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Tamarkin's construction is equivariant with respect to the action of the Grothendieck–Teichmueller group

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Abstract Recall that Tamarkin's construction (Hinich, Forum Math 15(4):591–614, 2003, arXiv:math.QA/0003052; Tamarkin, 1998, arXiv:math/9803025) gives us a map from the set of Drinfeld associators to the set of homotopy classes of L_∞ quasi-isomorphisms for Hochschild cochains of a polynomial algebra. Due to results of Drinfeld (Algebra i Analiz 2(4):149–181, 1990) and Willwacher Invent Math 200(3):671–760, 2015 both the source and the target of this map are equipped with natural actions of the Grothendieck–Teichmueller group GRT_1. In this paper, we use the result from Paljug (JHRS, 2015, arXiv:1305.4699) to prove that this map from the set of Drinfeld associators to the set of homotopy classes of L_∞ quasi-isomorphisms for Hochschild cochains is GRT_1-equivariant.

Keywords Formality theorems · Algebraic operads · Associators

Contents

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1 Introduction

Let $\mathbb{K}$ be a field of characteristic zero, $A = \mathbb{K}[x^1, x^2, \dots, x^d]$ be the algebra of functions on the affine space $\mathbb{K}^d$, and V_A be the algebra of polyvector fields on $\mathbb{K}^d$. Let us recall that Tamarkin's construction [15,24] gives us a map from the set of Drinfeld associators to the set of homotopy classes of L_∞ quasi-isomorphisms from V_A to the Hochschild cochain complex $C^\bullet(A) := C^\bullet(A, A)$ of A.

In paper [27], among proving many other things, Thomas Willwacher constructed a natural action of the Grothendieck–Teichmueller group GRT_1 from [11] on the set of homotopy classes of L_∞ quasi-isomorphisms from V_A to $C^\bullet(A)$. On the other hand, it is known [11] that the group GRT_1 acts simply transitively on the set of Drinfeld associators.

The goal of this paper is to prove GRT_1-equivariance of the map resulting from Tamarkin's construction using Theorem 4.3 from [22]. We should remark that the statement about GRT_1-equivariance of Tamarkin's construction was made in [27] (see the last sentence of Sect. 10.2 in [27, Version 3]) in which the author stated that "it is easy to see". The modest goal of this paper is to convince the reader that this statement can indeed be proved easily. However, the proof requires an additional tool developed in [22].

In this paper, we also prove various statements related to Tamarkin's construction [15,24] which are "known to specialists" but not proved in the literature in the desired generality. In fact, even the formulation of the problem of GRT_1-equivariance of Tamarkin's construction requires some additional work.

In this paper, Tamarkin's construction is presented in the slightly more general setting of graded affine space versus the particular case of the usual affine space. Thus, A is always the free (graded) commutative algebra over $\mathbb{K}$ in variables $x^1, x^2, \dots, x^d$ of (not necessarily zero) degrees $t_1, t_2, \dots, t_d$, respectively. Furthermore, V_A denotes the Gerstenhaber algebra of polyvector fields on the corresponding graded affine space, i.e.

$$V_A := S_A\left(\mathbf{s}\,\mathrm{Der}_{\mathbb{K}}(A)\right),$$

where $\mathrm{Der}_{\mathbb{K}}(A)$ denotes the A-module of derivations of A, $\mathbf{s}$ is the operator which shifts the degree up by 1, and $S_A(M)$ denotes the free (graded) commutative algebra on the A-module M.

The paper is organized as follows. In Sect. 2, we briefly review the main part of Tamarkin's construction and prove that it gives us a map $\mathfrak{T}$ [see Eq. (2.20)] from the set of homotopy classes of certain quasi-isomorphisms of dg operads to the set of homotopy classes of L_∞ quasi-isomorphisms for Hochschild cochains of A.

In Sect. 3, we introduce a (prounipotent) group which is isomorphic (due to Willwacher's theorem [27, Theorem 1.2]) to the prounipotent part GRT_1 of the Grothendieck–Teichmueller group GRT introduced in [11] by Drinfeld. We recall from [27] the actions of the group (isomorphic to GRT_1) both on the source and the target of the map $\mathfrak{T}$ (2.20). Finally, we prove the main result of this paper (see Theorem 3.3) which says that Tamarkin's map $\mathfrak{T}$ [see Eq. (2.20)] is GRT_1-equivariant.

In Sect. 4, we recall how to use the map $\mathfrak{T}$ [see Eq. (2.20) from Sect. 2], a specific solution of Deligne's conjecture on the Hochschild complex, and the formality of the operad of little discs [25] to construct a map from the set of Drinfeld associators to the set of homotopy classes of L_∞ quasi-isomorphisms for Hochschild cochains of A. Finally, we deduce, from Theorem 3.3, GRT_1-equivariance of the resulting map from the set of Drinfeld associators. The latter statement (see Corollary 4.1 in Sect. 4) can be deduced from what is written in [27] and Theorem 3.3 given in Sect. 3. However, we decided to add Sect. 4 just to make the story more complete.

Appendices, at the end of the paper, are devoted to proofs of various technical statements used in the body of the paper.

