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When every flat ideal is finitely projective

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Abstract In this paper, we study the class of rings in which every flat ideal is finitely projective. We investigate the stability of this property under localizations and homomorphic images, and its transfer to various contexts of constructions such as direct products, amalgamation of rings $A \bowtie^f J$, and trivial ring extensions. Our results generate examples which enrich the current literature with new and original families of non-coherent rings that satisfy this property.

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الملخص

في هذه الورقة، ندرس صف الحلقات التي يكون فيها كل مثاليٍّ بسيطٍ إسقاطياً بشكل منتهٍ. نبحث استقرار هذه الخاصية تحت تأثير المَوْضَعة والصور التشاكلية، وانتقالها إلى عدة سياقات من التركيبات مثل الجداءات المباشرة وحلقات الدمج $A \bowtie^f J$، والتمديدات التافهة للحلقات. تولد نتائجنا أمثلة تغني البحث العلمي الحالي بعائلات جديدة وأصيلة من الحلقات غير المتساوقة التي تحقق هذه الخاصية.

1 Introduction

All rings considered in this paper are assumed to be commutative with identity elements and all modules are unitary.

We start by recalling a few definitions. Azumaya [2] generalized the concept of projectivity of modules to finitely projective and gave an interesting study of finitely projective modules. An R-module M is called finitely projective if, for any finitely generated submodule N, the inclusion map $N \to M$ factors through a free module F. Note that Jones [15] use the term f-projective, Mao and Ding [20] and Simson [25] uses the term $\aleph_{-1}$-projective. It is well known that every projective module is finitely projective and any finitely generated finitely projective module is projective and also every finitely projective module is flat. The following diagram of implications summarizes the relations between them:

$$M \text{ is projective } \Longrightarrow M \text{ is finitely projective} \Longrightarrow M \text{ is flat.}$$

But these are not generally reversible, e.g., the rationals are finitely projective as $\mathbb{Z}$-module, though not projective. Let F be any field, $R := \prod_{n\in\mathbb{N}} F$ and $K := \oplus_{n\in\mathbb{N}} F$. Then R/K is R-flat since R is regular, but

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R/K is not finitely projective by [15, page 1611]. An interesting study of rings over which every flat module is finitely projective is done by Shenglin [24].

In this paper, we are interested in those rings over which every flat ideal is finitely projective. We call such ring an FFP-ring. In particular, perfect rings and hereditary rings are FFP-rings. Also, all Noetherian rings and Prüfer rings are FFP-rings. See for instance [2,6,15,24].

Let A and B be rings, J an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In [10], the amalgamation of A with B along J with respect to f is the sub-ring of $A \times B$ defined by:

$$A \bowtie^f J := \{(a, f(a) + j)\ ;\ a \in A, j \in J\}.$$

This construction is a generalization of the amalgamated duplication of a ring along an ideal introduced and studied in [7–9]. This construction has been studied, in the general case, by D'Anna and Fontana in [9].

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') := (aa', ae' + a'e)$. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R. Recall that prime (resp., maximal) ideals of R have the form $p \propto E$, where p is a prime (resp., maximal) ideal of A [1, Theorem 3.2]. Suitable background on commutative trivial ring extensions is demonstrated in [1,12,14,16].

The purpose of this paper is to give some simple methods to construct FFP-rings. For this, we investigate the stability of the FFP-property under localization and homomorphic image, and its transfer to various contexts of constructions such as direct products, amalgamation of rings $A \bowtie^f J$, and trivial ring extensions. Our results generate original examples which enrich the current literature with new families of non-coherent rings satisfying the FFP-property.

2 Main results

Let R be a commutative ring. We will use the following notations and basic notions:

$Z(R) := \{a \in R/ax = 0 \text{ for some } 0 \neq x \in R\}$ denotes the set of zero divisors of R.

$\mathrm{Nil}(R) := \{a \in R/a^n = 0 \text{ for some positive integer } n\}$ denotes the set of nilpotent elements of R.

$\mathrm{Min}(R) := \{P \in \mathrm{spec}(R)/\ P \text{ is a minimal prime ideal of } R\}$ denotes the set of minimal prime ideals of R.

$Q(R)$ denotes the total ring of quotients of R, that is, the localization of R by the set of all its non-zero divisors.

Let E be an R-module, $\mathrm{fd}_R(E)$ denotes the usual flat dimension of E.

A non-zero divisor of R will be called a regular element, and an ideal of R which contains a regular element will be called a regular ideal.

