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RESEARCH ARTICLE

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Almost principal ideals in R[x]

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Abstract For an integral domain R with quotient field K, an upper-type ideal of $R[\mathrm{x}]$ is an ideal of the form $I_f = f(\mathrm{x})K[\mathrm{x}] \cap R[\mathrm{x}]$ for some polynomial $f(\mathrm{x}) \in K[\mathrm{x}]\backslash K$. Clearly, $I_f = I_{rf}$ for each nonzero $r \in R$. Hence one can always choose $f(\mathrm{x})$ from $R[\mathrm{x}]$. Such an ideal I_f is said to be almost principal if there is a nonzero element $s \in R$ such that $sI_f \subseteq f(\mathrm{x})R[\mathrm{x}]$. If each upper-type ideal of $R[\mathrm{x}]$ is almost principal, then $R[\mathrm{x}]$ is said to be an almost principal ideal domain. The primary objective of this paper is to provide a unifying technique for verifying that certain types of domains always have corresponding polynomial rings that are almost principal ideal domains. These same techniques also will be used to show that certain types of upper-type ideals are always almost principal.

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الملخص

لأي حلقة صحيحة R مع حقل خوارج قسمة K، مثالي النوع العلوي هو أي مثالي من حلقة كثيرة الحدود $R[x]$ يمكن كتابته على الشكل $I_f = fK[x] \cap R[x]$، حيث $f \in K[x]\backslash K$ إحدى كثيرات الحدود. من الواضح أن $I_{rf} = I_f$ لأي عنصر غير صفري $r \in R$. لذلك، يمكن دائماً أن يتم اختيار $f(x)$ من $R[x]$. يُدْعى مثل هذا المثالي I_f رئيساً تقريباً إذا وجد عنصر غير صفري $s \in R$ بحيث يكون $sI_f \subseteq f(x)\, R[x]$. إذا كان كل مثالي من النوع العلوي من $R[x]$ رئيساً تقريباً، فإن $R[x]$ تدعى حلقة صحيحة رئيسة تقريباً. الهدف الأساس لهذه الورقة البحثية هو تقديم تقنية للتحقق من أن أنواعاً معينة من الحلقات الصحيحة لها دائماً حلقات كثيرات حدود رئيسة تقريباً. سيتم استخدام نفس تلك التقنيات لتبيين أن أنواعاً معينة من المثاليات من النوع العلوي هي دائماً رئيسة تقريباً.

1 Introduction

Let R be an integral domain with quotient field K and integral closure R'. For a polynomial $f(\mathrm{x}) \in K[\mathrm{x}]$, we let $C(f)$ denote the R-fractional ideal generated by the coefficients of $f(\mathrm{x})$. If $f(\mathrm{x})$ is not a constant, then the ideal $I_f := f(\mathrm{x})K[\mathrm{x}] \cap R[\mathrm{x}]$ has the property that $I_f \cap R = (0)$. Note that $I_{rf} = I_f$ for each nonzero element $r \in R$. Hence, with regard to the ideal I_f one may always assume $f(\mathrm{x}) \in R[\mathrm{x}]$. We refer to these ideals as *upper-type* ideals (of $R[\mathrm{x}]$). If $f(\mathrm{x})$ is irreducible over K, then I_f is a prime ideal of $R[\mathrm{x}]$ and is said

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to be an *upper to zero*. To reflect the fact that such an ideal is prime, the standard notation for an upper to zero is P_f. Throughout the paper, we will assume that $f(\mathrm{x}) \in R[\mathrm{x}]$.

A nonzero ideal J of $R[\mathrm{x}]$ is said to be *almost principal* if there is a nonzero element $r \in R$ and a polynomial $f(\mathrm{x}) \in J$ of positive degree such that $rJ \subseteq f(\mathrm{x})R[\mathrm{x}]$ (see, for example, [10]). Necessarily, if J is almost principal, then $J \cap R = (0)$ and $JK[\mathrm{x}] \neq K[\mathrm{x}]$. In fact, if $rJ \subseteq f(\mathrm{x})R[\mathrm{x}]$, then $JK[\mathrm{x}] = f(\mathrm{x})K[\mathrm{x}]$ and therefore $J \subseteq I_f$. Note that if J is a nonzero ideal of $R[\mathrm{x}]$ with $J \cap R = (0)$, then there is a polynomial $g(\mathrm{x}) \in J$ of positive degree such that $JK[\mathrm{x}] = g(\mathrm{x})K[\mathrm{x}]$. If the corresponding ideal I_g is almost principal, then certainly J is almost principal.

The polynomial ring $R[\mathrm{x}]$ is said to be an *almost principal ideal domain* if each nonzero ideal of $R[\mathrm{x}]$ with proper extension to $K[\mathrm{x}]$ is almost principal [10, p. 65]. Using [10, Lemmas 1.5 and 1.6] (and the fact that $K[\mathrm{x}]$ is a UFD), one can easily show that $R[\mathrm{x}]$ is an almost principal ideal domain if and only if each upper to zero is almost principal. In [16, Proposition 3.3], Johnson showed that if R is either an integrally closed domain or a Noetherian domain, then $R[\mathrm{x}]$ is an almost principal ideal domain. Also if $(R : R')$ is not the zero ideal, then R is an almost principal ideal domain [10, Theorem 1.2]. In Theorem 3.1, we extend Johnson's result about integrally closed domains to those that are seminormal. Specifically, if R is seminormal, then $R[\mathrm{x}]$ is an almost principal ideal domain. Also we show that if R has the radical trace property (for each noninvertible ideal I, $I(R : I)$ is a radical ideal), then $R[\mathrm{x}]$ is an almost principal ideal domain (see Theorem 3.7).

A primary objective of this paper is to establish a general sufficient condition for an upper-type ideal to be almost principal and then connect this condition to the known types of almost principal ideal domains and to the known types of almost principal uppers. In this regard, we provide two "mega theorems", one based on a certain type of ideal related to a particular upper-type ideal, and the other based on a certain type of overring related to a particular upper-type ideal. The first of these results is Theorem 2.6 which states that a given upper-type ideal I_f is almost principal if there is a nonzero ideal B of R such that $C(f)C(h) \subseteq (B : B)$ for all $f(\mathrm{x})h(\mathrm{x}) \in I_f$. Theorem 2.12 (the second mega theorem) has a similar proof. Specifically, we show that for a given upper-type ideal I_f, if there is an overring T of R such that $(R : T)$ is nonzero and $f(\mathrm{x})K[\mathrm{x}] \cap T[\mathrm{x}]$ is an almost principal ideal of $T[\mathrm{x}]$, then I_f is almost principal. Moreover, it is enough to have that $I_f T[\mathrm{x}]$ is almost principal as an ideal of $T[\mathrm{x}]$. The converse holds as well (when $(R : T) \neq (0)$)—if I_f is almost principal, then both $f(\mathrm{x})K[\mathrm{x}] \cap T[\mathrm{x}]$ and $I_f T[\mathrm{x}]$ are almost principal. In contrast, the rings in Example 3.12 show that when $(R : T) = (0)$, having $I_f T[\mathrm{x}]$ almost principal does not imply that I_f will be almost principal. Also in Sect. 2 (= the "tools" section), we derive several other sufficient conditions for an upper-type ideal to be almost principal. Each of these is based on the first of the two mega theorems. In Sect. 3 we show how the two mega theorems and other results in Sect. 2 are connected to the known types of almost principal ideal domains and known types of almost principal upper-type ideals. In particular, Corollary 2.7 is used several times in Sect. 3 (and once in Sect. 2). With regard to examples of almost principal ideals, we know of none where this status cannot be established using the basic tools we present in Sect. 2 together with the "applications" in Sect. 3. In several cases, it is possible to show that if I_f is almost principal and $g(\mathrm{x}) \in R[\mathrm{x}]$ is such that $C(g) = C(f)$, then I_g is almost principal.

A key tool in our approach to employing Theorem 2.6 is the Dedekind–Mertens content formula [8, Theorem 28.1]: for a pair of polynomials $f(\mathrm{x}), g(\mathrm{x}) \in K[\mathrm{x}]$, $C(f)^m C(fg) = C(f)^m C(f)C(g)$ for all $m \geq deg(g)$. Generally, we apply the formula to the case where $f(\mathrm{x}) \in R[\mathrm{x}]$ and $f(\mathrm{x})g(\mathrm{x}) \in I_f$. In this case, $C(f)C(g)$ is contained in $(C(f)^m : C(f)^m)$ as $C(fg) \subseteq R$. Based on Theorem 2.6, if we can find a nonzero ideal B such that $(B : B)$ contains $(C(f)^m : C(f)^m)$ for each positive integer m, then the upper-type ideal I_f is almost principal (see Corollary 2.7). By switching the roles of $f(\mathrm{x})$ and $g(\mathrm{x})$, we have $C(g)^n C(fg) = C(g)^n C(g)C(f)$ for all $f(\mathrm{x})g(\mathrm{x}) \in I_f$ where $n = deg(f)$. Using this we are able to establish Corollary 2.10 (and the converse of Theorem 2.6): I_f is almost principal if and only if there is a nonzero ideal B of R such that $C(f)C(g) \subseteq (B : B)$ for all $f(\mathrm{x})g(\mathrm{x}) \in I_f$.

Recall that for a nonzero fractional ideal J of R, $J_v := (R : (R : J))$ and $J_t := \bigcup\{I_v \mid (0) \neq I \subseteq J$, with I finitely generated$\}$. It is always the case that $J \subseteq J_t \subseteq J_v$, with J divisorial if $J = J_v$, and J a (fractional) t-ideal if $J = J_t$. The domain R can be realized as the intersection $\bigcap\{R_M \mid M \in tMax(R)\}$ where $tMax(R)$ denotes the set of (necessarily prime) integral proper ideals that are maximal with respect to being t-ideals. A property we will find useful with regard to the v-operation is that for a nonzero ideal I, $I \subseteq \bigcap\{IR_M \mid M \in tMax(R)\} \subseteq I_v$; to verify the second containment, simply note that if $q \in (R : I)$, then $qIR_M \subseteq R_M$ for all $M \in tMax(R)$ which implies q multiplies $\bigcap\{IR_M \mid M \in tMax(R)\}$ into R. We will make use of this containment in the proofs of Theorems 3.10 and 3.17. Also the v-operation satisfies the following distributive closure property: for nonzero (fractional) ideals I and J, $(IJ)_v = (IJ_v)_v = (I_v J_v)_v$.