Remark 1.1 While this paper was in preparation, the 4-th version of preprint [27] appeared on arXiv.org. In Remark 10.1 of [27, Version 4], Willwacher gave a sketch of admittedly more economic proof of equivariance of Tamarkin's construction with respect to the action of GRT_1.

1.1 Notation and conventions

The ground field $\mathbb{K}$ has characteristic zero. For most of algebraic structures considered here, the underlying symmetric monoidal category is the category $\mathsf{Ch}_{\mathbb{K}}$ of unbounded cochain complexes of $\mathbb{K}$-vector spaces. We will frequently use the ubiquitous combination "dg" (differential graded) to refer to algebraic objects in $\mathsf{Ch}_{\mathbb{K}}$. For a cochain complex V we denote by $\mathbf{s}V$ (resp. by $\mathbf{s}^{-1}V$) the suspension (resp. the desuspension) of V. In other words,

$$(\mathbf{s}V)^\bullet = V^{\bullet-1}, \quad \left(\mathbf{s}^{-1}V\right)^\bullet = V^{\bullet+1}.$$

Any $\mathbb{Z}$-graded vector space V is tacitly considered as the cochain complex with the zero differential. For a homogeneous vector v in a cochain complex or a graded vector space the notation $|v|$ is reserved for its degree.

The notation S_n is reserved for the symmetric group on n letters and $\mathrm{Sh}_{p_1,\dots,p_k}$ denotes the subset of $(p_1,\dots,p_k)$-shuffles in S_n, i.e. $\mathrm{Sh}_{p_1,\dots,p_k}$ consists of elements $\sigma \in S_n$, $n = p_1 + p_2 + \cdots + p_k$ such that

$$\begin{gathered}\sigma(1) < \sigma(2) < \cdots < \sigma(p_1),\\ \sigma(p_1+1) < \sigma(p_1+2) < \cdots < \sigma(p_1+p_2),\\ \cdots\\ \sigma(n-p_k+1) < \sigma(n-p_k+2) < \cdots < \sigma(n).\end{gathered}$$

We tacitly assume the Koszul sign rule. In particular,

$$(-1)^{\varepsilon(\sigma; v_1,\dots,v_m)}$$

will always denote the sign factor corresponding to the permutation $\sigma \in S_m$ of homogeneous vectors $v_1, v_2, \dots, v_m$. Namely,

$$(-1)^{\varepsilon(\sigma; v_1,\dots,v_m)} := \prod_{(i<j)} (-1)^{|v_i||v_j|}, \tag{1.1}$$

where the product is taken over all inversions $(i < j)$ of $\sigma \in S_m$.

For a pair V, W of $\mathbb{Z}$-graded vector spaces we denote by

$$\mathrm{Hom}(V, W)$$

the corresponding inner-hom object in the category of $\mathbb{Z}$-graded vector spaces, i.e.

$$\mathrm{Hom}(V, W) := \bigoplus_m \mathrm{Hom}^m_{\mathbb{K}}(V, W), \tag{1.2}$$

where $\mathrm{Hom}^m_{\mathbb{K}}(V, W)$ consists of $\mathbb{K}$-linear maps $f : V \to W$ such that

$$f(V^\bullet) \subset W^{\bullet+m}.$$

For a commutative algebra B and a B-module M, the notation $S_B(M)$ (resp. $\underline{S}_B(M)$) is reserved for the symmetric B-algebra (resp. the truncated symmetric B-algebra) on M, i.e.

$$S_B(M) := B \oplus M \oplus S^2_B(M) \oplus S^3_B(M) \oplus \cdots,$$

and

$$\underline{S}_B(M) := M \oplus S^2_B(M) \oplus S^3_B(M) \oplus \cdots.$$

For an A_∞-algebra $\mathcal{A}$, the notation $C^\bullet(\mathcal{A})$ is reserved for the Hochschild cochain complex of $\mathcal{A}$ with coefficients in $\mathcal{A}$.

We denote by Com (resp. Lie, Ger) the operad governing commutative (and associative) algebras without unit (resp. the operad governing Lie algebras, Gerstenhaber algebras[1] without unit). Furthermore, we denote by coCom the cooperad which is obtained from Com by taking the linear dual. The coalgebras over coCom are cocommutative (and coassociative) coalgebras without counit.

The notation Cobar is reserved for the cobar construction [5, Section 3.7].

For an operad (resp. a cooperad) P and a cochain complex V we denote by $P(V)$ the free P-algebra (resp. the cofree[2] P-coalgebra) generated by V:

[1] See, for example, Appendix A in [9].

[2] We tacitly assume that all coalgebras are nilpotent.

$$P(V) := \bigoplus_{n\ge 0} \big(P(n)\otimes V^{\otimes n}\big)_{S_n}. \tag{1.3}$$

For example,

$$\mathsf{Com}(V) = \mathsf{coCom}(V) = \underline{S}(V).$$

We denote by Λ the underlying collection of the endomorphism operad

$$\mathsf{End}_{\mathbf{s}\,\mathbb{K}}$$

of the one-dimensional space $\mathbf{s}\,\mathbb{K}$ placed in degree 1. The n-the space of Λ is

$$\Lambda(n) = \mathrm{sgn}_n \otimes \mathbf{s}^{1-n},$$

where sgn_n denotes the sign representation of the symmetric group S_n. Recall that Λ is naturally an operad and a cooperad.

