^1$-index with reverse edge orientations is large". Comparing with the statement of Theorem 1 of [3] (and the arguments for its proof), one can see that a part of this sentence, namely "which are separated by non-degenerate $\xi$-pieces", has been omitted unfortunately from the statement of Theorem 1' in [3].

This flow allows, see [3, pp 562–567], to overcome the issue of transversality in the variational problem defined by $(J, C_\beta)$, with the sole use of companions to existing families or with the transformation of single steady $\pm v$-jumps representing $*$s into families.

Over this process, given two different $*$s, $*_1$ and $*_2$, and two connected components $\mathcal{C}_1$ and $\mathcal{C}_2$ in the configuration space over which $*_1$ and $*_2$ are families, we know that either $\mathcal{C}_1 \cap \mathcal{C}_2$ is empty or one of these sets is contained in the other one: this rule is natural because the underlying problem is variational in nature; but it is harder to verify over the creation of companions in the resolution of the transversality issues.

The compactness arguments of [3] are rooted in the observation stated above. Namely, assuming that, inside some characteristic $\xi$-piece, some $*$, e.g, $*_1$, around which a hole flow has been built, is becoming a family, the deformation cannot continue as such. There is the need to shift to another $*$, e.g. $*_2$, on this characteristic piece. This switch is possible as long as there is another characteristic $\xi$-piece of $H_0^1$-index $i_k^0$, supporting $(i_0^k - 1)*$s, one of them, which we denote $*_3$, being reduced to a single $\pm v$-jump. Indeed, over the transition, as $*_1$ becomes a family, $*_3$ remains a single $\pm v$-jump.

There are two limitations to this compactness argument:

First, this argument requires the existence of another characteristic $\xi$-piece of positive strict $H_0^1$-index. This is needed if we want to find such a $*_3$.

Second, this $*_3$ must be a single $\pm v$-jump over the transition (as $*_1$ becomes a family).

This second limitation can be removed within the context of the exotic contact structure of J.Gonzalo and F.Varela because, if $*_3$ is a family-as well as all $*$s over this second characteristic $\xi$-piece-the violation of the Fredholm assumption allows on one side of $*_3$ to define a decreasing deformation that does not increase the number of zeros of $b$ (unless some sign repetition occurs among the $i_0^j - 1$ $*$s f this characteristic piece; then, the number of zeros of $b$ has to decrease below the maximal number of zeros allowed; this is straightforward in the case of an even number of families equal to $2k$. It is only slightly more complicated in the case of $2k + 1$ families; it then follows from the edge orientations of each characteristic piece with respect to its strict $H_0^1$-index $i_0^j$. If $i_0^j - 1$ is even, the edge orientations are opposed, there must be a sign-repetition outside the characteristic piece. If $i_0^j - 1$ is odd, the edge orientations are the same; again, we must have an outside repetition in the sign distribution).

The first limitation remains though.

However, in the framework of the contact structure of J. Gonzalo and F. Varela, we can overcome in another way both of these limitations and we do not need to assume the existence of an additional characteristic $\xi$-piece.

Indeed, to define a global flow that allows to switch between a Hole flow centered around $*_1$ and a Hole flow centered around $*_2$, whatever the locations of $*_1$ and $*_2$ are on this characteristic $\xi$-piece, we need to be able to switch between two distinct, not comparable (because of the a priori upper-bound on the number of zeros) Hole flows that this characteristic $\xi$-piece supports.

The violation of the Fredholm assumption allows to complete the switch. The process is as follows: we are given two consecutive $\pm v$-jumps that we can assume to be “steady”, with a definite orientation; the orientations of the two $\pm v$-jumps do not coincide.

Let us say that the first one (starting from the left) is a positive $v$-jump, whereas the second one is a negative one.

A1

The first Hole flow introduces a positive $v$-jump between these two $\pm v$-jumps, whereas the second Hole flow introduces a negative $-v$-jump between these two $\pm v$-jumps.

HF1

A2

HF2

A3

The location of these $\pm v$-jumps is such that, introduced together, they increase by 2 the number of zeros of $b$.