Recall that R is called semi-hereditary if every finitely generated ideal of R is projective and is said to have weak global dimension ≤ 1 (i.e., $\mathrm{wdim}(R) \leq 1$) if every finitely generated ideal of R is flat. A semi-hereditary ring R has $\mathrm{wdim}(R) \leq 1$. In the domain context, all these conditions coincide with the definition of a Prüfer domain. Glaz [11, Example 3.2.1] provides examples of non-semi-hereditary ring of wdim ≤ 1. See for instance [3,4,11]. These examples are not FFP-rings as shown by the following result.

Proposition 2.1 *Any FFP-ring of* wdim ≤ 1 *is semi-hereditary.*

Proof Let R be an FFP-ring with $\mathrm{wdim}(R) \leq 1$ and let I be an finitely generated ideal of R. Then I is flat since $\mathrm{wdim}(R) \leq 1$ and so I is finitely projective since R is an FFP-ring. Therefore, I is projective since it is finitely generated and finitely projective, as desired. □

Now, we give a class of FFP-rings.

Proposition 2.2 *Any coherent ring is an FFP-ring.*

Proof Assume that R is a coherent ring and we must show that it is an FFP-ring. Let J be a flat ideal of R and I be a finitely generated sub-ideal of J. Then I is a finitely presented since R is coherent. Hence, the inclusion map $I \longrightarrow J$ factors through a free module by [5, Theorem 1], as desired. □

The converse does not hold in general (see Examples 2.20 and 2.21 below).

Proposition 2.3 *Let $R \to S$ be an injective flat ring homomorphism. If S is an FFP-ring, then so is R.*

Proof Assume that S is an FFP-ring and let I be a flat ideal of R. Then, $I \otimes_R S = IS$ is a flat ideal of S and so $I \otimes_R S$ is a finitely projective ideal of S (since S is an FFP-ring). Hence, I is a finitely projective ideal of R by [6, lemma 5]. It follows that R is an FFP-ring. □

Corollary 2.4 *Any domain is an FFP-ring.*

Corollary 2.5 *Let R be a ring and $R[X]$ be the polynomial ring over R. If $R[X]$ is an FFP-ring, then so is R.*

Corollary 2.6 *Let R be a ring and I be a flat ideal of R. If $R \bowtie I$ is an FFP-ring, then so is R.*

Proof By Proposition 2.3, since if I is a flat ideal of R, then $R \bowtie I$ is faithfully flat R-module. □

Corollary 2.7 *Let R be a commutative ring and let S be a set of regular element of R. Then if $S^{-1}(R)$ is an FFP-ring, then so is R.*

Proposition 2.8 *Let R be a ring. If R is reduced and $Min(R)$ is compact, then $R[X]$ and R are FFP-ring.*

Proof If R is reduced and $\mathrm{Min}(R)$ is compact, then $Q(R[X])$ the total ring of quotients of $R[X]$ is Von Neumann regular by [21, Corollary 1, Page 270]. Hence, $R[X]$ is an FFP-ring by Proposition 2.3 and so is R by Corollary 2.5. □

Recall that a ring R is a PP-ring (or weak Baer ring) if principal ideals of R are projective.

Corollary 2.9 *Any PP-ring is an FFP-ring.*

Proof Let R be a PP-ring. Then $Q(R)$, the total ring quotients of R, is a Von Neumann regular ring by [13, Theorem 2.11]. Hence, R is an FFP-ring by Proposition 2.3. □

Next, we study the transfer of the FFP-property to direct products.

Theorem 2.10 *Let $(R_i)_{i=1,\ldots,n}$ be a family of commutative rings. Then $R =: \prod_{i=1}^{n} R_i$ is an FFP-ring if and only if so is R_i for each $i = 1, \ldots, n$.*

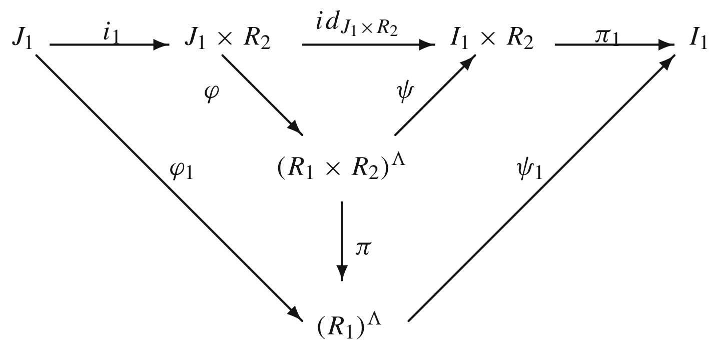
The Proof of the Theorem involves the following Lemma.

