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Quasi-elliptic cohomology and its power operations

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Abstract Quasi-elliptic cohomology is a variant of Tate K-theory. It is the orbifold K-theory of a space of constant loops. For global quotient orbifolds, it can be expressed in terms of equivariant K-theories. In this paper we show how this theory is equipped with power operations. We also prove that the Tate K-theory of symmetric groups modulo a certain transfer ideal classify the finite subgroups of the Tate curve.

Keywords Tate curve · Power operation · Elliptic cohomology · Representation theory

Mathematics Subject Classification Primary 55

1 Introduction

An elliptic cohomology theory is an even periodic multiplicative generalized cohomology theory whose associated formal group is the formal completion of an elliptic curve. It is an old idea of Witten, as shown in [17], that the elliptic cohomology of a space X is related to the $\mathbb{T}$-equivariant K-theory of the free loop space $LX = \mathbb{C}^\infty(S^1, X)$ with the circle $\mathbb{T}$ acting on LX by rotating loops.

It is surprisingly difficult to make this precise, especially if one wishes to consider equivariant generalization of this construction. In this case the loop space LX with the natural rotation action is a rich orbifold. In this paper we offer a new formulation

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between the loop space and Tate K-theory via a new theory which we call quasi-elliptic cohomology.

Tate K-theory is the generalized elliptic cohomology associated to the Tate curve. The Tate curve $Tate(q)$ is an elliptic curve over $\mathrm{Spec}\mathbb{Z}((q))$, which is classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity. A good reference for $Tate(q)$ is Section 2.6 of [1]. We give a sketch of it in Sect. 6.1. The relation between Tate K-theory and string theory is better understood than for most known elliptic cohomology theories. The definition of G-equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold, which is formulated explicitly in Section 2, [10]. Its relation with string theory and loop space makes Tate K-theory itself a distinctive subject to study.

The idea of quasi-elliptic cohomology is motivated by Ganter's construction of Tate K-theory. It is not an elliptic cohomology but from it we can recover the Tate K-theory. This new theory can be interpreted in a neat form by equivariant K-theories, which makes many constructions on it easier and more natural than those on the Tate K-theories. Some formulations can be generalized to other equivariant cohomology theories. In addition, quasi-elliptic cohomology provides a method that reduces facts such as the classification of geometric structures on the Tate curve into questions in representation theory.

1.1 Loop space

Quasi-elliptic cohomology is modelled on a version of equivariant loop space. For background on orbifolds and Lie groupoids, we refer the readers to Sections 2, 3, [18,23].

For any compact Lie group G and a manifold X with a smooth G-action, there is a Lie groupoid $X/\!\!/G$ which is explained in detail in Chapter 11, [6]. Smooth unbased loops in the orbifold $X/\!\!/G$ carries a lot of structure: on the one hand, it includes loops represented by smooth maps $\gamma : \mathbb{R} \longrightarrow X$ such that $\gamma(t+1) = \gamma(t)g$ for some $g \in G$; other than the group action by the loop group $LG := \mathbb{C}^{\infty}(S^1, G)$, the loop space also has the circle action by rotation. Lerman discussed thoroughly in Section 3, [18] that the strict 2-category of Lie groupoids can be embedded into a weak 2-category whose objects are Lie groupoids, 1-morphisms are bibundles and 2-morphisms equivariant diffeomorphisms between bibundles. Thus, the free loop space of an orbifold M is the category of bibundles from the trivial groupoid $S^1/\!\!/*$ to the Lie groupoid M. We will write

$$Loop_1(X/\!\!/G) := Bibun(S^1/\!\!/*, X/\!\!/G),$$

which is discussed in Definition 2.2. In Definition 2.3, we extend $Loop_1(X/\!\!/G)$ to a groupoid $Loop_1^{ext}(X/\!\!/G)$ by adding rotations as morphisms.

Especially we are interested in the ghost loops groupoid $GhLoop(X/\!\!/G)$, which is defined to be the full subgroupoid of $Loop_1^{ext}(X/\!\!/G)$ consisting of objects (π, f) with the image of f contained in a single G-orbit. Ghost loops are introduced by Rezk in his unpublished manuscript [26]. Another reference is Section 2.1.3, [12]. This groupoid has several good properties. They are computed locally in X. For instance, if $X = U \cup V$ where U and V are G-invariant open subsets, then

$$GhLoop(X/\!\!/G) \cong GhLoop(U/\!\!/G) \cup_{GhLoop((U\cap V)/\!\!/G)} GhLoop(V/\!\!/G).$$

So it satisfies a kind of Mayer–Vietoris property. In addition, if H is a closed subgroup of G and X is the quotient space G/H, $GhLoop(X/\!\!/G)$ is equivalent to $GhLoop(\mathrm{pt}/\!\!/H)$. In other words, it has the change-of-group property.

When G is finite, $GhLoop(X/\!\!/G)$ is isomorphic to the full subgroupoid $\Lambda(X/\!\!/G)$ of $Loop_1^{ext}(X/\!\!/G)$ consisting of constant loops. This groupoid $\Lambda(X/\!\!/G)$ can be regarded as an extended version of the inertia groupoid $I(X/\!\!/G)$. Please see Definition 3.7 for inertia groupoid.

1.2 Quasi-elliptic cohomology

For any compact orbifold groupoid $\mathbb{G}$, the orbifold K-theory $K_{orb}(\mathbb{G})$ is defined to be the Grothendieck ring of isomorphism classes of $\mathbb{G}$-vector bundles on $\mathbb{G}$. In particular, $K_{orb}(X/\!\!/G)$ is $K_G(X)$. A reference for orbifold K-theory is Chapter 3, [3] and a reference for equivariant K-theory is [27].

Quasi-elliptic cohomology $QEll^*(X/\!\!/G)$ is defined to be the orbifold K-theory of a subgroupoid $\Lambda(X/\!\!/G)$ of $GhLoop(X/\!\!/G)$ consisting of constant loops. When G is a finite group, $QEll^*_G(X)$ can be expressed in terms of the equivariant K-theory of X and its subspaces as

$$QEll^*_G(X) := K_{orb}(GhLoop(X/\!\!/G)) \cong \prod_{\sigma\in G_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) = \left(\prod_{\sigma\in G} K^*_{\Lambda_G(\sigma)}(X^\sigma)\right)^G, \tag{1.1}$$

where G_{conj} is a set of representatives of G-conjugacy classes in G. The group $\Lambda_G(\sigma) := C_G(\sigma)\times\mathbb{R}/\langle(\sigma,-1)\rangle$ acts on the fixed point space X^σ by $[g,t]\cdot m = g\cdot m$. In a coming paper by the author [13], we will present the construction of $QEll^*_G(X)$ for any compact Lie group G.

$QEll_G(X)$ has the structure of a $\mathbb{Z}[q^\pm]$-algebra. We have

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X). \tag{1.2}$$

We formulate the K*ü*nneth map, restriction map, change of group isomorphism and transfer for $QEll$. In general, if H^* is an equivariant cohomology theory, then the functor

$$X/\!\!/G \mapsto H^*(GhLoop(X/\!\!/G))$$

gives a new equivariant cohomology theory. Moreover, for each global cohomology theory, we can formulate a new global cohomology theory via the ghost loops.

1.3 Power operation

One significant feature of quasi-elliptic cohomology is that it has power operations, which was first observed by Ganter, as shown in [10,11]. In Sect. 4 we construct the total power operation of quasi-elliptic cohomology. It satisfies the axioms for equivariant power operations that Ganter gave in Definition 4.3 in [9]. For more details, please see Theorem 4.12.

The power operation $\{\mathbb{P}_n\}_{n\geq 0}$ mixes the power operation in K-theory with the natural operations of dilating and rotating loops. The key point of the construction of the power operation is an intermediate groupoid $d_{(\underline{g},\sigma)}(X)$ with $(\underline{g},\sigma)\in G\wr\Sigma_n$. It is constructed from $\Lambda(X/\!\!/G)$ and isomorphic to $(X^{\times n})^{(\underline{g},\sigma)}/\!\!/\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$. For more details of the construction, please see Sect. 4.2.

We illustrate what this power operation looks like by examples. Let G be the trivial group and X a space. Let $(-)_k$ denote the rescaling map defined in (4.11).

When $n=2$, $\mathbb{P}_{(\underline{1},(1)(1))}(x)=x\boxtimes x$ and $\mathbb{P}_{(\underline{1},(12))}(x)=(x)_2$.

When $n=3$, $\mathbb{P}_{(\underline{1},(1)(1)(1))}(x)=x\boxtimes x\boxtimes x$, $\mathbb{P}_{(\underline{1},(12)(1))}(x)=(x)_2\boxtimes x$, and $\mathbb{P}_{(\underline{1},(123))}(x)=(x)_3$.

In these cases, the number of factors corresponds to the number of cycles in the permutation and the rescaling map corresponds to the length of each cycle. For more examples please see Example 4.13.

For any equivariant cohomology theory $\{H^*_G(-)\}_G$ with an H_∞-structure in Ganter's sense, we can formulate a power operation for the equivariant cohomology theories

$$\mathbb{H}^*_G(-):=\prod_{\sigma\in G_{conj}} H^*_{\Lambda_G(\sigma)}(-)^{\sigma}$$

in the same way.

In addition, we can formulate the total power operation for the orbifold quasi-elliptic cohomology in the sense of Definition 3.9, [11]. The construction of the power operation is shown in Sect. 5.3.

1.4 Classification of the finite subgroups of the Tate curve

Though the general formulas for the power operations in $QEll_G$ are complicated, to understand it, it is useful to consider special cases. It is already illuminating to consider the case that X is a point and G is the trivial group, the power operation has a neat form, as shown in Example 4.13. It has a natural interpretation in terms of the Tate elliptic curve.

In Sect. 6.3 applying the power operation we prove that the Tate K-theory of symmetric groups modulo the transfer ideal classifies the finite subgroups of the Tate curve, which is analogous to the principal result in Strickland [28] that the Morava E-theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of the formal group $\mathbb{G}_E$.

The finite subgroups of the Tate curve are classified by

$$\prod_{d|N} \mathbb{Z}((q))[q']/\langle q^d - q'^{\frac{N}{d}}\rangle.$$

First we prove the parallel conclusion for quasi-elliptic cohomology.

Theorem 1.1

$$QEll^0_{\Sigma_N}(pt)/\mathcal{I}^{QEll}_{tr} \cong \prod_{d|N} \mathbb{Z}[q^{\pm}][q']/\langle q^d - q'^{\frac{N}{d}}\rangle, \tag{1.3}$$

where $\mathcal{I}^{QEll}_{tr}$ is the transfer ideal defined in (6.4) *and q' is the image of q under the power operation $\mathbb{P}_N$.*

Then applying the relationship between $QEll^*$ and Tate K-theory, we obtain the main theorem.

Theorem 1.2 *The Tate K-theory of symmetric groups modulo the transfer ideal I^{Tate}_{tr} defined in* (6.3) *classifies finite subgroups of the Tate curve. Explicitly,*

$$(K^0_{Tate})_{\Sigma_N}(pt)/I^{Tate}_{tr} \cong \prod_{d|N} \mathbb{Z}((q))[q']/\langle q^d - q'^{\frac{N}{d}}\rangle, \tag{1.4}$$

where q' is the image of q under the power operation P^{Tate} constructed in Definition 5.10, [10].

Moreover, via the isomorphism in Theorem 1.1, we can define a ring homomorphism

$$\begin{aligned}\overline{P}_N : QEll_G(X) &\xrightarrow{\mathbb{P}_N} QEll_{G\wr\Sigma_N}(X^{\times N}) \xrightarrow{res} QEll_{G\times\Sigma_N}(X^{\times N})\\ &\xrightarrow{diag^*} QEll_{G\times\Sigma_N}(X) \cong QEll_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} QEll_{\Sigma_N}(\mathrm{pt})\\ &\longrightarrow QEll_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} QEll_{\Sigma_N}(\mathrm{pt})/\mathcal{I}^{QEll}_{tr},\end{aligned}$$

as shown in Proposition 6.5. Under the identification (1.2), it extends uniquely to the ring homomorphism

$$\overline{P^{string}}_N : (K_{Tate})_G(X) \longrightarrow (K_{Tate})_G(X) \otimes_{\mathbb{Z}((q))} (K_{Tate})_{\Sigma_N}(\mathrm{pt})/I^{Tate}_{tr}$$

constructed in Section 5.4, [10]. In [14] we construct the universal finite subgroup of the Tate curve via the operation $\overline{P}_N$.

2 Models for orbifold loops and ghost loops

To understand $QEll_G^*(X)$, it is essential to understand the orbifold loop space. In this section, we will describe several models for the loop space of $X/\!\!/G$. In Definition 2.2 we discuss $Loop_1(X/\!\!/G)$ and introduce another model $Loop_2(X/\!\!/G)$ in Definition 2.4.

The groupoid structure of $Loop_1(X/\!\!/G)$ generalizes $Map(S^1, X)/\!\!/G$, which is a subgroupoid of it. Other than the G-action, we also consider the rotation by the circle group $\mathbb{T}$ on the objects and form the groupoids $Loop_1^{ext}(X/\!\!/G)$ and $Loop_2^{ext}(X/\!\!/G)$. The groupoid $Loop_2^{ext}(X/\!\!/G)$ has a skeleton

$$\mathcal{L}(X/\!\!/G) := \coprod_g {}_1\mathcal{L}_g X/\!\!/L_g^1 G \rtimes \mathbb{T},$$

where each ${}_1\mathcal{L}_g X = \mathrm{Map}_{\mathbb{Z}/l\mathbb{Z}}(\mathbb{R}/l\mathbb{Z}, X)$ with l the order of g is equipped with an evident $C_G(g)$-action. $\mathcal{L}(X/\!\!/G)$ has the same space of objects as the groupoid $L(X/\!\!/G)$ discussed in Definition 2.3, [21], from which equivariant Tate K-theory is defined. It has richer morphisms. The circle group $\mathbb{T}$ acts on $\mathbb{R}/l\mathbb{Z}$ by rotation, and so in principle on the orbifold ${}_1\mathcal{L}_g X$.

The key groupoid $\Lambda(X/\!\!/G)$ in the construction of quasi-elliptic cohomology is the full subgroupoid of $\mathcal{L}(X/\!\!/G)$ consisting of the constant loops. In order to unravel the relevant notations in the construction of $QEll_G^*(X)$, we study the orbifold loop space in Sects. 2.1.2 and 2.1.3.

In Sect. 2.1.1 we define $Loop_1(X/\!\!/G)$. In Sect. 2.1.2 we interpret the enlarged groupoid $Loop_1^{ext}(X/\!\!/G)$ and introduce a skeleton $\mathcal{L}(X/\!\!/G)$ of it. In Sect. 2.1.3 we show the construction of quasi-elliptic cohomology by ghost loops. In Sect. 3.1 we show the representation ring of $\Lambda_G(g)$. In Sect. 3.2 we introduce the construction of quasi-elliptic cohomology first in terms of orbifold K-theory and then equivariant K-theory. We show the properties of the theory in Sect. 3.3.

2.1 Loop space

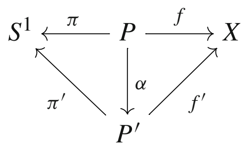
2.1.1 Bibundles

A standard reference for groupoids and bibundles is Sections 2 and 3, [18]. For each pair of Lie groupoids $\mathbb{H}$ and $\mathbb{G}$, the bibundles from $\mathbb{H}$ to $\mathbb{G}$ are defined in Definition 3.25, [18]. The category $Bibun(\mathbb{H}, \mathbb{G})$ has bibundles from $\mathbb{H}$ to $\mathbb{G}$ as the objects and bundle maps as the morphisms.

Example 2.1 ($Bibun(S^1/\!\!/*, */\!\!/G)$) According to the definition, a bibundle from $S^1/\!\!/*$ to $*/\!\!/G$ with G a Lie group is a smooth manifold P together with two maps $\pi : P \longrightarrow S^1$ a smooth principal G-bundle and the constant map $r : P \longrightarrow *$. So a bibundle in this case is equivalent to a smooth principal G-bundle over S^1. The morphisms in $Bibun(S^1/\!\!/*, */\!\!/G)$ are bundle isomorphisms.

Definition 2.2 ($Loop_1(X/\!\!/G)$) Let G be a Lie group acting smoothly on a manifold X. We use $Loop_1(X/\!\!/G)$ to denote the category $Bibun(S^1/\!\!/*, X/\!\!/G)$, which generalizes

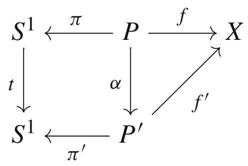
Example 2.1. Each object consists of a smooth manifold P and two structure maps $P \xrightarrow{\pi} S^1$ a smooth principal G-bundle and $f : P \longrightarrow X$ a G-equivariant map. We use the same symbol P to denote both the object and the smooth manifold when there is no confusion. A morphism is a G-bundle map $\alpha : P \longrightarrow P'$ making the diagram below commute.



Thus, the morphisms in $Loop_1(X/\!\!/G)$ from P to P' are bundle isomorphisms.

Only the G-action on X is considered in $Loop_1(X/\!\!/G)$. We add the rotations by adding more morphisms into the groupoid.

Definition 2.3 $(Loop_1^{ext}(X/\!\!/G))$ Let $Loop_1^{ext}(X/\!\!/G)$ denote the groupoid with the same objects as $Loop_1(X/\!\!/G)$. Each morphism consists of the pair (t, α) where $t \in \mathbb{T}$ is a rotation and α is a G-bundle map. They make the diagram below commute.



The groupoid $Loop_1(X/\!\!/G)$ is a subgroupoid of $Loop_1^{ext}(X/\!\!/G)$.

2.1.2 Another model for orbifold loop space

We give an equivalent description of the groupoids discussed in Sect. 2.1.1. The new models $Loop_2(X/\!\!/G)$ and $Loop_2^{ext}(X/\!\!/G)$ are more practicable to compute. We give a skeleton $\mathcal{L}(X/\!\!/G)$ of $Loop_2^{ext}(X/\!\!/G)$ when G is finite in Proposition 2.7.

Definition 2.4 $(Loop_2(X/\!\!/G))$ Let $Loop_2(X/\!\!/G)$ denote the groupoid whose objects are (σ, γ) with $\sigma \in G$ and $\gamma : \mathbb{R} \longrightarrow X$ a continuous map such that $\gamma(s+1) = \gamma(s)\cdot\sigma$, for any $s \in \mathbb{R}$. A morphism $\alpha : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ is a continuous map $\alpha : \mathbb{R} \longrightarrow G$ satisfying $\gamma'(s) = \gamma(s)\alpha(s)$. Note that $\alpha(s)\sigma' = \sigma\alpha(s+1)$, for any $s \in \mathbb{R}$.

Moreover, we can extend the groupoid $Loop_2(X/\!\!/G)$ by adding the rotations.

Definition 2.5 $(Loop_2^{ext}(X/\!\!/G))$
Let $Loop_2^{ext}(X/\!\!/G)$ denote the groupoid with the same objects as $Loop_2(X/\!\!/G)$. A morphism $(\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ consists of the pair (α, t) with $\alpha : \mathbb{R} \longrightarrow G$ a continuous map and $t \in \mathbb{R}$ satisfying $\gamma'(s) = \gamma(s-t)\alpha(s-t)$. Note that $(\alpha, t+1)$ and $(\alpha\sigma', t)$ are the same morphism and each morphism can be represented by a pair (α, t) with $t \in [0, 1)$.
$Loop_2(X/\!\!/G)$ is a subgroupoid of $Loop_2^{ext}(X/\!\!/G)$.

Lemma 2.6 *The groupoid $Loop_1^{ext}(X/\!\!/G)$ is isomorphic to a full subgroupoid of $Loop_2^{ext}(X/\!\!/G)$.*

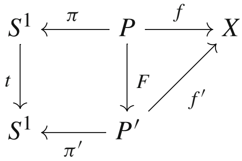
Proof Define a functor

$$F : Loop_1^{ext}(X/\!\!/G) \longrightarrow Loop_2^{ext}(X/\!\!/G)$$

by sending an object

$$S^1 \xleftarrow{\ \pi\ } P \xrightarrow{\ f\ } X$$

to (σ, γ) with $\gamma(s) := f([s, e])$ and $\sigma = \gamma(0)^{-1}\gamma(1)$ and sending a morphism



to $(\alpha, t) : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$ with $\alpha(s) := F([s, e])^{-1}$.