HF1 + HF2

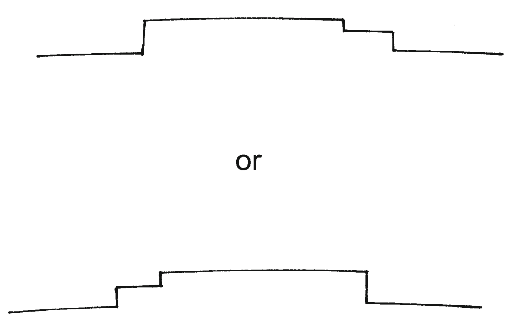
A4

The violation of the Fredholm condition allows to find a way out, a way to glue these two flows:

Let us assume that, e.g. the points of the characteristic $\xi$-piece between the two $\pm v$-jumps are in $A^+$.

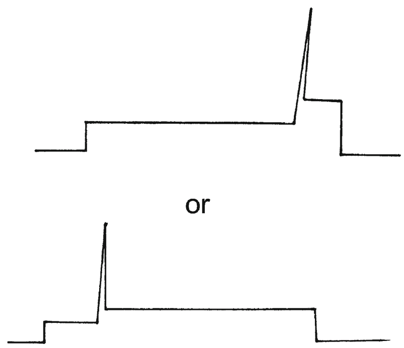
There is no loss of generality in this assumption, for two reasons; first, we can redistribute the rotation along the $\xi$-piece using the technique of [2, pp 81–102], so that all the nodes of this characteristic $\xi$-piece are within a definite region where the Fredholm condition is violated, e.g. in $A^+$; this is always possible in the framework of the contact structure of J. Gonzalo and F. Varela since all points are generically then either in $A^+$ or in $A^-$. This generical assumption holds after perturbation. The contribution of the critical points at infinity is computed after this perturbation. Second, if the points to the right of the first $v$-jump are not in $A^+$, but are in $A^-$, then we move our construction to the left of this first $v$-jump. We may assume that it is "steady" (otherwise, there are "open holes" and the flow can be extended as above).

Coming back to our earlier configurations



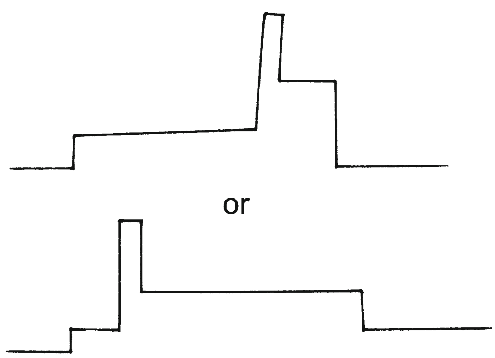

A5

We need to switch, without adding zeros to $b$, between these two configurations. We build for this a very thin "positive Dirac mass" over each of these configurations:




A6

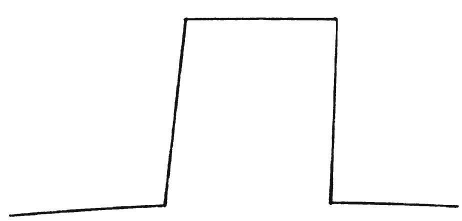
Once the "Dirac mass" reaches an appropriate size, we "open it up", inserting at its top a small $\xi$-piece: $J$ decreases.




A7

“Opening up” more, we eventually “absorb” the $\pm v$-jump in the large oscillation (all the points on the base curve are in fact very close, after the redistribution of the $v$-rotation over this $\xi$-piece that we have performed).

Eventually, we reach



A8

This curve is the same for both configurations. We have, therefore, glued and we have switched between Hole Flows without adding zeros to $b$.

If we exit the framework of the first exotic contact structure of J. Gonzalo and F. Varela and we consider a more general one, these arguments extend under minimal additional assumptions. For example, we could assume that the Fredholm condition is violated as soon as the characteristic $\xi$-piece is large enough. We would then be reduced to curves having several $\xi$-pieces (their number would have to tend to $\infty$ with the index); all these $\xi$-pieces would have to live within the region of the manifold $M$ where the Fredholm condition is not violated. We need then to assume that there are no such closed curves, critical at infinity, for large enough indexes. This assumption does not seem to be stringent, but it is only natural that we check it against several examples before concluding that it is a good one.

15.3 Getting rid of hypotheses (A), (2B) and (2B)’ of [3]:

We now get rid of the Hypotheses (A), (2B) and (2B)’ of [3].