2 Technical tools

Recall that if a nonzero ideal J of $R[\mathrm{x}]$ is almost principal, then there is a nonzero element $s \in R$ and a polynomial $f(\mathrm{x}) \in J$ of positive degree such that $sJ \subseteq f(\mathrm{x})R[\mathrm{x}]$. Necessarily, $f(\mathrm{x})$ has smallest positive degree for a nonzero element of J. Hence each element of J has the form $f(\mathrm{x})h(\mathrm{x})$ for some $h(\mathrm{x}) \in K[\mathrm{x}]$. It follows that $sh(\mathrm{x}) \in R[\mathrm{x}]$. Conversely, if $g(\mathrm{x}) \in J$ has smallest positive degree among the nonzero elements of J, then $JK[\mathrm{x}] = g(\mathrm{x})K[\mathrm{x}]$ and so each element of J has the form $g(\mathrm{x})k(\mathrm{x})$ for some polynomial $k(\mathrm{x}) \in K[\mathrm{x}]$. If we have a nonzero element $t \in R$ such that $tk(\mathrm{x}) \in R[\mathrm{x}]$ for all such $k(\mathrm{x})$, then $tJ \subseteq g(\mathrm{x})R[\mathrm{x}]$.

In addition to the standard notation of I_f for the contraction of $f(\mathrm{x})K[\mathrm{x}]$ to $R[\mathrm{x}]$, we let $G_f := \{h(\mathrm{x}) \in K[\mathrm{x}] \mid f(\mathrm{x})h(\mathrm{x}) \in I_f\}$ and let $H_f := \bigcup\{C(h) \mid h(\mathrm{x}) \in G_f\}$. Note that $G_f = (R[\mathrm{x}] : f(\mathrm{x}))$, so it is a fractional ideal of $R[\mathrm{x}]$. Also, from the proof of Lemma 2.4, it is easy to see that $H_f = C(G_f)$. Thus, one could restate Lemma 2.4 as I_f is almost principal if and only if $C(G_f)$ is a fractional ideal of R.

A "best case" for I_f to be almost principal is for it to actually be principal. While this is rather rare, it always occurs if $C(f)$ is a principal ideal of R.

Lemma 2.1 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$. If $C(f)$ is a principal ideal of R, then I_f is principal.*

Proof If $C(f) = bR$ for some (nonzero) $b \in R$, then $g(\mathrm{x}) := b^{-1}f(\mathrm{x}) \in I_f$ has unit content in R and $I_g = I_f$. Using the content formula we have $C(h) = C(g)C(h) = C(gh) \subseteq R$ for each $h(\mathrm{x}) \in G_g$. It follows that $I_f = I_g = g(\mathrm{x})R[\mathrm{x}]$ is principal. □

The converse of Lemma 2.1 does not hold. For example, if $C(f) \subsetneq C(f)_v = R$, then $I_f = f(\mathrm{x})R[\mathrm{x}]$ (see, for example, [22, Lemma 1]).

For ease of reference we recall the following results from [10]. We provide proofs as they suggest ways to verify that a particular upper-type ideal is almost principal.

Lemma 2.2 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$.*

(1) *If $f(\mathrm{x}) = g(\mathrm{x})h(\mathrm{x})$ with both $g(\mathrm{x}), h(\mathrm{x}) \in R[\mathrm{x}]$ of positive degree, then I_f is almost principal if and only if both I_g and I_h are almost principal* (cf. [10, Lemma 1.5]).
(2) *If $f(\mathrm{x}) \in J$ and $J \subseteq I_f$ with I_f almost principal, then J is almost principal* (cf. [10, p. 65]).
(3) *If I_f is almost principal and $I_f \subseteq B$ where $B \cap R = (0)$, then B is almost principal* (cf. [10, Lemma 1.6]).

Proof For (1), let $h(\mathrm{x})k(\mathrm{x}) \in I_h$. Then $f(\mathrm{x})k(\mathrm{x}) = g(\mathrm{x})h(\mathrm{x})k(\mathrm{x}) \in I_f$. If I_f is almost principal, then there is a nonzero element $r \in R$ such that $rk(\mathrm{x}) \in R[\mathrm{x}]$. Clearly, the same r works for I_h. Hence I_h is almost principal. A similar proof shows that I_g is almost principal.

For the reverse implication, one has nonzero elements $s, t \in R$ such that $sI_g \subseteq g(\mathrm{x})R[\mathrm{x}]$ and $tI_h \subseteq h(\mathrm{x})R[\mathrm{x}]$. If $f(\mathrm{x})k(\mathrm{x}) \in I_f \subsetneq I_g$, then $sg(\mathrm{x})h(\mathrm{x})k(\mathrm{x})$ is in $g(\mathrm{x})R[\mathrm{x}]$ and this implies $sh(\mathrm{x})k(\mathrm{x})$ is in I_h. Hence $tsk(\mathrm{x}) \in R[\mathrm{x}]$ and therefore I_f is almost principal when both I_g and I_h are almost principal.

For (2), suppose $f(\mathrm{x}) \in J \subseteq I_f$ with I_f almost principal. Then obviously $sJ \subseteq sI_f$ for each nonzero $s \in R$. So I_f almost principal implies J is almost principal.

For (3), suppose B is an nonzero ideal of $R[\mathrm{x}]$ such that $B \cap R = (0)$ and let $k(\mathrm{x})$ be a nonzero polynomial of minimal degree that is contained in B. If $I_f \subseteq B$, then we also have $I_f \subseteq I_k$. In $K[\mathrm{x}]$, $k(\mathrm{x})$ divides $f(\mathrm{x})$ and since nonzero constant multiples produce the same upper-type ideal, we may assume $f(\mathrm{x}) = k(\mathrm{x})s(\mathrm{x})$ for some polynomial $s(\mathrm{x}) \in R[\mathrm{x}]$. Applying (1), if I_f is almost principal, then so is I_k and therefore so is B by (2). □

With regard to the statement in (1), the same conclusion holds if all we have is $f(\mathrm{x}) = g(\mathrm{x})h(\mathrm{x})$ for some polynomials $g(\mathrm{x})$ and $h(\mathrm{x})$ in $K[\mathrm{x}]$. Simply choose nonzero elements $r, s \in R$ such that $rg(\mathrm{x}), sh(\mathrm{x}) \in R[\mathrm{x}]$. Then $I_f = I_{rsf}$, $I_g = I_{rg}$ and $I_h = I_{sh}$ with $rsf(\mathrm{x}) = (rg(\mathrm{x}))(sh(\mathrm{x}))$.

The following theorem can be easily proved by combining the results in Lemma 2.2 with the fact that $K[\mathrm{x}]$ is a UFD (see [10, p. 68]).

Theorem 2.3 *The following are equivalent for a domain R.*

(1) *$R[\mathrm{x}]$ is an almost principal ideal domain.*
(2) *Each upper-type ideal of $R[\mathrm{x}]$ is almost principal.*
(3) *Each upper to zero of $R[\mathrm{x}]$ is almost principal.*

The first new result suggests at least some connection with ideals of R and almost principal upper-type ideals of $R[\mathrm{x}]$.

Lemma 2.4 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$. Then the corresponding set H_f is an R-submodule of K. Moreover, I_f is almost principal if and only if H_f is a fractional ideal of R.*

Proof A useful trick when dealing with contents of polynomials is that for a pair of (nonzero) polynomials $r(\mathrm{x}), s(\mathrm{x}) \in K[\mathrm{x}]$ the polynomial $t_k(\mathrm{x}) := r(\mathrm{x}) + \mathrm{x}^k s(\mathrm{x})$ has content equal to $C(r) + C(s)$ whenever $k > deg(r)$. Certainly, if both $f(\mathrm{x})r(\mathrm{x})$ and $f(\mathrm{x})s(\mathrm{x})$ are in I_f, then so is $f(\mathrm{x})t_k(\mathrm{x})$ (for each k). It follows that H_f is an R-submodule of K.

As H_f is a fractional ideal if and only if there is a nonzero element $b \in R$ such that $bH_f \subseteq R$, it is clear that I_f is almost principal if and only if H_f is a fractional ideal of R. □

For an ideal J of $R[\mathrm{x}]$, the proof of Lemma 2.4 shows that we can define the content of J as (the ideal) $C(J) = \bigcup\{C(g) \mid g(\mathrm{x}) \in J\}$.

Lemma 2.5 *If I_f is an upper-type ideal of $R[\mathrm{x}]$, then $(R : C(I_f)) = (C(I_f) : C(I_f))$.*

Proof Let $q \in (R : C(I_f))$ and let $f(\mathrm{x})g(\mathrm{x})$ be a nonzero polynomial in I_f. Then $qC(fg) \subseteq R$. It follows that $qf(\mathrm{x})g(\mathrm{x}) \in R[\mathrm{x}]$ and thus $qf(\mathrm{x})g(\mathrm{x}) \in I_f$. Therefore $qC(fg) \subseteq C(I_f)$. As this occurs for each polynomial in I_f we have $qC(I_f) \subseteq C(I_f)$ and thus $(R : C(I_f)) = (C(I_f) : C(I_f))$. □

A nonzero ideal I of R is said to be a *trace ideal* if $(R : I) = (I : I)$. Thus the preceding lemma shows that the content of an upper-type ideal is always a trace ideal of R. A simple way to create a trace ideal is to take the product of a nonzero ideal and its dual (see [2, Proposition 7.2]). In [15, Proposition 2.2], Huckaba and Papick showed that if I is a nonzero ideal such that $(R : I)$ is a ring, then $(R : I) = (I_v : I_v)$ which means that I_v is a trace ideal. We make use of their result in the proof of Theorem 2.17 below.

The proofs for the two mega theorems are quite short, but we will show both theorems are quite useful. In particular, the first can be applied in many different situations.

Theorem 2.6 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$. If there is a nonzero ideal B of R such that $C(f)C(h) \subseteq (B : B)$ for all $h(\mathrm{x}) \in G_f$, then I_f is almost principal.*

Proof Suppose such an ideal B exists and let $r \in C(f)$ and $b \in B$ be nonzero elements. Then $rh(\mathrm{x}) \in (B : B)[\mathrm{x}]$ for all $h(\mathrm{x}) \in G_f$ and $b(B : B)[\mathrm{x}] \subseteq R[\mathrm{x}]$. It follows that $brh(\mathrm{x}) \in R[\mathrm{x}]$ and therefore I_f is almost principal. □

Corollary 2.7 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$. If there is a nonzero ideal B of R such that $(B : B) \supseteq (C(f)^m : C(f)^m)$ for each m, then I_f is almost principal.*

Proof Let $f(\mathrm{x})h(\mathrm{x}) \in I_f$. Then by the content formula, there is a nonnegative integer m such that $C(f)^m C(fh) = C(f)^m C(f)C(h)$. Thus $C(f)C(h) \subseteq (C(f)^m : C(f)^m) \subseteq (B : B)$. That I_f is almost principal now follows from Theorem 2.6. □

The converse of Theorem 2.6 holds as well. The following lemma will be useful.