For a (co)operad P, we denote by ΛP the (co)operad which is obtained from P by tensoring with Λ:

$$\Lambda P := \Lambda \otimes P.$$

It is clear that tensoring with

$$\Lambda^{-1} := \mathsf{End}_{\mathbf{s}^{-1}\,\mathbb{K}}$$

gives us the inverse of the operation $P \mapsto \Lambda P$.

For example, the dg operad $\mathrm{Cobar}(\Lambda\mathsf{coCom})$ governs L_∞-algebras and the dg operad

$$\mathrm{Cobar}(\Lambda^2\mathsf{coCom}) \tag{1.4}$$

governs $\Lambda\mathsf{Lie}_\infty$-algebras.

1.1.1 Ger_∞*-algebras and a basis in* $\mathsf{Ger}^\vee(n)$

Let us recall that Ger_∞-algebras (or homotopy Gerstenhaber algebras) are governed by the dg operad

$$\mathrm{Cobar}(\mathsf{Ger}^\vee), \tag{1.5}$$

where $\mathsf{Ger}^\vee$ is the cooperad which is obtained by taking the linear dual of $\Lambda^{-2}\mathsf{Ger}$.

For our purposes, it is convenient to introduce the free $\Lambda^{-2}\mathsf{Ger}$-algebra $\Lambda^{-2}\mathsf{Ger}(b_1, b_2, \ldots, b_n)$ in n auxiliary variables $b_1, b_2, \ldots, b_n$ of degree 0 and identify the n-th space $\Lambda^{-2}\mathsf{Ger}(n)$ of $\Lambda^{-2}\mathsf{Ger}$ with the subspace of $\Lambda^{-2}\mathsf{Ger}(b_1, b_2, \ldots, b_n)$ spanned by $\Lambda^{-2}\mathsf{Ger}$-monomials in which each variable b_j appears exactly once. For example, $\Lambda^{-2}\mathsf{Ger}(2)$ is spanned by the monomials b_1b_2 and $\{b_1, b_2\}$ of degrees 2 and 1, respectively.

Let us consider the ordered partitions of the set $\{1, 2, \ldots, n\}$

$$\{i_{11}, i_{12}, \ldots, i_{1p_1}\} \sqcup \{i_{21}, i_{22}, \ldots, i_{2p_2}\} \sqcup \cdots \sqcup \{i_{t1}, i_{t2}, \ldots, i_{tp_t}\} \tag{1.6}$$

satisfying the following properties:

- for each $1 \le \beta \le t$ the index $i_{\beta p_\beta}$ is the biggest among $i_{\beta 1}, \ldots, i_{\beta p_\beta}$
- $i_{1p_1} < i_{2p_2} < \cdots < i_{tp_t}$ (in particular, $i_{tp_t} = n$).

It is clear that the monomials

$$\{b_{i_{11}}, \ldots, \{b_{i_{1(p_1-1)}}, b_{i_{1p_1}}\}..\} \ldots \{b_{i_{t1}}, \ldots, \{b_{i_{t(p_t-1)}}, b_{i_{tp_t}}\}..\} \tag{1.7}$$

corresponding to all ordered partitions (1.6) satisfying the above properties form a basis of the space $\Lambda^{-2}\mathsf{Ger}(n)$.

In this paper, we use the notation

$$\Big(\{b_{i_{11}}, \ldots, \{b_{i_{1(p_1-1)}}, b_{i_{1p_1}}\}..\} \ldots \{b_{i_{t1}}, \ldots, \{b_{i_{t(p_t-1)}}, b_{i_{tp_t}}\}..\}\Big)^* \tag{1.8}$$

for the elements of the dual basis in $\mathsf{Ger}^\vee(n) = \big(\Lambda^{-2}\mathsf{Ger}(n)\big)^*$.

1.1.2 The dg operad Braces

In this brief subsection, we recall the dg operad Braces from [9, Section 9] and [17].[3]

Following [9], we introduce, for every $n \ge 1$, the auxiliary set $\mathcal{T}(n)$. An element of $\mathcal{T}(n)$ is a planted[4] planar tree T with the following data

- a partition of the set $V(T)$ of vertices

$$V(T) = V_{\text{lab}}(T) \sqcup V_\nu(T) \sqcup V_{root}(T)$$

  into the singleton $V_{root}(T)$ consisting of the root vertex, the set $V_{\text{lab}}(T)$ consisting of n vertices which we call *labeled*, and the set $V_\nu(T)$ consisting of vertices which we call *neutral*;
- a bijection between the set $V_{\text{lab}}(T)$ and the set $\{1, 2, \ldots, n\}$.

We require that each element T of $\mathcal{T}(n)$ satisfies this condition.

Condition 1.2 *Every neutral vertex of T has at least 2 incoming edges.*

Elements of $\mathcal{T}(n)$ are called *brace trees*.

For $n \ge 1$, the vector space $\mathsf{Braces}(n)$ consists of all finite linear combinations of brace trees in $\mathcal{T}(n)$. To define a structure of a graded vector space on $\mathsf{Braces}(n)$, we declare that each brace tree $T \in \mathcal{T}(n)$ carries degree

$$|T| = 2|V_\nu(T)| - |E(T)| + 1, \tag{1.9}$$

where $|V_\nu(T)|$ denotes the total number of neutral vertices of T and $|E(T)|$ denotes the total number of edges of T.