Observe that if two consecutive $*$s on a given $\xi$-piece are families, then we can use the violation of the Fredholm assumption and the fact that $S^3 \setminus (T_0 \cup \{\bar{y} = \bar{y}_0\}) = A^+ \cup A^-$ for the contact structure of Gonzalo–Varela to define a decreasing deformation. The argument is straightforward, but for the fact that it might involve a re-scaling of the $v$-rotation along the $\xi$-piece so that all the old nodal zones along this $\xi$-piece are in the region where the points of this $\xi$-piece are either in $A^+$ or $A^-$, depending on the circumstances.

This extends easily, see Sect. 8, to the case where there is one single interior $*$ that is a family.

It follows that Hypotheses (A), (2B) and (2B)’ of [3] are not needed here, except in the case where there is *no* interior $*$. Since we are assuming on these $\xi$-pieces that there are $(i_0^j - 1)*$s, this means that we are considering characteristic $\xi$-pieces of strict $H_0^1$-index 1.

Our arguments above have led us to deformation arguments in all cases where we had at least one interior $*$ (observe that there is no need to switch $*$s if there is precisely one interior $*$ on a characteristic $\xi$-piece) on a given characteristic $\xi$-piece.

We are left with characteristic $\xi$-pieces of strict $H_0^1$-index 0 or 1. We also have non-characteristic $\xi$-pieces; the arguments for characteristic $\xi$-pieces extend to the non-degenerate ones; we used this several times in our work, see, e.g. [2, pp 79–102], also Sect. 8 and Sect. 9.2 above. However, the arguments that we develop now are insensitive to the fact that the $\xi$-pieces are or are not characteristic.

Therefore, for simplicity, we assume in the sequel that all our $\xi$-pieces are characteristic of strict $H_0^1$-index 0 or 1. We claim that we can assume, over every configuration outside a stratified set of codimension one or more in the space of configurations, that one $*$ is a single $\pm v$-jump; this allows to proceed with the deformation arguments introduced above (Hole flow or decreasing normal, or violation of the Fredholm assumption related to this $*$).

Let us first consider the case when the dominating periodic orbits are of even index $2k$ so that there are $2k$ $*$s to track. These $*$s can become families through three distinct processes: they can become families because warranting transversality would imply that a companion is added to a given $*$; they can become a family because a given flow-line reaches a false critical point at infinity, with a characteristic $\xi$-piece having an “ill-oriented normal” [3, p 483]. This characteristic $\xi$-piece can then either be “sign-false” or “sign-true”, [3, p 483].

Let us, given a $*$ that we would like to keep as a single $\pm v$-jump, consider each of these instances:

Transversality is overcome using the new Hole flow; this flow might involve the introduction of companions. However, when transversality is violated, the counting performed in [3, p 567], and also above leads always to the same conclusion: there is an “open hole” backed by a “steady $*$”. If this $*$ is an interior $*$, we may use the new Hole Flow on it to decrease $J_\infty$ past $x^\infty$ unless it is very close to a new nodal line. In such a case, we might need, if we were to use the flow on this $*$, to have to use companions. However, then, there is one constraint on the configuration and, therefore, there is another “hole”, with a steady $*$ behind it that we can use. On this other $*$, which we do not have to spare, we can use companions and we can decrease $J_\infty$ even if it is on or close to a new nodal zone.

If the only inside $*$ is the $*$ that we want to spare, we may use companions to both edges when this $*$ is on or close to a new nodal line.

Therefore, the only case that we need to study is the case when the “open hole” is related to an edge and this edge is represented by the $*$ which we want to spare; we then hit a contradiction in our construction process. We now use companions to inside $*$s and each $\xi$-piece has two edges. On this $\xi$-piece, we can define two distinct new Hole flows related to each of them. For example, above, we were using companions to the inside $*$ closest to the right edge, as we were viewing holes to the left of the $*$s. Extending our construction, there is always an “open hole”; we view now “open holes” from right to left, starting from the right edge, or from left to right, starting from the left edge, whatever is more convenient to us. In the first case we might have to introduce companions to the right edge, but only to the $*$ of that edge and not to any other $*$, whereas in the second case, we might have to introduce companions to the left edge and not to any other edge.

We thus choose from the onset the edge that is not the $*$ that we want to spare; this is possible since this $*$, being a single $\pm v$-jump, cannot represent both edges.

This takes care of the transversality issues.