Lemma 2.8 *Let $f(\mathrm{x}) \in R[\mathrm{x}]$ be a polynomial with positive degree. Then for each positive integer m and each element $s \in H_f^m$, there is a polynomial $g(\mathrm{x}) \in G_f$ such that $s \in C(g)^m$.*

Proof For $s \in H_f^m$, we have $s = a_1 + a_2 + \cdots + a_n$ where each a_i is a product of m elements of H_f, specifically we can write a_i as a product $a_i = a_{i,1}a_{i,2}\cdots a_{i,m}$ with each $a_{i,j} \in H_f$. By the proof of Lemma 2.4, there is a polynomial $g(\mathrm{x}) \in G_f$ such that each $a_{i,j}$ is in $C(g)$. Hence $s \in C(g)^m$. □

No matter whether H_f is a fractional ideal of R or simply an R-module, $(H_f^m : H_f^m)$ is a ring that contains R for each $m \geq 1$. Moreover, $(H_f^m : H_f^m) = (tH_f^m : tH_f^m)$ for each nonzero $t \in R$. In the next theorem, we show that if $k = deg(f)$, then $(H_f^k : H_f^k)$ contains $C(f)C(h)$ for each $h(\mathrm{x}) \in G_f$.

Theorem 2.9 *Let $f(\mathrm{x}) \in R[\mathrm{x}]$ be a polynomial with positive degree k. Then $C(f)C(h) \subseteq (H^k : H^k)$ for all $h(\mathrm{x}) \in G_f$.*

Proof Let $s \in H_f^k$ and let $h(\mathrm{x}) \in G_f$. By Lemma 2.8, there is a polynomial $g(\mathrm{x}) \in G_f$ such that $s \in C(g)^k$. Next, let $t(\mathrm{x}) := h(\mathrm{x}) + \mathrm{x}^n g(\mathrm{x})$ where $n = 1 + deg(h)$. Then $t(\mathrm{x}) \in G_f$ with $C(t) = C(h) + C(g)$ and thus $s \in C(t)^k$. By the content formula $C(t)^k C(ft) = C(t)^k C(t) C(f) = C(t)^k (C(h) + C(g)) C(f) \supseteq C(t)^k C(h) C(f)$. As $C(ft) \subseteq R$ and $s \in C(t)^k \subseteq H_f^k$, we have $sC(f)C(h) \subseteq H_f^k$. Therefore $C(f)C(h) \subseteq (H_f^k : H_f^k)$. □

Corollary 2.10 *For an upper-type ideal I_f of $R[\mathrm{x}]$, I_f is almost principal if and only if there is a nonzero ideal B of R such that $C(f)C(h) \subseteq (B : B)$ for each $h(\mathrm{x}) \in G_f$.*

Proof If such an ideal B exists, then I_f is almost principal by Theorem 2.6. For the converse, if I_f is almost principal, then there is a nonzero element $r \in R$ such that $rH_f \subseteq R$. Hence $B := r^k H_f^k$ is a nonzero ideal of R. By Theorem 2.9, $C(f)C(h) \subseteq (H_f^k : H_f^k) = (B : B)$ for all $h(\mathrm{x}) \in G_f$. □

For a pair of nonzero ideals A and B of R, $(A + B : A + B)$ contains $(A : A) \bigcap (B : B)$. Hence if both A and B are such that $C(f)C(h) \subseteq (A : A)$ and $C(f)C(h) \subseteq (B : B)$ for all $h(\mathrm{x}) \in G_f$, then $C(f)C(h) \subseteq (A + B : A + B)$.

Theorem 2.11 *If I_f is almost principal, then there is a unique largest ideal B of R (that might be R) such that $C(f)C(h) \subseteq (B : B)$ for all $h(\mathrm{x}) \in G_f$, moreover, B is both divisorial and a trace ideal.*

Proof Assume I_f is almost principal and let $\mathcal{S} := \{A \mid C(f)C(h) \subseteq (A : A)$ for all $h(\mathrm{x}) \in G_f\}$. Let $\{A_\alpha\}$ be a chain of ideals in $\mathcal{S}$ (under set containment) and let $A := \bigcup A_\alpha$. For $d \in A$, there is an α such that $d \in A_\alpha$. Thus $dC(f)C(h) \subseteq A_\alpha \subseteq A$. It follows that $C(f)C(h) \subseteq (A : A)$ which puts A in $\mathcal{S}$. Therefore, Zorn's Lemma applies to the set $\mathcal{S}$. Since $\mathcal{S}$ is closed with respect to finite sums, there is a unique largest ideal B in $\mathcal{S}$. Since $(BB^{-1} : BB^{-1}) = (R : BB^{-1})$ contains $(B : B)$, B is trace ideal of R. By [15, Proposition 2.2], we also have that $B = B_v$ since $(R : B)$ a ring implies it is equal to $(B_v : B_v)$. □

Theorem 2.12 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$. If T is an overring of R such that $(R : T) \neq (0)$, then the following are equivalent:*

(1) *I_f is almost principal as an ideal of $R[\mathrm{x}]$,*
(2) *$f(\mathrm{x})K[\mathrm{x}] \bigcap T[\mathrm{x}]$ is almost principal as an ideal of $T[\mathrm{x}]$, and*
(3) *$I_f T[\mathrm{x}]$ is almost principal as an ideal of $T[\mathrm{x}]$.*

Proof Assume $R \subseteq T$ with $(R : T) \neq (0)$. To avoid the trivial case, we assume the containment is proper. Since $f(\mathrm{x}) \in I_f$, (2) implies (3) is from Lemma 2.2(2).

For (3) implies (1), assume $I_f T[\mathrm{x}]$ is almost principal. Then there is a nonzero element $t \in T$ such that $th(\mathrm{x}) \in T[\mathrm{x}]$ for each $h(\mathrm{x}) \in G_f$. For a nonzero $r \in (R : T)$, we have $rth(\mathrm{x}) \in R[\mathrm{x}]$. Hence I_f is almost principal.

Finally assume I_f is almost principal and let $J_f := f(\mathrm{x})K[\mathrm{x}] \bigcap T[\mathrm{x}]$. As above, let r be a nonzero element in $(R : T)$. Then $rJ_f \subseteq R[\mathrm{x}]$ which puts rJ_f in I_f. Since I_f is almost principal, there is a nonzero element $s \in R$ such that $sI_f \subseteq f(\mathrm{x})R[\mathrm{x}]$. It follows that $rsJ_f \subseteq f(\mathrm{x})T[\mathrm{x}]$ and therefore J_f is an almost principal upper-type ideal of $T[\mathrm{x}]$. □

Without the assumption that $(R : T)$ is nonzero, the equivalences in Theorem 2.12 do not hold. However, both (1) and (2) imply (3) no matter whether $(R : T)$ is nonzero or not. Lemma 2.2(2) takes care of (2) implies (3), and if $s \in R \backslash \{0\}$ is such that $sI_f \subseteq f(\mathrm{x})R[x]$, then certainly we also have $sI_f T[\mathrm{x}] \subseteq f(\mathrm{x})T[\mathrm{x}]$. See Example 3.12 below for the failure of the other implications.

There are at least some general cases where knowing that I_f is almost principal as an ideal of $R[\mathrm{x}]$ implies $f(\mathrm{x})K[\mathrm{x}] \bigcap T[\mathrm{x}]$ is almost principal as an ideal of $T[\mathrm{x}]$. For example, this holds in the case $T = R_S$ for some multiplicative set S. Hence if I_f is almost principal, then $f(\mathrm{x})K[\mathrm{x}] \bigcap R_M[\mathrm{x}]$ is almost principal in $R_M[\mathrm{x}]$ for each maximal ideal M. It is not clear whether the converse holds or not. However, it does hold under the additional assumption that R has finite character (each nonzero nonunit is contained in only finitely many maximal ideals).

Theorem 2.13 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$ and let S be a multiplicative subset of R. If I_f is almost principal as an ideal of $R[\mathrm{x}]$, then the upper-type ideal $f(\mathrm{x})K[\mathrm{x}] \bigcap R_S[\mathrm{x}]$ of $R_S[\mathrm{x}]$ is almost principal.*

Proof Clearly, $I_f R_S[\mathrm{x}] \subseteq f(\mathrm{x})K[\mathrm{x}] \cap R_S[\mathrm{x}]$. For the reverse containment, note that if $f(\mathrm{x})h(\mathrm{x}) \in R_S[\mathrm{x}]$, then there is an element $s \in S$ such that $sf(\mathrm{x})h(\mathrm{x}) \in R[\mathrm{x}]$. Hence $sf(\mathrm{x})h(\mathrm{x}) \in I_f$. As s is unit of $R_S[\mathrm{x}]$, we have $f(\mathrm{x})h(\mathrm{x}) \in I_f R_S[\mathrm{x}]$. Therefore $I_f R_S[\mathrm{x}]$ is the upper-type ideal of $R_S[\mathrm{x}]$ corresponding to $f(\mathrm{x})$. By the argument given above, if I_f is almost principal as an ideal of $R[\mathrm{x}]$, then $I_f R_S[\mathrm{x}] = f(\mathrm{x})K[\mathrm{x}] \cap R_S[\mathrm{x}]$ is almost principal as an ideal of $R_S[\mathrm{x}]$. □

Corollary 2.14 *If I_f is almost principal as an ideal of $R[\mathrm{x}]$, then $f(\mathrm{x})K[\mathrm{x}] \cap R_M[\mathrm{x}]$ is almost principal as an ideal of $R_M[\mathrm{x}]$ for each maximal ideal M of R.*[1]

Theorem 2.15 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$. If R has finite character, then I_f is almost principal as an ideal of $R[\mathrm{x}]$ if and only if $f(\mathrm{x})K[\mathrm{x}] \cap R_M[\mathrm{x}]$ is almost principal as an ideal of $R_M[\mathrm{x}]$ for each maximal ideal M of R.*[1]

Proof Suppose $f(\mathrm{x})K[\mathrm{x}] \cap R_M[\mathrm{x}]$ is almost principal for each maximal ideal M of R. If $C(f) = R$, then $I_f = f(\mathrm{x})R[\mathrm{x}]$ by the proof of Lemma 2.1. Thus we may assume $C(f)$ is a proper ideal of R. Since R has finite character, at most finitely many maximal ideals contain $C(f)$, say $M_1, M_2, \ldots, M_n$. Then for each M_i, there is a nonzero ideal B_i of R such that $C(f)C(h) \subseteq (B_i R_{M_i} : B_i R_{M_i})$ for each $h(\mathrm{x}) \in G_f$ (Theorem 2.10). Let $B := B_1 B_2 \cdots B_n$. Then we have $C(f)C(h)BR_{M_i} \subseteq BR_{M_i}$ for each i. For those maximal ideals N of R that do not contain $C(f)$ (if any), we have $C(f)R_N = R_N$ and from the content formula, must then have $C(h) \subseteq R_N$ for each $h(\mathrm{x}) \in G_f$. Thus $C(f)C(h)BR_M \subseteq BR_M$ for each maximal ideal M and each polynomial $h(\mathrm{x}) \in G_f$. It follows that $C(f)C(h)B \subseteq B$ and therefore I_f is almost principal as an ideal of $R[\mathrm{x}]$ by Theorem 2.6. □

For an alternate proof of the previous theorem, one could instead pick nonzero elements $r_1, r_2, \ldots, r_n \in R$ such that $r_i I_f \subseteq f(\mathrm{x})R_{M_i}[\mathrm{x}]$ for each M_i. Then by checking locally, we would find that $rI_f \subseteq f(\mathrm{x})R[\mathrm{x}]$ for $r := r_1 r_2 \cdots r_n (\in R \backslash \{0\})$.