F is a fully faithful functor but not essentially surjective. □

Therefore, the groupoid $Loop_2^{ext}(X/\!\!/G)$ contains all the information of $Loop_1^{ext}(X/\!\!/G)$. Next we will show a skeleton of this larger groupoid when G is finite. Before that, we introduce some symbols.

Let $k \geq 0$ be an integer and g an element in the compact Lie group G. Let $L_g^k G$ denote the twisted loop group

$$\{\gamma : \mathbb{R} \longrightarrow G | \gamma(s+k) = g^{-1}\gamma(s)g\}. \tag{2.1}$$

The multiplication of it is defined by

$$(\delta \cdot \delta')(t) = \delta(t)\delta'(t), \quad \text{for any} \delta, \delta' \in L_g^k G, \text{ and } t \in \mathbb{R}. \tag{2.2}$$

The identity element e is the constant map sending all the real numbers to the identity element of G. We extend this group by adding the rotations. Let $L_g^k G \rtimes \mathbb{T}$ denote the group with elements (γ, t), $\gamma \in L_g^k G$ and $t \in \mathbb{T}$. The multiplication is defined by

$$(\gamma, t) \cdot (\gamma', t') := (s \mapsto \gamma(s)\gamma'(s+t), t+t'). \tag{2.3}$$

The set of constant maps $\mathbb{R} \longrightarrow G$ in $L_g^k G$ is a subgroup of it, i.e. the centralizer $C_G(g)$. When G is finite, $L_g^k G = C_G(g)$.

When G is finite, the objects of $Loop_2(X/\!\!/G)$ can be identified with the space

$$\coprod_{g \in G} {}_1\mathcal{L}_g X$$

where

$$_k\mathcal{L}_g X := \mathrm{Map}_{\mathbb{Z}/l\mathbb{Z}}(\mathbb{R}/kl\mathbb{Z}, X), \tag{2.4}$$

and l is the order of the element g. The cyclic group $\mathbb{Z}/l\mathbb{Z}$ is isomorphic to the subgroup $k\mathbb{Z}/kl\mathbb{Z}$ of $\mathbb{R}/kl\mathbb{Z}$. The isomorphism $\mathbb{Z}/l\mathbb{Z} \longrightarrow k\mathbb{Z}/kl\mathbb{Z}$ sends the generator [1] corresponding to 1 to the generator $[k]$ of $k\mathbb{Z}/kl\mathbb{Z}$ corresponding to k. $k\mathbb{Z}/kl\mathbb{Z}$ acts on $\mathbb{R}/kl\mathbb{Z}$ by group multiplication. Thus, via the isomorphism, $\mathbb{Z}/l\mathbb{Z}$ acts on $\mathbb{R}/kl\mathbb{Z}$. $\mathbb{Z}/l\mathbb{Z}$ is also isomorphic to the cyclic group $\langle g\rangle$ by identifying the generater [1] with g. So it acts on X via the G-action on it.

$_1\mathcal{L}_g X /\!\!/ L^1_g G$ is a full subgroupoid of $Loop_2(X/\!\!/G)$. Moreover, $_1\mathcal{L}_g X/\!\!/ L^1_g G \rtimes \mathbb{T}$ is a full subgroupoid of $Loop^{ext}_2(X/\!\!/G)$ where $L^k_g G \rtimes \mathbb{T}$ acts on $_k\mathcal{L}_g X$ by

$$\delta\cdot(\gamma,t) := (s\mapsto \delta(s+t)\cdot\gamma(s+t)),\ \text{for any } (\gamma,t)\in L^k_g G\rtimes\mathbb{T},\ \text{and } \delta\in {}_k\mathcal{L}_g X. \tag{2.5}$$

The action by g on $_k\mathcal{L}_g X$ coincides with that by $k\in\mathbb{R}$. So we have the isomorphism

$$L^k_g G\rtimes\mathbb{T} = L^k_g G\rtimes \mathbb{R}/\langle(\overline{g}, -k)\rangle, \tag{2.6}$$

where $\overline{g}$ represents the constant loop $\mathbb{T}\longrightarrow\{g\}\subseteq G$.

In fact we have already proved Proposition 2.7.

Proposition 2.7 *Let G be a finite group. The groupoid*

$$\mathcal{L}(X/\!\!/G) := \coprod_{[g]} {}_1\mathcal{L}_g X/\!\!/L^1_g G\rtimes\mathbb{T}$$

is a skeleton of $Loop^{ext}_2(X/\!\!/G)$, where the coproduct goes over conjugacy classes in $\pi_0 G$.

Next we show the physical meaning of $L^1_\sigma G$. Recall that the gauge group of a principal bundle is defined to be the group of its vertical automorphisms. The readers may refer [22] for more details. For a G-bundle $P\longrightarrow S^1$, let $L_P G$ denote its gauge group.

We have the well-known facts below.

Lemma 2.8 *The principal G-bundles over S^1 are classified up to isomorphism by homotopy classes*

$$[S^1, BG]\cong \pi_0 G/conj.$$

Up to isomorphism every principal G-bundle over S^1 is isomorphic to one of the forms $P_\sigma \longrightarrow S^1$ with $\sigma\in G$ and

$$P_\sigma := \mathbb{R}\times G/(s+1, g)\sim(s,\sigma g).$$

A complete collection of isomorphism classes is given by a choice of representatives for each conjugacy class of $\pi_0 G$.

For the gauge group $L_{P_\sigma}G$ we have the conclusion below.

Proposition 2.9 *For the bundle $P_\sigma \longrightarrow S^1$, $L_{P_\sigma}G$ is isomorphic to the twisted loop group $L^1_\sigma G$.*

Proof Each automorphism f of the bundle $P_\sigma \longrightarrow S^1$ has the form

$$\begin{array}{ccc} P_\sigma & \xrightarrow{[s,g]\mapsto[s,\gamma_f(s)g]} & P_\sigma \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{=} & S^1 \end{array} \tag{2.7}$$

for some $\gamma_f : \mathbb{R} \longrightarrow G$. The morphism is well-defined if and only if $\gamma_f(s+1) = \sigma^{-1}\gamma_f(s)\sigma$. So we get a well-defined map

$$F : L_{P_\sigma}G \longrightarrow L^1_\sigma G,\ f \mapsto \gamma_f.$$

It is a bijection. Moreover, by the property of group action, F sends the identity map to the constant map $\mathbb{R} \longrightarrow G$, $s \mapsto e$, which is the trivial element in $L^1_\sigma G$, and for two automorphisms f_1 and f_2 at the object, $F(f_1 \circ f_2) = \gamma_{f_1} \cdot \gamma_{f_2}$. So $L_{P_\sigma}G$ is isomorphic to $L^1_\sigma G$. □

2.1.3 Ghost loops

Let G be a compact Lie group and X a G-space. In this section we introduce a subgroupoid $GhLoop(X/\!\!/G)$ of $Loop_1^{ext}(X/\!\!/G)$, which can be computed locally.

Definition 2.10 (*Ghost loops*) The groupoid of ghost loops is defined to be the full subgroupoid $GhLoop(X/\!\!/G)$ of $Loop_1^{ext}(X/\!\!/G)$ consisting of objects $S^1 \leftarrow P \stackrel{\widetilde{\delta}}{\rightarrow} X$ such that $\widetilde{\delta}(P) \subseteq X$ is contained in a single G-orbit.

For a given $\sigma \in G$, define the space

$$GhLoop_\sigma(X/\!\!/G) := \{\delta \in {}_1\mathcal{L}_\sigma X | \delta(\mathbb{R}) \subseteq G\delta(0)\}. \tag{2.8}$$

We have a corollary of Proposition 2.7 below.

Proposition 2.11 *$GhLoop(X/\!\!/G)$ is equivalent to the groupoid*

$$\Lambda(X/\!\!/G) := \coprod_{[\sigma]} GhLoop_\sigma(X/\!\!/G)/\!\!/L^1_\sigma G \rtimes \mathbb{T}$$

where the coproduct goes over conjugacy classes in $\pi_0 G$.

Example 2.12 If G is a finite group, it has the discrete topology. In this case, LG consists of constant loops and, thus, is isomorphic to G. The space of objects of $GhLoop(X/\!\!/G)$ can be identified with X. For $\sigma \in G$ and any integer k, $L^k_\sigma G$ can be identified with $C_G(\sigma)$; $L^k_\sigma G \rtimes \mathbb{T} \cong C_G(\sigma) \times \mathbb{R}/\langle(\sigma, -k)\rangle$; and $GhLoop_\sigma(X/\!\!/G)$ can be identified with X^σ.

Unlike true loops, ghost loops have the property that they can be computed locally, as shown in the lemma below. The proof is left to the readers.

Proposition 2.13 *If $X = U \cup V$ where U and V are G-invariant open subsets, then $GhLoop(X /\!\!/ G)$ is isomorphic to the fibred product of groupoids*

$$GhLoop(U /\!\!/ G) \cup_{GhLoop((U\cap V)/\!/G)} GhLoop(V /\!\!/ G).$$

Thus, the ghost loop construction satisfies Mayer–Vietoris property. Moreover, it has the change-of-group property.

Proposition 2.14 *Let H be a closed subgroup of G. It acts on the space of left cosets G/H by left multiplication. Let pt denote the single point space with the trivial H-action. Then we have the equivalence of topological groupoids between $Loop_1^{ext}((G/H) /\!\!/ G)$ and $Loop_1^{ext}(pt /\!\!/ H)$. Especially, there is an equivalence between the groupoids $GhLoop((G/H) /\!\!/ G)$ and $GhLoop(pt /\!\!/ H)$.*

Proof First we define a functor $F : Loop_1^{ext}((G/H) /\!\!/ G) \longrightarrow Loop_1^{ext}(\mathrm{pt} /\!\!/ H)$ sending an object $S^1 \leftarrow P \overset{\widetilde{\delta}}{\rightarrow} G/H$ to $S^1 \leftarrow Q \rightarrow \{eH\} = \mathrm{pt}$ where $Q \longrightarrow eH$ is the constant map, and $Q \longrightarrow S^1$ is the pull back bundle

$$\begin{array}{ccc} Q & \longrightarrow & \{eH\} \\ \downarrow & & \downarrow \\ P & \longrightarrow & G/H. \end{array}$$

It sends a morphism

$$\begin{array}{ccccc} P' & \longrightarrow & P & \longrightarrow & G/H \\ \downarrow & & \downarrow & & \\ S^1 & \longrightarrow & S^1 & & \end{array}$$

to the morphism

$$\begin{array}{ccccc} Q' & \longrightarrow & Q & \longrightarrow & \{eH\} \\ \downarrow & & \downarrow & & \downarrow \\ P' & \longrightarrow & P & \longrightarrow & G/H \\ \downarrow & & \downarrow & & \\ S^1 & \longrightarrow & S^1 & & \end{array}$$

where all the squares are pull-back.

In addition, we can define a functor $F' : Loop_1^{ext}(\mathrm{pt}/\!\!/H) \longrightarrow Loop_1^{ext}((G/H)/\!\!/G)$ sending an object $S^1 \leftarrow Q \rightarrow \mathrm{pt}$ to $S^1 \leftarrow G \times_H Q \rightarrow G \times_H \mathrm{pt} = G/H$ and sending a morphism

$$\begin{array}{ccc} Q' & \longrightarrow & Q \\ \downarrow & & \downarrow \\ S^1 & \longrightarrow & S^1 \end{array}$$

to

$$\begin{array}{ccccc} G \times_H Q' & \longrightarrow & G \times_H Q & \longrightarrow & G \times_H \mathrm{pt} = G/H \\ \downarrow & & \downarrow & & \\ S^1 & \longrightarrow & S^1 & & \end{array}$$

$F \circ F'$ and $F' \circ F$ are both identity maps. So the topological groupoids $Loop_1^{ext}((G/H)/\!\!/G)$ and $Loop_1^{ext}(\mathrm{pt}/\!\!/H)$ are equivalent.

We can prove the equivalence between $GhLoop((G/H)/\!\!/G)$ and $GhLoop(\mathrm{pt}/\!\!/H)$ in the same way. □

Remark 2.15 In general, if H^* is an equivariant cohomology theory, Proposition 2.14 implies the functor

$$X/\!\!/G \mapsto H^*(GhLoop(X/\!\!/G))$$

gives a new equivariant cohomology theory. When H^* has the change of group isomorphism, so does $H^*(GhLoop(-))$.

3 Quasi-elliptic cohomology $QEll_G^*$

Unless otherwise indicated, we assume G is a finite group and X is a G-space in the rest part of the paper. The main references for Sect. 3 are Rezk's unpublished work [25] and the author's PhD thesis [12]. The construction of the theory $QEll_G^*$ for any compact Lie group G will be shown in the paper [13]. In Sect. 3.2 we define $QEll_G^*$ and prove some of its main properties. Before that we discuss in Sect. 3.1 the complex representation ring of

$$\Lambda_G(g) := L_g^1 G \rtimes \mathbb{T} \cong C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle, \tag{3.1}$$

which is a factor of $QEll_G^*(\mathrm{pt})$. We assume familiarity with [5,27].

3.1 Preliminary: representation ring of $\Lambda_G(g)$

Let $q : \mathbb{T} \longrightarrow U(1)$ be the isomorphism $t \mapsto e^{2\pi it}$. The complex representation ring $R\mathbb{T}$ is $\mathbb{Z}[q^{\pm}]$.

We have an exact sequence

$$1 \longrightarrow C_G(g) \longrightarrow \Lambda_G(g) \xrightarrow{\pi} \mathbb{T} \longrightarrow 0$$

where the first map is $g \mapsto [g, 0]$ and the second map is

$$\pi([g, t]) = e^{2\pi it}. \tag{3.2}$$

The map $\pi^* : R\mathbb{T} \longrightarrow R\Lambda_G(g)$ equips the representation ring $R\Lambda_G(g)$ the structure as an $R\mathbb{T}$-module.

There is a relation between the complex representation ring of $C_G(g)$ and that of $\Lambda_G(g)$, which is shown as Lemma 1.2 in [25] and Lemma 2.4.1 in [12].

Lemma 3.1 *The $R\mathbb{T}$-module $R\Lambda_G(g)$ with the action defined by $\pi^* : R\mathbb{T} \longrightarrow R\Lambda_G(g)$ is a free module.*

In particular, there is an $R\mathbb{T}$-basis of $R\Lambda_G(g)$ given by irreducible representations $\{V_\lambda\}$, such that restriction $V_\lambda \mapsto V_\lambda|_{C_G(g)}$ to $C_G(g)$ defines a bijection between $\{V_\lambda\}$ and the set $\{\lambda\}$ of irreducible representations of $C_G(g)$.

Proof Let l be the order of g. Note that $\Lambda_G(g)$ is isomorphic to

$$C_G(g) \times \mathbb{R}/l\mathbb{Z}/\langle(g, -1)\rangle.$$

Thus, it is the quotient of the product of two compact Lie groups.

Let $\lambda : C_G(g) \longrightarrow GL(n, \mathbb{C})$ be an n-dimensional $C_G(g)$-representation with representation space V and $\eta : \mathbb{R} \longrightarrow GL(n, \mathbb{C})$ be a representation of $\mathbb{R}$ such that $\lambda(g)$ acts on V via scalar multiplication by $\eta(1)$. Define a n-dimensional $\Lambda_G(g)$-representation $\lambda \odot_{\mathbb{C}} \eta$ with representation space V by

$$\lambda \odot_{\mathbb{C}} \eta([h, t]) := \lambda(h)\eta(t). \tag{3.3}$$

Any irreducible n-dimensional representation of the quotient group $\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle(g, -1)\rangle$ is an irreducible n-dimensional representation of the product $C_G(g) \times \mathbb{R}$. And any finite dimensional irreducible complex representation of the product of two compact Lie groups is the tensor product of an irreducible representation of each factor. So any irreducible representation of the quotient group $\Lambda_G(g)$ is the tensor product of an irreducible representation λ of $C_G(g)$ with representation space V and an irreducible representation η of $\mathbb{R}$. Any irreducible complex representation η of $\mathbb{R}$ is one dimensional. So the representation space of $\lambda \odot_{\mathbb{C}} \eta$ is still V. $\eta(1)^l = I$. We need $\eta(1) = \lambda(g)$. So $\eta(1) = e^{\frac{2\pi ik}{l}}$ for some $k \in \mathbb{Z}$. So

$$\eta(t) = e^{\frac{2\pi i(k+lm)t}{l}}.$$

Any $m \in \mathbb{Z}$ gives a choice of η in this case. And η is a representation of $\mathbb{R}/l\mathbb{Z} \cong \mathbb{T}$.

Therefore, we have a bijective correspondence between

1. isomorphism classes of irreducible $\Lambda_G(g)$-representation ρ, and
2. isomorphism classes of pairs (λ, η) where λ is an irreducible $C_G(g)$-representation and $\eta : \mathbb{R} \longrightarrow \mathbb{C}^*$ is a character such that $\lambda(g) = \eta(1)I$. $\lambda = \rho|_{C_G(g)}$.

Then as a corollary, the $R\mathbb{T}$-module $R\Lambda_G(g)$ with the $R\mathbb{T}$-action defined by $\pi^* : R\mathbb{T} \longrightarrow R\Lambda_G(g)$

$\pi^* : R\mathbb{T} \longrightarrow R\Lambda_G(g)$ exhibits $R\Lambda_G(g)$ as a free $R\mathbb{T}$-module. □

Remark 3.2 We can make a canonical choice of $\mathbb{Z}[q^{\pm}]$-basis for $R\Lambda_G(g)$. For each irreducible G-representation $\rho : G \longrightarrow Aut(G)$, write $\rho(\sigma) = e^{2\pi i c} id$ for $c \in [0, 1)$, and set $\chi_\rho(t) = e^{2\pi i c t}$. Then the pair (ρ, χ_ρ) corresponds to a unique irreducible $\Lambda_G(g)$-representation

$$\rho \odot_{\mathbb{C}} \chi_\rho([h, t]) := \rho(h)\chi_\rho(t). \tag{3.4}$$

Example 3.3 ($G = \mathbb{Z}/N\mathbb{Z}$) Let $G = \mathbb{Z}/N\mathbb{Z}$ for $N \geq 1$, and let $\sigma \in G$. Given an integer $k \in \mathbb{Z}$ which projects to $\sigma \in \mathbb{Z}/N\mathbb{Z}$, let x_k denote the representation of $\Lambda_G(\sigma)$ defined by

$$\Lambda_G(\sigma) = (\mathbb{Z} \times \mathbb{R})/(\mathbb{Z}(N, 0) + \mathbb{Z}(k, 1)) \xrightarrow{[a,t]\mapsto[(kt-a)/N]} \mathbb{R}/\mathbb{Z} = \mathbb{T} \xrightarrow{q} U(1). \tag{3.5}$$

$R\Lambda_G(\sigma)$ is isomorphic to the ring $\mathbb{Z}[q^{\pm}, x_k]/(x_k^N - q^k)$.

Example 3.4 ($G = \Sigma_3$) $G = \Sigma_3$ has three conjugacy classes represented by 1, (12), (123) respectively.

$\Lambda_{\Sigma_3}(1) = \Sigma_3 \times \mathbb{T}$, thus, $R\Lambda_{\Sigma_3}(1) = R\Sigma_3 \otimes R\mathbb{T} = \mathbb{Z}[X, Y]/(XY - Y, X^2 - 1, Y^2 - X - Y - 1) \otimes \mathbb{Z}[q^{\pm}]$ where X is the sign representation on Σ_3 and Y is the standard representation.