For “sign-false” characteristic pieces, we observe that the strict $H_0^1$-unstable manifold of such characteristic piece involves $i_0^j + 1$ sign changes that are represented by an alternating sequence of $(i_0^j + 2)$ $\pm v$-jumps (including edges). Therefore, it can be represented by $(i_0^j + 2)$ distinct$*$s (assuming that the two edges correspond to two different $*$s) and the $i_0^j$ interior $*$s are then single $\pm v$-jumps: we can change the representation of $*$s as we bypass this false critical point at infinity and adopt on the related decreasing flow-lines the $*$s of its strict unstable manifold in between the $\pm v$-jumps corresponding to this sign-false characteristic piece. The definitions of the families of $*$s adjust in a natural way within this framework. Along the full half-unstable manifold, an additional $\pm v$-jump, a companion to one edge, has been introduced. If $i_0^j$ is one or more, this edge has become a family, but just nearby, there is another (maybe new) $*$ that is a single $\pm v$-jump. The dividing line is provided by the strict $H_0^1$-unstable manifold of this characteristic $\xi$-piece. If $i_0^j$ is zero, then the two edges have opposite orientation. Here, we have to use the violation of the Fredholm assumption and the fact that every point of $S^3$ not on $T_0$ or on $\{\bar{y} = \bar{y}_0\}$ is either in $A^+$ or in $A^-$ in a strong sense, see Sect. 8, Propositions 8.1, 8.2, 8.3: namely, the Fredholm assumption is violated using "Dirac masses" (here "Dirac masses" as emphasized in Sect. 8 are either back and forth or forth and back runs along $v$) as large as we please.

One of the edges of this $\xi$-piece of index 0 (this works also for $\xi$-pieces that are non-degenerate of $H_0^1$-index 0) corresponds, at least for one configuration, to the "spared" $*$. Assume it runs along $+v$, ending at $x^-$, which is, therefore, the starting point of the $\xi$-piece of index 0 that we have singled out. Since this configuration is assumed not to correspond to a drop in the number of sign changes, this edge is also preceded with an edge having the reverse orientation; therefore, this edge is oriented along $-v$. If $x^-$ is in $A^-$ (in the strong sense defined above), then we can use the violation of the Fredholm assumption on the previous $\xi$-piece, with a "negative" "Dirac mass", at a point on this $\xi$-piece close to the starting point of the edge abutting at $x^-$ with its "positive $v$-jump" containing the edge abutting at $x^-$ (case of strict $H_0^1$-index equal to zero) or nearby (case of strict $H_0^1$-index equal to 1) and conclude. Observe that if this previous $\xi$-piece is of strict $H_0^1$-index 1, then there must be an inside $*$ living on it: the lack of companions for the ending edge does not allow to represent the strict unstable manifold with companions for both edges, the companions of one edge are missing; therefore, the result has a definite sign and cannot create an unstable direction. Re-scaling the $v$-rotation on this $\xi$-piece, we may assume that the node corresponding to the strict $H_0^1$-index is close to the right edge of the $\xi$-piece and is, therefore, in $A^-$ as well. We may then view the negative $v$-jump of the "Dirac mass" as a companion to the negative right edge, whereas its positive $v$-jump is represented by $*$.

Otherwise, $x^-$ is in $A^+$ in a strong sense and we can introduce a "positive Dirac mass" on the $\xi$-piece of index 0, again with its positive $v$-jump containing the edge and conclude. Again, we have not introduced any companion to the "spared $*$". The result follows in this case.

For "sign-true" characteristic $\xi$-pieces, either they are of strict $H_0^1$-index 1 or more. We can then introduce our "ill-oriented normal" on whatever edge that does not correspond to the $*$ which we want to spare, unless the two edges are the same (not the same $*$, they are the same). Then, $x^\infty$ has a single $\xi$-piece that is characteristic. Its $H_0^1$-index is large; our compactness/violation of the Fredholm assumption[15] arguments above apply.

If they are of $H_0^1$-index zero, then either they correspond to two different $*$s and whatever normal is introduced, we attribute it to the other edge than the one corresponding to the spared $*$ (the two edges have the same orientation); or they correspond to the same $*$ and, then, this is not the spared $*$ since this spared $*$ is a single $\pm v$-jump.