The next lemma is derived from [14, Proposition 2.5].

Lemma 2.16 *Let $\{I_\alpha\}$ be a descending chain of divisorial trace ideals of a domain R. If $I := \bigcap I_\alpha \neq (0)$, then I is a divisorial trace ideal of R.*

Proof Since each I_α is divisorial, $I \subseteq I_v \subseteq (I_\alpha)_v = I_\alpha$. Thus $I = I_v$. Also $(R : I) \supseteq (R : I_\alpha) = (I_\alpha : I_\alpha)$ for each I_α. Since the family $\{I_\alpha\}$ forms a descending chain, $\bigcup (R : I_\alpha)$ is a ring. Therefore $(R : I) = (I : I)$ by [14, Proposition 2.5]. □

For nonzero ideals I and J of R, the fact that $I \cdot J(R : IJ)$ is contained in R puts $J(R : IJ)$ in $(R : I)$ and thus we have $IJ(R : IJ) \subseteq I(R : I)$. With regard to powers of I, we have $(I(R : I))^{m+1} \subseteq I^{m+1}(R : I^{m+1}) \subseteq I^m(R : I^m)$ for all positive integers m. It follows that $\sqrt{I^m(R : I^m)} = \sqrt{I(R : I)}$ and $\sqrt{(I^m(R : I^m))_v} = \sqrt{(I(R : I))_v}$ for all m, the latter equality from the fact that $(AB)_v = (AB_v)_v = (A_v B_v)_v$ for all fractional ideals A and B of R.

Theorem 2.17 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$ and for each $m \geq 1$, let $J_m := C(f)^m(R : C(f)^m)$. If $\bigcap (J_m)_v \neq (0)$, then I_f is almost principal.*

Proof Let $B := \bigcap (J_m)_v$ and assume $B \neq (0)$. From the discussion above, we have $(J_{m+1})_v \subseteq (J_m)_v$ for each $m \geq 1$. Also as each J_m is a trace ideal, so is each $(J_m)_v$ [15, Proposition 2.2]. Moreover $(R : J_m) = (J_m : J_m) \supseteq (C(f)^m : C(f)^m)$ for each $m \geq 1$. Hence by Lemma 2.16, B is a divisorial trace ideal, necessarily with $(B : B) = (R : B) \supseteq (R : J_m)$ for each m. That I_f is almost principal now follows from Corollary 2.7. □

It is not clear whether the converse of Theorem 2.17 holds or not. If is does, then for any pair of polynomials $f(\mathrm{x}), g(\mathrm{x}) \in R[\mathrm{x}]$ with $C(f) = C(g)$, I_f is almost principal if and only if I_g is almost principal—the proof follows easily from the fact that $C(f) = C(g)$ implies $(C(f)^m(R : C(f)^m))_v = (C(g)^m(R : C(g)^m))_v$ for each $m \geq 1$.

Corollary 2.18 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$. If $C(f)(R : C(f)) = C(f)^n(R : C(f)^n)$ for infinitely many positive integers n, then I_f is almost principal.*

[1] Corollary 2.14 and Theorem 2.15 provide a partial answer to a question posed by Marco Fontana at AUS-ICMS'10: The First International Conference on Mathematics and Statistics, held March 18–21, 2010 at the American University of Sharjah, Sharjah, UAE.

Proof Since $C(f)^{m+1}(R : C(f)^{m+1}) \subseteq C(f)^m(R : C(f)^m)$ for all positive integers m, if $C(f)(R : C(f)) = C(f)^n(R : C(f)^n)$ for infinitely many positive integers n, then we have equality for all positive integers and therefore I_f is almost principal. □

We will make use of the following result from [6] when dealing with integrally closed domains and seminormal domains.

Theorem 2.19 (cf. [6, Corollary 3.4]) *If I is a nonzero ideal of a seminormal domain R such that $(R : I)$ is a ring, then $(R : I) = (R : \sqrt{I}) = (\sqrt{I} : \sqrt{I})$.*

A simple corollary is the following.

Corollary 2.20 *If I is a nonzero ideal of a seminormal domain R and J is the radical of $I(R : I)$, then $(J : J) \supseteq (I^m : I^m)$ for each positive integer m.*

Proof For each $m \geq 1$, the ideal $J_m := I^m(R : I^m)$ is a trace ideal with $\sqrt{J_m} = J$ (where $J = \sqrt{I(R : I)}$). In general, all we can say is that $(R : J) = (J : J)$ and $(R : J_m) = (J_m : J_m) \supseteq (I^m : I^m)$ for each $m \geq 1$. But when R is seminormal, we have $(J : J) = (J_m : J_m) \supseteq (I^m : I^m)$ for each m. □

3 Applications

We start by showing that if R is seminormal, then $R[\mathrm{x}]$ is an almost principal ideal domain.

Theorem 3.1 *If R is seminormal, then each upper-type ideal of $R[\mathrm{x}]$ is almost principal and $R[\mathrm{x}]$ is an almost principal ideal domain.*

Proof Assume R is seminormal and let I_f be an upper-type ideal of $R[\mathrm{x}]$. Also let $J := C(f)(R : C(f))$ and $B := \sqrt{J}$. By Corollary 2.20, $(B : B) \supseteq (C(f)^m : C(f)^m)$ for each $m \geq 1$. That I_f is almost principal now follows from Corollary 2.7. Apply Theorem 2.3 to see that $R[\mathrm{x}]$ is an almost principal ideal domain. □

Recall that a domain R is said to be n-root closed for some positive integer $n \geq 2$ if $x^n \in R$ for $x \in K$, implies $x \in R$. It is easy to check that an n-root closed domain is seminormal as each positive integer $n \geq 2$ can be written in the form $n = 2i + 3j$ for some nonnegative integers i and j. With regard to the next corollary, the integrally closed case was originally established by Johnson [16, Proposition 3.3].

Corollary 3.2 *If R is either n-root closed for some $n \geq 2$ or integrally closed, then $R[\mathrm{x}]$ is an almost principal ideal domain.*

According to [10, Theorem 1.2], if $(R : R')$ is nonzero, then $R[\mathrm{x}]$ is an almost principal ideal domain. Since an intersection of seminormal domains is seminormal and an intersection of n-root closed domains is n-root closed, each domain R is contained in a unique smallest seminormal overring and a unique n-root closed overring. The *seminormalization* is the smallest seminormal domain between R and K and the *total n-root closure* is likewise the smallest n-root closed domain between R and K (see [21] and [1], respectively, for constructions of the seminormalization and total n-root closure as unions of overrings). Since an n-root closed domain is seminormal, the n-root closure contains the seminormalization. Thus using Theorem 2.12, it suffices to have a nonzero conductor for R and its seminormalization. We provide an alternate proof of this using Theorem 2.6.

Corollary 3.3 *Let S be the seminormalization of R. If $(R : S)$ is nonzero, then $R[\mathrm{x}]$ is an almost principal ideal domain.*

With regard to this corollary (and the proof we will present), please note that for a given upper-type ideal I_f, we can describe at least one of the nonzero ideals B of R that has the property that $C(f)C(h) \subseteq (B : B)$ for each $h(\mathrm{x}) \in G_f$. An interesting phenomenon here is that B is defined in terms of the trace of $C(f)$ and the conductor of S into R, so the same ideal B will work for quite a large number of upper-type ideals, including all of the form I_b for $b(\mathrm{x}) \in R[\mathrm{x}]$ such that $C(b) = C(f)$. As we will be dealing with contents in both R and S, we use C_R to denote the content with respect to R and C_S for the content with respect to S.

Proof Assume S has a nonzero conductor into R and let $A := (R : S)$. Also let I_f be an upper-type ideal of $R[\mathrm{x}]$ and let J_f be the corresponding upper-type ideal of $S[\mathrm{x}]$. From the proof of Theorem 3.1, the ideal $B' := \sqrt{C_S(f)(S : C_S(f))}$ has the property that $C_S(f)C_S(h) \subseteq (B' : B')$ for all polynomials $h(\mathrm{x}) \in K[\mathrm{x}]$ such that $f(\mathrm{x})h(\mathrm{x}) \in J_f$, so certainly for all $h(\mathrm{x}) \in G_f$. The product $B := B'A$ is an ideal of both R and S with $(B' : B') \subseteq (AB' : AB')$. Thus $C_R(f)C_R(h) \subseteq (B : B)$, and we have that I_f is almost principal by Theorem 2.6. It follows that $R[\mathrm{x}]$ is an almost principal ideal domain. □

Suppose the domain D is seminormal but not integrally closed and $(D : D') = (0)$. Then the polynomial ring $D[\mathrm{x}]$ is seminormal with integral closure $D'[\mathrm{x}]$ such that $(D[\mathrm{x}] : D'[\mathrm{x}]) = (0)$. Also $D[\mathrm{x}]$ is the seminormalization of $D[\mathrm{x}^2, \mathrm{x}^3]$, obviously with x^2 a nonzero conductor of $D[\mathrm{x}]$ into $D[\mathrm{x}^2, \mathrm{x}^3]$. Thus Corollary 3.3 provides a nontrivial extension of [10, Theorem 1.2]. For a specific example, start with a field F and two countably infinite sets of algebraically independent indeterminates $\{\mathrm{Y}_n\}$ and $\{\mathrm{Z}_n\}$ (indexed over the positive integers). Then let $D := F[\{\mathrm{Y}_n, \mathrm{Y}_n\mathrm{Z}_n^k, \mathrm{Z}_n^{k(n+1)} \mid 1 \le k,\ 1 \le n\}]$. The integral closure of D is $D' = F[\{\mathrm{Y}_n, \mathrm{Z}_n \mid 1 \le n\}]$ and D is seminormal, but there is no positive integer $n \ge 2$ such that D is n-root closed. Also $(D : D') = (0)$.

Proposition 1.4 of [10] states that if J is a nonzero ideal of $R[\mathrm{x}]$ such that $JK[\mathrm{x}] \ne K[\mathrm{x}]$ and J can be generated by a set of polynomials of bounded degree, then J is almost principal. We provide an alternate proof of this using Theorem 2.6. Later we derive a somewhat improved version in terms of a bound on the number of generators for the contents.