Lemma 2.11 *Let R_1 and R_2 be two rings and let E_1 and E_2 be two modules over R_1 and R_2, respectively. Then $fd_{R_1 \times R_2}(E_1 \times E_2) = sup\{fd_{R_1}(E_1), fd_{R_2}(E_2)\}$.*

Proof By [17, Lemma 2.5]. □

Proof of Theorem 2.10 The proof is done by induction on n and it suffices to check it for $n = 2$. Assume that $(R_1 \times R_2)$ is an FFP-ring and we must show that R_i is an FFP-ring for $i = 1, 2$.

Let I_1 be a flat ideal of R_1 and J_1 be a finitely generated sub-ideal of I_1. Then, $I_1 \times R_2$ is a flat ideal of $R_1 \times R_2$ which is an FFP-ring (by Lemma 2.11) and $J_1 \times R_2$ is a finitely generated sub-ideal of $I_1 \times R_2$. Then, there exists a free $(R_1 \times R_2)$-module F ($F \simeq (R_1 \times R_2)^{\Lambda}$ for some finite index set Λ), a morphism $\varphi : J_1 \times R_2 \longrightarrow F$ and a morphism $\psi : F \longrightarrow I_1 \times R_2$ such that $\psi \circ \varphi := id_{J_1 \times R_2}$. Consider the morphism ψ_1: $R_1^{\Lambda} \longrightarrow I_1$, where $\psi_1((x_i)_{i\in\Lambda}) = \pi_1(\psi((x_i, 0)_{i\in\Lambda}))$, for every $(x_i)_{i\in\Lambda} \in R_1^{\Lambda}$ and a morphism φ_1: $J_1 \longrightarrow R_1^{\Lambda}$ defined by the following diagram:



where $\pi((x_i, y_i)_{i\in\Lambda}) = (x_i)_{i\in\Lambda}$ for every $(x_i, y_i)_{i\in\Lambda} \in (R_1 \times R_2)^\Lambda$. Then, $\psi_1 \circ \varphi_1 = \psi_1 \circ \pi \circ \varphi \circ i_1 = \pi_1 \circ \psi \circ \varphi \circ i_1 = \pi_1 \circ id_{J_1 \times R_2} \circ i_1 = id_{J_1}$, and so I_1 is finitely projective. Hence, R_1 is an FFP-ring.

Also, R_2 is an FFP-ring by the same argument as R_1.

Conversely, we assume that R_1 and R_2 are FFP-rings. Note that I is an ideal of $R_1 \times R_2$ if and only if $I := I_1 \times I_2$ for some ideals I_1, I_2 of R_1 and R_2, respectively. On the other hand, I is flat if and only if I_i is R_i-flat for every $i = 1, 2$ (by Lemma 2.11). Let $J_1 \times J_2$ be a finitely generated sub-ideal of $I_1 \times I_2$. Then J_1 (respectively J_2) is finitely generated sub-ideal of I_1 (respectively I_2). Since R_i is an FFP-ring, then, there exists a free R_i-module F_i, a morphism $\varphi_i : J_i \longrightarrow F_i$ and a morphism $\psi_i : F_i \longrightarrow I_i$ such that $\psi_i o \varphi_i := id_{J_i}$. Considers the morphism φ and ψ defined by: φ: $J_1 \times J_2 \longrightarrow F_1 \times F_2$, where $\varphi(x, y) := (\varphi_1(x), \varphi_2(y))$ and $\psi : F_1 \times F_2 \longrightarrow I_1 \times I_2$, where $\psi(x, y) := (\psi_1(x), \psi_2(y))$. Then $\psi \circ \varphi := id_{J_1 \times J_2}$. Therefore, $I_1 \times I_2$ is finitely projective, which completes the proof. □

Example 2.12 Let R_1 be a non-FFP-ring, R_2 be any ring and let $R := R_1 \times R_2$. Then R is not an FFP-ring.

We combine Proposition 2.3 with [10, Proposition 3.1] to get the transfer of the FFP-property to the amalgamation $A \bowtie^f J$.