In the case where the dominating periodic orbits are of odd index $2k + 1$, we can define a set of "dividing lines" of codimension one or more, across which there are recognizable definite repetitions in the sign-distribution of the $*$s. Unless the number of sign-changes drops below $2k$, we can choose a $*$ to spare as a single $\pm v$-jump over the configurations outside these "dividing lines": we can choose one $*$ over the maximal domain where it is a single $\pm v$-jump and where it is not involved in a repetition; on the complement domain, we choose another $*$, separated from the first $*$ by a large number ($k$ is large) of other $*$s. In this way, if a "sign-true" false critical point at infinity is encountered and the edges of the characteristic piece involve these two $*$s, we can use a decreasing normal that is a companion to an intermediate $*$. The arguments above then proceed.

### 15.4 Convex-combination of the semi-flows:

We indicate in what follows how to build a global deformation out of the various pieces that we have defined for it. Some technical details in the glueing combination of the flow in $C_\beta$ with the $H_0^1$-flow , that require special care, are left out here and will appear in [13].

[15] Observe that the violation of the Fredholm assumption involving a given $*$ can be assumed not to introduce companions to this given $*$: it can be built with the use of the sole $\pm v$-jump corresponding to this $*$ and another $\pm v$-jump of reverse orientation.

The construction of our deformation has two essential pieces. One is the $Z_\nu$-semi-flow of [4], and the other one is the $H_0^1$-semi-flow of [2] and [4]. A natural question is to understand how they can be convex-combined into the same global semi-flow. The construction of each of them separately is clear from [2] and [4], although the arguments of [4] would certainly gain in being rewritten, with several misprints removed. Also after having read pp 1–91 of [4], the reader is advised to jump to the pp 184–186 "a direct way to reach the $\nu$ or $\tilde{\nu}$-stretched curves". pp 91–183 can be skipped without serious damage to the understanding. This takes care of the $Z_\nu$-semi-flow. [2] and the present paper gives all the necessary estimates (they can be improved) for the $H_0^1$-semi-flow.

The convex-combination of these semi-flows is not obvious because they (a priori) require different spaces for their definition. For the $Z_\nu$-flow, the $v$-component of $\dot{x}$, $\dot{x}$ being the tangent vector to the curve $x$ of $C_\beta$, which we usually denote $b$ needs to be $H^1$. Using for a short time the regularizing semi-flow that has $\eta = b$, see [4], we can assume that $b$ verifies this assumption. For the $H_0^1$-semi-flow, we need to have defined nearly large $\pm v$-jumps and between them nearly $\xi$-pieces. The $H_0^1$-semi-flow "slides" then the ends of the nearly $\xi$-pieces along the nearly $\pm v$-jumps (suitably extended) and seeks to transform the nearly $\xi$-piece in a genuine $\xi$-piece. It is called an $H_0^1$-semi-flow because the $w$-component of the generalized $(H^{-1})$ tangent vector that defines it, $\eta$ is $H_0^1$, $\eta$ being zero at both ends of each nearly $\xi$-piece. Just as for the $Z_\nu$-flow of [4], this $H_0^1$-semi-flow admits a "compactification", an approximation by a finite dimensional, compact, locally Lipschitz vector field, see for $Z_\nu$ pp 59–70 of [2], the flow $Z_\epsilon$ defined using $\eta = \Phi_\epsilon(b)$ in particular. This compactification can be completed for the $H_0^1$-semi-flow as well so that one could think that the convex-combination of $Z_\nu$ with the finite-dimensional Lipschitz vector field becomes possible. Only that this approximation lives once these nearly large $\pm v$-pieces are well-defined and extended.

We would hope that the $Z_\nu$-semi-flow would bring us to such curves that would have definite large almost $\pm v$-jumps. This flow almost "does this job": $b$ is driven through this semi-flow to be close at the blow-up time, in the $L^1$-sense, to the following profile: in this profile, almost Dirac masses for $b$ arise on very short periods of time (they arise as "plateaux" where $b$ is very large, almost constant and equal to $|b|_\infty$); they are then followed by very small pieces of curves (they arise over sets of measure $O(\frac{1}{|b|_\infty^N})$) where $b$ decreases to "plateaux" where it takes the value $\pm\nu$, only to fall, again very fast, as fast as above, to 0 or to $-\nu$ and then rise again very fast for the next positive or negative Dirac mass. The difference between $b$ and such a profile is as small as we please in the $L^1$-sense; certainly we may assume that it is $O(\frac{1}{|b|_\infty^N})$, $N$ as large as we please. It follows that the use of the regularizing semi-flow that has $\eta = b$, of which we spoke above, would, in a very short time, transform the estimate of difference between $b$ and its limit profile from an $L^1$-estimate into a $C^2$-estimate. The convex-combination with the $H_0^1$-flow could then be completed.