Theorem 3.4 (cf. [10, Proposition 1.4]) *Let J be a nonzero ideal of $R[\mathrm{x}]$ such that $JK[\mathrm{x}] \ne K[\mathrm{x}]$. If J can be generated by a set of polynomials of bounded degree, then J is almost principal.*

Proof Let $f(\mathrm{x}) \in J$ have the smallest positive degree of all nonzero elements of J. Then $J \subseteq I_f$. Thus if J can be generated by a set of polynomials of bounded degree, there is nonnegative integer n and a nonempty set of polynomials $\mathcal{G} := \{g_\alpha(\mathrm{x})\}_{\alpha\in\mathcal{A}} \subseteq G_f$ such that $\{f(\mathrm{x})g_\alpha(\mathrm{x})\}_{\alpha\in\mathcal{A}}$ generates J and $deg(g_\alpha) \le n$ for each $\alpha \in \mathcal{A}$.

From the content formula, we have $C(f)^nC(fg_\alpha) = C(f)^nC(f)C(g_\alpha)$ for each $g_\alpha(\mathrm{x}) \in \mathcal{G}$. Therefore $C(f)C(g_\alpha) \subseteq (C(f)^n : C(f)^n)$.

Let $h(\mathrm{x}) \in K[\mathrm{x}]$ be such that $f(\mathrm{x})h(\mathrm{x}) \in J$. Then there are polynomials $g_{\alpha_1}(\mathrm{x}), g_{\alpha_2}(\mathrm{x}), \ldots, g_{\alpha_k}(\mathrm{x})$ in $\mathcal{G}$ and corresponding polynomials $r_1(\mathrm{x}), r_2(\mathrm{x}), \ldots, r_k(\mathrm{x})$ in $R[\mathrm{x}]$ such that $h(\mathrm{x}) = \sum r_i(\mathrm{x})g_{\alpha_i}(\mathrm{x})$. Since each $r_i(\mathrm{x})$ is in $R[\mathrm{x}]$, $C(h) \subseteq C(g_{\alpha_1}) + C(g_{\alpha_2}) + \cdots + C(g_{\alpha_k})$. Hence $C(f)C(h) \subseteq (C(f)^n : C(f)^n)$. That J is almost principal now follows from Theorem 2.6. □

Corollary 3.5 (cf. [16, Proposition 3.3]) *If R is a Noetherian domain, then $R[\mathrm{x}]$ is an almost principal ideal domain.*

A domain that satisfies the ascending chain condition on divisorial ideals is said to be a *Mori domain*. A useful property of Mori domains is that they have *finite t-character*; i.e., each nonzero nonunit is contained in only finitely many maximal t-ideals. With regard to polynomial rings, if a Mori domain R contains an uncountable field, then the corresponding polynomial ring $R[\mathrm{x}]$ is a Mori domain (see both [3] and [19, Theorem 3.15]). Also, if R is an integrally closed Mori domain, then $R[\mathrm{x}]$ is a Mori domain [18, §3, Théorème 5]. In contrast, Roitman provides a general scheme for taking an arbitrary countable field F and finding a Mori domain R that contains F such that the corresponding polynomial ring $R[\mathrm{x}]$ is not a Mori domain [20, Theorem 8.4]. Another type of Mori domain where the corresponding polynomial ring is a Mori domain is a *strong Mori domain*. For our purposes all we need to know about strong Mori domains is that a domain R is a strong Mori domain if and only if it has finite t-character and R_M is Noetherian for each maximal t-ideal M of R (see, [23, Theorem 1.9]). If R is a strong Mori domain, then not only is $R[\mathrm{x}]$ a Mori domain, but a strong Mori domain as well [23, Theorem 1.13]. Moreover, each upper-type ideal of $R[\mathrm{x}]$ is almost principal when R is a strong Mori domain. This conclusion can be derived from a result due to Gabelli in combination with several from [10]. Specifically, Gabelli proved that if $R[\mathrm{x}]$ is a Mori domain, then $(R[\mathrm{x}] : P_f)$ properly contains $(P_f : P_f)$ for each upper to zero P_f (see, [7, Proposition 1.2 and Remark 1.4]); that P_f is almost principal then follows from [10, Proposition 1.15]. Thus from Theorem 2.3, if $R[\mathrm{x}]$ is a Mori domain, then it is an almost principal ideal domain. In particular, if R is a strong Mori domain, then $R[\mathrm{x}]$ is an almost principal ideal domain. We provide an alternate proof of the latter statement using the tools developed in Sect. 2. At the end of the paper we give a direct proof that if $R[\mathrm{x}]$ is a Mori domain, then each nonzero ideal of $R[\mathrm{x}]$ with proper extension to $K[\mathrm{x}]$ is almost principal.

Theorem 3.6 *If R is a strong Mori domain, then $R[\mathrm{x}]$ is an almost principal ideal domain.*

Proof Assume R is a strong Mori domain. It suffices to show that each upper-type ideal I_f is almost principal. By finite t-character, at most finitely many maximal t-ideals contain $C(f)$ and each of these will contain $C(f)_v$. Let $M_1, M_2, \ldots, M_n$ be the maximal t-ideals that contain $C(f)$ (if any). For any other maximal t-ideal M, $C(f)R_M = R_M$. Thus by the content formula, we have $h(\mathrm{x}) \in R_M[\mathrm{x}]$ for all polynomials $h(\mathrm{x}) \in G_f$ (and all $M \notin \{M_1, M_2, \ldots, M_n\}$). Hence $I_f R_M[\mathrm{x}] = f(\mathrm{x})R_M[\mathrm{x}]$.

Since R_N is Noetherian for each maximal t-ideal N, $I_f R_{M_i}[\mathrm{x}]$ is finitely generated for each M_i. Since there are only finitely many M_i's, there is a finite set of polynomials $\{g_1(\mathrm{x}), g_2(\mathrm{x}), \ldots, g_k(\mathrm{x})\} \subseteq G_f$ such that the set $\{f(\mathrm{x}), f(\mathrm{x})g_1(\mathrm{x}), f(\mathrm{x})g_2(\mathrm{x}), \ldots, f(\mathrm{x})g_k(\mathrm{x})\}$ generates $I_f R_N[\mathrm{x}]$ for each maximal t-ideal N. Let $m := max\{deg(g_j(\mathrm{x})) \mid 1 \le j \le k\}$. Since $C(f)$ is finitely generated, $(C(f)^r R_N : C(f)^r R_N) = (C(f)^r : C(f)^r)R_N$ for each maximal t-ideal N and each positive integer r.

Let $h(\mathrm{x}) \in G_f$. As noted above, if M is a maximal t-ideal that does not contain $C(f)$, then $h(\mathrm{x}) \in R_M[\mathrm{x}]$ and $C(f)C(h) \subseteq (C(f)^m : C(f)^m)R_M = R_M$.

For the M_i's, we have $C(f)C(h) \subseteq (C(f)^m R_{M_i} : C(f)^m R_{M_i})$ by the proof of Theorem 3.4. Hence $C(f)C(h)C(f)^m \subseteq \bigcap\{C(f)^m R_N \mid N \in tMax(R)\}$. As noted earlier, while the intersection $\bigcap\{C(f)^m R_N \mid N \in tMax(R)\}$ may be larger than $C(f)^m$, it will be contained in $(C(f)^m)_v$. Thus the distributive closure property of the v-operation yields $C(f)C(h) \subseteq ((C(f)^m)_v : (C(f)^m)_v)$. Hence I_f is almost principal by Theorem 2.6. □

Recall that a domain R is said to have the radical trace property if $I(R : I)$ is a radical ideal for each nonzero noninvertible ideal I of R [12]. The following result also appears in [17].

Theorem 3.7 *If R has the radical trace property, then $R[\mathrm{x}]$ is an almost principal ideal domain.*

Proof Assume R has the radical trace property and let I_f be an upper-type ideal of $R[\mathrm{x}]$. Then for each integer $m \ge 1$, $C(f)^m(R : C(f)^m)$ is a radical ideal of R (possibly equal to R). As noted earlier, for any nonzero ideal J, $J(R : J)$ has the same radical as $J^m(R : J^m)$ for each positive integer m. In our situation, $C(f)(R : C(f)) = C(f)^m(R : C(f)^m)$ for all $m \ge 1$ (see, [12, Remark 2.13(b)]). Thus I_f is almost principal by your choice of Corollary 2.7 (using $B = C(f)(R : C(f))$), Theorem 2.17 or Corollary 2.18. Therefore $R[\mathrm{x}]$ is an almost principal ideal domain. □

In [11], Heinzer and Huneke established an improved bound on the power needed to ensure equality in the content formula. For a finitely generated nonzero fractional ideal B of a domain R, let $\mu_R(B)$ be the smallest integer k such that BR_M can be generated by k elements (or fewer) for each maximal ideal M. The Heinzer–Huneke bound is given in terms of $\mu_R(C(g))$.

Theorem 3.8 (cf. [11, Theorem 2.1]) *Let R be an integral domain and let $g(\mathrm{x}) \in K[\mathrm{x}]$ be a nonzero polynomial. If $\mu_R(C(g)) = m$, then $C(f)^{m-1}C(fg) = C(f)^m C(g)$ for all polynomials $f(\mathrm{x}) \in K[\mathrm{x}]$.*

Our next seven applications of the tools provided in Sect. 2 deal with upper-type ideals that contain individual polynomials of a special type with regard to content. As with several of the applications so far, some of the results are not new, others generalize known types of almost principal ideals. The next two theorems are of the latter type.

The following theorem provides a way to relax the degree bound condition provided in Theorem 3.4. We do not know of a specific example that satisfies the sufficient condition given below that does not also satisfy one of the (seemingly) simpler sufficient conditions already discussed. However, it can be used to provide an alternate proof of Theorem 3.17 below.

For $f(\mathrm{x}) \in R[\mathrm{x}]\backslash R$, we let $\mathcal{F}_n(f) := \{C(g) \mid g(\mathrm{x}) \in G_f$ with $C(g)$ n-generated$\}$. We say that H_f is *n-quasigenerated* if for each $h(\mathrm{x}) \in G_f$, there are polynomials $g_1(\mathrm{x}), g_2(\mathrm{x}), \ldots, g_m(\mathrm{x}) \in G_f$ such that each $C(g_i)$ is in $\mathcal{F}_n(f)$ and $C(h) \subseteq C(g_1) + C(g_2) + \cdots + C(g_m)$. The following theorem makes use of a "t-local" version of n-quasigeneration. For each maximal t-ideal M_β, we let $\mathcal{F}_{\beta,n}(f) := \{C(g)R_{M_\beta} \mid g(\mathrm{x}) \in G_f$ with $C(g)R_{M_\beta}$ n-generated$\}$. Note that if $f(\mathrm{x})h(\mathrm{x}) \in R_{M_\beta}[\mathrm{x}]$, then there is an element $s \in R\backslash M_\beta$ such that $sh(\mathrm{x}) \in G_f$. Thus the set $\mathcal{F}_{\beta,n}(f)$ can also be defined in terms of polynomials that multiply $f(\mathrm{x})$ into $R_{M_\beta}[\mathrm{x}]$. With all of this in mind we say that H_f is *t-locally n-quasigenerated* if $H_f R_{M_\beta}$ is n-quasigenerated for each maximal t-ideal M_β.