3 In paper [17], the dg operad Braces is called the “minimal operad”.

4 Recall that a *planted* tree is a rooted tree whose root vertex has valency 1.

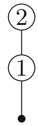

Fig. 1 A brace tree $T \in \mathcal{T}(2)$

Fig. 2 A brace tree $T_{21} \in \mathcal{T}(2)$

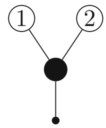

Fig. 3 A brace tree $T_{\cup} \in \mathcal{T}(2)$

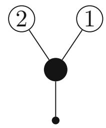

Fig. 4 A brace tree $T_{\cup^{opp}} \in \mathcal{T}(2)$

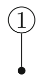

Fig. 5 The brace tree $T_{\mathsf{id}} \in \mathcal{T}(1)$

Examples of brace trees in $\mathcal{T}(2)$ (and hence vectors in $\mathsf{Braces}(2)$) are shown on Figs. 1, 2, 3 and 4.

According to (1.9), the brace trees T and T_{21} on Figs. 1 and 2, respectively, carry degree -1 and the brace trees $T_{\cup}$, $T_{\cup^{opp}}$ on Figs. 3 and 4, respectively, carry degree 0.

Condition 1.2 implies that $\mathcal{T}(1)$ consists of exactly one brace tree T_{id} shown on Fig. 5.

Hence we have $\mathsf{Braces}(1) = \mathbb{K}$.

Finally, we set $\mathsf{Braces}(0) = \mathbf{0}$.

For the definition of the operadic multiplications on Braces, we refer the reader to[5] [9, Section 8] and, in particular, Example 8.2. For the definition of the differential on Braces, we refer the reader to [9, Section 8.1] and, in particular, Example 8.4.

Let us also recall that the dg operad Braces acts naturally on the Hochschild cochain complex $C^{\bullet}(\mathcal{A})$ of any A_{∞}-algebra $\mathcal{A}$. For example, if T (resp. T_{21}) is the brace tree shown on Fig. 1 (resp. Fig. 2), then the expression

$$T(P_1, P_2) + T_{21}(P_1, P_2), \quad P_1, P_2 \in C^{\bullet}(\mathcal{A})$$

coincides (up to a sign factor) with the Gerstenhaber bracket of P_1 and P_2. Similarly, if $T_{\cup}$ is the brace tree shown on Fig. 3, then the expression

[5] Strictly speaking Braces is a suboperad of the dg operad defined in [9, Section 8].

$$T_{\cup}(P_1, P_2), \quad P_1, P_2 \in C^{\bullet}(\mathcal{A})$$

coincides (up to a sign factor) with the cup product of P_1 and P_2.

For the precise construction of the action of Braces on $C^{\bullet}(\mathcal{A})$, we refer the reader to [9, Appendix B].

2 Tamarkin's construction in a nutshell

Various solutions of Deligne's conjecture on the Hochschild cochain complex [3,4,8,17,21,23,26] imply that the dg operad Braces is quasi-isomorphic to the dg operad

$$C_{-\bullet}(E_2, \mathbb{K})$$

of singular chains for the little disc operad E_2.

Combining this statement with the formality [18,25] for the dg operad $C_{-\bullet}(E_2, \mathbb{K})$, we conclude that the dg operad Braces is quasi-isomorphic to the operad Ger. Hence there exists a quasi-isomorphism of dg operads

$$\Psi : \mathsf{Ger}_{\infty} \to \mathsf{Braces} \tag{2.1}$$

for which the vector[6] $\Psi(\mathbf{s}(b_1 b_2)^*)$ is cohomologous to the sum $T + T_{21}$ and the vector $\Psi(\mathbf{s}\{b_1, b_2\}^*)$ is cohomologous to

$$\frac{1}{2}(T_{\cup} + T_{\cup^{opp}}),$$

where T (resp. T_{21}, $T_{\cup}$, $T_{\cup^{opp}}$) is the brace tree depicted on Fig. 1 (resp. Figs. 2, 3, 4).

Replacing Ψ by a homotopy equivalent map we may assume, without loss of generality, that

$$\Psi(\mathbf{s}(b_1 b_2)^*) = T + T_{21}, \quad \Psi(\mathbf{s}\{b_1, b_2\}^*) = \frac{1}{2}(T_{\cup} + T_{\cup^{opp}}). \tag{2.2}$$

So from now on we will assume that the map Ψ (2.1) satisfies conditions (2.2).

Since the dg operad Braces acts on the Hochschild cochain complex $C^{\bullet}(\mathcal{A})$ of an A_{∞}-algebra $\mathcal{A}$, the map Ψ equips the Hochschild cochain complex $C^{\bullet}(\mathcal{A})$ with a structure of a Ger_{∞}-algebra. We will call it *Tamarkin's* Ger_{∞}*-structure* and denote by

$$C^{\bullet}(\mathcal{A})^{\Psi}$$

the Hochschild cochain complex of $\mathcal{A}$ with the Ger_{∞}-structure coming from Ψ.