However, the semi-flow having $\eta = b$, if we were to use it without further restriction, blows up too often, too fast. Even tamed into $\frac{b}{1+|b|_\infty^{1000}}$, there are not enough estimates on the curves subject to its associated evolution equation.

For some curves, carrying "enough energy" in their nearly $\xi$-pieces (derived after the use of the $Z_\nu$-semi-flow) another "flow" can be used, cautiously, and it will provide this regularizing effect, whereas it will not move the nearly large $\pm v$-pieces much.

This semi-flow is the same than the $Z_\nu$-flow. It is used on the curves to which the semi-flow $Z_\nu$ leads at the blow-up time, that is the curves having $b$ in the $L^1$-sense close (close as above) to one of the profiles defined above. With respect to the $Z_\nu$-semi-flow, there are two modifications: first $\nu$ is replaced by $\frac{\nu}{2}$ and, second, the support of the main part of this semi-flow lies within the nearly $\xi$-pieces defined by this profile.

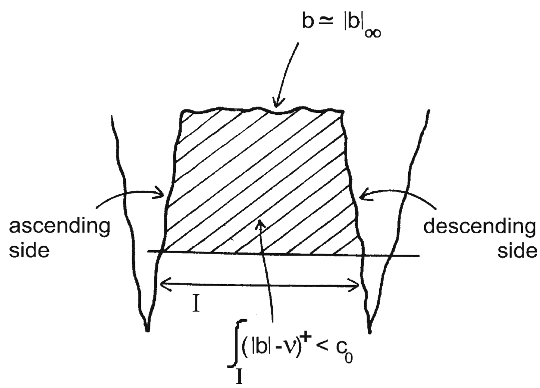
It is not difficult to see then that if $b$, on these nearly $\xi$-pieces, is close to a profile containing a $\pm\nu$-"plateau" having a measure that could be $O(\frac{1}{|b|_\infty^{N_0}})$, $N_0$ large, but would not be $O(\frac{1}{|b|_\infty^N})$, $N$ much larger, as prescribed for an upper-bound between $b$ and its limit profile in the $L^1$-sense, then the use of the $Z_{\frac{\nu}{2}}$-semi-flow within this "plateau" would provide a rate of decrease in $\int_0^1 \alpha_x(\dot{x})$ that would maybe be $O(\frac{1}{|b|_\infty^{N_0}})$, $N_0$ large, but would not be $O(\frac{1}{|b|_\infty^N})$. This would allow for a use of a sizable fraction $cb$ of the regularizing flow, that is $c$ would also be maybe $O(\frac{1}{|b|_\infty^{N_0+1}})$, $N_0$ large, but would not be $O(\frac{1}{|b|_\infty^N})$. In addition, this semi-flow would mainly act *within* the nearly $\xi$-pieces of the curve. Its action, therefore, on the nearly $v$-pieces would be essentially reduced to the action of the generalized tangent vector defined by $\eta = cb$ (there is an additional time translation, required to keep the $\xi$-component of $\dot{x}$ time (of the curve-independent), that is to the regularizing semi-flow. Skipping details (that require complete proofs), this semi-flow would transform the $L^1$-estimate on $b$ with respect to its profile on the nearly $\pm v$-pieces into a $C^2$-estimate and the convex-combination with the $H_0^1$-semi-flow could be completed.

We are left then with curves that do not have enough "energy" in their nearly $\xi$-pieces to induce a regularizing effect on the large nearly $\pm v$-pieces. Using the same line of thought, we can use the $Z_\nu$-flow on these nearly $\pm v$-pieces exclusively (with the additional, as tiny as we please and need, $cb$ see [4] acting also on the nearly $\xi$-pieces) so that the curves will enter the set where there is enough "energy" inside the nearly $\xi$-pieces to regularize the large nearly $\pm v$-pieces.