Theorem 3.9 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$. If there is a positive integer n such that H_f is t-locally n-quasigenerated, then I_f is almost principal.*

The key steps for this proof make use of the Heinzer–Huneke bound for the content formula discussed above and the fact that $I \subseteq \bigcap\{IR_M \mid M \in tMax(R)\} \subseteq I_v$ for all nonzero ideals I of R.

Proof Suppose H_f is t-locally n-quasigenerated for some positive integer n and let $h(\mathrm{x})$ be a polynomial in G_f. Then for each maximal t-ideal M_β, there are polynomials $g_1(\mathrm{x}), g_2(\mathrm{x}), \ldots, g_m(\mathrm{x}) \in \mathcal{F}_{\beta,n}$ such that $C(h) \subseteq C(g_1)R_{M_\beta} + C(g_2)R_{M_\beta} + \cdots + C(g_m)R_{M_\beta}$. Using the Heinzer–Huneke bound for the content formula, we have $C(f)^{n-1}C(fg_i)R_{M_\beta} = C(f)^{n-1}C(f)C(g_i)R_{M_\beta}$ for each i. Thus $C(f)C(g_i)R_{M_\beta} \subseteq (C(f)^{n-1}R_{M_\beta} : C(f)^{n-1}R_{M_\beta})$. Hence $C(f)^{n-1}C(f)C(h)R_{M_\beta} \subseteq C(f)^{n-1}R_{M_\beta}$. As this containment holds for each maximal t-ideal, we have $C(f)C(h) \cdot (C(f)^{n-1})_v \subseteq (C(f)^{n-1})_v$, equivalently $C(f)C(h) \subseteq ((C(f)^{n-1})_v : (C(f)^{n-1})_v)$. That I_f is almost principal follows from Theorem 2.6. □

Corollary 3.10 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$. If there is an integer n such that for each $M \in tMax(R)$, $I_fR_M[\mathrm{x}]$ can be generated by a set of elements $\{f(\mathrm{x})g_\alpha(\mathrm{x})\} \subseteq R_M[\mathrm{x}]$ (possibly dependent on M) where each $C(g_\alpha)R_M$ is n-generated, then I_f is almost principal.*

Proof If $h(\mathrm{x}) := b_1(\mathrm{x})g_1(\mathrm{x}) + b_2(\mathrm{x})g_2(\mathrm{x}) + \cdots + b_m(\mathrm{x})g_m(\mathrm{x})$ for polynomials $b_i(\mathrm{x}) \in R_{M_\beta}$ and $g_i(\mathrm{x}) \in G_f$ with $C(g_i)R_{M_\beta} \in \mathcal{F}_{\beta,n}$, then certainly we have $C(h) \subseteq C(g_1)R_{M_\beta} + C(g_2)R_{M_\beta} + \cdots + C(g_m)R_{M_\beta}$. Thus if each $h(\mathrm{x}) \in G_f$ can be generated in this way for each maximal t-ideal M_β, then I_f is almost principal by Theorem 3.9. □

An alternate (seemingly more restrictive) sufficient condition to invoke Theorem 3.9 would be that for each polynomial $h(\mathrm{x}) \in G_f$ and each maximal t-ideal M_β, there is a polynomial $g_\beta(\mathrm{x}) \in G_f$ such that $C(h) \subseteq C(g_\beta)R_{M_\beta}$ with $C(g_\beta)R_{M_\beta} \in \mathcal{F}_{\beta,n}$.

Recall that an upper to zero P_f is said to be *rational* if $f(\mathrm{x}) = a\mathrm{x} + b$ for some nonzero $a \in K$. Using the notion of a "spotty" element, Hamann, Houston and Johnson were able to demonstrate a type of rational upper to zero that is never almost principal—specifically, see the example labeled *Arnold's Example* and *Theorem 2.1* both on page 73 of [10]. In contrast, they showed that if a/b is almost integral over R (and thus not spotty), then P_f is almost principal [10, Theorem 2.4]. We provide a proof of this same result using Theorem 2.17. Our result is more general as we actually consider the case of an upper-type ideal I_f where $f(\mathrm{x})$ is simply a nonconstant polynomial whose content is two-generated (with no bound on degree or number of nonzero coefficients, but with an "almost integral" relation for some pair of generators).

Theorem 3.11 *Let I_f be an upper-type ideal of $R[\mathrm{x}]$ where $C(f) = aR + bR$ for some nonzero elements $a, b \in R$. If $t = a/b$ is almost integral over R, then I_f is almost principal.*

Proof Assume $t = a/b$ is almost integral over R and let $p \in R$ be a nonzero element such that $pt^n \in R$ for all $n \geq 1$. Then $pa^ib^j/b^n = p(a^i/b^i) = pt^i \in R$ for all pairs of nonnegative integers i and j such that $i + j = n$. Hence $p/b^n \in (R : C(f)^n)$ for all n. It follows that $p \in J_n := C(f)^n(R : C(f)^n)$ for all n. Thus $B = \bigcap(J_n)_v$ is not the zero ideal. That I_f is almost principal now follows from Theorem 2.17. □

As an application of Theorem 3.11, we recall [10, Example 2.8] and contrast this with a slight variation in the aforementioned "Arnold's Example" from [10].

Example 3.12 Let $R := F[\mathrm{w}, \mathrm{y}, \{\mathrm{yz}^{2^n} \mid 0 \leq n\}]$, $S := F[\mathrm{w}, \mathrm{y}, \mathrm{yz}]$ and $T := F[\mathrm{w}, \mathrm{y}, \{\mathrm{wz}^n, \mathrm{yz}^{2^n} \mid 0 \leq n\}]$ where F is a field and w, y and z are indeterminates. Also let $f(\mathrm{x}) := \mathrm{yzx} - \mathrm{y}$ with $I_f := f(\mathrm{x})F(\mathrm{w}, \mathrm{y}, \mathrm{z})[\mathrm{x}] \cap R[\mathrm{x}]$, $J_f := f(\mathrm{x})K[\mathrm{x}] \cap S[\mathrm{x}]$ and $B_f := f(\mathrm{x})F(\mathrm{w}, \mathrm{y}, \mathrm{z})[\mathrm{x}] \cap T[\mathrm{x}]$. Then I_f is not almost principal but both J_f and B_f are almost principal. Moreover $I_fT[\mathrm{x}]$ and $J_fR[\mathrm{x}]$ are almost principal.

As in the proof of Corollary 3.3, we use C_R for contents with respect to R and C_T for contents with respect to T.

Proof The domain S is integrally closed (it is isomorphic to $F[\mathrm{w}, \mathrm{y}, \mathrm{z}]$). Hence each upper-type ideal of $S[\mathrm{x}]$ is almost principal. In particular, J_f is almost principal. Clearly, if $s \in S\backslash\{0\}$ is such that $sJ_f \subseteq f(\mathrm{x})S[\mathrm{x}]$, then $sJ_fR[\mathrm{x}] \subseteq f(\mathrm{x})R[\mathrm{x}]$. Thus $J_fR[\mathrm{x}]$ is almost principal (as an ideal of $R[\mathrm{x}]$). On the other hand, the indeterminate z is *spotty* over R: y multiplies infinitely many different powers of z into R, but no nonzero element of R multiplies every positive power of z into R. In contrast, $\mathrm{wz}^k \in T$ for all $k \geq 1$. In some sense, there is no significant difference between the rings $(C_R(f)^m : C_R(f)^m)$ and $(C_T(f)^m : C_T(f)^m)$ for a given $m \geq 1$ (other than the latter will contain polynomial expressions that include terms from $\mathrm{wz}F[\mathrm{w}, \mathrm{y}, \mathrm{z}]$). But when one considers the ideals $C_R(f)^m(R : C_R(f)^m)$ and $C_T(f)^m(T : C_T(f)^m)$, one finds that the intersection $\bigcap C_R(f)^m(R : C_R(f)^m)$ is the zero ideal while $\bigcap C_T(f)^m(T : C_T(f)^m)$ contains W. With regard to the latter conclusion, simply check that W/Y^m is in $(T : C_T(f)^m)$ for each m (as in the proof of Theorem 3.11). A simple way to see that $\bigcap C_R(f)^m(R : C_R(f)^m)$ is the zero ideal is to make use of Theorem 2.17 and the fact that I_f is not almost principal (as we show below).

To see that I_f is not almost principal, we make use of Houston's proof of [13, Proposition 1.2]. For a given positive integer n, the polynomial $h_n(\mathrm{X}) = \mathrm{Z}^{2^n-1}\mathrm{X}^{2^n-1} + \mathrm{Z}^{2^n-2}\mathrm{X}^{2^n-2} + \cdots + \mathrm{ZX} + 1$ is such that $f(\mathrm{X})h_n(\mathrm{X}) = \mathrm{YZ}^{2^n}\mathrm{X}^{2^n} - \mathrm{Y} \in I_f$. Since Z is spotty over R, no nonzero element of R multiplies each $h_n(\mathrm{X})$ into $R[\mathrm{X}]$. Hence I_f is not almost principal. On the other hand $\mathrm{W}h_n(\mathrm{X}) \in T[\mathrm{X}]$ for each n. Of course, since $\mathrm{W} \in \bigcap C_T(f)^m(T : C_T(f)^m)$, B_f is almost principal by Theorem 2.17. As $I_f K[\mathrm{X}] = B_f K[\mathrm{X}]$, $I_f T[\mathrm{X}]$ is almost principal as well. □

A finitely generated ideal I is said to be *prestable*, if for each prime ideal P (equivalently, each maximal ideal P) there is a positive integer n such that $I^{2n}R_P = dI^nR_P$ for some $d \in I^n$ (see [5]). There are several ways to characterize finitely generated prestable ideals. Relevant to the study here is their connection with a certain type of upper-type ideals.