$C_{\Sigma_3}((12)) = \langle(12)\rangle = \Sigma_2$, thus, $\Lambda_{\Sigma_3}((12)) \cong \Lambda_{\Sigma_2}((12))$. So we have

$$R\Lambda_{\Sigma_3}((12)) \cong R\Lambda_{\Sigma_2}((12)) = \mathbb{Z}[q^{\pm}, x_1]/(x_1^2 - q) \cong \mathbb{Z}[q^{\pm\frac{1}{2}}].$$

$C_{\Sigma_3}(123) = \langle(123)\rangle = \mathbb{Z}/3\mathbb{Z}$, thus, $\Lambda_{\Sigma_3}((123)) \cong \Lambda_{\mathbb{Z}/3\mathbb{Z}}(1)$. So we have

$$R\Lambda_{\Sigma_3}((123)) \cong \mathbb{Z}[q^{\pm}, x_1]/(x_1^3 - q) \cong \mathbb{Z}[q^{\pm\frac{1}{3}}].$$

Moreover, we have the conclusion below about the relation between the induced representations $Ind|_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(-)$ and $Ind|_{C_H(\sigma)}^{C_G(\sigma)}(-)$.

Lemma 3.5 *Let H be a subgroup of G and σ an element of H. Let m denote $[C_G(\sigma) : C_H(\sigma)]$. Let V denote a $\Lambda_H(\sigma)$-representation $\lambda \odot_{\mathbb{C}} \chi$ with λ a $C_H(\sigma)$-representation, χ a $\mathbb{R}$-representation and $\odot_{\mathbb{C}}$ defined in (3.4).*

(i)

$$res_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \eta) = (res_{C_H(\sigma)}^{C_G(\sigma)}\lambda) \odot_{\mathbb{C}} \eta. \tag{3.6}$$

(ii) *The induced representation*

$$Ind_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \chi)$$

is isomorphic to the $\Lambda_G(\sigma)$*-representation*

$$(Ind_{C_H(\sigma)}^{C_G(\sigma)}\lambda) \odot_{\mathbb{C}} \chi.$$

Their underlying vector spaces are both $V^{\oplus m}$*.*

Thus, the computation of both $Ind_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \chi)$ *and* $res_{\Lambda_H(\sigma)}^{\Lambda_G(\sigma)}(\lambda \odot_{\mathbb{C}} \eta)$ *can be reduced to the computation of representations of finite groups.*

The proof is straightforward and left to the readers.

Let k be any integer. Next we describe the relation between

$$\Lambda_G^k(g) := L_g^k G \rtimes \mathbb{T} \cong C_G(g) \times \mathbb{R}/\langle (g, -k)\rangle \tag{3.7}$$

and $\Lambda_G(g)$, which gives the relation between their representation rings.

There is an exact sequence

$$1 \longrightarrow C_G(g) \xrightarrow{g \mapsto [g,0]} \Lambda_G^k(g) \xrightarrow{\pi_k} \mathbb{R}/k\mathbb{Z} \longrightarrow 0$$

where the second map $\pi_k : \Lambda_G^k(g) \longrightarrow \mathbb{R}/k\mathbb{Z}$ is $\pi_k([g, t]) = e^{2\pi i t}$.

Let $q^{\frac{1}{k}} : \mathbb{R}/k\mathbb{Z} \longrightarrow U(1)$ denote the composition

$$\mathbb{R}/k\mathbb{Z} \xrightarrow{t \mapsto \frac{t}{k}} \mathbb{R}/\mathbb{Z} \xrightarrow{q} U(1).$$

The representation ring $R(\mathbb{R}/k\mathbb{Z})$ is $\mathbb{Z}[q^{\pm\frac{1}{k}}]$.

Analogous to Lemma 3.1, we have the conclusion about $R\Lambda_G^k(g)$ below.

Lemma 3.6 *The map* $\pi_k^* : R(\mathbb{R}/k\mathbb{Z}) \longrightarrow R\Lambda_G^k(g)$ *exhibits it as a free* $\mathbb{Z}[q^{\pm\frac{1}{k}}]$*-module. There is a* $\mathbb{Z}[q^{\pm\frac{1}{k}}]$*-basis of* $R\Lambda_G^k(g)$ *given by irreducible representations* $\{\rho_k\}$ *such that the restrictions* $\rho_k|_{C_G(g)}$ *of them to* $C_G(g)$ *are precisely the* $\mathbb{Z}$*-basis of* $RC_G(g)$ *given by irreducible representations.*

In other words, any irreducible $\Lambda_G^k(g)$*-representation has the form* $\rho \odot_{\mathbb{C}} \chi$ *where* ρ *is an irreducible representation of* $C_G(g)$*,* $\chi : \mathbb{R}/k\mathbb{Z} \longrightarrow GL(n, \mathbb{C})$ *such that* $\chi(k) = \rho(g)$*, and*

$$\rho \odot_{\mathbb{C}} \chi([h, t]) := \rho(h)\chi(t), \textit{for any } [h, t] \in \Lambda_G^k(g). \tag{3.8}$$

$R\Lambda_G^k(g)$ is a $\mathbb{Z}[q^{\pm}]$-module via the inclusion $\mathbb{Z}[q^{\pm}] \longrightarrow \mathbb{Z}[q^{\pm\frac{1}{k}}]$.

By Lemma 3.6, we can make a $\mathbb{Z}[q^{\pm\frac{1}{k}}]$-basis $\{\rho \odot_{\mathbb{C}} \chi_{\rho,k}\}$ for $R\Lambda_G^k(g)$ with each $\rho : G \longrightarrow Aut(G)$ an irreducible G-representation and $\chi_{\rho,k}(t) = e^{2\pi i \frac{ct}{k}}$ with $c \in$

$[0, 1)$ such that $\rho(\sigma) = e^{2\pi i c} id$. This collection $\{\rho \odot_{\mathbb{C}} \chi_{\rho,k}\}$ gives a $\mathbb{Z}[q^{\pm\frac{1}{k}}]$-basis of $R\Lambda_G^k(g)$.

There is a group isomorphism $\alpha_k : \Lambda_G^k(g) \longrightarrow \Lambda_G(g)$ sending $[g, t]$ to $[g, \frac{t}{k}]$. Observe that there is a pullback square of groups

$$\begin{array}{ccc} \Lambda_G^k(g) & \xrightarrow{\alpha_k} & \Lambda_G(g) \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi} \\ \mathbb{R}/k\mathbb{Z} & \xrightarrow{t \mapsto \frac{t}{k}} & \mathbb{R}/\mathbb{Z} \end{array} \qquad (3.9)$$

So we have the commutative square of a pushout square in the category of λ-rings.

$$\begin{array}{ccc} R\Lambda_G^k(g) & \longleftarrow & R\Lambda_G(g) \\ \uparrow & & \uparrow \\ R(\mathbb{R}/k\mathbb{Z}) & \longleftarrow & R\mathbb{T} \end{array} \qquad (3.10)$$

It gives a canonical isomorphism of λ-rings $R\Lambda_G(g) \longrightarrow R\Lambda_G^k(g)$ sending q to $q^{\frac{1}{k}}$. A good reference for λ-rings is Chapters 1 and 2, [29].

3.2 Quasi-elliptic cohomology

In this section we introduce the definition of quasi-elliptic cohomology $QEll_G^*$ in terms of orbifold K-theory, and then express it via equivariant K-theory. We assume familiarity with [27]. The reader may read Chapter 3 in [3,23] for a reference of orbifold K-theory.

When G is finite, quasi-elliptic cohomology is defined from the ghost loops in Definition 2.10. By Proposition 2.11 and Example 2.12, we can see the groupoid $GhLoop(X/\!\!/G)$ is equivalent to the disjoint union of some translation groupoids. Before describing this equivalent groupoid $\Lambda(X/\!\!/G)$ in detail, we recall what inertia groupoid is. A reference for that is Section 4, [20].

Definition 3.7 Let $\mathbb{G}$ be a groupoid. The inertia groupoid $I(\mathbb{G})$ of $\mathbb{G}$ is defined as follows.

An object a is an arrow in $\mathbb{G}$ such that its source and target are equal. A morphism v joining two objects a and b is an arrow v in $\mathbb{G}$ such that

$$v \circ a = b \circ v.$$

In other words, b is the conjugate of a by v, $b = v \circ a \circ v^{-1}$.

Let X a G-space.

Example 3.8 The inertia groupoid $I(X/\!\!/G)$ is the groupoid with

objects: the space $\coprod_{g\in G} X^g$

morphisms: the space $\coprod_{g,g'\in G} C_G(g,g')\times X^g$ where $C_G(g,g')=\{\sigma\in G|g'\sigma=\sigma g\}\subseteq G$.

For $x\in X^g$ and $(\sigma,g)\in C_G(g,g')\times X^g$, $(\sigma,g)(x)=\sigma x\in X^{g'}$.

Definition 3.9 The groupoid $\Lambda(X/\!\!/G)$ has the same objects as $I(X/\!\!/G)$ but richer morphisms

$$\coprod_{g,g'\in G}\Lambda_G(g,g')\times X^g$$

where $\Lambda_G(g,g')$ is the quotient of $C_G(g,g')\times\mathbb{R}$ under the equivalence

$$(x,t)\sim(gx,t-1)=(xg',t-1).$$

For an object $x\in X^g$ and a morphism $([\sigma,t],g)\in\Lambda_G(g,g')\times X^g$, $([\sigma,t],g)(x)=\sigma x\in X^{g'}$. The composition of the morphisms is defined by

$$[\sigma_1,t_1][\sigma_2,t_2]=[\sigma_1\sigma_2,t_1+t_2]. \tag{3.11}$$

Definition 3.10 The quasi-elliptic cohomology $QEll^*_G(X)$ is defined to be $K^*_{orb}(GhLoop(X/\!\!/G))\cong K^*_{orb}(\Lambda(X/\!\!/G))$.

We can unravel the definition and express it via equivariant K-theory.

Let $\sigma\in G$. The fixed point space X^σ is a $C_G(\sigma)$-space. We can define a $\Lambda_G(\sigma)$-action on X^σ by

$$[g,t]\cdot x:=g\cdot x.$$

Then we have

Proposition 3.11

$$QEll^*_G(X)=\prod_{g\in G_{conj}}K^*_{\Lambda_G(g)}(X^g)=\left(\prod_{g\in G}K^*_{\Lambda_G(g)}(X^g)\right)^G, \tag{3.12}$$

where G_{conj} is a set of representatives of G-conjugacy classes in G.

Thus, for each $g\in\Lambda_G(g)$, we can define the projection

$$\pi_g: QEll^*_G(X)\longrightarrow K^*_{\Lambda_G(g)}(X^g).$$

For the singe point space, we have

$$QEll^0_G(\text{pt})\cong\prod_{g\in G_{conj}}R\Lambda_G(g). \tag{3.13}$$

We have the ring homomorphism

$$\mathbb{Z}[q^{\pm}] = K^0_{\mathbb{T}}(\text{pt}) \xrightarrow{\pi^*} K^0_{\Lambda_G(g)}(\text{pt}) \longrightarrow K^0_{\Lambda_G(g)}(X)$$

where $\pi : \Lambda_G(g) \longrightarrow \mathbb{T}$ is the projection defined in (3.2) and the second is via the collapsing map $X \longrightarrow \text{pt}$. So $QEll^*_G(X)$ is naturally a $\mathbb{Z}[q^{\pm}]$-algebra.

3.3 Properties

In this section we discuss some properties of $QEll^*_G$, including the restriction map, the Künneth map on it, its tensor product and the change-of-group isomorphism.

Since each homomorphism $\phi : G \longrightarrow H$ induces a well-defined homomorphism $\phi_\Lambda : \Lambda_G(\tau) \longrightarrow \Lambda_H(\phi(\tau))$ for each τ in G, we can get the proposition below directly.

Proposition 3.12 *For each homomorphism $\phi : G \longrightarrow H$, it induces a ring map*

$$\phi^* : QEll^*_H(X) \longrightarrow QEll^*_G(\phi^* X)$$

characterized by the commutative diagrams

$$\begin{array}{ccc} QEll^*_H(X) & \xrightarrow{\phi^*} & QEll^*_G(\phi^* X) \\ \pi_{\phi(\tau)} \downarrow & & \pi_\tau \downarrow \\ K^*_{\Lambda_H(\phi(\tau))}(X^{\phi(\tau)}) & \xrightarrow{\phi^*_\Lambda} & K^*_{\Lambda_G(\tau)}(X^{\phi(\tau)}) \end{array} \tag{3.14}$$

for any $\tau \in G$. So $QEll^*_G$ is functorial in G.

Moreover, we can define Künneth map of quasi-elliptic cohomology induced from that on equivariant K-theory.

Let G and H be two finite groups. X is a G-space and Y is a H-space. Let $\sigma \in G$ and $\tau \in H$. Let $\Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$ denote the fibered product of the morphisms

$$\Lambda_G(\sigma) \xrightarrow{\pi} \mathbb{T} \xleftarrow{\pi} \Lambda_H(\tau).$$

It is isomorphic to $\Lambda_{G\times H}(\sigma, \tau)$ under the correspondence

$$([\alpha, t], [\beta, t]) \mapsto [\alpha, \beta, t].$$

Consider the composition below

$$\begin{aligned} T : & K_{\Lambda_G(\sigma)}(X^\sigma) \otimes K_{\Lambda_H(\tau)}(Y^\tau) \longrightarrow K_{\Lambda_G(\sigma)\times\Lambda_H(\tau)}(X^\sigma \times Y^\tau) \xrightarrow{res} \\ & K_{\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)}(X^\sigma \times Y^\tau) \xrightarrow{\cong} K_{\Lambda_{G\times H}(\sigma,\tau)}((X \times Y)^{(\sigma,\tau)}), \end{aligned}$$

where the first map is the Künneth map of equivariant K-theory, the second is the restriction map and the third is the isomorphism induced by the group isomorphism $\Lambda_{G\times H}(\sigma,\tau)\cong\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)$.

For any $g\in G$, let 1 denote the trivial line bundle over X^g and let q denote the line bundle $1\odot_{\mathbb{C}} q$ over X^g. The map T above sends both $1\otimes q$ and $q\otimes 1$ to q. So we get the well-defined map

$$K^*_{\Lambda_G(\sigma)}(X^\sigma)\otimes_{\mathbb{Z}[q^\pm]}K^*_{\Lambda_H(\tau)}(Y^\tau)\longrightarrow K_{\Lambda_{G\times H}(\sigma,\tau)}((X\times Y)^{(\sigma,\tau)}). \tag{3.15}$$

Definition 3.13 The tensor produce of quasi-elliptic cohomology is defined by

$$QEll^*_G(X)\otimes_{\mathbb{Z}[q^\pm]}QEll^*_H(Y)\cong\prod_{\sigma\in G_{conj},\ \tau\in H_{conj}}K^*_{\Lambda_G(\sigma)}(X^\sigma)\otimes_{\mathbb{Z}[q^\pm]}K^*_{\Lambda_H(\tau)}(Y^\tau). \tag{3.16}$$

The direct product of the maps defined in (3.15) gives a ring homomorphism

$$QEll^*_G(X)\otimes_{\mathbb{Z}[q^\pm]}QEll^*_H(Y)\longrightarrow QEll^*_{G\times H}(X\times Y),$$

which is the Künneth map of quasi-elliptic cohomology.

By Lemma 3.1 we have

$$QEll^*_G(\mathrm{pt})\otimes_{\mathbb{Z}[q^\pm]}QEll^*_H(\mathrm{pt})=QEll^*_{G\times H}(\mathrm{pt}).$$

More generally, we have the proposition below.

Proposition 3.14 *Let X be a $G\times H$-space with trivial H-action and let pt be the single point space with trivial H-action. Then we have*

$$QEll_{G\times H}(X)\cong QEll_G(X)\otimes_{\mathbb{Z}[q^\pm]}QEll_H(pt).$$

Especially, if G acts trivially on X, we have

$$QEll_G(X)\cong QEll(X)\otimes_{\mathbb{Z}[q^\pm]}QEll_G(pt).$$

Here $QEll^*(X)$ is $QEll^*_{\{e\}}(X)=K^*_{\mathbb{T}}(X)$.

Proof

$$\begin{aligned}QEll_{G\times H}(X)&=\prod_{\substack{g\in G_{conj}\\ h\in H_{conj}}}K_{\Lambda_{G\times H}(g,h)}(X^{(g,h)})\cong\prod_{\substack{g\in G_{conj}\\ h\in H_{conj}}}K_{\Lambda_G(g)\times_{\mathbb{T}}\Lambda_H(h)}(X^g)\\&\cong\prod_{\substack{g\in G_{conj}\\ h\in H_{conj}}}K_{\Lambda_G(g)}(X^g)\otimes_{\mathbb{Z}[q^\pm]}K_{\Lambda_H(h)}(\mathrm{pt})=QEll_G(X)\otimes_{\mathbb{Z}[q^\pm]}QEll_H(\mathrm{pt}).\end{aligned}$$

□

Proposition 3.15 *If G acts freely on X,*

$$QEll_G^*(X) \cong QEll_e^*(X/G).$$

Proof Since G acts freely on X,

$$X^\sigma = \begin{cases} \emptyset, & \text{if } \sigma \neq e; \\ X, & \text{if } \sigma = e. \end{cases}$$

Thus, $QEll_G^*(X) \cong \prod_{\sigma \in G_{conj}} K^*_{\Lambda_G(\sigma)/C_G(\sigma)}(X^\sigma / C_G(\sigma)) \cong K^*_{\mathbb{T}}(X/G)$.

Since $\mathbb{T}$ acts trivially on X, we have $K^*_{\mathbb{T}}(X/G) = QEll_e^*(X/G)$ by definition. It is isomorphic to $K^*(X/G) \otimes R\mathbb{T}$. □

We also have the change-of-group isomorphism as in equivariant K-theory.

Let H be a subgroup of G and X a H-space. Let $\phi : H \longrightarrow G$ denote the inclusion homomorphism. The change-of-group map $\rho_H^G : QEll_G^*(G \times_H X) \longrightarrow QEll_H^*(X)$ is defined as the composite

$$\rho_H^G : QEll_G^*(G \times_H X) \xrightarrow{\phi^*} QEll_H^*(G \times_H X) \xrightarrow{i^*} QEll_H^*(X) \tag{3.17}$$

where ϕ^* is the restriction map and $i : X \longrightarrow G \times_H X$ is the H-equivariant map defined by $i(x) = [e, x]$.

Proposition 3.16 *The change-of-group map*

$$\rho_H^G : QEll_G^*(G \times_H X) \longrightarrow QEll_H^*(X)$$

defined in (3.17) *is an isomorphism.*

Proof For any $\tau \in H_{conj}$, there exists a unique $\sigma_\tau \in G_{conj}$ such that $\tau = g_\tau \sigma_\tau g_\tau^{-1}$ for some $g_\tau \in G$. Consider the maps

$$\Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau \xrightarrow{[[a,t],x] \mapsto [a,x]} (G \times_H X)^\tau \xrightarrow{[u,x] \mapsto [g_\tau^{-1} u, x]} (G \times_H X)^\sigma. \tag{3.18}$$

The first map is $\Lambda_G(\tau)$-equivariant and the second is equivariant with respect to the homomorphism $c_{g_\tau} : \Lambda_G(\sigma) \longrightarrow \Lambda_G(\tau)$ sending $[u, t] \mapsto [g_\tau u g_\tau^{-1}, t]$. Taking a coproduct over all the elements $\tau \in H_{conj}$ that are conjugate to $\sigma \in G_{conj}$ in G, we get an isomorphism

$$\gamma_\sigma : \coprod_\tau \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau \longrightarrow (G \times_H X)^\sigma$$

which is $\Lambda_G(\sigma)$-equivariant with respect to c_{g_τ}. Then we have the map

$$\gamma := \prod_{\sigma \in G_{conj}} \gamma_\sigma : \prod_{\sigma \in G_{conj}} K^*_{\Lambda_G(\sigma)}(G \times_H X)^\sigma \longrightarrow \prod_{\sigma \in G_{conj}} K^*_{\Lambda_G(\sigma)} \left(\coprod_\tau \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^\tau \right) \tag{3.19}$$

It is straightforward to check the change-of-group map coincide with the composite

$$QEll^*_G(G \times_H X) \xrightarrow{\gamma} \prod_{\sigma \in G_{conj}} K^*_{\Lambda_G(\sigma)}\left(\coprod_{\tau} \Lambda_G(\tau) \times_{\Lambda_H(\tau)} X^{\tau}\right) \longrightarrow \prod_{\tau \in H_{conj}} K^*_{\Lambda_H(\tau)}(X^{\tau})$$
$$= QEll^*_H(X)$$

with the second map the change-of-group isomorphism in equivariant K-theory. □

4 Power operation

In Sect. 4.2 we define power operations for equivariant quasi-elliptic cohomology $QEll^*_G(-)$. We show in Theorem 4.12 that they satisfy the axioms that Ganter established in Definition 4.3, [9] for equivariant power operations.