We can also proceed differently: we use the $Z_\nu$-semi-flow with a prescribed large value $M$ for $|b|_\infty$, see [4] this semi-flow controls $|b|_\infty$. The curves reach the set $\mathcal{V}_\epsilon$ where $b$ is $L^1$-close to one of the profile, $|b|_\infty \leq 2M$. How close is measured by a small constant $\epsilon = O(\frac{1}{|b|_\infty^N})$, $N$ large. In $\mathcal{V}_{\frac{\epsilon}{2}}$ ($|b|_\infty \leq 4M$), we use the tamed regularizing semi-flow that has $\eta = \frac{b}{1+|b|_\infty^{1000}}$. We convex-combine $Z_\nu$ and this semi-flow in between $\mathcal{V}_\epsilon$ and $\mathcal{V}_{\frac{\epsilon}{2}}$. Defining a yet smaller $\mathcal{V}_{\frac{\epsilon}{4}}$, $J(x) = \int_0^1 \alpha_x(\dot{x})$ decreases at a rate bounded away from zero over the curves that stay outside of this set. The (semi)-flow-lines, starting from $\mathcal{V}_\epsilon$, will then not enter $\mathcal{V}_{\frac{\epsilon}{4}}$ unless the $v$-component, $b$, of their tangent vector $\dot{x}$, has now been regularized. The convex-combination can be completed now.

There are three additional observations that we wish to make in order to conclude this sub-section:

First, with just the use of the $Z_\nu$-flow of [4], $b$ has "plateaux" where it is essentially equal to $\pm|b|_\infty$. It can "depart" over a "plateau" from this top value and oscillate fast downwards. However, if there are two such oscillations and if in between, the "mass" of $b$, that is the integral of $|b|$ from the "ascending side"




of the first oscillation to the descending side of the second one (assuming $b$ is here locally essentially equal to $= +|b|_\infty$) is less than a fixed positive constant $c_{10}$, see [4, p 25], but larger than some $O(\frac{1}{|b|_\infty^{N_0}})$, then the flow $Z_\nu$ can still be used, with a sizable decrease for $J$. Therefore, along the "large" nearly $\pm v$-pieces at the blow-up time, these "sharp downwards" oscillations are "scarce". They are separated by sizable (of length $\geq c_{10}$)nearly $\pm v$-pieces. We cannot state that $b$ is close $C^1$ on these $\pm v$-pieces to $\pm|b|_\infty$, but it is certainly $C^0$-close. We can then pick up a "mesh" of points over the curves that are sitting over these (relatively)large nearly $\pm v$-pieces and use these points to define (we might need to extend suitably these nearly $\pm v$-pieces beyond the parts defined by the curve itself so that the end-points can move freely, this is not needed *inside* the (large) nearly $\pm v$-pieces, but it is needed near their edge), without further regularization, the $H_0^1$-semi-flow in a way that can be convex-combined with the $Z_\nu$-semi-flow.

Second, we can use Lemmas 13.1 and 13.2 on these large $\pm v$-pieces (maybe interrupted with these "scarce" downwards oscillations), once the "mesh" of points is given. If we take enough of these points so that they are separated by nearly $\pm v$-pieces of length $\ell \leq c_0$ and if $b$ does not change sign in between, then Lemma 13.2 gives us an algorithm, with $b\eta \geq 0$—hence with a process along which $L$ and $L^*$ do not decrease and $J$ decreases—by which the curve is replaced by two $\pm v$-pieces (of the same orientation then the initial one) separated by a tiny $\xi$-piece. We thus build a family of tiny $\xi$-pieces and other large, but not so large $\pm v$-pieces. We can then use the flow of Lemmas 11.1 and 11.4(i) (with restrictions removed, see Lemma 11.8) and reduce the number of these tiny $\xi$-pieces. The only restriction is the restriction over $b$ not to change sign over these intervals. $b$ might have zeros, but they are in finite number and the "mesh" of points can be refined; the length of the nearly $\pm v$-pieces can be decreased as we "approach" a zero of $b$ so that the process will be carried everywhere on the large nearly $\pm v$-pieces except in tiny neighborhoods, as small as we please, of the zeros of $b$.