The choice of the homotopy class of Ψ (2.1) (and hence the choice of Tamarkin's Ger_{∞}-structure) is far from unique. In fact, it follows from [27, Theorem 1.2] that, the set of homotopy classes of maps (2.1) satisfying conditions (2.2) form a torsor for an infinite dimensional pro-algebraic group.

[6] Here, we use basis (1.8) in $\mathsf{Ger}^{\vee}(n)$.

A simple degree bookkeeping in Braces shows that for every $n \geq 3$

$$\Psi(\mathbf{s}(b_1 b_2 \cdots b_n)^*) = 0. \tag{2.3}$$

Combining this observation with (2.2) we see that any Tamarkin's Ger_∞-structure on $C^\bullet(\mathcal{A})$ satisfies the following remarkable property:

Property 2.1 The $\Lambda\mathsf{Lie}_\infty$ part of Tamarkin's Ger_∞-structure on $C^\bullet(\mathcal{A})$ coincides with the $\Lambda\mathsf{Lie}$-structure given by the Gerstenhaber bracket on $C^\bullet(\mathcal{A})$.

From now on, we only consider the case when $\mathcal{A} = A$, i.e. the free (graded) commutative algebra over $\mathbb{K}$ in variables $x^1, x^2, \ldots, x^d$ of (not necessarily zero) degrees $t_1, t_2, \ldots, t_d$, respectively. Furthermore, V_A denotes the Gerstenhaber algebra of polyvector fields on the corresponding graded affine space, i.e.

$$V_A := S_A\left(\mathbf{s}\,\mathrm{Der}_{\mathbb{K}}(A)\right).$$

It is known[7] [16] that the canonical embedding

$$V_A \hookrightarrow C^\bullet(A) \tag{2.4}$$

is a quasi-isomorphism of cochain complexes, where V_A is considered with the zero differential. In this paper, we refer to (2.4) as the *Hochschild–Kostant–Rosenberg embedding*.

Let us now consider the Ger_∞-algebra $C^\bullet(A)^\Psi$ for a chosen map Ψ (2.1). By the first claim of Corollary 6.4 from Appendix B, there exists a Ger_∞-quasi-isomorphism

$$U_{\mathsf{Ger}} : V_A \rightsquigarrow C^\bullet(A)^\Psi \tag{2.5}$$

whose linear term coincides with the Hochschild–Kostant–Rosenberg embedding.

Restricting U_{Ger} to the $\Lambda^2\mathsf{coCom}$-coalgebra

$$\Lambda^2\mathsf{coCom}(V_A)$$

and taking into account Property 2.1 we get a $\Lambda\mathsf{Lie}_\infty$-quasi-isomorphism

$$U_{\mathsf{Lie}} : V_A \rightsquigarrow C^\bullet(A) \tag{2.6}$$

of (dg) $\Lambda\mathsf{Lie}$-algebras.

Thus we deduced the main statement of Tamarkin's construction [24] which can be summarized as

[7] Paper [16] treats only the case of usual (not graded) affine algebras. However, the proof of [16] can be generalized to the graded setting in a straightforward manner.

Theorem 2.2 (Tamarkin [24]) *Let A (resp. V_A) be the algebra of functions (resp. the algebra of polyvector fields) on a graded affine space. Let us consider the Hochschild cochain complex $C^\bullet(A)$ with the standard $\Lambda\mathsf{Lie}$-algebra structure. Then, for every map of dg operads Ψ (2.1), there exists a $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism*

$$U_{\mathsf{Lie}} : V_A \rightsquigarrow C^\bullet(A) \tag{2.7}$$

which can be extended to a Ger_∞ quasi-isomorphism

$$U_{\mathsf{Ger}} : V_A \rightsquigarrow C^\bullet(A)^\Psi$$

where V_A carries the standard Gerstenhaber algebra structure. □

Remark 2.3 In this paper we tacitly assume that the linear part of every $\Lambda\mathsf{Lie}_\infty$ (resp. Ger_∞) quasi-isomorphism from V_A to $C^\bullet(A)$ (resp. $C^\bullet(A)^\Psi$) coincides with the Hochschild–Kostant–Rosenberg embedding of polyvector fields into Hochschild cochains.

Since the above construction involves several choices it leaves the following two obvious questions:

Question A Is it possible to construct two homotopy inequivalent $\Lambda\mathsf{Lie}_\infty$-quasi-isomorphisms (2.6) corresponding to the same map Ψ (2.1)? And if no then

Question B Are $\Lambda\mathsf{Lie}_\infty$-quasi-isomorphisms U_{Lie} and $\widetilde{U}_{\mathsf{Lie}}$ (2.6) homotopy equivalent if so are the corresponding maps of dg operads Ψ and $\widetilde{\Psi}$ (2.1)?

The (expected) answer (NO) to Question A is given in the following proposition:

Proposition 2.4 *Let Ψ a map of dg operads (2.1) satisfying (2.2) and*

$$U_{\mathsf{Lie}}, \widetilde{U}_{\mathsf{Lie}} : V_A \rightsquigarrow C^\bullet(A) \tag{2.8}$$

be $\Lambda\mathsf{Lie}_\infty$ quasi-morphisms which extend to Ger_∞ quasi-isomorphisms

$$U_{\mathsf{Ger}}, \widetilde{U}_{\mathsf{Ger}} : V_A \rightsquigarrow C^\bullet(A)^\Psi \tag{2.9}$$

respectively. Then U_{Lie} is homotopy equivalent to $\widetilde{U}_{\mathsf{Lie}}$.