The power operation of quasi-elliptic cohomology is of the form

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)_{conj}} \mathbb{P}_{(\underline{g},\sigma)} :$$
$$QEll^*_G(X) \longrightarrow QEll^*_{G\wr\Sigma_n}(X^{\times n}) = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)_{conj}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}),$$

where $\mathbb{P}_n$ maps a bundle over the groupoid

$$\Lambda(X/\!\!/G)$$

to a bundle over

$$\Lambda(X^{\times n}/\!\!/(G \wr \Sigma_n)),$$

and each $\mathbb{P}_{(\underline{g},\sigma)}$ maps a bundle over

$$\Lambda(X/\!\!/G)$$

to a $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$-bundle over the space $(X^{\times n})^{(\underline{g},\sigma)}/\!\!/\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$.

We construct each $\mathbb{P}_{(\underline{g},\sigma)}$ as the composition below.

$$QEll^*_G(X) \xrightarrow{U^*} K^*_{orb}(\Lambda^1_{(\underline{g},\sigma)}(X)) \xrightarrow{(\,)^{\Lambda}_k} K^*_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X)) \tag{4.1}$$

$$\xrightarrow{\boxtimes} K^*_{orb}(d_{(\underline{g},\sigma)}(X)) \xrightarrow{f^*_{(\underline{g},\sigma)}} K^*_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}), \tag{4.2}$$

where $k \in \mathbb{Z}$ and $(i_1, \ldots i_k)$ goes over all the k-cycles of σ. We explain the first three functors in detail in Sect. 4.2. In Sect. 4.1 we construct the isomorphism $f_{(\underline{g},\sigma)}$ between the groupoid

$$\Lambda(X^{\times n}/\!\!/(G \wr \Sigma_n))$$

and the groupoid $d((X/\!\!/G)\wr\Sigma_n)$ constructed in Definition 4.5. With it, it is convenient to construct the explicit formula of the power operation.

4.1 Loop space of symmetric power

4.1.1 The groupoid $d((X/\!\!/G) \wr \Sigma_n)$

For an introduction of actions of wreath product $G \wr \Sigma_n$ on $X^{\times n}$ and symmetric power $\mathbb{G}\wr\Sigma_n$ of a groupoid $\mathbb{G}$, we refer the readers to Section 4.1, [10]. The symmetric power $(X/\!\!/G) \wr \Sigma_n$ is isomorphic to $X^{\times n}/\!\!/(G \wr \Sigma_n)$.

Before introducing the groupoid $d((X/\!\!/G) \wr \Sigma_n)$, we need to introduce several ingredients.

Definition 4.1 $(\Lambda^k(X/\!\!/G))$ The groupoid $\Lambda^k(X/\!\!/G)$ has the same objects as $\Lambda(X/\!\!/G)$ but different morphisms

$$\coprod_{g,g'\in G} \Lambda^k_G(g, g') \times X^g$$

where $\Lambda^k_G(g, g')$ is the quotient of $C_G(g, g') \times \mathbb{R}$ under the equivalence

$$(x, t) \sim (gx, t - k) = (xg', t - k).$$

For an object $x \in X^g$ and a morphism $([\sigma, t], g) \in \Lambda^k_G(g, g') \times X^g$, $([\sigma, t], g)(x) = \sigma x \in X^{g'}$. The composition of the morphisms is defined by

$$[\sigma_1, t_1][\sigma_2, t_2] = [\sigma_1\sigma_2, t_1 + t_2]. \tag{4.3}$$

Definition 4.2 (*Fibred wreath product*) The groupoid $\Lambda^k(X/\!\!/G) \wr_{\mathbb{T}} \Sigma_N$ is defined to be the subgroupoid of the symmetric power $\Lambda^k(X/\!\!/G)\wr\Sigma_N$ with the same objects but only those morphisms

$$(([b_1, t_1], \cdots [b_N, t_N], \tau), x)$$

with all the t_js having the same image under the quotient map $\mathbb{R}/k\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$.

The isotropy group of each object in $\prod_1^N X^g$ is $\Lambda^k_G(g) \wr_{\mathbb{T}} \Sigma_N$.

Let Y be an H-space.

Definition 4.3 (Fibred product and fibred coproduct) The groupoid

$$\left(\Lambda^{k_1}(X/\!\!/G) \wr_{\mathbb{T}} \Sigma_{N_1}\right) \times_{\mathbb{T}} \left(\Lambda^{k_2}(Y/\!\!/H) \wr_{\mathbb{T}} \Sigma_{N_2}\right)$$

is defined to be the subgroupoid of $\Lambda^{k_1}(X/\!\!/G) \wr_{\mathbb{T}} \Sigma_{N_1} \times \Lambda^{k_2}(Y/\!\!/H) \wr_{\mathbb{T}} \Sigma_{N_2}$ with the same objects but only those morphisms

$$\big((([g_1, t_{1,1}], \ldots [g_{N_1}, t_{1,N_1}], \sigma_1), x), (([h_1, t_{2,1}], \ldots [h_{N_2}, t_{2,N_2}], \sigma_2), y)\big)$$

with all the t_{i,j_i}s having the same image under the quotient map $\mathbb{R}/k_i\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$, for $i = 1, 2$ and $j_i = 1, \ldots N_i$.

The isotropy group of each object in $\prod_1^{N_1} X^g \prod_1^{N_2} Y^h$ is

$$\big(\Lambda_G^{k_1}(g) \wr_{\mathbb{T}} \Sigma_{N_1}\big) \times_{\mathbb{T}} \big(\Lambda_H^{k_2}(h) \wr_{\mathbb{T}} \Sigma_{N_2}\big).$$

We can define the fibred coproduct $\big(\Lambda^{k_1}(X/\!\!/G) \wr_{\mathbb{T}} \Sigma_{N_1}\big) \coprod_{\mathbb{T}} \big(\Lambda^{k_2}(Y/\!\!/H) \wr_{\mathbb{T}} \Sigma_{N_2}\big)$ in the same way.

Let $\sigma \in \Sigma_n$ correspond to the partition $n = \sum_k kN_k$, i.e. it has N_k k-cycles. Assume that for each cycle $(i_1, \ldots i_k)$ of σ, $i_1 < i_2 \cdots < i_k$.

For $(\underline{g}, \sigma) \in G \wr \Sigma_n$, we consider the orbits of the bundle $G \times \underline{\underline{n}} \longrightarrow \underline{\underline{n}}$ under the action by $(\underline{g}, \sigma)$. The orbits of $\underline{\underline{n}}$ under the action by σ corresponds to the cycles in the cycle decomposition of σ. The bundle $G \times \underline{\underline{n}} \longrightarrow \underline{\underline{n}}$ is the disjoint union of the G-bundles

$$\coprod_{(i_1 \ldots i_k)} (G \times \{i_1, \ldots i_k\} \longrightarrow \{i_1, \ldots i_k\})$$

where $(i_1, \ldots i_k)$ goes over all the cycles of σ. Each bundle $G \times \{i_1, \ldots i_k\} \longrightarrow \{i_1, \ldots i_k\}$ is an orbit of $G \times \underline{\underline{n}} \longrightarrow \underline{\underline{n}}$ under the action by $(\underline{g}, \sigma)$.

Let $C_G(g, g')$ denote $\{x \in G | gx = xg'\}$. Two G-subbundles

$$G \times \{i_1, \ldots i_k\} \longrightarrow \{i_1, \ldots i_k\} \text{ and } G \times \{j_1, \ldots j_m\} \longrightarrow \{j_1, \ldots j_m\}$$

are $(\underline{g}, \sigma)$-equivariant equivalent if and only if $k = m$ and $C_G(g_{i_k} \cdots g_{i_1}, g_{j_k} \cdots g_{j_1})$ is nonempty. For each k-cycle $i = (i_1, \ldots i_k)$ of σ, let W_i^σ denote the set of all the G-subbundles $G \times \{j_1, \ldots j_m\} \longrightarrow \{j_1, \ldots j_m\}$ that are $(\underline{g}, \sigma)$-isomorphic to $G \times \{i_1, \ldots i_k\} \longrightarrow \{i_1, \ldots i_k\}$. There is a bijection between W_i^σ and the set

$$\{j = (j_1, \ldots j_k) \mid (j_1, \ldots j_k) \text{ is a k-cycle of } \sigma \text{ and } C_G(g_{i_k} \cdots g_{i_1}, g_{j_k} \cdots g_{j_1}) \text{ is nonempty}\}.$$

Let M_i^σ denote the size of the set W_i^σ. Let $\alpha_1^i, \alpha_2^i, \ldots \alpha_{M_i^\sigma}^i$ denote all the elements of the set W_i^σ. Obviously, $i = (i_1, \ldots i_k)$ is in W_i^σ. So we can assume it is α_1^i.

For any k-cycle i and m-cycle j of σ, if $k = m$ and $C_G(g_{i_k} \cdots g_{i_1}, g_{j_k} \cdots g_{j_1})$ is nonempty, W_i^σ and W_j^σ are the same set. Otherwise, they are disjoint. The set of all the k-cycles of σ can be divided into the disjoint union of several W_i^σs. We can pick a set of representatives θ_k of k-cycles of σ such that the set of k-cycles of σ equals the disjoint union

$$\coprod_{i \in \theta_k} W_i^{\sigma}.$$

Definition 4.4 $(d_{(\underline{g},\sigma)}(X))$ The groupoid $d_{(\underline{g},\sigma)}(X)$ is defined to be a full subgroupoid of $\prod_{k}{}^{\mathbb{T}} \prod_{i \in \theta_k}{}^{\mathbb{T}} \Lambda^k(X /\!\!/ G) \wr_{\mathbb{T}} \Sigma_{M_i^{\sigma}}$ with objects the points of the space

$$\prod_k \prod_{(i_1,\dots i_k)} X^{g_{i_k}\cdots g_{i_1}},$$

where the second product goes over all the k-cycles of σ.

Definition 4.5 $(d((X /\!\!/ G) \wr \Sigma_n))$ The groupoid $d((X /\!\!/ G) \wr \Sigma_n)$ is defined to be

$$\coprod_{(\underline{g},\sigma)}{}^{\mathbb{T}} d_{(\underline{g},\sigma)}(X)$$

where $(\underline{g}, \sigma)$ goes over $(G \wr \Sigma_n)_{conj}$.

Proposition 4.6 *Each $d_{(\underline{g},\sigma)}(X)$ is isomorphic to the translation groupoid*

$$\left(\prod_k \prod_{(i_1,\dots i_k)} X^{g_{i_k}\cdots g_{i_1}}\right) /\!\!/ \left(\prod_k{}^{\mathbb{T}} \prod_{j \in \theta_k}{}^{\mathbb{T}} \Lambda_G^k(\alpha_j) \wr_{\mathbb{T}} \Sigma_{M_j^{\sigma}}\right)$$

where $\alpha_j = g_{j_k} \dots g_{j_1}$ with $j = (j_1, \dots j_k)$.

The proof is straightforward.

To study $K_{orb}(d_{(\underline{g},\sigma)}(X))$, we start by studying the representation ring of the wreath product

$$\prod_k \prod_{j \in \theta_k} \Lambda_G^k(\alpha_j) \wr \Sigma_{M_j^{\sigma}}.$$

Theorem 4.7 gives all the irreducible representations of a wreath product. It is Theorem Theorem 4.3.34 in [15].

Theorem 4.7 *Let $\{\rho_k\}_1^N$ be a complete family of irreducible representations of G and let V_k be the corresponding representation space for ρ_k. Let (n) be a partition of n. $(n) = (n_1, \dots n_N)$. Let $D_{(n)}$ be the representation*

$$\rho_1^{\otimes n_1} \otimes \cdots \otimes \rho_N^{\otimes n_N}$$

of $G^{\times N}$ on $V_1^{\otimes n_1} \otimes \cdots \otimes V_N^{\otimes n_N}$. Let $\Sigma_{(n)} = \Sigma_{n_1} \times \cdots \times \Sigma_{n_N}$.

Let $(D_{(n)})^{\sim}$ be the extension of $D_{(n)}$ from $G^{\times n}$ to $G \wr \Sigma_{(n)}$ defined by

$$(D_{(n)})^{\sim}((g_{1,1}, \dots g_{1,n_1}, \dots g_{N,1}, \dots g_{N,n_N}; \sigma))$$

$$(v_{1,1}\otimes\cdots\otimes v_{1,n_1}\otimes\cdots\otimes v_{N,1}\otimes\cdots\otimes v_{N,n_N})$$
$$=\bigotimes_{k=1}^{N}\rho_k(g_{k,1})v_{k,\sigma_k^{-1}(1)}\otimes\cdots\otimes\rho_k(g_{k,n_k})v_{k,\sigma_k^{-1}(n_k)},$$

where $\sigma=\sigma_1\times\cdots\times\sigma_N$ with each $\sigma_k\in\Sigma_{n_k}$.

Let D_τ with $\tau\in R\Sigma_{(n)}$ be the representation of $G\wr\Sigma_{(n)}$ defined by

$$D_\tau((g_{1,n_1},\dots g_{N,n_N};\sigma)):=\tau(\sigma). \tag{4.4}$$

Then,

$$\{Ind|_{G\wr\Sigma_{(n)}}^{G\wr\Sigma_n}(D_{(n)})^{\sim}\otimes D_\tau|(n)=(n_1,\dots n_N)\ \textit{goes over all the partitions};$$
$$\tau\ \textit{goes over all the irreducible representations of}\ \Sigma_{(n)}.\}$$

goes over all the irreducible representations of $G\wr\Sigma_n$ nonrepeatedly.

The proof of Theorem 4.8 is analogous to that of Theorem 4.7 in [15], applying Clifford's theory in [7,8]. Note that

$$\{\rho_1\otimes_{\mathbb{Z}[q^\pm]}\cdots\otimes_{\mathbb{Z}[q^\pm]}\rho_n\mid \text{Each }\rho_j\text{ is an irreducible representation of }\Lambda_G(\sigma).\}$$

goes over all the irreducible representations of the fibred product

$$\Lambda_G(\sigma)\times_{\mathbb{T}}\cdots\times_{\mathbb{T}}\Lambda_G(\sigma).$$

Theorem 4.8 *Let $\{\rho_k\}_1^N$ be a basis of the $\mathbb{Z}[q^\pm]$-module $R\Lambda_G(\sigma)$ and let V_k be the corresponding representation space for ρ_k. Let (n) be a partition of n. $(n)=(n_1,\dots n_N)$. Let $D_{(n)}^{\mathbb{T}}$ be the $\Lambda_G(\sigma)^{\times_{\mathbb{T}}n}$-representation*

$$\rho_1^{\otimes_{\mathbb{Z}[q^\pm]}n_1}\otimes_{\mathbb{Z}[q^\pm]}\cdots\otimes_{\mathbb{Z}[q^\pm]}\rho_N^{\otimes_{\mathbb{Z}[q^\pm]}n_N}$$

on the space $V_1^{\otimes n_1}\otimes\cdots\otimes V_N^{\otimes n_N}$. Let $\Sigma_{(n)}=\Sigma_{n_1}\times\cdots\times\Sigma_{n_N}$.

Let $(D_{(n)}^{\mathbb{T}})^{\sim}$ be the extension of $D_{(n)}$ from $\Lambda_G(\sigma)^{\times_{\mathbb{T}}n}$ to $\Lambda_G(\sigma)\wr_{\mathbb{T}}\Sigma_{(n)}$ defined by

$$(D_{(n)}^{\mathbb{T}})^{\sim}(([g_{1,1},t],\dots[g_{1,n_1},t],\dots[g_{N,1},t],\dots[g_{N,n_N},t];\sigma))$$
$$(v_{1,1}\otimes\cdots\otimes v_{1,n_1}\otimes\cdots\otimes v_{N,1}\otimes\cdots\otimes v_{N,n_N})$$
$$=\bigotimes_{\mathbb{Z}[q^\pm]}\rho_k([g_{k,1},t])v_{k,\sigma_k^{-1}(1)}\otimes_{\mathbb{Z}[q^\pm]}\cdots\otimes_{\mathbb{Z}[q^\pm]}\rho_k([g_{k,n_k},t])v_{k,\sigma_k^{-1}(n_k)},$$

where k is from 1 to N and $\sigma=\sigma_1\times\cdots\times\sigma_N$ with each $\sigma_k\in\Sigma_{n_k}$.

Let $D_\tau^{\mathbb{T}}$ with $\tau\in R\Sigma_{(n)}$ be the representation of $\Lambda_G(\sigma)\wr_{\mathbb{T}}\Sigma_{(n)}$ defined by

$$D_\tau^{\mathbb{T}}(([g_{1,n_1},t],\dots[g_{N,n_N},t];\sigma)):=\tau(\sigma). \tag{4.5}$$

Then,

$$\{Ind|^{\Lambda_G(\sigma)\wr_{\mathbb{T}}\Sigma_n}_{\Lambda_G(\sigma)\wr_{\mathbb{T}}\Sigma_{(n)}}(D^{\mathbb{T}}_{(n)})^{\sim}\otimes D^{\mathbb{T}}_{\tau} \mid (n)=(n_1,\dots n_N) \text{ goes over all the partitions};$$
$$\tau \text{ goes over all the irreducible representations of } \Sigma_{(n)}.\}$$

goes over all the irreducible representation nonrepeatedly of $\Lambda_G(\sigma)\wr_{\mathbb{T}}\Sigma_n$.

From Theorem 4.7, the representation ring of each $\Lambda^k_G(\alpha_j)\wr\Sigma_{M^\sigma_j}$ is a $\mathbb{Z}[q^{\pm\frac{1}{k}}]$-module. Thus, the representation ring of each $\Lambda^k_G(\alpha_j)\wr\Sigma_{M^\sigma_j}$ is a $\mathbb{Z}[q^{\pm}]$-module via the map

$$\mathbb{Z}[q^{\pm}]\longrightarrow\mathbb{Z}[q^{\pm\frac{1}{k}}],\quad q\mapsto q^{\pm\frac{1}{k}}.$$

The representation ring

$$R(\prod_k\prod_{j\in\theta_k}\Lambda^k_G(\alpha_j)\wr\Sigma_{M^\sigma_j})\cong\bigotimes_k\bigotimes_{j\in\theta_k}R(\Lambda^k_G(\alpha_j)\wr\Sigma_{M^\sigma_j})$$

is a $\mathbb{Z}[q^{\pm}]$-module. So is $R(\prod_k{}_{\mathbb{T}}\prod_{j\in\theta_k}{}_{\mathbb{T}}\Lambda^k_G(\alpha_j)\wr_{\mathbb{T}}\Sigma_{M^\sigma_j})$.

Moreover, $K_{orb}(d_{(\underline{g},\sigma)}(X))$ is a $\mathbb{Z}[q^{\pm}]$-module via the map

$$R\left(\prod_k{}_{\mathbb{T}}\prod_{j\in\theta_k}{}_{\mathbb{T}}\Lambda^k_G(\alpha_j)\wr_{\mathbb{T}}\Sigma_{M^\sigma_j}\right)\cong K^0_{orb}(d_{(\underline{g},\sigma)}(\mathrm{pt}))\longrightarrow K^0_{orb}(d_{(\underline{g},\sigma)}(X)),\quad (4.6)$$

which is induced by $X\longrightarrow\mathrm{pt}$.

4.1.2 The isomorphism $f_{(\underline{g},\sigma)}$

Before we show in Theorem 4.10 that the groupoids $\Lambda(X^{\times n}/\!\!/(G\wr\Sigma_n))$ and $d((X/\!\!/G)\wr\Sigma_n)$ are isomorphic, we recall some properties of $C_{G\wr\Sigma_n}((\underline{g},\sigma),(\underline{g}',\sigma'))$.