Proposition 2.13 *Let A and B be rings, $f : A \longrightarrow B$ be a ring homomorphism, and let J be an ideal of B such that J and $f^{-1}(J)$ are regular ideals of B and A, respectively. If $Q(A)$ and $Q(B)$ are FFP-rings, then so is $A \bowtie^f J$.*

Proof By [10, Proposition 3.1], we have $Q(A \bowtie^f J) := Q(A) \times Q(B)$. Then $Q(A \bowtie^f J)$ is an FFP-ring if and only if so are $Q(A)$ and $Q(B)$ (by Theorem 2.10). Hence, $A \bowtie^f J$ is an FFP-ring by Proposition 2.3. □

Corollary 2.14 *Let A be a ring and I be a regular ideal of A. Then $A \bowtie I$ is an FFP-ring if so is $Q(A)$.*

Proposition 2.13 enables us to construct other classes of FFP-rings.

Example 2.15 Let A and B be two domains, and let J be an ideal of B. Then $A \bowtie^f J$ is an FFP-ring.
In particular, if I is an ideal of a domain A, then $R := A \bowtie I$ is an FFP-ring.

Now, we study the transfer of the FFP-property between a ring A and the trivial ring extension of A by E, where E is an A-module. The main result (Theorem 2.16) enriches the literature with original examples of (non-coherent) FFP-rings. Recall that if E is an A-module, then $Z(E) := \{a \in A$ such that $ae := 0$ for some $0 \neq e \in E\}$.

Theorem 2.16 *Let A be a ring, let E be an A-module and let $R := A \propto E$ be a trivial ring extension of A by E. Then:*

(1) *Assume that E is a flat A-module or an ideal of A. If R is an FFP-ring, then so is A.*
(2) *Assume that A is a domain and E is an A-module such that $Z(E) := 0$ [In particular, if E is a K-vector space, where $K := qf(A)$ is the quotient field of A]. Then R is an FFP-ring.*

(3) *Assume that A is a domain and E is a divisible A-module. Then R is an FFP-ring.*

(4) *Let (A, M) be a local ring and let $R := A \propto E$ be the trivial ring extension of A by an A-module E such that $ME = 0$. Then R is an FFP-ring if so is A.*

Before proving Theorem 2.16, we establish the following lemmas.

Lemma 2.17 [19, Theorem 7(2)] *Let A be a ring, let E be an A-module and let $R := A \propto E$ be a trivial ring extension of A by E. If $J := I \propto E$ (where I is a non-zero ideal of A) is a flat ideal of R, then I is a flat ideal of A.*

Lemma 2.18 *Let $T = K \propto E$ be the trivial ring extension of a field K by a K-vector space E. Then T is an FFP-ring.*

Proof T is the only non-zero flat ideal of T by [1, Corollary 3.4] since K is a field. Hence, T is an FFP-ring. □

An R-module M is called P-flat if, for any $(s, x) \in R \times M$ such that $sx = 0$, $x \in (0 : s)M$, where $(0 : s) = \mathrm{Ann}_R(s)$. If M is flat, then M is naturally P-flat. In the domain case, P-flat is equivalent to torsion free and when R is an arithmetical ring, i.e., the lattice formed by its ideals is distributive, then any P-flat module is flat (by [6, p. 236]). Also, every P-flat cyclic module is flat (by [6, Proposition 1(2)]). See for instance [6,22].

Before proving Theorem 2.16, we also need the following lemma of independent interest.

Lemma 2.19 *Let A be a domain, E be an A-module, $F(\neq 0)$ be a sub-module of E and $R := A \propto E$ be a trivial ring extension of A by E. Then $0 \propto F$ is not a P-flat R-module.*

Proof Let $F(\neq 0)$ be a sub-module of E. Two cases are then possible:

Case 1: $Z(F) := 0$. Let $(0, 0) \neq (0, f) \in 0 \propto F$ and $(0, 0) \neq (0, e) \in 0 \propto F$. Then, $(0, f)(0, e) = (0, 0)$ and $(0 : (0, e)) = 0 \propto E$ since $Z(F) = 0$. Then $(0, f) \notin (0 : (0, e))(0 \propto F) = (0 \propto E)(0 \propto F) = 0$. Thus, $0 \propto F$ is not a P-flat R-module.

Case 2: $Z(F) \neq 0$. Let $0 \neq d \in Z(F)$ and $0 \neq f \in F$ such that $df := 0$. Hence, $(d, 0)(0, f) := (0, 0)$ and $(0 : (d, 0)) \subseteq 0 \propto E$ and so $(0, f) \notin (0 : (d, 0))(0 \propto F) := 0$. Therefore, $0 \propto F$ is not a P-flat R-module, as desired. □

Proof of Theorem 2.16 (1) Assume that R is an FFP-ring. We have two cases.