Proof This statement is essentially a consequence of general Corollary 6.4 from Appendix B.2.

Indeed, the second claim of Corollary 6.4 implies that Ger_∞-morphisms (2.9) are homotopy equivalent. Hence so are their restrictions to the $\Lambda^2\mathsf{coCom}$-coalgebra

$$\Lambda^2\mathsf{coCom}(V_A)$$

which coincide with U_{Lie} and $\widetilde{U}_{\mathsf{Lie}}$, respectively. □

The expected answer (YES) to Question B is given in the following addition to Theorem 2.2:

Theorem 2.5 *The homotopy type of* U_{Lie} (2.6) *depends only on the homotopy type of the map* Ψ (2.1).

Proof Let Ψ and $\widetilde{\Psi}$ be maps of dg operads (2.1) satisfying (2.2) and let

$$U_{\mathsf{Lie}} : V_A \rightsquigarrow C^{\bullet}(A) \tag{2.10}$$
$$\widetilde{U}_{\mathsf{Lie}} : V_A \rightsquigarrow C^{\bullet}(A) \tag{2.11}$$

be $\Lambda\mathsf{Lie}_{\infty}$ quasi-morphisms which extend to Ger_{∞} quasi-isomorphisms

$$U_{\mathsf{Ger}} : V_A \rightsquigarrow C^{\bullet}(A)^{\Psi}, \quad \text{and} \quad \widetilde{U}_{\mathsf{Ger}} : V_A \rightsquigarrow C^{\bullet}(A)^{\widetilde{\Psi}} \tag{2.12}$$

respectively. Our goal is to show that if Ψ is homotopy equivalent to $\widetilde{\Psi}$ then U_{Lie} is homotopy equivalent to $\widetilde{U}_{\mathsf{Lie}}$.

Let us denote by $\Omega^{\bullet}(\mathbb{K})$ the dg commutative algebra of polynomial forms on the affine line with the canonical coordinate t.

Since quasi-isomorphisms $\Psi, \widetilde{\Psi} : \mathsf{Ger}_{\infty} \to \mathsf{Braces}$ are homotopy equivalent, we have[8] a map of dg operads

$$\mathfrak{H} : \mathsf{Ger}_{\infty} \to \mathsf{Braces} \otimes \Omega^{\bullet}(\mathbb{K}) \tag{2.13}$$

such that

$$\Psi = p_0 \circ \mathfrak{H}, \quad \text{and} \quad \widetilde{\Psi} = p_1 \circ \mathfrak{H},$$

where p_0 and p_1 are the canonical maps (of dg operads)

$$p_0, p_1 : \mathsf{Braces} \otimes \Omega^{\bullet}(\mathbb{K}) \to \mathsf{Braces},$$
$$p_0(v) := v\,\Big|_{dt=0,\ t=0}, \quad p_1(v) := v\,\Big|_{dt=0,\ t=1}.$$

The map $\mathfrak{H}$ induces a Ger_{∞}-structure on $C^{\bullet}(A) \otimes \Omega^{\bullet}(\mathbb{K})$ such that the evaluation maps (which we denote by the same letters)

$$\begin{aligned} p_0 &: C^{\bullet}(A) \otimes \Omega^{\bullet}(\mathbb{K}) \to C^{\bullet}(A)^{\Psi}, \quad p_0(v) := v\big|_{dt=0,\ t=0}, \\ p_1 &: C^{\bullet}(A) \otimes \Omega^{\bullet}(\mathbb{K}) \to C^{\bullet}(A)^{\widetilde{\Psi}}, \quad p_1(v) := v\big|_{dt=0,\ t=1}. \end{aligned} \tag{2.14}$$

are strict quasi-isomorphisms of the corresponding Ger_{∞}-algebras.

So, in this proof, we consider the cochain complex $C^{\bullet}(A) \otimes \Omega^{\bullet}(\mathbb{K})$ with the Ger_{∞}-structure coming from $\mathfrak{H}$ (2.13). The same degree bookkeeping argument in Braces shows that[9]

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[8] For justification of this step see, for example, [5, Section 5.1].

[9] Here, we use basis (1.8) in $\mathsf{Ger}^{\vee}(n)$.

$$\mathfrak{H}(\mathbf{s}(b_1 b_2 \cdots b_n)^*) = 0. \tag{2.15}$$

Hence, the $\Lambda\mathsf{Lie}_\infty$ part of the Ger_∞-structure on $C^\bullet(A) \otimes \Omega^\bullet(\mathbb{K})$ coincides with the $\Lambda\mathsf{Lie}$-structure given by the Gerstenhaber bracket extended from $C^\bullet(A)$ to $C^\bullet(A) \otimes \Omega^\bullet(\mathbb{K})$ to by $\Omega^\bullet(\mathbb{K})$-linearity.