$(\underline{h},\tau)$ is in $C_{G\wr\Sigma_n}((\underline{g},\sigma),(\underline{g}',\sigma'))$ if and only if $\tau\sigma'=\sigma\tau$ and $g_{\sigma(\tau(i))}h_{\tau(i)}=h_{\tau(\sigma'(i))}g'_{\sigma'(i)}$, $\forall i$. We can reinterpret these two conditions. Since $\tau\in C_{\Sigma_n}(\sigma,\sigma')$, τ maps a k-cycle $i=(i_1,\dots i_k)$ of σ' to a k-cycle $j=(j_1,\dots j_k)$ of σ. τ will still used to denote its map on the cycles, such as $\tau(r)=s$. For each $l\in\mathbb{Z}/k\mathbb{Z}$, let $\tau(i_l)=j_{l+m_i}$ where m_i depends only on τ and the cycle i. Then, the second conditions can be expressed as

$$\forall l\in\mathbb{Z}/k\mathbb{Z},\ g_{j_l}h_{j_{l-1}}=h_{j_l}g'_{i_{l-m_i}}.\quad (4.7)$$

From this equivalence, we can induce that the element

$$h_{j_k}g'^{-1}_{i_{1-m_i}}\dots g'^{-1}_{i_{k-1}}g'^{-1}_{i_k}=g^{-1}_{j_1}\dots g^{-1}_{j_{m_i}}h_{j_{m_i}}$$

maps $g_{j_k}\dots g_{j_1}$ to $g'_{i_k}\dots g'_{i_1}$ by conjugation. In other words,

$$\beta^{\underline{h},\tau}_{j,i} := h_{j_k} g'^{-1}_{i_{1-m_i}} \dots g'^{-1}_{i_{k-1}} g'^{-1}_{i_k} \tag{4.8}$$

is an element in $C_G(g_{j_k}\dots g_{j_1}, g'_{i_k}\dots g'_{i_1})$. Thus, $C_G(g_{j_k}\dots g_{j_1}, g'_{i_k}\dots g'_{i_1})$ is nonempty.

First we show each component $(X^{\times n})^{(\underline{g},\sigma)}/\!\!/\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$ is isomorphic to the groupoid $d_{(\underline{g},\sigma)}(X)$. We construct a functor

$$f_{(\underline{g},\sigma)} : (X^{\times n})^{(\underline{g},\sigma)}/\!\!/\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma) \longrightarrow d_{(\underline{g},\sigma)}(X).$$

It sends a point

$$x = (x_1,\dots x_n) \in (X^{\times n})^{(\underline{g},\sigma)}$$

to

$$\prod_k \prod_{(i_1,\dots i_k)} x_{i_k}.$$

Note that $x_{i_k} = x_{i_1} g_{i_1} = \cdots = x_{i_{k-1}} g_{i_{k-1}} \dots g_{i_1}$.

Let $[(\underline{h},\tau), t] \in \Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$. Let τ send the k-cycle $i = (i_1,\dots i_k)$ of σ to a k-cycle $j = (j_1,\dots j_k)$ of σ and $\tau(i_1) = j_{1+m_i}$. We have

$$f_{(\underline{g},\sigma)}(\gamma\cdot[(\underline{h},\tau),t_0]) = \prod_k \prod_{(i_1,\dots i_k)} x_{j_{m_i}} h_{j_{m_i}} = \prod_k \prod_{(i_1,\dots i_k)} x_{j_k}\cdot\beta^{\underline{h},\tau}_{j,i},$$

where $\beta^{\underline{h},\tau}_{j,i}$ is the symbol defined in (4.8). So $f_{(\underline{g},\sigma)}$ maps the morphism $[(\underline{h},\tau),t]$ to

$$\times_k \times_{i\in\theta_k} ([\beta^{\underline{h},\tau}_{\tau(1),1}, m_1 + t], \dots [\beta^{\underline{h},\tau}_{\tau(M^\sigma_i),M^\sigma_i}, m_{M^\sigma_i} + t], \tau|_{W^\sigma_i})$$

where $\tau|_{W^\sigma_i}$ denotes the permutation induced by τ on the set $W^\sigma_i = \{\alpha^i_1, \alpha^i_2, \dots \alpha^i_{M^\sigma_i}\}$, $\tau^{-1}(j)$ is short for $\tau^{-1}(\alpha^i_j)$ and $\tau(j_l) = \tau(j)_{l+m_j}$.

It sends the identity map $[(1,\dots,1,\mathrm{Id}),0]$ to the identity

$$\times_k \times_{i\in\theta_k} ([1,0],\dots[1,0],\mathrm{Id}),$$

and preserves composition of morphisms. So it is well-defined.

Theorem 4.9 *The two groupoids $(X^{\times n})^{(\underline{g},\sigma)}/\!\!/\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$ and $d_{(\underline{g},\sigma)}(X)$ are isomorphic. Thus, this isomorphism induces a $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$-action on the space*

$$\prod_k \prod_{(i_1,\dots i_k)} X^{g_{i_k}\cdots g_{i_1}}.$$

Proof We construct the inverse functor

$$J_{(\underline{g},\sigma)} : d_{(\underline{g},\sigma)}(X) \longrightarrow (X^{\times n})^{(\underline{g},\sigma)} /\!\!/ \Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$$

of $f_{(\underline{g},\sigma)}$. For an object $\times_k \times_{i\in\theta_k} \nu_{i,k}$ in $d_{(\underline{g},\sigma)}(X)$, $J_{(\underline{g},\sigma)}(\times_k \times_{i\in\theta_k} \nu_{i,k}) = \{\nu_m\}_1^n$ with $\nu_{i_k} = \nu_{i,k}|_{[0,1]}$ and $\nu_{i_s}(t) := \nu_{i,k}(s+t)g_{i_1}^{-1}\dots g_{i_s}^{-1}$.

Let

$$\prod_k \prod_{i\in\theta_k} ((u_1^i, m_1'^i), (u_2^i, m_2'^i), \dots (u^i_{M_i^\sigma}, m'^i_{M_i^\sigma}), \varrho_i^k)$$

be a morphism in $d_{(\underline{g},\sigma)}(X)$. Let t be a representative of the image of $m_1'^i$ in $\mathbb{R}/\mathbb{Z}$. Then, each $m_k^i := m_k'^i - t$ is an integer.

When we know how $\tau \in C_{\Sigma_n}(\sigma)$ permutes the cycles of σ, whose information is contained in those $\varrho_i^k \in \Sigma_{M_i^\sigma}$, and the numbers $m_1^i, \dots m^i_{M_i^\sigma}$, we can get a unique τ. Explicitly, for any number $j_r = 1, 2 \dots n$, if j_r is in a k-cycle $(j_1, \dots j_k)$ of σ and it is in the set W_i^σ, then τ maps j_r to $\varrho_i^k(j)_{r+m_j^i}$, i.e. the $r+m_j^i$-th element in the cycle $\varrho_i^k(j)$ of σ.

For any $a \in W_i^\sigma$, $\forall k$ and i, we want $u_a^i = \beta^{\underline{h},\tau}_{\tau(a),a}$ for some $\underline{h}$. Thus,

$$h_{\tau(a)_k} = u_a^i g_{a_k} \dots g_{a_{1-m_a^i}}. \tag{4.9}$$

By (4.7) we can get all the other $h_{\tau(a)_j}$.

It can be checked straightforward that $J_{(\underline{g},\sigma)}$ is a well-defined functor. It does not depend on the choice of the representative t.

$J_{(\underline{g},\sigma)} \circ f_{(\underline{g},\sigma)} = \mathrm{Id}$; $f_{(\underline{g},\sigma)} \circ J_{(\underline{g},\sigma)} = \mathrm{Id}$. So the conclusion is proved. □

Then by Proposition 4.6, we get the main conclusion in Sect. 4.1.

Theorem 4.10 *The two groupoids $\Lambda((X/\!\!/G)\wr\Sigma_n)$ and $d((X/\!\!/G)\wr\Sigma_n)$ are isomorphic.*

The last conclusion in this section is some properties of the functor $f_{(\underline{g},\sigma)}$.

Proposition 4.11 (i) *If $\sigma = (1) \in \Sigma_1$, the morphism $f_{(g,(1))}$ is the identity map on $X^g /\!\!/ \Lambda_G(g)$.*

(ii) *Let $(\underline{g},\sigma) \in G\wr\Sigma_n$ and $(\underline{h},\tau) \in G\wr\Sigma_m$. The groupoids*

$$(X^{\times n})^{(\underline{g},\sigma)} /\!\!/ \Lambda_{G\wr\Sigma_n}(\underline{g},\sigma) \times_{\mathbb{T}} (X^{\times m})^{(\underline{h},\tau)} /\!\!/ \Lambda_{G\wr\Sigma_m}(\underline{h},\tau)$$

and

$$(X^{\times(n+m)})^{(\underline{g},\underline{h}\sigma\tau)} /\!\!/ \Lambda_{G\wr\Sigma_{n+m}}(\underline{g},\underline{h},\sigma\tau)$$

are isomorphic.

(iii) *$f_{(\underline{g},\sigma)}$ preserves Cartesian product of loops. The following diagram of groupoids commutes.*

$$\begin{array}{ccc}
(X^{\times n})^{(\underline{g},\sigma)} /\!\!/ \Lambda_{G\wr\Sigma_n}(\underline{g},\sigma) \times_{\mathbb{T}} (X^{\times m})^{(\underline{h},\tau)} /\!\!/ \Lambda_{G\wr\Sigma_m}(\underline{h},\tau) & \xrightarrow{\cong} & (X^{\times(n+m)})^{(\underline{g},\underline{h},\sigma\tau)} /\!\!/ \Lambda_{G\wr\Sigma_{n+m}}(\underline{g},\underline{h},\sigma\tau) \\
\downarrow{\scriptstyle f_{(\underline{g},\sigma)}\times f_{(\underline{h},\tau)}} & & \downarrow{\scriptstyle f_{(\underline{g},\underline{h},\sigma\tau)}} \\
d_{(\underline{g},\sigma)}(X) \times_{\mathbb{T}} d_{(\underline{h},\tau)}(X) & \xrightarrow{\cong} & d_{(\underline{g},\underline{h},\sigma\tau)}(X)
\end{array}$$

Proof (i) is indicated in the proof of Theorem 4.9.

(ii) We can define a functor Φ from

$$(X^{\times n})^{(\underline{g},\sigma)} /\!\!/ \Lambda_{G\wr\Sigma_n}(\underline{g},\sigma) \times_{\mathbb{T}} (X^{\times m})^{(\underline{h},\tau)} /\!\!/ \Lambda_{G\wr\Sigma_m}(\underline{h},\tau)$$

to $(X^{\times(n+m)})^{(\underline{g},\underline{h}\sigma\tau)} /\!\!/ \Lambda_{G\wr\Sigma_{n+m}}(\underline{g},\underline{h},\sigma\tau)$ sending an object (x_1, x_2) to (x_1, x_2) and a morphism $([\alpha, t], [\beta, t])$ to $[\alpha, \beta, t]$. It is straightforward to check Φ is an isomorphism between the groupoids.

(iii) The proof is left to the readers. □

4.2 Total power operation of $QEll^*_G$

In this section we construct the total power operations for quasi-elliptic cohomology and give its explicit formula in (4.17). We show in Theorem 4.12 that they satisfy the axioms that Ganter concluded in Definition 4.3, [9] for equivariant power operation.

We explain each map in the formula (4.1) and (4.2). The functor $U : \Lambda^1_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda(X/\!\!/G)$ is defined in (4.10). The pullback $(\ )^{\Lambda}_k$ is defined in (4.12). The external product $\boxtimes$ is explained in (4.16). The fourth is the pullback by $f_{(\underline{g},\sigma)}$.

The Functor U

For each $(\underline{g},\sigma) \in G\wr\Sigma_n$, $r \in \mathbb{Z}$, let $\Lambda^r_{(\underline{g},\sigma)}(X)$ denote the groupoid with objects

$$\coprod_k \coprod_{(i_1,\ldots i_k)} X^{g_{i_k}\cdots g_{i_1}}$$

where $(i_1, \ldots i_k)$ goes over all the k-cycles of σ, and with morphisms

$$\coprod_k \coprod_{(i_1,\ldots i_k),(j_1,\ldots j_k)} \Lambda^r_G(g_{i_k}\ldots g_{i_1}, g_{j_k}\ldots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}},$$

where $(i_1, \ldots i_k)$ and $(j_1, \ldots j_k)$ go over all the k-cycles of σ respectively. It may not be a subgroupoid of $\Lambda^r(X/\!\!/G)$ because there may be cycles $(i_1, \ldots i_k)$ and $(j_1, \ldots j_m)$ such that

$$g_{i_k}\cdots g_{i_1}=g_{j_m}\cdots g_{j_1}.$$

Let

$$U:\Lambda^1_{(\underline{g},\sigma)}(X)\longrightarrow \Lambda(X/\!\!/G) \tag{4.10}$$

denote the functor sending x in the component $X^{g_{i_k}\cdots g_{i_1}}$ to the x in the component $X^{g_{i_k}\cdots g_{i_1}}$ of $\Lambda(X/\!\!/G)$, and send each morphism

$$([h,t],x) \text{ in } \Lambda_G(g_{i_k}\ldots g_{i_1}, g_{j_k}\ldots g_{j_1})\times X^{g_{i_k}\cdots g_{i_1}}$$

to

$$([h,t],x) \text{ in } \Lambda_G(g_{i_k}\ldots g_{i_1}, g_{j_k}\ldots g_{j_1})\times X^{g_{i_k}\cdots g_{i_1}}.$$

In the case that $g_{i_k}\ldots g_{i_1}$ and $g_{j_k}\ldots g_{j_1}$ are equal, $([h,t],x)$ is an arrow inside a single connected component.

The Functor $(\ )_k$

For each integer k, there is a functor of groupoids $(\ )_k:\Lambda^k(X/\!\!/G)\longrightarrow\Lambda(X/\!\!/G)$ sending an object x to x and a morphism $([h,t_0],x)$ to $([h,\frac{t_0}{k}],x)$. The composition $((\ )_k)_r=(\ )_{kr}$.

The functor $(\ )_k$ gives a well-defined map

$$K_{orb}(\Lambda(X/\!\!/G))\longrightarrow K_{orb}(\Lambda^k(X/\!\!/G))$$

by pullback of bundles. We still use the symbol $(\ )_k$ to denote it when there is no confusion. For any $\Lambda(X/\!\!/G)$-vector bundle $\mathcal{V}$, S^1 acts on $(\mathcal{V})_k$ via

$$q^{\frac{1}{k}}:\mathbb{R}/k\mathbb{Z}\longrightarrow U(1)$$
$$a\mapsto e^{\frac{2\pi i a}{k}}.$$

If $\mathcal{V}$ has the decomposition $\mathcal{V}=\bigoplus_{j\in\mathbb{Z}}V_jq^j$, then

$$(\mathcal{V})_k=\bigoplus_{j\in\mathbb{Z}}V_jq^{\frac{j}{k}}. \tag{4.11}$$

The Functor $(\ )^{\Lambda}_k$

Let $\Lambda^{var}_{(\underline{g},\sigma)}(X)$ be the groupoid with the same objects as $\Lambda^1_{(\underline{g},\sigma)}(X)$ and morphisms

$$\coprod_k\ \coprod_{(i_1,\ldots i_k),(j_1,\ldots j_k)}\Lambda^k_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})\times X^{g_{i_k}\cdots g_{i_1}},$$

where $(i_1,\ldots i_k)$ and $(j_1,\ldots j_k)$ go over all the k-cycles of σ respectively.

We can define a similar functor

$$(\ )^{\Lambda}_{k}:\Lambda^{var}_{(\underline{g},\sigma)}(X)\longrightarrow\Lambda^{1}_{(\underline{g},\sigma)}(X)\tag{4.12}$$

that is identity on objects and sends each $[g,t]\in\Lambda^{k}_{G}(g_{i_k}\dots g_{i_1},g_{j_k}\dots g_{j_1})$ to $[g,\frac{t}{k}]\in\Lambda^{1}_{G}(g_{i_k}\dots g_{i_1},g_{j_k}\dots g_{j_1})$. We use the same symbol $(\ )^{\Lambda}_{k}$ to denote the pull back

$$K_{orb}(\Lambda^{1}_{(\underline{g},\sigma)}(X))\longrightarrow K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X)).\tag{4.13}$$

The external product $\boxtimes$

Let Y an H-space, $(\underline{g},\sigma)\in G\wr\Sigma_n$ and $(\underline{h},\tau)\in G\wr\Sigma_m$.

Each $K^{*}_{orb}(d_{(\underline{g},\sigma)}(X))$ is a $\mathbb{Z}[q^{\pm}]$-algebra, as shown in Sect. 4.1.1. The external product in the theory $K^{*}_{orb}(d_{(\underline{g},\sigma)}(-))$ is defined to be the tensor product of $\mathbb{Z}[q^{\pm}]$-algebras. The fibred product $d_{(\underline{g},\sigma)}(X)\times_{\mathbb{T}}d_{(\underline{h},\tau)}(X)$ has the same objects as $d_{(\underline{g},\underline{h},\sigma\tau)}(X)$ and is a subgroupoid of it.

So we have the Künneth map

$$K^{*}_{orb}(d_{(\underline{g},\sigma)}(X))\otimes_{\mathbb{Z}[q^{\pm}]}K^{*}_{orb}(d_{(\underline{h},\tau)}(X))\longrightarrow K^{*}_{orb}(d_{(\underline{g},\sigma)}(X)\times_{\mathbb{T}}d_{(\underline{h},\tau)}(X))\tag{4.14}$$

It is compatible with the Künneth map (3.15) of the quasi-elliptic cohomology in the sense that the diagram below commutes.

$$\begin{array}{ccc}
K^{*}_{orb}(d_{(\underline{g},\sigma)}(X))\underset{\mathbb{Z}[q^{\pm}]}{\otimes}K^{*}_{orb}(d_{(\underline{h},\sigma)}(X)) & \rightarrow & K^{*}_{orb}(d_{(\underline{g},\sigma)}(X)\times_{\mathbb{T}}d_{(\underline{h},\sigma)}(X))\\
\Big\downarrow{\scriptstyle f^{*}_{(\underline{g},\sigma)}\otimes_{\mathbb{Z}[q^{\pm}]}f^{*}_{(\underline{h},\sigma)}} & & \Big\downarrow{\scriptstyle f^{*}_{(\underline{(g,h)},\sigma)}}\\
K^{*}_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{n})^{(\underline{g},\sigma)})\underset{\mathbb{Z}[q^{\pm}]}{\otimes}K^{*}_{\Lambda_{H\wr\Sigma_n}(\underline{h},\sigma)}((Y^{m})^{(\underline{h},\sigma)}) & \rightarrow & K^{*}_{\Lambda_{(G\times H)\wr\Sigma_n}(\underline{(g,h)},\sigma)}((X\times Y)^{n})^{(\underline{(g,h)},\sigma)}
\end{array}\tag{4.15}$$

where the horizontal maps are Künneth maps.

If we have a vector bundle $E=\coprod_{k}\coprod_{(i_1,\dots i_k)}E_{g_{i_k}\cdots g_{i_1}}$ over $\Lambda^{1}_{(\underline{g},\sigma)}(X)$, the external product

$$\boxtimes_k\boxtimes_{(i_1,\dots i_k)}E_{g_{i_k}\cdots g_{i_1}}$$

is a vector bundler over $d_{(\underline{g},\sigma)}(X)$. This defines a map

$$K_{orb}(\Lambda^{1}_{(\underline{g},\sigma)}(X))\longrightarrow K_{orb}(d_{(\underline{g},\sigma)}(X))\tag{4.16}$$

Composing all the functors as in (4.1) and (4.2), we get the explicit formula of $\mathbb{P}_{(\underline{g},\sigma)}$

$$\mathbb{P}_{(\underline{g},\sigma)}(\mathcal{V})=f^{*}_{(\underline{g},\sigma)}(\boxtimes_k\boxtimes_{(i_1,\dots i_k)}(\mathcal{V}_{g_{i_k}\cdots g_{i_1}})_k).\tag{4.17}$$

$\mathbb{P}_{(\underline{g},\sigma)}$ is natural. If $(\underline{g},\sigma)$ and $(\underline{h},\tau)$ are conjugate in $G\wr\Sigma_n$, $\mathbb{P}_{(\underline{g},\sigma)}(\mathcal{V})$ and $\mathbb{P}_{(\underline{h},\tau)}(\mathcal{V})$ are isomorphic.