Case 1. If E is a flat A-module, the result follows clearly by Proposition 2.3.

Case 2. Assume now, that E is an ideal of A and let I be a flat ideal of A. Then $I \otimes_A R$ $(:= I \propto IE)$ (since I is flat and E is an ideal of A) is a flat ideal of R and so it is a finitely projective since R is an FFP-ring. Therefore, I is a finitely projective ideal of A by [6, Lemma 5] and so R is an FFP-ring.

(2) Assume that A is a domain and E is an A-module such that $Z(E) = 0$. The set $S := (A - \{0\}) \propto E$ is the set of regular elements of $A \propto E$ by [1, Theorem 3.5]. Hence, by [1, Theorem 4.1], $Q(A \propto E) \simeq S^{-1}(A) \propto S^{-1}(E) := K \propto S^{-1}(E)$, where $K = qf(A)$ is the quotient field of A. Therefore, $A \propto E$ is an FFP-ring by Lemma 2.18 and Proposition 2.3.

(3) Assume that A is a domain and E is a divisible A-module. Let J be a non-zero flat ideal of R, we need to prove that J is finitely projective. By [1, Corollary 3.4], $J := I \propto E$ or $J := 0 \propto E'$ for some ideal I of R or some submodule E' of E. Since $0 \propto E'$ is not flat since is not P-flat by Lemma 2.19, then $J := I \propto E$. Let L be a finitely generated sub-ideal of J. Two cases are possible:

Case 1: $L := 0 \propto E'$, where E' is a finitely generated sub-module of E. Thus, J is finitely projective. Indeed there exists a free A-module F and an epimorphism f such that $I \simeq F/kerf$. Consider the morphism φ: $F \otimes_A R \longrightarrow I \otimes_A R (\simeq I \propto E)$ defined by $\varphi := (f \otimes id_R)$. Then $\varphi \circ id_{(0 \propto E')} := id_{(0 \propto E')}$.

Case 2: $L := I' \propto E$, where I' is a finitely generated sub-ideal of I. Hence, I is a flat ideal of A (by Lemma 2.17) and so it is finitely projective (since A is an FFP-ring). Then there exists a free A-module F, a morphism φ_1: $I' \longrightarrow F$ and a morphism $\psi_1 : F \longrightarrow I$ such that $\psi_1 \circ \varphi_1 := id_{I'}$. Consider the morphisms φ and ψ defined by: φ: $I' \otimes_A R \longrightarrow F \otimes_A R$ such that $\varphi := (\varphi_1 \otimes id_R)$ and $\psi : F \otimes R \longrightarrow I \otimes_A R$, such that $\psi := (\psi_1 \otimes id_R)$, then $\psi \circ \varphi := (\psi_1 \otimes id_R) \circ (\varphi_1 \otimes id_R) := (\psi_1 \circ \varphi_1) \otimes id_R := id_{I'} \otimes id_R := id_{(I' \propto E)}$. Therefore, $I \propto E$ is finitely projective, as desired.

(4) Let (A, M) be a local FFP-ring, $R := A \propto E$ be the trivial ring extension of A by an A-module E such that $ME := 0$ and let J be a flat ideal of R. By [23, Lemma 2.1], we may assume that $J(M \propto E) := J$. Then $J := J(M \propto E) \subseteq (M \propto E)(M \propto E) := M^2 \propto 0$. Hence, $J := I \propto 0$ for some ideal I of A. We have $J \otimes_R A \cong J \otimes_R R/(0 \propto E) \cong J/J(0 \propto E) \cong I \propto 0/(I \propto 0)(0 \propto E) := I \propto 0$. So, I is a flat ideal of A since J is a flat ideal of R. Hence, I is a finitely projective ideal of A since A is an FFP-ring. We claim that J is a finitely projective ideal of R.
Indeed, let $I' \propto 0$ be a finitely generated sub-ideal of J, where I' is a finitely generated sub-ideal of I. Since I is finitely projective and $(I \propto 0 := J) \cong I$, then there exists a free A-module F, a morphism φ_1: $I' \propto 0 \longrightarrow F$ and a morphism $\psi_1 : F \longrightarrow I \propto 0$ such that $\psi - 1 \circ \varphi_1 := id_{I' \propto 0}$. Consider the morphisms f, φ and ψ defined by: $f : F \longrightarrow F \otimes_A R$ such that $f(x) := x \otimes 1_R$, $\varphi : I' \propto 0 \longrightarrow F$ such that $\varphi := (f o \varphi_1)$, and $\psi : F \otimes_A R \longrightarrow I \propto 0$ such that $\psi_1 := (\psi \circ f)$. Then $\psi \circ \varphi := \psi \circ (f o \varphi_1) := \psi - 1 \circ \varphi_1 := id_{(I' \propto 0)}$. Therefore, $J := I \propto 0$ is finitely projective and this completes the Proof of Theorem 2.16. □