Theorem 3.13 *The following are equivalent for a finitely generated nonzero ideal J of a domain R.*

(1) *If $f(\mathrm{X}) \in R[\mathrm{X}]$ is a polynomial with $C(f) = J$, then the upper-type ideal I_f contains a polynomial with unit content.*
(2) *There is a polynomial $f(\mathrm{X}) \in R[\mathrm{X}]$ with $C(f) = J$ such that the upper-type ideal I_f contains a polynomial with unit content.*
(3) *JR' is an invertible ideal of R', the integral closure of R.*
(4) *There is a positive integer m such that J^m is an invertible ideal of $(J^m : J^m)$.*
(5) *There is a positive integer m such that for each maximal ideal M of R, there is an element $t \in J^m$ such that $J^{2m} = tJ^mR_M$.*
(6) *J is prestable.*

Proof Let $J := a_0R + a_1R + \cdots + a_nR$ and let $a(x) := \sum_i a_i x^i$. Then certainly $C(a) = J$. We also have $J^k = C(a)^k$ for each positive integer k. For convenience we let $T_k := (J^k : J^k)$.

It is clear that (1) implies (2). Suppose $f(\mathrm{X}) \in R[\mathrm{X}]$ is such that $C(f) = J$ and I_f contains a polynomial with unit content. Let $g(\mathrm{X}) \in K[\mathrm{X}]$ be such that $f(\mathrm{X})g(\mathrm{X}) \in I_f$ with $C(fg) = R$. By the content formula, there is an integer m such that $C(f)^mC(f)C(g) = C(f)^mC(fg) = C(f)^m$. It follows that $C(f)C(g) \subseteq (C(f)^m : C(f)^m) \subseteq R'$. Thus $C(g) \subseteq (R' : C(f))$ and we have that $C(f)R' = JR'$ is an invertible ideal of R'. Hence (2) implies (3).

Since $T_k \subseteq R'$ for each k, it is clear that (4) implies (3). To see that (3) implies both (1) and (4), let $f(\mathrm{X}) := f_k\mathrm{X}^k + f_{k-1}\mathrm{X}^{k-1} + \cdot f_0 \in R[\mathrm{X}]$ be such that $C(f) = J$. Then $J^m = C(f)^m$ for each positive integer m. If JR' is invertible as an ideal of R', then there are elements $t_0, t_1, \ldots, t_k \in (R' : C(f)R')$ such that $\sum t_i f_i = 1$. Since $U(R')$ is the saturation of $U(R)$ in $R'[\mathrm{X}]$ ([9, Theorem 3]), there is a polynomial $v(\mathrm{X}) \in U(R')$ such that $f(\mathrm{X})t(\mathrm{X})v(\mathrm{X}) \in U(R)$. Obviously, $f(\mathrm{X})t(\mathrm{X})v(\mathrm{X}) \in I_f$, so (3) implies (1). Also by the content formula, there is a positive integer m such that $C(f)^mC(f)C(tv) = C(f)^mC(ftv) = C(f)^m$. It follows that $C(tv) \subseteq (T_m : J)$ with $JC(tv) = T_m$ since $R = C(ftv) \subseteq JC(tv) \subseteq T_m$. Hence JT_m is an invertible ideal of T_m, and therefore $J^m = J^mT_m$ is invertible as an ideal of $T_m = (J^m : J^m)$.

Next we show that (4) implies (5). Assume J^m is invertible as an ideal of $T_m = (J^m : J^m)$. Let $J^m = b_1R + b_2R + \cdots + b_nR$. Then there are elements $t_1, t_2, \ldots, t_n \in (T_m : J^m)$ such that $\sum t_ib_i = 1$. It follows that J^m is an invertible ideal of $R[\{t_ib_j\}]$, a ring that is contained in T_m. But if this is the case, $R[\{t_ib_j\}] = (J^m : J^m) = T_m$, and therefore T_m is a finite R-module. As such, each maximal ideal of R is contained in only finitely many maximal ideals of T_m. It follows that $(J^mR_M : J^mR_M) = (T_m)_M$ is a semilocal domain with J^mR_M invertible in $(T_m)_M$ for each maximal ideal M of R. Hence J^mR_M is a principal ideal of $(J^mR_M : J^mR_M)$. We have $J^mR_M = t(J^mR_M : J^mR_M)$ for some $t \in J^m$ and from this we obtain $J^{2m}R_M = tJ^mR_M(J^mR_M : J^mR_M) = tJ^mR_M$.

It is clear that (5) implies (6). To complete the proof we show that (6) implies (3). Let M' be a maximal ideal of R' and let $M := M' \cap R$. Then there is a positive integer k such that $J^{2k}R_M = tJ^kR_M$ for some element $t \in J^k$. Thus $t^{-1}J^k(J^kR_M) = J^kR_M$ and from this we have that t^{-1} is in the dual of J^kR_M with respect to the ring $(J^kR_M : J^kR_M)$. Since $t \in J^k$, $J^kR_M = t(J^kR_M : J^kR_M)$. The domain R'_M is the integral closure of R_M, so $J^kR'_M$ is a principal ideal of R'_M. Hence $JR'_{M'}$ is a principal ideal of $R'_{M'}$. Since J is finitely generated, JR' is an invertible ideal of R'. □

Theorem 3.14 *If I_f is an upper-type ideal that contains a polynomial with unit content, then I_f is almost principal.*

Proof For positive integers $n \leq m$, $(C(f)^n : C(f)^n) \subseteq (C(f)^m : C(f)^m)$. Also by Theorem 3.13, if I_f contains a polynomial with unit content, then there is an integer m such that $C(f)^m$ is invertible as an ideal of $T_m := (C(f)^m : C(f)^m)$. For all positive integers $1 \leq j \leq m \leq k$, we have that $C(f)^k$ is an invertible ideal of T_m and therefore $(C(f)^j : C(f)^j) \subseteq T_m = (C(f)^k : C(f)^k)$. That I_f is almost principal follows from Corollary 2.7. □

We can easily establish the following corollary using the proof of Theorem 3.14.

Corollary 3.15 *If $f(\mathrm{x}), b(\mathrm{x}) \in R[\mathrm{x}]$ are such that $C(f) = C(b)$ and I_f contains a polynomial with unit content, then I_b is almost principal.*

Proof From the proof of Theorem 3.14, if I_f contains a polynomial with unit content, then there is an integer n such that $(C(f)^n : C(f)^n) = (C(f)^k : C(f)^k)$ for all $k \geq n$. As $C(f) = C(b)$, we also have $(C(b)^k : C(b)^k) = (C(f)^n : C(f)^n)$ for all $k \geq n$. Thus I_b is almost principal by Corollary 2.7. □

For an alternate proof of this corollary, one can also show directly that I_b will contain a polynomial with unit content. Using the notation of the proof of Theorem 3.14, $C(f)T = C(b)T$ is an invertible ideal of $T := (C(f)^n : C(f)^n)$. Thus one can find a polynomial $t(\mathrm{x}) \in (T : C(b)T)$ such that $b(\mathrm{x})t(\mathrm{x})$ has unit content in T. Next note that T is contained in the integral closure of R. Thus from the proof of [9, Theorem 3], there is a polynomial $v(\mathrm{x}) \in T[\mathrm{x}]$ such that $b(\mathrm{x})t(\mathrm{x})v(\mathrm{x}) \in I_b$ has unit content in R. Yet another proof can be obtained by noting that the polynomial $b(\mathrm{x})t(\mathrm{x})$ is contained in the upper-type ideal $J_b = b(\mathrm{x})K[\mathrm{x}] \cap T[\mathrm{x}]$ of $T[\mathrm{x}]$. Hence by Theorem 3.14, J_b is almost principal in $T[\mathrm{x}]$. Since $(R : T) \supseteq C(f)^n \neq (0)$, I_b is almost principal in $R[\mathrm{x}]$ by Theorem 2.12.

Corollary 3.16 *If R is a domain whose integral closure is a Prüfer domain, then $R[\mathrm{x}]$ is an almost principal ideal domain.*

Proof By [4, Lemma 2.2], the integral closure of R is a Prüfer domain if and only if each upper to zero of $R[\mathrm{x}]$ contains a polynomial with unit content. Thus if the integral closure of R is a Prüfer domain, then $R[\mathrm{x}]$ is an almost principal ideal domain [10, p. 68]. □

For the next result we provide three proofs. The first has at least some of the flavor of the proof given for [10, Proposition 1.8], but also differs substantially. This proof makes direct use of the definition of almost principal ideals. The second makes use of [10, Lemma 1.5] and relies (at least somewhat) on Corollary 2.7. The third makes direct use of Theorem 2.6 and provides a great deal more information about the polynomials in I_f. In particular, the third proof enables us to extend the conclusion of being almost principal to I_b for any $b(\mathrm{x}) \in R[\mathrm{x}]$ where $C(b) = C(f)$. Note that [10, Proposition 1.8] is stated only for an upper to zero, but the proof provided is valid for any upper-type ideal.

Theorem 3.17 (cf. [10, Proposition 1.8]) *If I_f is an upper-type ideal that contains a polynomial $f(\mathrm{x})g(\mathrm{x})$ such that $(R : C(fg)) = R$, then I_f is almost principal.*

Proof Suppose $g(\mathrm{x}) \in G_f$ is such that $(R : C(fg)) = R$. Then (equivalently) $C(fg)_v = R$. For the first proof we let r be a nonzero element of R such that $rg(\mathrm{x}) \in R[\mathrm{x}]$. We will show that $rh(\mathrm{x}) \in R[\mathrm{x}]$ for all $h(\mathrm{x}) \in G_f$.

First proof: Apply the content formula to the product $rf(\mathrm{x})g(\mathrm{x})h(\mathrm{x})$ (factored as $(f(\mathrm{x})g(\mathrm{x})) \cdot (rh(\mathrm{x}))$) to obtain $C(fg)^m C(rfgh) = C(fg)^m C(fg)C(rh)$ for some m. It follows that $C(fg)C(rh) \subseteq (C(fg)^m : C(fg)^m)$. Note that we have $(C(fg)^m : C(fg)^m) = R$ since $(R : C(fg)) = R$ implies $(R : C(fg)^k) = R$ for all $k \geq 1$. Thus $C(rh) \subseteq (R : C(fg)) = R$, and therefore $rh(\mathrm{x}) \in R[\mathrm{x}]$. Hence I_f is almost principal.

Second proof: For this version, we first show that $I_{fg}\ (= I_{f(rg)})$ is almost principal, then apply [10, Lemma 1.5] to get that I_f is almost principal. As noted in the first proof, $(R : C(fg)) = R$ implies $(C(fg)^k : C(fg)^k) = R$ for all k. Thus I_{fg} is almost principal by Corollary 2.7. Hence I_f is almost principal.