Theorem 4.12 *The family of maps*

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)_{conj}} \mathbb{P}_{(\underline{g},\sigma)} : QEll^*_G(X) \longrightarrow QEll^*_{G\wr\Sigma_n}(X^{\times n}),$$

satisfy

(i) $\mathbb{P}_1 = Id$, $\mathbb{P}_0(x) = 1$.

(ii) *Let* $x \in QEll^*_G(X)$, $(\underline{g},\sigma) \in G\wr\Sigma_n$ *and* $(\underline{h},\tau) \in G\wr\Sigma_m$. *The external product of two power operations*

$$\mathbb{P}_{(\underline{g},\sigma)}(x) \boxtimes \mathbb{P}_{(\underline{h},\tau)}(x) = res|^{\Lambda_{G\wr\Sigma_{m+n}}(\underline{g},\underline{h};\sigma\tau)}_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{G\wr\Sigma_m}(\underline{h},\tau)} \mathbb{P}_{(\underline{g},\underline{h};\sigma\tau)}(x).$$

(iii) *The composition of two power operations is*

$$\mathbb{P}_{((\underline{h},\tau);\sigma)}(\mathbb{P}_m(x)) = res|^{\Lambda_{G\wr\Sigma_{mn}}(\underline{h},(\underline{\tau},\sigma))}_{\Lambda_{(G\wr\Sigma_m)\wr\Sigma_n}((\underline{h},\tau);\sigma)} \mathbb{P}_{(\underline{h},(\underline{\tau},\sigma))}(x)$$

where $(\underline{h,\tau}) \in (G\wr\Sigma_m)^{\times n}$, *and* $\sigma \in \Sigma_n$. $(\underline{\tau},\sigma)$ *is in* $\Sigma_m\wr\Sigma_n$, *thus, can be viewed as an element in* Σ_{mn}.

(iv) $\mathbb{P}$ *preserves external product. For* $\underline{(g,h)} = ((g_1,h_1),\dots(g_n,h_n)) \in (G\times H)^{\times n}$, $\sigma \in \Sigma_n$,

$$\mathbb{P}_{(\underline{(g,h)},\sigma)}(x\boxtimes y) = res|^{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{H\wr\Sigma_n}(\underline{h},\sigma)}_{\Lambda_{(G\times H)\wr\Sigma_n}(\underline{(g,h)},\sigma)} \mathbb{P}_{(\underline{g},\sigma)}(x)\boxtimes\mathbb{P}_{(\underline{h},\sigma)}(y).$$

Proof We check each one respectively.

(i) When $n = 1$, all the cycles of a permutation is 1-cycle. $(\ )_1$ and the homeomorphism $f_{(g,(1))}$ are both identity maps. Directly from the formula (4.17), $\mathbb{P}_1(x) = x$.

(ii)

$$\begin{aligned}
&\mathbb{P}_{(\underline{g},\sigma)}(x) \boxtimes \mathbb{P}_{(\underline{h},\tau)}(x)\\
&= f^*_{(\underline{g},\sigma)}(\boxtimes_k \boxtimes_{(i_1,\dots i_k)} (x_{g_{i_k}\dots g_{i_1}})_k) \boxtimes f^*_{(\underline{h},\tau)}(\boxtimes_j \boxtimes_{(r_1,\dots r_j)} (x_{h_{r_j}\dots h_{r_1}})_j)\\
&= res|^{\Lambda_{G\wr\Sigma_{m+n}}(\underline{g},\underline{h};\sigma\tau)}_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{G\wr\Sigma_m}(\underline{h},\tau)} f^*_{(\underline{g},\underline{h};\sigma\tau)}((\boxtimes_k \boxtimes_{(i_1,\dots i_k)} (x_{g_{i_k}\dots g_{i_1}})_k)\\
&\boxtimes (\boxtimes_j \boxtimes_{(r_1,\dots r_j)} (x_{h_{r_j}\dots h_{r_1}})_j)).
\end{aligned}$$

where $(i_1,\dots i_k)$ goes over all the k-cycles of σ and $(r_1,\dots r_j)$ goes over all the j-cycles of τ and $(\ )_k$ is the map cited in (4.11). The second step is from Proposition 4.11 (iii).

$$f^*_{(\underline{g},\underline{h};\sigma\tau)}((\boxtimes_k \boxtimes_{(i_1,\dots i_k)} (x_{g_{i_k}\dots g_{i_1}})_k) \boxtimes (\boxtimes_j \boxtimes_{(r_1,\dots r_j)} (x_{h_{r_j}\dots h_{r_1}})_j))$$

is exactly

$$\mathbb{P}_{(\underline{g},\underline{h};\sigma\tau)}(x).$$

(iii) Recall that for an element $(\underline{\tau}, \sigma) = (\tau_1, \dots \tau_n, \sigma) \in \Sigma_{mn}$, it acts on the set with mn elements

$$\{(i, j)|1 \leq i \leq n, 1 \leq j \leq m\}$$

in this way:

$$(\underline{\tau}, \sigma) \cdot (i, j) = (\sigma(i), \tau_{\sigma(i)}(j)).$$

That also shows how to view it as an element in Σ_{mn}.

Then for any integer q,

$$(\underline{\tau}, \sigma)^q \cdot (i, j) = (\sigma^q(i), \tau_{\sigma^q(i)}\tau_{\sigma^{q-1}(i)} \dots \tau_{\sigma(i)}(j)). \tag{4.18}$$

To find all the cycles of $(\underline{\tau}, \sigma)$ is exactly to find all the orbits of the action by $(\underline{\tau}, \sigma)$. If i belongs to an s-cycle of σ and j belongs to a r-cycle of $\tau_{\sigma^s(i)}\tau_{\sigma^{s-1}(i)} \dots \tau_{\sigma(i)}$, then the orbit containing (i, j) has sr elements by (4.18). In other words, $(i_1, \dots i_s)$ is an s-cycle of σ and $(j_1, \dots j_r)$ is a r-cycle of $\tau := \tau_{i_s} \dots \tau_{i_1}$ if and only if

$$\begin{pmatrix} (i_1, \tau_{i_1}(j_{r-1}))(i_2, \tau_{i_2}\tau_{i_1}(j_{r-1})) \dots (i_s, j_r) \\ (i_1, \tau_{i_1}(j_{r-2}))(i_2, \tau_{i_2}\tau_{i_1}(j_{r-2})) \dots (i_s, j_{r-1}) \\ \dots \\ (i_1, \tau_{i_1}(j_1))(i_2, \tau_{i_2}\tau_{i_1}(j_1)) \dots (i_s, j_2) \\ (i_1, \tau_{i_1}(j_r))(i_2, \tau_{i_2}\tau_{i_1}(j_r)) \dots (i_s, j_1) \end{pmatrix}$$

is an sr-cycle of $(\underline{\tau}, \sigma)$.

$$\begin{aligned}
&\mathbb{P}_{(\underline{(\underline{h},\tau)};\sigma)}(\mathbb{P}_m(x)) \\
&= f^*_{(\underline{(\underline{h},\tau)};\sigma)}[\boxtimes_k \boxtimes_{(i_1,\dots i_k)} (\mathbb{P}_{((\underline{h}_{i_k},\tau_{i_k})\dots(\underline{h}_{i_1},\tau_{i_1}))}(x))_k] \\
&= f^*_{(\underline{(\underline{h},\tau)};\sigma)}[\boxtimes_k \boxtimes_{(i_1,\dots i_k)} [f^*_{((\underline{h}_{i_k},\tau_{i_k})\dots(\underline{h}_{i_1},\tau_{i_1}))}(\boxtimes_r \boxtimes_{(j_1,\dots j_r)} (x_{H_{\underline{i},\underline{j}}})_r)]_k] \\
&= \left(f^*_{(\underline{(\underline{h},\tau)};\sigma)} \circ \prod_{k,(i_1,\dots i_k)} f^*_{((\underline{h}_{i_k},\tau_{i_k})\dots(\underline{h}_{i_1},\tau_{i_1}))} \right) [\boxtimes_{k,(i_1,\dots i_k)} \boxtimes_{r,(j_1,\dots j_r)} (x_{H_{\underline{i},\underline{j}}})_{kr}] \\
&= f^*_{(\underline{\underline{h}},(\underline{\tau},\sigma))}[\boxtimes_{k,(i_1,\dots i_k)} \boxtimes_{r,(j_1,\dots j_r)} (x_{H_{\underline{i},\underline{j}}})_{kr}]
\end{aligned}$$

where

$$
\begin{aligned}
H_{\underline{i}\,\underline{j}} := {} & h_{i_k,j_1} h_{i_{k-1},\tau_{i_k}^{-1}(j_1)} \dots h_{i_1,(\tau_{i_k}\dots\tau_{i_2})^{-1}(j_1)} \\
& h_{i_k,j_2} h_{i_{k-1},\tau_{i_k}^{-1}(j_2)} \dots h_{i_1,(\tau_{i_k}\dots\tau_{i_2})^{-1}(j_2)} \\
& \dots \\
& h_{i_k,j_r} h_{i_{k-1},\tau_{i_k}^{-1}(j_r)} \dots h_{i_1,(\tau_{i_k}\dots\tau_{i_2})^{-1}(j_r)} \\
= {} & h_{i_k,j_1} h_{i_{k-1},\tau_{i_{k-1}}\dots\tau_{i_2}\tau_{i_1}(j_1)} \dots h_{i_1,\tau_{i_1}(j_r)} \\
& h_{i_k,j_2} h_{i_{k-1},\tau_{i_{k-1}}\dots\tau_{i_2}\tau_{i_1}(j_2)} \dots h_{i_1,\tau_{i_1}(j_1)} \\
& \dots \\
& h_{i_k,j_r} h_{i_{k-1},\tau_{i_{k-1}}\dots\tau_{i_2}\tau_{i_1}(j_{r-1})} \dots h_{i_1,\tau_{i_1}(j_{r-1})}
\end{aligned}
$$

where $(i_1, \dots i_k)$ goes over all the k-cycles of $\sigma \in \Sigma_m$ and $(j_1, \dots j_r)$ goes over all the r-cycles of $\tau_{i_k} \dots \tau_{i_1} \in \Sigma_n$. The last step is by Proposition 4.11 in [10].

$$
f^*_{(\underline{\underline{h}},(\underline{\tau},\sigma))}[\boxtimes_{k,(i_1,\dots i_k)} \boxtimes_{r,(j_1,\dots j_r)} (x_{H_{\underline{i},\underline{j}}})_{kr}]
$$

is the same space as $\mathbb{P}_{(\underline{\underline{h}},(\underline{\tau},\sigma))}(x)$, but the action is restricted by

$$
res|^{\Lambda_{G\wr\Sigma_{mn}}(\underline{\underline{h}},(\underline{\tau},\sigma))}_{\Lambda_{(G\wr\Sigma_m)\wr\Sigma_n}((\underline{\underline{h},\tau});\sigma)}.
$$

(iv)We have

$$
\mathbb{P}_{(\underline{(g,h)},\sigma)}(x \boxtimes y) = f^*_{(\underline{(g,h)},\sigma)}(\boxtimes_k \boxtimes_{(i_1,\dots i_k)} ((x \boxtimes y)_{(g_{i_k}\dots g_{i_1},h_{i_k}\dots h_{i_1})})_k)
$$

$$
= f^*_{(\underline{(g,h)},\sigma)}(\boxtimes_k \boxtimes_{(i_1,\dots i_k)} (x_{g_{i_k}\dots g_{i_1}})_k \boxtimes (y_{h_{i_k}\dots h_{i_1}})_k)
$$

$$
= f^*_{(\underline{(g,h)},\sigma)}(\boxtimes_k \boxtimes_{(i_1,\dots i_k)} (x_{g_{i_k}\dots g_{i_1}})_k) \boxtimes (\boxtimes_j \boxtimes_{(r_1,\dots r_j)} (y_{h_{r_j}\dots h_{r_1}})_j)
$$

$$
= res|^{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{H\wr\Sigma_n}(\underline{h},\sigma)}_{\Lambda_{(G\times H)\wr\Sigma_n}(\underline{(g,h)},\sigma)} (f^*_{(\underline{g},\sigma)} \times f^*_{(\underline{h},\sigma)})(\boxtimes_k \boxtimes_{(i_1,\dots i_k)} (x_{g_{i_k}\dots g_{i_1}})_k)
\boxtimes (\boxtimes_j \boxtimes_{(r_1,\dots r_j)} (y_{h_{r_j}\dots h_{r_1}})_j)
$$

$$
= res|^{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{H\wr\Sigma_n}(\underline{h},\sigma)}_{\Lambda_{(G\times H)\wr\Sigma_n}(\underline{(g,h)},\sigma)} f^*_{(\underline{g},\sigma)}[\boxtimes_k \boxtimes_{(i_1,\dots i_k)} (x_{g_{i_k}\dots g_{i_1}})_k]
\boxtimes f^*_{(\underline{h},\sigma)}[\boxtimes_j \boxtimes_{(r_1,\dots r_j)} (y_{h_{r_j}\dots h_{r_1}})_j],
$$

where $(i_1, \dots i_k)$ goes over all the k-cycles of σ and $(r_1, \dots r_j)$ goes over all the j-cycles of σ. It equals to

$$res|^{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)\times_{\mathbb{T}}\Lambda_{H\wr\Sigma_n}(\underline{h},\sigma)}_{\Lambda_{(G\times H)\wr\Sigma_n}(\underline{g},\sigma)}\mathbb{P}_{(\underline{g},\sigma)}(x)\boxtimes\mathbb{P}_{(\underline{h},\sigma)}(y).$$

□

Example 4.13 Let G be the trivial group and X a space. Let $\sigma \in \Sigma_n$. Then $QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. The functor $f_{(\underline{1},\sigma)}$ gives the homeomorphism

$$(X^{\times n})^{(\underline{1},\sigma)} \cong \prod_k \prod_{(i_1,\dots i_k)} X,$$

where the second direct product goes over all the k-cycles of σ. By (4.17), the power operation is

$$\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\dots i_k)} (x)_k.$$

When $n = 2$, $\mathbb{P}_{(\underline{1},(1)(1))}(x) = x \boxtimes x$ and $\mathbb{P}_{(\underline{1},(12))}(x) = (x)_2$.

When $n = 3$, $\mathbb{P}_{(\underline{1},(1)(1)(1))}(x) = x \boxtimes x \boxtimes x$, $\mathbb{P}_{(\underline{1},(12)(1))}(x) = (x)_2 \boxtimes x$, and $\mathbb{P}_{(\underline{1},(123))}(x) = (x)_3$.

When $n = 4$, $\mathbb{P}_{(\underline{1},(1)(1)(1)(1))}(x) = x \boxtimes x \boxtimes x \boxtimes x$, $\mathbb{P}_{(\underline{1},(12))}(x) = (x)_2 \boxtimes x \boxtimes x$, $\mathbb{P}_{(\underline{1},(123))}(x) = (x)_3 \boxtimes x$, $\mathbb{P}_{(\underline{1},(1234))}(x) = (x)_4$, and $\mathbb{P}_{(\underline{1},(12)(34))}(x) = (x)_2 \boxtimes (x)_2$. Note that there is a Σ_2-action permuting the two $(x)_2$ in $\mathbb{P}_{(\underline{1},(12)(34))}(x)$.

Remark 4.14 We have the relation between equivariant Tate K-theory and quasi-elliptic cohomology

$$QEll_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong (K_{Tate})_G(X). \tag{4.19}$$

It extends uniquely to a power operation for Tate K-theory

$$QEll_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q))$$

which is the stringy power operation P^{string}_n constructed in Definition 5.10, [10]. It is elliptic in the sense of [2].

5 Orbifold quasi-elliptic cohomology and its power operation

The elliptic cohomology of orbifolds involves a rich interaction between the orbifold structure and the elliptic curve. Ganter explores this interaction in the case of the Tate curve in [11], describing K_{Tate} for an orbifold X in terms of the equivariant K-theory and the groupoid structure of X.

In Sect. 5.1 we give a description of orbifold quasi-elliptic cohomology. In Sect. 5.2 we discuss the inertia groupoid of symmetric power and the groupoids needed for the construction of the power operation in Sect. 5.3.

5.1 Definition

We have two ways to define orbifold quasi-elliptic cohomology. The first one is motivated by Ganter's definition of orbifold Tate K-theory in Section 2, [11]. The other one is a generalization of the definition of quasi-elliptic cohomology in Sect. 3.2.

We consider the category of groupoids $\mathcal{G}pd$ as a 2-category with small topological groupoids as the objects and with

$$1\mathrm{Hom}(X, Y) = Fun(X, Y).$$

This 2-category is different from that in Section 3 [18]. Let $\mathcal{G}pd^{cen}$ denote the 2-category of centers of groupoids defined in Section 2, [11]. Ganter constructed in Example 2.3 [11] a 2-functor for any $k \in \mathbb{Z}$

$$\mathcal{G}pd \longrightarrow \mathcal{G}pd^{cen}$$
$$X \mapsto (I(X), \xi^k)$$

where ξ^k is the center element of the inertia groupoid $I(X)$ sending (x, g) to (x, g^k). We use ξ to denote ξ^1.

Let $\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi} I(X)$ denote the groupoid

$$(\mathrm{pt}/\!\!/\mathbb{R}) \times I(X)/\sim$$

with $\sim$ generated by $1 \sim \xi$.

Definition 5.1 For any topological groupoid X, the quasi-elliptic cohomology $QEll^*(X)$ is the orbifold K-theory

$$K^*_{orb}(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi} I(X)). \tag{5.1}$$

In other words, for a topological groupoid X, $QEll(X)$ is defined to be a subring of $K_{orb}(X)[\![q^{\pm\frac{1}{|\xi|}}]\!]$ that is the Grothendieck group of finite sums

$$\sum_{a\in\mathbb{Q}} V_a q^a$$

satisfying:

for each $a \in \mathbb{Q}$, the coefficient V_a is an $e^{2\pi ia} -$ eigenbundle of ξ.

In the global quotient case,

$$QEll^*(X/\!\!/G) = QEll^*_G(X).$$

In addition, for any topological groupoid X, we can also consider the category

$$Loop_1(X) := Bibun(S^1/\!\!/*, X)$$

and formulate $Loop_1^{ext}(X)$ by adding the rotation action by circle, as the construction in Sect. 2.1.2. Afterwards we can construct the subgroupoid $\Lambda(X)$ of $Loop_1^{ext}(X)$ consisting of the constant loops, which is isomorphic to $\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi} I(X)$. So in this way we give an equivalent definition of orbifold quasi-elliptic cohomology.

5.2 Symmetric powers of orbifolds and its inertia groupoid

In this section we introduce the groupoids necessary for the construction of the power operation. In Lemmas 5.3, 5.4 and 5.5 we show the relation between them.

For groupoids like $\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X$, instead of the total symmetric power (Definition 3.1, [11]) $S(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)$, we consider a subgroupoid

$$S^R(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)$$

of it.

Definition 5.2 (The groupoid $S^R(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)$) Let

$$\rho_k : \mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X \longrightarrow \mathrm{pt}/\!\!/(\mathbb{R}/\mathbb{Z})$$

be the functor sending all the objects to the single point, and an arrow

$$[g, t]$$

to

$$t \bmod \mathbb{Z}.$$

Let $\times_{\mathbb{R}}(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)$ denote the limit of the diagram of groupoids

$$\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X \xrightarrow{\rho_k} \mathrm{pt}/\!\!/(\mathbb{R}/\mathbb{Z}) \xleftarrow{\rho_k} \mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X .$$

Let

$$\times^n_{\mathbb{R}}(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)$$

denote the limit of n morphisms ρ_ks. It inherits a Σ_n-action on it by permutation from that on the product $(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)^{\times n}$.