Third, the displacement of the nearly $\pm v$-pieces transversally to $v$ is "small" through the $Z_{\frac{v}{2}}$-semi-flow when its use is concentrated, as above, "inside" the nearly $\xi$-pieces. Indeed, then the displacement of these nearly $\pm v$-pieces is due to $\lambda\xi + \eta w$, with $\eta = cb$ and $\lambda + \bar{\mu}\eta = \int_0^t b\eta - t\int_0^1 b\eta = O(\frac{\partial a}{\partial s})$. $c$ is so small that $cb$ is also $O(\frac{\partial a}{\partial s})$. It is in fact, for $c$ small enough, $o(\frac{\partial a}{\partial s})$. $\lambda$ is not necessarily $o(\frac{\partial a}{\partial s})$, but this is due to the term $t\int_0^1 b\eta$. This term is in fact, see [4, p 121 and p 124], due to a time re-parametrization required to keep $a$ constant. If we remove this time re-parametrization along the curve, we find the displacement transverse to $v$ to be $o(\frac{\partial a}{\partial s})$. With some further work, this can probably be transformed into an estimate on the transversal displacement of these large nearly $\pm v$-pieces of curves: the $\xi$-component of $\dot{x}$ along them is $O(\frac{1}{|b|_\infty})$, so that the additional (with respect to the estimates introduced above, transversally to $v$)displacement transversally to $\dot{x}$ is $O(\frac{c\dot{b}+\lambda b}{a|b|_\infty})$. $\dot{b}$, after regularization, should be $O(|b|_\infty^{N_0})$ and the argument should proceed, yielding a very precise convergence of all pieces of the curves under deformation.

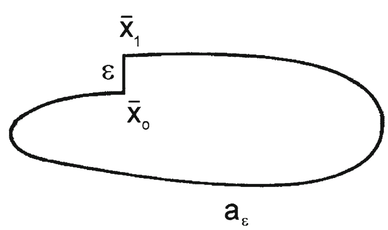
This concludes our observations about the convex-combination of the $Z_v$ and the $H_0^1$-semi-flows.

## 16 Appendix: Critical points at infinity collapsing with degenerating periodic orbits

We prove in this Appendix that the critical points at infinity collapsing with two degenerating periodic orbits, as they come together and cancel, see [2, pp 103–107], are not in $\Gamma_2$. This result was used in Sect. 10, to prove that the homology was invariant through "Fredholm" deformation.

**Proposition 16.1** *Near a degeneracy involving a periodic orbit of index $m$ with a periodic orbit of index $(m-1)$, there is, in the vicinity of the degeneracy, no critical point at infinity in $\Gamma_2$.*

*Proof* Assume that there is such a critical point at infinity:




Then the $\xi$-piece of length $a_c$ must correspond to a $v$-rotation of $k\pi$ in the $\xi$-transport, where $k$ is an integer equal to $m$ or $(m-1)$. The $v$-piece from $\bar{x}_0$ to $\bar{x}_1$ is small, of size $\epsilon$.

Let $\ell$ denote the transport map around this critical point at infinity, which we assume to be in $\Gamma_2$. Computing its differential along $\xi$, $v$, $[\xi, v]$, we find ($\gamma_s$ is the one-parameter group of $v$, $\phi_s$ is the one-parameter group of $\xi$):

$$d\ell(\xi) - \xi = d\gamma_\epsilon \circ d\phi_{a_c}(\xi) - \xi = d\gamma_\epsilon(\xi) - \xi$$

The transport equations in the $(\xi, v, -[\xi, v])$-frame ($z = \lambda\xi + \mu v - \eta[\xi, v]$) read

$$\dot{\eta} = -d\beta(\dot{x}, [\xi, v])\eta - \lambda;\ \dot{\lambda} = \eta$$

Therefore,

$$d\gamma_\epsilon(\xi) - \xi = -\frac{\epsilon^2}{2}\xi + \epsilon(1 + o(1))[\xi, v] + O(\epsilon^3) + \nu v$$
$$d\phi_{a_c}(v) = \theta v, |\theta| \text{ close to } 1, \theta \neq 1$$
$$d\phi_{a_c}(v) - v = (\theta - 1)v$$

Since $d\alpha(d\phi_{a_c}(v), d\phi_{a_c}([\xi, v])) = d\alpha(v, [\xi, v]) = -1$,

$$d\phi_{a_c}([\xi, v] = \frac{1}{\theta}[\xi, v]$$

and

$$d\gamma_\epsilon \circ d\phi_{a_c}(-[\xi, v]) + [\xi, v] = \frac{\epsilon}{\theta}\xi - \left(\frac{1}{\theta} - 1 + O(\epsilon)\right)[\xi, v] + h.o$$

Since $\det d\ell - \mathrm{Id}$ should be zero, since $\theta$ is not 1, we find that

$$\frac{\epsilon}{\theta(\frac{1}{\theta} - 1 + O(\epsilon))} = -\frac{\epsilon(1 + o(1))}{2}$$

that is $\theta = 3 + o(1)$, a contradiction. □



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