Since the canonical embedding

$$P \mapsto P \otimes 1 : C^\bullet(A) \hookrightarrow C^\bullet(A) \otimes \Omega^\bullet(\mathbb{K})$$

is a quasi-isomorphism of cochain complexes, Corollary 6.4 from Appendix B.2 implies that there exists a Ger_∞ quasi-isomorphism

$$U^{\mathfrak{H}}_{\mathsf{Ger}} : V_A \rightsquigarrow C^\bullet(A) \otimes \Omega^\bullet(\mathbb{K}), \tag{2.16}$$

where V_A is considered with the standard Gerstenhaber structure.

Since the $\Lambda\mathsf{Lie}_\infty$ part of the Ger_∞-structure on $C^\bullet(A) \otimes \Omega^\bullet(\mathbb{K})$ coincides with the standard $\Lambda\mathsf{Lie}$-structure, the restriction of $U^{\mathfrak{H}}_{\mathsf{Ger}}$ to the $\Lambda^2\mathsf{coCom}$-coalgebra $\Lambda^2\mathsf{coCom}(V_A)$ gives us a homotopy connecting the $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism

$$p_0 \circ U^{\mathfrak{H}}_{\mathsf{Ger}}\Big|_{\Lambda^2\mathsf{coCom}(V_A)} : V_A \rightsquigarrow C^\bullet(A) \tag{2.17}$$

to the $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism

$$p_1 \circ U^{\mathfrak{H}}_{\mathsf{Ger}}\Big|_{\Lambda^2\mathsf{coCom}(V_A)} : V_A \rightsquigarrow C^\bullet(A), \tag{2.18}$$

where p_0 and p_1 are evaluation maps (2.14).

Let us now observe that $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphisms (2.17) and (2.18) extend to Ger_∞ quasi-isomorphisms

$$p_0 \circ U^{\mathfrak{H}}_{\mathsf{Ger}} : V_A \rightsquigarrow C^\bullet(A)^{\Psi}, \quad \text{and} \quad p_1 \circ U^{\mathfrak{H}}_{\mathsf{Ger}} : V_A \rightsquigarrow C^\bullet(A)^{\widetilde{\Psi}} \tag{2.19}$$

respectively. Hence, by Proposition 2.4, $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism (2.17) is homotopy equivalent to (2.10) and $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism (2.18) is homotopy equivalent to (2.11).

Thus $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphisms (2.10) and (2.11) are indeed homotopy equivalent. □

The general conclusion of this section is that Tamarkin's construction [15,24] gives us a map

$$\mathfrak{T} : \pi_0\,(\mathsf{Ger}_\infty \to \mathsf{Braces}) \to \pi_0\left(V_A \rightsquigarrow C^\bullet(A)\right) \tag{2.20}$$

from the set $\pi_0\,(\mathsf{Ger}_\infty \to \mathsf{Braces})$ of homotopy classes of operad morphisms (2.1) satisfying conditions (2.2) to the set $\pi_0\,(V_A \rightsquigarrow C^\bullet(A))$ of homotopy classes of $\Lambda\mathsf{Lie}_\infty$-morphisms from V_A to $C^\bullet(A)$ whose linear term is the Hochschild–Kostant–Rosenberg embedding.

3 Actions of GRT_1

Let $\mathcal{C}$ be a coaugmented cooperad in the category of graded vector spaces and $\mathcal{C}_\circ$ be the cokernel of the coaugmentation. We assume that $\mathcal{C}(0) = \mathbf{0}$ and $\mathcal{C}(1) = \mathbb{K}$.

Let us denote by

$$\mathrm{Der}'\left(\mathrm{Cobar}(\mathcal{C})\right) \tag{3.1}$$

the dg Lie algebra of derivation $\mathcal{D}$ of $\mathrm{Cobar}(\mathcal{C})$ satisfying the condition

$$p_{\mathbf{s}\,\mathcal{C}_\circ} \circ \mathcal{D} = 0, \tag{3.2}$$

where $p_{\mathbf{s}\,\mathcal{C}_\circ}$ is the canonical projection $\mathrm{Cobar}(\mathcal{C}) \to \mathbf{s}\,\mathcal{C}_\circ$. Conditions $\mathcal{C}(0) = \mathbf{0}, \mathcal{C}(1) = \mathbb{K}$ and (3.2) imply that $\mathrm{Der}'\left(\mathrm{Cobar}(\mathcal{C})\right)^0$ and $H^0\left(\mathrm{Der}'(\mathrm{Cobar}(\mathcal{C}))\right)$ are pronilpotent Lie algebras.

In this paper, we are mostly interested in the case when $\mathcal{C} = \Lambda^2\mathsf{coCom}$ and $\mathcal{C} = \mathsf{Ger}^\vee$. The corresponding dg operads $\Lambda\mathsf{Lie}_\infty = \mathrm{Cobar}(\Lambda^2\mathsf{coCom})$ and $\mathsf{Ger}_\infty = \mathrm{Cobar}(\mathsf{Ger}^\vee)$ govern $\Lambda\mathsf{Lie}_\infty$ and Ger_∞ algebras, respectively.

A simple degree bookkeeping shows that

$$\mathrm{Der}'(\Lambda\mathsf{Lie}_\infty)^{\le 0} = \mathbf{0}, \tag{3.3}$$

i.e. the dg Lie algebra $\mathrm{Der}'(\Lambda\mathsf{Lie}_\infty)$ does not have non-zero elements in degrees ≤ 0. In particular, the Lie algebra $H^0\left(\mathrm{Der}'(\Lambda\mathsf{Lie}_\infty)\right)$ is zero.