Let $S^R_n(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)$ denote the groupoid with the same objects as

$$\times^n_{\mathbb{R}}(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)$$

and morphisms of the form $([g_1, t_1], \dots [g_n, t_n]; \sigma)$ with $([g_1, t_1], \dots [g_n, t_n])$ a morphism in $\times^n_{\mathbb{R}}(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)$ and $\sigma \in \Sigma_n$. This new groupoid $S^R_n(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)$ is a subgroupoid of

$$(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X) \wr \Sigma_n.$$

Define

$$S^R(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X) := \coprod_{n\geq 0} S^R_n(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X). \tag{5.2}$$

The triple

$$(S^R(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X), *, (\,))$$

is a symmetric monoid where $*$ is the concatenation and the unit $(\,)$ is the unique object in $X \wr \Sigma_0$. $S^R(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} X)$ is the symmetric product that we will use to formulate the power operation.

Lemma 5.3 *Let $\Phi_k(X)$ denote the groupoid in Definition* 3.3, [11], *and $\phi_k \in$ Center(Φ_k) denote the restriction of $S_k(\xi)$ to Φ_k. For each integer $k \geq 1$, there is an equivalence between*

$$pt/\!\!/\mathbb{R} \times_{1\sim\phi_k} \Phi_k(X)$$

and the groupoid $pt/\!\!/\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I(X)[\xi^{\frac{1}{k}}]$ which identifies ϕ_k with $\xi^{\frac{1}{k}}$. Here $\xi^{\frac{1}{k}}$ is an added element such that the composition of k $\xi^{\frac{1}{k}}$s is ξ.

Proof We can define a functor

$$A_k : \mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\phi_k} \Phi_k(X) \longrightarrow \mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I(X)[\xi^{\frac{1}{k}}]$$

by sending an object $(\underline{x}, \underline{g}, (12\dots k))$ to $(x_1, g_k \dots g_1)$ and sending a morphism $[\underline{h}, (12\dots k)^m, t]$ to

$$[h_k g^{-1}_{1-m} \dots g^{-1}_{k-1} g^{-1}_k, m + t].$$

Recall $h_k g^{-1}_{1-m} \dots g^{-1}_{k-1} g^{-1}_k$ conjugates $g_k \dots g_1$ to itself. It is the element

$$\beta^{\underline{h},\mathrm{Id}}_{(12\dots k),(12\dots k)}$$

defined in (4.8). The functor A_k is an isomorphism, as implied in the proof of Theorem 4.9. □

Let $\Phi(X) := \coprod_{k\geq 1} \Phi_k(X)$. Let $\phi := \coprod_{k\geq 1} \phi_k \in \mathrm{Center}(\Phi)$ denote the restriction of $S(\xi)$ to Φ.

Theorem 4.9 can be reinterpreted as Lemma 5.4.

Lemma 5.4 *The groupoid* $S^R(\coprod_k pt/\!\!/\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I(X)[\xi^{\frac{1}{k}}])$ *is equivalent to*

$$pt/\!\!/\mathbb{R} \times_{1\sim S(\xi)} I(S(X)).$$

The proof is similar to that of Theorem 4.9.

Lemma 5.5 *We have an equivalence of groupoids*

$$Q^R : S^R(pt/\!\!/\mathbb{R} \times_{1\sim\phi} \Phi(X)) \longrightarrow pt/\!\!/\mathbb{R} \times_{1\sim S(\xi)} I(S(X)),$$

which is natural in X *and satisfies*

$$Q^R S^R(\phi) = S(\xi) Q^R.$$

Proof Let I be the inclusion

$$\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\phi} \Phi(X) \longrightarrow \mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim S(\xi)} I(S(X)).$$

Let ϵ be the counit of the adjunction $(S, *, (\ )) \dashv$ forget. Let Q denote the composition

$$S(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\phi} \Phi(X)) \xrightarrow{S(I)} S(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim S(\xi)} I(S(X))) \xrightarrow{\epsilon} \mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim S(\xi)} I(S(X)).$$

Let Q^R be the restriction of Q to the subgroupoid $S^R(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\phi} \Phi(X))$, i.e. the composition

$$\begin{aligned} Q^R : S^R(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\phi} \Phi(X)) &\xrightarrow{S^R(I)} S^R(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim S(\xi)} I(S(X))) \\ &\xrightarrow{\text{restriction of } \epsilon} \mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim S(\xi)} I(S(X)). \end{aligned}$$

The essential image of I consists exactly of the indecomposable objects of $\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim S(\xi)} I(S(X))$, thus, both Q and Q^R are essentially surjective.

Q is not fully faithful but Q^R is. This is why we need the product S^R instead of S. □

5.3 Power operation for orbifold quasi-elliptic cohomology

In this section we construct the total power operation for the orbifold quasi-elliptic cohomology

$$P^{Ell} : QEll(X) \longrightarrow QEll(SX)$$

in (5.6), which satisfy the axioms that Ganter formulated in Definition 3.9, [11] for power operations for orbifold theories. The power operation we constructed in Sect. 4.2 is a special case of it for G-spaces.

Example 5.6 We can construct Atiyah's power operation for orbifold quasi-elliptic cohomology.

Let V be an orbifold vector bundle over the orbifold

$$\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi} I(X),$$

thus, V represents an element in $QEll(X)$. Then

$$P_n(V) := V^{\otimes_{\mathbb{Z}[q^{\pm}]} n}$$

is an orbifold vector bundle over

$$S^R(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi} I(X)) \cong \mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi} SI(X).$$

So $P_n(V)$ is in $QEll^*(S(X))$.

$P = (P_n)_{n\geq 0}$ satisfies the axioms of a total power operation.

Before the construction of the power operation of $QEll$, we introduce several maps necessary for the construction of the power operation.

Let X be an orbifold groupoid and $k \geq 1$ an integer. We define the map

$$s_k : K_{orb}(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi} I(X)) \longrightarrow K_{orb}(\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} I(X)) \tag{5.3}$$

$$\left[\sum V_a q^a\right] \mapsto \left[\sum V_a q^{\frac{a}{k}}\right] \tag{5.4}$$

and

$$\coprod_k s_k : K_{orb}(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi} I(X)) \longrightarrow K_{orb}(\coprod_k (\mathrm{pt}/\!\!/\mathbb{R} \times_{k\sim\xi} I(X))). \tag{5.5}$$

The functor

$$(\ )_k : \Lambda_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda^1_{(\underline{g},\sigma)}(X)$$

defined in (4.11) is a special local case of s_k when X is a G-space and $(\underline{g}, \sigma)$ is fixed.

Let $\theta : QEll(X) \longrightarrow K_{orb}(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\phi} \Phi(X))$ be the additive operation whose k-th component is $A_k^* \circ s_k$, where A_k is the equivalence defined in Lemma 5.3.

Now we are ready to define the total power operation P^{Ell} of $QEll^*$ as the composition below:

$$\begin{array}{c} QEll(X) \xrightarrow{\theta} K_{orb}(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\phi} \Phi(X)) \xrightarrow{P} K_{orb}(S^R(\mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\phi} \Phi(X))) \\ \Big\downarrow (Q^{R*})^{-1} \\ QEll(SX). \end{array} \tag{5.6}$$

Theorem 5.7 *P^{Ell} satisfies the axioms of a total power operation in Definition* 3.9 [11].

Proof From the definition of P^{Ell}, we can see it is a well-defined natural transformation $QEll \Rightarrow QEll \circ S$ and is a comodule over the comonad $(-) \circ S$.

In addition, the functor θ has the property of additivity

$$\theta : QEll(X \sqcup Y) \longrightarrow QEll(\Phi(X) \sqcup \Phi(Y))$$
$$(a, b) \mapsto (\theta(a), \theta(b)).$$

The power operation P defined in Example 5.6 has the exponential property. Therefore, P^{Ell} has the exponential property. So P^{Ell} is a total power operation. □

Remark 5.8 Let $X /\!\!/ G$ be a quotient orbifold. The power operation we construct in Sect. 4.1 for quotient orbifolds is in fact the one below.

$$\mathbb{P} : QEll^*(X/\!\!/G) \xrightarrow{\coprod_k s_k} K^*_{orb}(\coprod_k \mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I(X/\!\!/G)[\xi^{\frac{1}{k}}]) \xrightarrow{P}$$

$K^*_{orb}(S^R(\coprod_k \mathrm{pt}/\!\!/\mathbb{R} \times_{1\sim\xi^{\frac{1}{k}}} I(X/\!\!/G)[\xi^{\frac{1}{k}}])) \xrightarrow{J^*} QEll^*(S(X/\!\!/G))$ where J is constructed from the functors $J_{(\underline{g},\sigma)}$ in the proof of Theorem 4.9.

For global quotient orbifolds, P^{Ell} and $\mathbb{P}$ are the same up to isomorphism. The diagram

$$\begin{array}{ccc}
QEll^*(X/\!\!/G) & & QEll^*(S(X/\!\!/G)) \\
\downarrow \theta & & \uparrow (Q^{R*})^{-1} \\
K_{orb}(\mathrm{pt}/\!\!/\mathbb{R} \underset{1\sim\phi}{\times} I(\Phi(X/\!\!/G))) & \xrightarrow{P} & K_{orb}(S^R(\mathrm{pt}/\!\!/\mathbb{R} \underset{1\sim\phi}{\times} \Phi(X/\!\!/G))) \\
\uparrow \coprod_k A^*_k & & \uparrow S^R(\coprod_k A^*_k) \\
K_{orb}(\coprod_k \mathrm{pt}/\!\!/\mathbb{R} \underset{1\sim\xi^{\frac{1}{k}}}{\times} I(X/\!\!/G)[\xi^{\frac{1}{k}}]) & \xrightarrow{P} & K_{orb}(S^R(\coprod_k \mathrm{pt}/\!\!/\mathbb{R} \underset{1\sim\xi^{\frac{1}{k}}}{\times} I(X/\!\!/G)[\xi^{\frac{1}{k}}]))
\end{array}$$

commutes. The vertical maps $\coprod_k A^*_k$ and $S^R(\coprod_k A^*_k)$ are both equivalences of groupoids. The horizontal maps are the power operation defined in Example 5.6.

6 Finite subgroups of the Tate curve

Strickland showed in [28] that the quotient of the Morava E-theory of the symmetric group by a certain transfer ideal can be identified with the product of rings $\prod_{k\geq 0} R_k$

where each R_k classifies subgroup-schemes of degree p^k in the formal group associated to $E^0\mathbb{C}P^\infty$. In this section we prove similar conclusions for Tate K-theory and quasi-elliptic cohomology. The main conclusion for Sect. 6 is Theorem 6.4.

6.1 Background

In this section we introduce the Tate curve and its finite subgroups. The main references are Section 2.6, [1] and Sections 8.7, 8.8, [16].

An elliptic curve over the complex numbers $\mathbb{C}$ is a connected Riemann surface, i.e. a connected compact 1-dimensional complex manifold, of genus 1. By the uniformization theorem every elliptic curve over $\mathbb{C}$ is analytically isomorphic to a 1-dimensional complex torus, and can be expressed as

$$\mathbb{C}^*/q^{\mathbb{Z}}$$

with $q \in \mathbb{C}$ and $0 < |q| < 1$, where $\mathbb{C}^*$ is the multiplicative group $\mathbb{C}\backslash\{0\}$.

The Tate curve $Tate(q)$ is the elliptic curve

$$E_q : y^2 + xy = x^3 + a_4 x + a_6$$

whose coefficients are given by the formal power series in $\mathbb{Z}((q))$

$$a_4 = -5\sum_{n\geqslant 1} n^3 q^n/(1-q^n) \qquad a_6 = -\frac{1}{12}\sum_{n\geqslant 1}(7n^5+5n^3)q^n/(1-q^n).$$

Before we talk about the torsion part of $Tate(q)$, we recall a smooth one-dimensional commutative group scheme T over $\mathbb{Z}[q^{\pm}]$. It sits in a short exact sequence of group-schemes over $\mathbb{Z}[q^{\pm}]$

$$0 \longrightarrow \mathbb{G}_m \longrightarrow T \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

The N-torsion points $T[N]$ of it is the disjoint union of N schemes $T_0[N], \ldots T_{N-1}[N]$, where

$$T_i[N] = \mathrm{Spec}(\mathbb{Z}[q^{\pm}][x]/(x^N - q^i)).$$

It fits into a short exact sequence

$$0 \longrightarrow \mu_N \xrightarrow{a_N} T[N] \xrightarrow{b_N} \mathbb{Z}/N\mathbb{Z} \longrightarrow 0,$$

The canonical extension structure on $T(N)$ is compatible with an alternating paring of $\mathbb{Z}[q^{\pm}]$-group schemes $e_N : T(N) \times T(N) \longrightarrow \mu_N$ in the sense that

$$e_N(a_N(x), y) = x^{b_N(y)}, \text{for any } \mathbb{Z}[q^{\pm}]-\text{algebra } R \text{ and any } x \in \mu_N(R).$$

We have the conclusion below, which is Theorem 8.7.5, [16].

Theorem 6.1 *There exists a faithfully flat* $\mathbb{Z}[q^{\pm}]$*-algebra* R*, an elliptic curve* E/R*, and an isomorphism of ind-group-schemes over* R

$$T_{torsion} \otimes_{\mathbb{Z}[q^{\pm}]} R \xrightarrow{\sim} E_{tors},$$

such that for every $N \geq 1$*, the isomorphism on* N*-division points* $T[N] \otimes R \xrightarrow{\sim} E[N]$ *is compatible with* e_N*-pairings.*

Thus, we have the unique isomorphism of ind-group-schemes on $\mathbb{Z}((q))$

$$T_{torsion} \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \xrightarrow{\sim} Tate(q)_{tors}.$$

The isomorphism is compatible with the canonical extension structure: for each $N \geq 1$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_N & \longrightarrow & T[N] & \longrightarrow & \mathbb{Z}/N\mathbb{Z} & \longrightarrow & 0 \\ & & \Big\downarrow = & & \Big\downarrow \cong & & \Big\downarrow = & & \\ 0 & \longrightarrow & \mu_N & \longrightarrow & Tate(q)[N] & \longrightarrow & \mathbb{Z}/N\mathbb{Z} & \longrightarrow & 0 \end{array}$$

Therefore, $Tate(q)[N]$ is isomorphic to the disjoint union

$$\coprod_{k=0}^{N-1} \mathrm{Spec}(\mathbb{Z}((q))[x]/(x^N - q^k)).$$

In addition, we have the question how to classify all the finite subgroups of $Tate(q)$. As shown in Proposition 6.5.1, [16], the ring O_{Sub_n} that classifies subgroups of $Tate(q)$ of order n exists. To give a description of it, first we describe the isogenies for the analytic Tate curve over $\mathbb{C}$.

Let (d, e) be a pair of positive integers such that $N = de$ and q' a nonzero complex number such that $q^d = q'^e$. The map

$$\psi_d : \mathbb{C}^*/q^{\mathbb{Z}} \longrightarrow \mathbb{C}^*/q'^{\mathbb{Z}}$$
$$x \mapsto x^d$$

is well-defined since $\psi_d(q^{\mathbb{Z}}) \subseteq q'^{\mathbb{Z}}$. The kernel of ψ_d is

$$\{\mu_d^n q^{\frac{m}{e}} q^{\mathbb{Z}} | n, m \in \mathbb{Z}\}$$

where μ_d is a d-th primitive root of 1 and $q^{\frac{1}{e}}$ is a e-th primitive root of q. Its order is N. In fact

$$\{\mathrm{Ker}\psi_d | \ d \text{ divides } N \text{ and } d \geq 1\}$$

gives all the subgroups of $\mathbb{C}^*/q^{\mathbb{Z}}$ of order N.

Proposition 6.2 *For each pair of number (d, e), there exists an isogeny*

$$\Psi_{d,e} : Tate((q)) \longrightarrow Tate((q'))$$

of the elliptic curves over O_{Sub_n} such that its kernel is the universal subgroup.

We have

$$O_{Sub_n} \otimes \mathbb{C} = \prod_{N=de} \mathbb{C}((q))[q']/\langle q^d - q'^e \rangle.$$

Moreover, we have the conclusion below.

Proposition 6.3 *The finite subgroups of the Tate curve are the kernels of isogenies.*

6.2 Formulas for induction

Before the main conclusion, we introduce the induction formula for quasi-elliptic cohomology. The induction formula for Tate K-theory is constructed in Section 2.3.3, [11].

Let $H \subseteq G$ be an inclusion of finite groups and X be a G-space. Then we have the inclusion of the groupoids

$$j : X/\!\!/H \longrightarrow X/\!\!/G.$$

Let $a' = \prod_{\sigma \in H_{conj}} a'_\sigma$ be an element in $QEll_H(X) = \prod_{\sigma \in H_{conj}} K_{\Lambda_H(\sigma)}(X^\sigma)$ where σ goes over all the conjugacy classes in H. The finite covering map

$$f' : \Lambda(G \times_H X/\!\!/G) \longrightarrow \Lambda(X/\!\!/G)$$

is defined by sending an object $(\sigma, [g, x])$ to (σ, gx) and a morphism $([g', t], (\sigma, [g, x]))$ to $([g', t], (gx, \sigma))$. The transfer of quasi-elliptic cohomology

$$\mathcal{I}_H^G : QEll_H(X) \longrightarrow QEll_G(X)$$

is defined to be the composition

$$QEll_H(X) \xrightarrow{\cong} QEll_G(G \times_H X) \longrightarrow QEll_G(X) \tag{6.1}$$

where the first map is the change-of-group isomorphism and the second is the finite covering.

Thus

$$\mathcal{I}_H^G(a')_g = \sum_r r \cdot a'_{r^{-1}gr}$$

where r goes over a set of representatives of $(G/H)^g$, in other words, $r^{-1}gr$ goes over a set of representatives of conjugacy classes in H conjugate to g in G.

$$\mathcal{I}_H^G(a')_g = \begin{cases} Ind_{\Lambda_H}^{\Lambda_G}(a'_g) & \text{if } g \text{ is conjuate to some element } h \text{ in } H; \\ 0 & \text{if there is no element conjugate to } g \text{ in } H. \end{cases} \tag{6.2}$$

There is another way to describe the transfer, which is shown in Rezk's unpublished work [25] for quasi-elliptic cohomology. The transfer of Tate K-theory can be described similarly.

6.3 The main theorem

Theorem 6.4 gives a classification of finite subgroups of the Tate curve and a similar conclusion for the quasi-elliptic cohomology. We prove it in this section by representation theory. We assume the readers are familiar with the transfer ideal I_{tr} of equivariant K-theory. References for that include Chapter II, [19] and Section 1.8, [24].

Let N be an integer. Analogous to the transfer ideal I_{tr} of equivariant K-theory, we can define the transfer ideal for Tate K-theory

$$I_{tr}^{Tate} := \sum_{\substack{i+j=N,\\ N>j>0}} \text{Image}[I_{\Sigma_i\times\Sigma_j}^{\Sigma_N} : K_{Tate}(\text{pt}/\!\!/\Sigma_i \times \Sigma_j) \longrightarrow K_{Tate}(\text{pt}/\!\!/\Sigma_N)] \tag{6.3}$$

where I_H^G is the transfer map of K_{Tate} along $H \hookrightarrow G$ defined in Proposition 2.23, [11], and the transfer ideal for quasi-elliptic cohomology

$$\mathcal{I}_{tr}^{QEll} := \sum_{\substack{i+j=N,\\ N>j>0}} \text{Image}[\mathcal{I}_{\Sigma_i\times\Sigma_j}^{\Sigma_N} : QEll(\text{pt}/\!\!/\Sigma_i \times \Sigma_j) \longrightarrow QEll(\text{pt}/\!\!/\Sigma_N)] \tag{6.4}$$

with $\mathcal{I}_H^G$ the transfer map of $QEll$ along $H \hookrightarrow G$ defined in (6.1).

Theorem 6.4 *The Tate K-theory of symmetric groups modulo the transfer ideal I_{tr}^{Tate} classifies the finite subgroups of the Tate curve. Explicitly,*

$$(K_{Tate})_{\Sigma_N}(pt)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[q']/\langle q^d - q'^e\rangle, \tag{6.5}$$

where q' is the image of q under the power operation P^{Tate} constructed in Definition 3.15, [11]. *The product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.*

We have the analogous conclusion for quasi-elliptic cohomology.

$$QEll_{\Sigma_N}(pt)/\mathcal{I}_{tr}^{QEll} \cong \prod_{N=de} \mathbb{Z}[q^{\pm}][q']/\langle q^d - q'^e\rangle, \tag{6.6}$$

where q' is the image of q under the power operation $\mathbb{P}_N$ constructed in Sect. 4.2. The product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

We show the proof of (6.6). The proof of (6.5) is similar.