Theorem 2.16 gives new and original examples of (non-coherent) FFP-rings.

Example 2.20 Let A be a domain which is not Prüfer, $K := qf(A)$, and let $R := A \propto K$ be the trivial ring extension of A by K. Then:

(1) R is an FFP-ring by Theorem 2.16(2).

(2) R is not coherent by [16, Theorem 2.8(1)]. In particular, R is non-Noetherian.

Example 2.21 Let (A, M) be a local domain and let $R := A \propto (A/M)^{\Lambda}$, where Λ is an infinite set, be the trivial ring extension of A by the A-module $(A/M)^{\Lambda}$. Then:

(1) R is an FFP-ring by Theorem 2.16(4).

(2) R is not coherent by [18, Theorem 2.1]. In particular, R is non-Noetherian.

Example 2.22 (1) $\mathbb{Z} \propto \mathbb{Q}/\mathbb{Z}$ is an FFP-ring.

(2) If G is a divisible abelian group, then $\mathbb{Z} \propto G$ is an FFP-ring.

(3) Let R be a domain and G be a divisible abelian group, then $R \propto Hom_{\mathbb{Z}}(R, G)$ is an FFP-ring.

(4) If R is a Dedekind ring and A is a torsion-free R-module, then $R \propto Ext^1_R(A, B)$ is an FFP-ring, for every R-module B.

Proof (1) The rationals $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module, then $\mathbb{Q}/\mathbb{Z}$ is a divisible $\mathbb{Z}$-module. Hence, $\mathbb{Z} \propto \mathbb{Q}/\mathbb{Z}$ is an FFP-ring.

(2) If G is a divisible abelian group, then G is a divisible $\mathbb{Z}$-module. Hence, $\mathbb{Z} \propto G$ is an FFP-ring by Theorem 2.16.

(3) Let R be a domain and G be a divisible abelian group, then $Hom_{\mathbb{Z}}(R, G)$ is an injective R-module by [22, Lemma 3.37]. Then, $R \propto Hom_{\mathbb{Z}}(R, G)$ is an FFP-ring by Theorem 2.16.

(4) By [22, Proposition 8.2] and Theorem 2.16. □

Our next (and last) result establishes the transfer of the FFP property to a particular homomorphic image.

Proposition 2.23 *Let R be a ring and let I be a pure ideal of R. If R is an FFP-ring, then so is R/I.*

Proof Let R be an FFP-ring and let J/I be a flat ideal of R/I. Then J is a flat ideal of R (using the exact sequence: $0 \longrightarrow I \longrightarrow J \longrightarrow J/I \longrightarrow 0$ where I and J/I are flat R-modules since I is a pure ideal of R). So J is finitely projective. The epimorphism $p : J \longrightarrow J/I$ is finitely split, since I is a pure sub-module of J (see [2] page 114). Hence, by [2, Corollary 13] J/I is finitely projective R-module. Let K/I be a finitely generated sub-ideal of J/I, there exists a free R-module F, a morphism $\varphi : K/I \longrightarrow F$ and a morphism $\psi : F \longrightarrow J/I$ such that $\psi \circ \varphi := id_{K/I}$. It follows that $((\psi \otimes 1_{R/I}) \circ (\varphi \otimes 1_{R/I})(x) := x)$ for all $x \in K/I$ since R/I is a flat R-module. We get that J/I is finitely projective ideal of R/I. Hence, R/I is an FFP-ring. □

The converse does not hold in general as the following examples shows.

Example 2.24 Let R be a ring and let m be a prime ideal of R. Then R/m is always an FFP-ring.

Example 2.25 $\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ are FFP-rings, but $6\mathbb{Z}$ is a non-pure finitely generated ideal of $\mathbb{Z}$.

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