Third proof: Our final version makes direct use of Theorem 2.6 and the proof of Theorem 3.14. Since $C(fg)_v = R$, no maximal t-ideal contains $C(fg)$. From the content formula we have $C(f)^m C(fg) = C(f)^m C(f)C(g)$ for $m = deg(g)$. For the remainder of the proof, we let $T := (C(f)^m : C(f)^m)$. Let M be a maximal t-ideal of R. Then $C(fg)R_M = R_M$. As in the proof of Theorem 3.14, $C(f)$ extends to an invertible ideal in $(C(f)^m R_M : C(f)^m R_M) = T_{R\setminus M}$ and therefore $T_{R\setminus M} = (C(f)^k R_M : C(f)^k R_M) = (C(f)^k : C(f)^k)R_M$ for all $k \geq m$. Let $h(\mathrm{x}) \in G_f$. Then there is an integer $k \geq m$ such that $C(f)^k C(fh) = C(f)^k C(f)C(h)$. Thus $C(f)C(h) \in T_{R\setminus M} = (C(f)^m : C(f)^m)R_M$. Hence $C(f)C(h)C(f)^m \subseteq C(f)^m R_M$. It follows that $C(f)C(h)C(f)^m \subseteq (C(f)^m)_v$ which implies $C(f)C(h) \subseteq ((C(f)^m)_v : (C(f)^m)_v)$. Therefore I_f is almost principal by Theorem 2.6. □

Corollary 3.18 *If $f(\mathrm{x}), b(\mathrm{x}) \in R[\mathrm{x}]$ are such that $C(f) = C(b)$ and I_f contains a polynomial $f(\mathrm{x})g(\mathrm{x})$ such that $(R : C(fg)) = R$, then I_b is almost principal.*

Proof As in the proof of Theorem 3.17, we let $T := (C(f)^m : C(f)^m)$ where $m = deg(g)$. Then $T_{R\setminus M} = (C(f)^k R_M : C(f)^k R_M) = (C(b)^k : C(b)^k)R_M$ for all $k \geq m$. Continuing as above, we have $C(b)C(d) C(b)^m \subseteq (C(b)^m)_v$ for all $d(\mathrm{x}) \in G_b$. Hence $C(b)C(d) \subseteq ((C(b)^m)_v : (C(b)^m)_v)$ and therefore I_b is almost principal by Theorem 2.6. □

As in the case that $f(\mathrm{x})g(\mathrm{x}) \in I_f$ has unit content, if instead all we know is that $(R : C(fg)) = R$ and $b(\mathrm{x})$ is such that $C(b) = C(f)$, then I_b will contain a polynomial $b(\mathrm{x})h(\mathrm{x})$ for which $(R : C(bh)) = R$ (which provides an alternate proof of Corollary 3.18 directly from the statement of Theorem 3.17 rather than its proof).

Theorem 3.19 *For polynomials $f(\mathrm{x}), b(\mathrm{x}) \in R[\mathrm{x}]$ with $C(f) = C(b)$, if I_f contains a polynomial $f(\mathrm{x})g(\mathrm{x})$ such that $(R : C(fg)) = R$, then there is a polynomial $b(\mathrm{x})h(\mathrm{x}) \in I_b$ such that $(R : C(bh)) = R$.*

Proof For each maximal t-ideal M_α, let $T_\alpha := (C(f)^m : C(f)^m)R_{M_\alpha}$ where $m = deg(g)$. As M_α does not contain $C(fg)$, $C(f)T_\alpha = C(b)T_\alpha$ is an invertible ideal of T_α (from the proof of Theorem 3.14). Hence for a given M_α, there is a polynomial $t_\alpha(\mathrm{x})$ such that $b(\mathrm{x})t_\alpha(\mathrm{x})$ has unit content in T_α. Since $C(f)^m$ is finitely generated, T_α is contained in the integral closure of R_{M_α}. Thus from the proof of [9, Theorem 3], there is a polynomial $v_\alpha(\mathrm{x}) \in T_\alpha[\mathrm{x}]$ such that $b(\mathrm{x})t_\alpha(\mathrm{x})v_\alpha(\mathrm{x}) \in R_{M_\alpha}[\mathrm{x}]$ has unit content in R_{M_α}. It follows that there is an element $s_\alpha \in R\setminus M_\alpha$ such that $s_\alpha b(\mathrm{x})t_\alpha(\mathrm{x})v_\alpha(\mathrm{x}) \in R[\mathrm{x}]$, necessarily with content not contained in M_α. Hence no maximal t-ideal (of R) contains $C(I_b)$. Therefore $C(I_b)_t = R$ which implies there is a polynomial $b(\mathrm{x})h(\mathrm{x}) \in I_b$ such that $(R : C(bh)) = R$. □

We can extend Theorem 3.17 to an upper-type ideal I_f that contains a polynomial $f(\mathrm{x})g(\mathrm{x})$ such that the ideal $C(fg)(R : C(fg))$ has a trivial dual. Unlike Theorems 3.14 and 3.17 and the corresponding corollaries, we have not been able to extend the conclusion to I_b when $C(b) = C(f)$.

Theorem 3.20 *If I_f is an upper-type ideal that contains a polynomial $f(\mathrm{x})g(\mathrm{x})$ such that $(C(fg)(R : C(fg))_v = R$, then I_f is almost principal.*

Proof Suppose $g(\mathrm{x}) \in G_f$ is such that $(C(fg)(R : C(fg))_v = R$. As in the second proof of Theorem 3.17, it suffices to show that I_{fg} is almost principal. As noted earlier, we have $(C(fg)(R : C(fg))^m \subseteq C(fg)^m(R : C(fg)^m)$ for all $m \geq 1$. Hence the distributive closure property of the v-operation implies $(C(fg)^m(R : C(fg)^m))_v = R$ for all m. That I_{fg} is almost principal follows from Theorem 2.17, and so I_f is almost principal by [10, Lemma 1.5]. □

For our last two (complete) results, we have had to make direct use of the definition of almost principal ideals in order to provide proofs.

For an alternate proof that $R[\mathrm{x}]$ is an almost principal ideal domain when R is a strong Mori domain, one may use the next theorem and Corollary 3.5.

Theorem 3.21 *Let R be a domain with finite t-character and let I_f be an upper-type ideal of $R[\mathrm{x}]$. Then I_f is almost principal as an ideal of $R[\mathrm{x}]$ if and only if $I_f R_M[\mathrm{x}]$ is almost principal as an ideal of $R_M[\mathrm{x}]$ for each maximal t-ideal M of R.*

Proof If I_f is almost principal as an ideal of $R[\mathrm{x}]$, then there is a nonzero element $r \in R$ such that $rI_f \subseteq f(\mathrm{x})R[\mathrm{x}]$. It follows that $rI_f R_M[\mathrm{x}] \subseteq f(\mathrm{x})R_M[\mathrm{x}]$ for each maximal t-ideal M of R. Hence $I_f R_M[\mathrm{x}]$ is almost principal as an ideal of $R_M[\mathrm{x}]$.

For the converse, assume $I_f R_M[\mathrm{x}]$ is almost principal for each maximal t-ideal M. Since R has finite t-character, at most finitely many maximal t-ideals contain $C(f)$. As in the proof of Theorem 3.6, if M is maximal t-ideal that does not contain $C(f)$, then $h(\mathrm{x}) \in R_M[\mathrm{x}]$ and $C(h) \subseteq R_M$ for each polynomial $h(\mathrm{x}) \in G_f$.

Let $M_1, M_2, \ldots, M_n$ be the maximal t-ideals that contain $C(f)$ (if any). Since $I_f R_{M_i}[x]$ is almost principal for each M_i, there is a nonzero element $r_i \in R$ such that $r_i h(x) \in R_{M_i}[x]$ for each $h(x) \in G_f$. Certainly, $r := r_1 r_2 \cdots r_n$ is such that $rh(x) \in R_{M_i}[x]$ for each M_i. It follows that $rC(h) \subseteq R_N$ for each maximal t-ideal N. As $R = \bigcap\{R_N \mid N \in tMax(R)\}$, $rC(h) \subseteq R$. Therefore I_f is almost principal. □

As noted above, one proof for the following theorem can be obtained by first applying [7, Proposition 1.2 and Remark 1.4] followed by [10, Proposition 1.15] to show that each upper to zero is almost principal (when $R[x]$ is a Mori domain), and then use Theorem 2.3 to see that each nonzero ideal of $R[x]$ with proper extension to $K[x]$ is almost principal. We give a direct proof that if $R[x]$ is a Mori domain, then each nonzero ideal of $R[x]$ with proper extension to $K[x]$ is almost principal.

Theorem 3.22 *Let R be a Mori domain. If $R[x]$ is a Mori domain, then it is an almost principal ideal domain.*

Proof Assume $R[x]$ is a Mori domain and let J be a nonzero ideal of $R[x]$ such that $JK[x] \neq K[x]$. Thus $JK[x] = f(x)K[x]$ for some polynomial $f(x)$ with positive degree, and we may further assume that $f(x)$ is in J. Since $R[x]$ is a Mori domain, there is finitely generated ideal $B := f_1(x)R[x] + f_2(x)R[x] + \cdots + f_n(x)R[x]$ contained in J with $B_v = J_v$. Since $JK[x] = f(x)K[x]$, there are polynomials $g_1(x), g_2(x), \ldots, g_n(x) \in K[x]$ such that $f_i(x) = f(x)g_i(x)$ for each i. Certainly, there is a nonzero element $s \in R$ such that $sg_i(x) \in R[x]$ for each g_i. It follows that $s/f(x) \in (R[x] : B) = (R[x] : J)$ and therefore $sJ \subseteq f(x)R[x]$. □

We close with a rather odd observation concerning the upper-type ideals of $R[x]$ when R is a Mori domain (but perhaps with $R[x]$ not Mori).

Remark 3.23 Let I_f be an upper-type ideal of $R[x]$ for a Mori domain R. Then $C(I_f)$ is a trace ideal (Lemma 2.5) and since R is a Mori domain, there is a finitely generated ideal $J \subseteq C(I_f)$ such that $J_v = C(I_f)_v$. Moreover, there is a polynomial $g(x) \in G_f$ with $J \subseteq C(fg) \subseteq C(I_f)$. Hence $C(fg)_v = C(I_{fg})_v = C(I_f)_v$ and $C(fg)_v$ is a trace ideal [15, Proposition 2.2]. By [10, Lemma 1.5], I_{fg} almost principal implies I_f almost principal. Thus, to simplify notation we assume we are in the case that $C(f)_v = C(I_f)_v$. If $C(f)_v = R$, I_f is almost principal by Theorem 3.17.

Let $h(x) \in G_f$ with $deg(h) = m$. Then $C(f)^m C(fh) = C(f)^m C(f)C(h)$ by the content formula. Note that since $C(f)_v = C(I_f)_v$, $C(fh) \subseteq C(fh)_v \subseteq C(f)_v$. Thus $C(h)(C(f)^{m+1})_v \subseteq (C(f)^m C(fh))_v \subseteq (C(f)^{m+1})_v$ and therefore $C(h)^k \subseteq ((C(I_f)^{m+1})_v : (C(I_f)^{m+1})_v)$ for all positive integers k. It follows that $h^k(x)I_f^{m+1} \subseteq I_f^{m+1}$. An unfortunate thing here is that the m is dependent on $h(x)$.

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