Proof of (6.6) We divide the elements in Σ_N into two cases.

Case I

The decomposition of σ has cycles of different length. For example, the element

$$(1\ 2)(3\ 4)(5\ 6)(7\ 8\ 9\ 10)(11\ 12\ 13\ 14)(15\ 16\ 17) \in \Sigma_{17}$$

is in this case and (1 2)(3 4)(5 6), (1 2 3 4 5)(6 7 8 9 10) are not.

Most elements in Σ_N belong to Case I. σ is not in this case if and only if it consists of cycles of the same length, such as (1 2)(3 4), (1 2 3), 1, (1 2 3)(4 5 6).

For those σ that belong to Case I, $\Lambda_{\Sigma_N}(\sigma) = \Lambda_{\Sigma_r\times\Sigma_{N-r}}(\sigma)$, so $Ind_{\Lambda_{\Sigma_r\times\Sigma_{N-r}}(\sigma)}^{\Lambda_{\Sigma_N}(\sigma)}$ is the identity map, so $K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt})$ is equal to $Ind_{\Lambda_{\Sigma_r\times\Sigma_{N-r}}(\sigma)}^{\Lambda_{\Sigma_N}(\sigma)} K_{\Lambda_{\Sigma_r\times\Sigma_{N-r}}(\sigma)}(\mathrm{pt})$. Thus, the summand corresponding to σ in $QEll(\mathrm{pt}/\!\!/\Sigma_N)$ is completely cancelled.

Case II

σ consists of cycles of the same length. In other words, it consists of d e-cycles with $N = de$.

The centralizer $C_{\Sigma_N}(\sigma) \cong C_e \wr \Sigma_d$, where $C_e = \mathbb{Z}/e\mathbb{Z}$ is the cyclic group with order e. We have

$$\Lambda_{\Sigma_N}(\sigma) \cong \Lambda_{\Sigma_e}(12\ldots e) \wr_{\mathbb{T}} \Sigma_d$$

is the subgroup of $\Lambda_{\Sigma_e}(12\ldots e) \wr \Sigma_d$ with elements of the form

$$([a_1, t], [a_2, t], \ldots [a_d, t]; \tau), \text{ with } a_1, \ldots a_d \in C_e,\ \tau \in \Sigma_d,\ t \in \mathbb{R}.$$

$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt})$ is the representation ring $R\Lambda_{\Sigma_N}(\sigma)$. According to Theorem 4.8, as a $\mathbb{Z}[q^{\pm}]$-module, it has the basis

$$\{Ind_{\Lambda_{\Sigma_e}(12\ldots e)\wr_{\mathbb{T}}\Sigma_{(d)}}^{\Lambda_{\Sigma_e}(12\ldots e)\wr_{\mathbb{T}}\Sigma_d}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d_1} \otimes_{\mathbb{Z}[q^{\pm}]}\cdots\otimes_{\mathbb{Z}[q^{\pm}]}(q^{\frac{a_r}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d_r}\otimes D_\tau \mid$$
$$(d) = (d_1, d_2, \ldots d_r) \text{ is a partition of } d.$$
$$a_1, a_2, \ldots a_r \text{ are in } \{0, 1, \ldots e-1\}.\ \tau \in R\Sigma_{(d)} \text{ is irreducible.}\}$$

where for each $a \in \mathbb{Z}$, $q^{\frac{a}{e}} : \Lambda_{C_e}((12\ldots e)) \longrightarrow U(1)$ is the map

$$q^{\frac{a}{e}}([(12\ldots e)^j, t]) = e^{2\pi i a\frac{j+t}{e}}. \tag{6.7}$$

Namely, it is the map x_1^a in the sense of Example 3.3.

For each partition (d) of d, if it has more than one cycle, $\Sigma_{(d)}$ is a subgroup of some $\Sigma_{d_1} \times \Sigma_{d-d_1}$ for some positive integer $0 < d_1 < d$. So for each

$$Ind^{\Lambda_{\Sigma_e}(12...e)\wr_{\mathbb{T}}\Sigma_d}_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_{(d)}}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d_1} \otimes_{\mathbb{Z}[q^{\pm}]} \cdots \otimes_{\mathbb{Z}[q^{\pm}]} (q^{\frac{a_r}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d_r} \otimes D_\tau$$

with $r \geq 2$, it is equal to

$$Ind^{\Lambda_{\Sigma_e}(12...e)\wr_{\mathbb{T}}\Sigma_d}_{\Lambda_{\Sigma_e}(12...e)\wr_{\mathbb{T}}(\Sigma_{d_1}\times\Sigma_{d-d_1})}(Ind^{\Lambda_{\Sigma_e}(12...e)\wr_{\mathbb{T}}(\Sigma_{d_1}\times\Sigma_{d-d_1})}_{\Lambda_{\Sigma_e}(12...e)\wr_{\mathbb{T}}\Sigma_{(d)}}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d_1} \otimes_{\mathbb{Z}[q^{\pm}]} \cdots \otimes_{\mathbb{Z}[q^{\pm}]} (q^{\frac{a_r}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d_r} \otimes D_\tau)$$

by the property of induced representation. Note that

$$\Lambda_{\Sigma_e}(12\ldots e) \wr_{\mathbb{T}} (\Sigma_{d_1} \times \Sigma_{d-d_1}) \cong \Lambda_{\Sigma_{d_1 e}\times\Sigma_{N-d_1 e}}(\sigma).$$

So

$$Ind^{\Lambda_{\Sigma_e}(12...e)\wr_{\mathbb{T}}(\Sigma_{d_1}\times\Sigma_{d-d_1})}_{\Lambda_{\Sigma_e}(12...e)\wr_{\mathbb{T}}\Sigma_{(d)}}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d_1} \otimes_{\mathbb{Z}[q^{\pm}]} \cdots \otimes_{\mathbb{Z}[q^{\pm}]} (q^{\frac{a_r}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d_r} \otimes D_\tau$$

is in $K_{\Lambda_{\Sigma_{d_1 e}\times\Sigma_{N-d_1 e}}(\sigma)}(\mathrm{pt})$, Thus, each base element with $r \geq 2$ is contained in the transfer ideal.

When $r = 1$, consider

$$(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d} \otimes D_\tau$$

with $\tau \in R\Sigma_d$. As indicated in Proposition 1.1 and Corollary 1.5 in [4], each τ, except the trivial representation of Σ_d, can be induced from a representation τ' in some $R(\Sigma_i \times \Sigma_{d-i})$ with $d > i > 0$.

Claim The representation

$$Ind^{\Lambda_{\Sigma_e}(12...e)\wr_{\mathbb{T}}\Sigma_d}_{\Lambda_{\Sigma_e}(12...e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}i} \otimes_{\mathbb{Z}[q^{\pm}]} (q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}(d-i)} \otimes D_{\tau'}$$

is isomorphic to

$$(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d} \otimes D_{Ind^{\Sigma_d}_{\Sigma_i\times\Sigma_{d-i}}\tau'},$$

which is

$$(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]}d} \otimes D_\tau.$$

To prove this, we consider a set $\{\tau_\alpha\}_{\alpha\in\Sigma_d/\Sigma_i\times\Sigma_{d-i}}$ of coset representatives. Then

$$\{\eta_\alpha := (1, \ldots 1; \tau_\alpha)\}_{\alpha\in\Sigma_d/\Sigma_i\times\Sigma_{d-i}}$$

is a set of coset representatives of

$$\big(\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}\Sigma_d\big)/\big(\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})\big).$$

Let W be a representation space of $\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})$, Then

$$Ind^{\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}\Sigma_d}_{\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})}W$$

is the direct product of $[\Sigma_d:\Sigma_i\times\Sigma_{d-i}]$ copies of W. For any element

$$H=(g_1,\dots g_d;\beta)\in\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}\Sigma_d,$$

and each $\alpha\in\Sigma_d/\Sigma_i\times\Sigma_{d-i}$, there is a unique $\alpha'\in\Sigma_d/\Sigma_i\times\Sigma_{d-i}$ and a unique

$$J_\alpha=(g'_1,\dots g'_d;\gamma_\alpha)\in\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})$$

such that $H\eta_\alpha=\eta_{\alpha'}J_\alpha$. Note that

$$g'_1,\dots g'_d$$

is a permutation of

$$g_1,\dots g_d.$$

So $(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^\pm]}d}(g'_1,\dots g'_d)=(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^\pm]}d}(g_1,\dots g_d)$. In addition, $\beta\tau_\alpha=\tau_{\alpha'}\gamma_\alpha$. Let

$$\prod_\alpha w_\alpha$$

be an element in

$$Ind^{\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}\Sigma_d}_{\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})}W.$$

We have

$$\begin{aligned}&\left(Ind^{\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}\Sigma_d}_{\Lambda_{\Sigma_e}(12\dots e)\wr_{\mathbb{T}}(\Sigma_i\times\Sigma_{d-i})}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^\pm]}i}\otimes_{\mathbb{Z}[q^\pm]}(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^\pm]}d-i}\otimes D_{\tau'}\right)(H)\left(\prod_\alpha w_\alpha\right)\\&=\prod_\alpha J_\alpha w_{\beta(\alpha)}=\prod_\alpha(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^\pm]}d}(g_1,\dots g_d)D_{\tau'}(1,\dots 1;\gamma_\alpha)(w_{\beta\alpha})\\&=(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^\pm]}d}(g_1,\dots g_d)\prod_\alpha\tau'(\gamma_\alpha)(w_{\beta\alpha})\\&=(q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^\pm]}d}(g_1,\dots g_d)(Ind^{\Sigma_d}_{\Sigma_i\times\Sigma_{d-i}}\tau')(\beta)\left(\prod_\alpha w_\alpha\right)\end{aligned}$$

$$= (q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]} d}(g_1, \dots g_d; \beta) D_{Ind^{\Sigma_d}_{\Sigma_i \times \Sigma_{d-i}} \tau'}(g_1, \dots g_d; \beta) \left(\prod_{\alpha} w_\alpha\right)$$

$$= \left((q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]} d} \otimes D_{Ind^{\Sigma_d}_{\Sigma_i \times \Sigma_{d-i}} \tau'}\right)(g_1, \dots g_d; \beta) \left(\prod_{\alpha} w_\alpha\right)$$

So the claim is proved.

Since

$$\{Ind^{\Sigma_d}_{\Sigma_i \times \Sigma_{d-i}} \tau' \mid \tau' \in R(\Sigma_i \times \Sigma_{d-i}) \text{ and } i = 1, 2, \dots d-1.\}$$

contains all the irreducible representation of Σ_d except the trivial representation, which is corresponding to the partition (d), thus, by the claim, $K_{\Lambda_{\Sigma_N}(\sigma)}(\text{pt})$ modulo the image of the transfer, is a $\mathbb{Z}[q^{\pm}]$-module generated by the equivalent classes represented by

$$\{((q^{\frac{a}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]} d})^{\sim} \mid a = 0, 1, \dots e-1\}. \tag{6.8}$$

For any a, $(q^{\frac{a}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]} d}$ is $(q^{\frac{1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]} d}$ to the a-th power. Note that, by (4.17), $(q^{\frac{1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]} d}$ is

$$q' := \mathbb{P}_\sigma(q).$$

To get the isomorphism (6.6), consider a map

$$\Psi : \mathbb{Z}[q^{\pm}][x] \longrightarrow K_{\Lambda_{\Sigma_N}(\sigma)}(\text{pt})/\mathcal{I}^{QEll}_{tr}$$

by sending q to q and x to q', which is a well-defined $\mathbb{Z}[q^{\pm}]$-homomorphism.

Since $q'^e = q^d$, $K_{\Lambda_{\Sigma_N}(\sigma)}(\text{pt})/\mathcal{I}^{QEll}_{tr}$ is a $\mathbb{Z}[q^{\pm}]$-module generated by

$$1, q', \dots q'^{e-1}.$$

So any element in it can be expressed as

$$\sum_{j=0}^{e-1} f_j(q) q'^j$$

where each $f_j(q)$ is in the polynomial ring $\mathbb{Z}[q^{\pm}]$. It is the image of

$$\sum_{j=0}^{e-1} f_j(q) x^j$$

in $\mathbb{Z}[q^{\pm}][x]$. So Ψ is surjective.

Then we study its kernel. If

$$F := \sum_{j=0}^{e-1} f_j(q)q'^j$$

is in $\mathcal{I}_{tr}^{QEll}$, then it is in $\mathbb{Z}[q^{\pm}]$. So we can assume $F = 0$.

For each element $[(a_1, \dots a_d; \beta), t]$ in $\Lambda_{\Sigma_N}(\sigma)$ with $(a_1, \dots a_d; \beta) \in C_{\Sigma_N}(\sigma)$,

$$q([(a_1, \dots a_d; \beta), t]) = e^{2\pi i t}, \tag{6.9}$$

$$q'([(a_1, \dots a_d; \beta), t]) = e^{\frac{2\pi i(a_1+\dots a_d+dt)}{e}}. \tag{6.10}$$

$$\begin{aligned} F([(a_1, \dots a_d; \beta), t]) &= \sum_{j=0}^{e-1} f_j(q)q'^j([(a_1, \dots a_d; \beta), t]) \\ &= \sum_{j=0}^{e-1} f_j(e^{2\pi i t})e^{\frac{2\pi i j(a_1+\cdots+a_d+dt)}{e}} \\ &= \sum_{j=0}^{e-1} f_j(e^{2\pi i t})e^{\frac{2\pi i j d t}{e}} e^{\frac{2\pi i j(a_1+\cdots+a_d)}{e}}. \end{aligned}$$

Let

$$F_j(t) := f_j(e^{2\pi i t})e^{\frac{2\pi i j d t}{e}}$$

be the complex-valued function in the variable t. Let α denote the number $e^{\frac{2\pi i}{e}}$. The integers

$$(a_1 + \cdots + a_d)$$

go over $0, 1, \dots e-1$. Consider the e equations

$$\sum_{j=0}^{e-1} F_j(t)\alpha^{jk} = 0, \text{ for } k = 0, 1, \dots e-1.$$

In other words,

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{e-1} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(e-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha^{e-1} & \alpha^{2(e-1)} & \cdots & \alpha^{(e-1)^2} \end{pmatrix} \begin{pmatrix} F_0(t) \\ F_1(t) \\ F_2(t) \\ \vdots \\ F_{e-1}(t) \end{pmatrix} = 0$$

The determinant of the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{e-1} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(e-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha^{e-1} & \alpha^{2(e-1)} & \cdots & \alpha^{(e-1)^2} \end{pmatrix}$$

is

$$\prod_{j=0}^{e-2} \prod_{k=j+1}^{e-1} (\alpha^k - \alpha^j). \tag{6.11}$$

When $\alpha = e^{\frac{2\pi i}{e}}$, each $(\alpha^k - \alpha^j)$ in the product (6.11) is nonzero, so for any e, the determinant is nonzero and the matrix is non-singular. So we get $F_j(t) = 0$ for any $t \in \mathbb{R}$ and $j = 0, 1, 2, \ldots e-1$.

So each $f_j(q)$ in F is the zero polynomial.

The kernel of Ψ is the ideal generated by $q'^e - q^d$. □

From the power operation of quasi-elliptic cohomology, we can construct a new operation for quasi-elliptic cohomology.

Proposition 6.5 *The composition*

$$\begin{aligned}
\overline{P}_N : QEll_G(X) &\xrightarrow{\mathbb{P}_N} QEll_{G\wr\Sigma_N}(X^{\times N}) \xrightarrow{res} QEll_{G\times\Sigma_N}(X^{\times N}) \\
&\xrightarrow{diag^*} QEll_{G\times\Sigma_N}(X) \cong QEll_G(X) \otimes_{\mathbb{Z}[q^\pm]} QEll_{\Sigma_N}(pt) \\
&\longrightarrow QEll_G(X) \otimes_{\mathbb{Z}[q^\pm]} QEll_{\Sigma_N}(pt)/\mathcal{I}_{tr}^{QEll} \\
&\cong QEll_G(X) \otimes_{\mathbb{Z}[q^\pm]} \prod_{N=de} \mathbb{Z}[q^\pm][q']/\langle q^d - q'^e \rangle
\end{aligned}$$

defines a ring homomorphism, where res is the restriction map by the inclusion

$$G \times \Sigma_N \hookrightarrow G \wr \Sigma_N,\ (g, \sigma) \mapsto (g, \ldots g; \sigma),$$

$diag$ is the diagonal map

$$X \longrightarrow X^{\times N},\ x \mapsto (x, \ldots x)$$

and the last map is the isomorphism (6.6).

Proof Let $V = \prod_{g\in G_{conj}} V_g \in QEll_G(X)$. Apply the explicit formula of the power operation in (4.17), the composition $diag^* \circ res \circ \mathbb{P}_N$ sends V to

$$\prod_{\substack{g\in G_{conj} \\ \sigma\in\Sigma_{N\,conj}}} \otimes_k \otimes_{(i_1,\ldots i_k)} V_{g^k} q^{\frac{1}{k}}$$

where $(i_1, \ldots i_k)$ goes over all the k-cycles of σ, and the tensor products are those of the $\mathbb{Z}[q^{\pm}]$-algebras. Then, as shown in the proof of (6.6), after taking the quotient by the transfer ideal $\mathcal{I}_{tr}^{QEll}$, all the factors in $diag^* \circ res \circ \mathbb{P}_N(V)$ are cancelled except those corresponding to the elements in $\Sigma_{N\,conj}$ with cycles of the same length. For the factor corresponding to the element $\sigma \in \Sigma_{N\,conj}$ with d e-cycles and $de = N$, the nontrivial part is $V_{g^e,d} \otimes_{\mathbb{Z}[q^{\pm}]} q'_{d,e}$ where $V_{g^e,d}$ is the fixed point space of $V_{g^e}^{\otimes_{\mathbb{Z}[q,q^{-1}]} d}$ by the permutations Σ_d and $q'_{d,e} = \mathbb{P}_\sigma(q) = (q^{\frac{1}{e}})^{\otimes_{\mathbb{Z}[q,q^{-1}]} d}$.

Thus,

$$\overline{P}_N(V) = \prod_{\substack{g \in G_{conj} \\ N=de}} V_{g^e,d} \otimes_{\mathbb{Z}[q^{\pm}]} q'_{d,e}. \tag{6.12}$$

Let V, W be two elements in $QEll_G(X)$. We have

$$\begin{aligned}
&(V \oplus W)_{g^e,d} = V_{g^e,d} \oplus W_{g^e,d} \text{ and } (V \otimes W)_{g^e,d} = V_{g^e,d} \otimes W_{g^e,d}. \\
&\overline{P}_N(V \oplus W) = \prod_{\substack{g \in G_{conj} \\ N=de}} (V \oplus W)_{g^e,d} \otimes_{\mathbb{Z}[q^{\pm}]} q'_{d,e} \\
&\qquad = \left(\prod_{\substack{g \in G_{conj} \\ N=de}} V_{g^e,d} \otimes_{\mathbb{Z}[q^{\pm}]} q'_{d,e} \right) \oplus \left(\prod_{\substack{g \in G_{conj} \\ N=de}} W_{g^e,d} \otimes_{\mathbb{Z}[q^{\pm}]} q'_{d,e} \right) \\
&\qquad = \overline{P}_N(V) \oplus \overline{P}_N(W).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{P}_N(V \otimes W) &= \prod_{\substack{g \in G_{conj} \\ N=de}} (V \otimes W)_{g^e,d} \otimes_{\mathbb{Z}[q^{\pm}]} q'_{d,e} \\
&= \left(\prod_{\substack{g \in G_{conj} \\ N=de}} V_{g^e,d} \otimes_{\mathbb{Z}[q^{\pm}]} q'_{d,e} \right) \otimes \left(\prod_{\substack{g \in G_{conj} \\ N=de}} W_{g^e,d} \otimes_{\mathbb{Z}[q^{\pm}]} q'_{d,e} \right) \\
&= \overline{P}_N(V) \otimes \overline{P}_N(W).
\end{aligned}$$

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