On the other hand, the Lie algebra

$$\mathfrak{g} = H^0\left(\mathrm{Der}'(\mathsf{Ger}_\infty)\right) \tag{3.4}$$

is much more interesting. According to Willwacher's theorem [27, Theorem 1.2], this Lie algebra is isomorphic to the pro-nilpotent part $\mathfrak{grt}_1$ of the Grothendieck–Teichmueller Lie algebra $\mathfrak{grt}$ [1, Section 4.2]. Hence, the group $\exp(\mathfrak{g})$ is isomorphic to the group $\mathsf{GRT}_1 = \exp(\mathfrak{grt}_1)$.

Let us now describe how the group $\exp(\mathfrak{g}) \cong \mathsf{GRT}_1$ acts both on the source and the target of Tamarkin's map $\mathfrak{T}$ (2.20).

3.1 The action of GRT_1 on π_0 ($\mathsf{Ger}_\infty \to \mathsf{Braces}$)

Let v be a vector of $\mathfrak{g}$ represented by a (degree zero) cocycle $\mathcal{D} \in \mathrm{Der}'(\mathsf{Ger}_\infty)$. Since the Lie algebra $\mathrm{Der}'(\mathsf{Ger}_\infty)^0$ is pro-nilpotent, $\mathcal{D}$ gives us an automorphism

$$\exp(\mathcal{D}) \tag{3.5}$$

of the operad Ger_∞.

Let Ψ be a quasi-isomorphism of dg operads (2.1). Due to Proposition B.2 in [22], the homotopy type of the composition

$$\Psi \circ \exp(\mathcal{D})$$

does not depend on the choice of the cocycle $\mathcal{D}$ in the cohomology class v. Furthermore, for every pair of (degree zero) cocycles $\mathcal{D}, \widetilde{\mathcal{D}} \in \mathrm{Der}'(\mathsf{Ger}_\infty)$ we have

$$\Psi \circ \exp(\mathcal{D}) \circ \exp(\widetilde{\mathcal{D}}) = \Psi \circ \exp\big(\mathrm{CH}(\mathcal{D}, \widetilde{\mathcal{D}})\big)\,,$$

where $\mathrm{CH}(x, y)$ denotes the Campbell–Hausdorff series in symbols x, y.

Thus the assignment

$$\Psi \to \Psi \circ \exp(\mathcal{D})$$

induces a *right* action of the group $\exp(\mathfrak{g})$ on the set $\pi_0\,(\mathsf{Ger}_\infty \to \mathsf{Braces})$ of homotopy classes of operad morphisms (2.1).

3.2 The action of GRT_1 on $\pi_0\,(V_A \rightsquigarrow C^\bullet(A))$

Let us now show that $\exp(\mathfrak{g}) \cong \mathsf{GRT}_1$ also acts on the set $\pi_0\,(V_A \rightsquigarrow C^\bullet(A))$ of homotopy classes of $\Lambda\mathsf{Lie}_\infty$-morphisms from V_A to $C^\bullet(A)$.

For this purpose, we denote by

$$\mathrm{Act}_{stan} : \mathsf{Ger}_\infty \to \mathsf{End}_{V_A} \tag{3.6}$$

the operad map corresponding to the standard Gerstenhaber structure on V_A.

Then, given a cocycle $\mathcal{D} \in \mathrm{Der}'(\mathsf{Ger}_\infty)$ representing $v \in \mathfrak{g}$, we may precompose map (3.6) with automorphism (3.5). This way, we equip the graded vector space V_A with a new Ger_∞-structure $Q^{\exp(\mathcal{D})}$ whose binary operations are the standard ones. Therefore, by Corollary 6.3 from Appendix B.1, there exists a Ger_∞ quasi-isomorphism

$$U_{\mathrm{corr}} : V_A \to V_A^{Q^{\exp(\mathcal{D})}} \tag{3.7}$$

from V_A with the standard Gerstenhaber structure to V_A with the Ger_∞-structure $Q^{\exp(\mathcal{D})}$.

Due to observation (3.3), the restriction of $\mathcal{D}$ onto the suboperad $\mathrm{Cobar}(\Lambda^2\mathsf{coCom}) \subset \mathrm{Cobar}(\mathsf{Ger}^\vee)$ is zero. Hence, for every degree zero cocycle $\mathcal{D} \in \mathrm{Der}'(\mathsf{Ger}_\infty)$, we have

$$\exp(\mathcal{D})\Big|_{\mathrm{Cobar}(\Lambda^2\mathsf{coCom})} = \mathrm{Id} : \mathrm{Cobar}(\Lambda^2\mathsf{coCom}) \to \mathrm{Cobar}(\Lambda^2\mathsf{coCom}). \tag{3.8}$$

Therefore the $\Lambda\mathsf{Lie}_\infty$-part of the Ger_∞-structure $Q^{\exp(\mathcal{D})}$ coincides with the standard $\Lambda\mathsf{Lie}$-structure on V_A given by the Schouten bracket. Hence the restriction of the

Ger_∞ quasi-isomorphism U_{corr} onto the $\Lambda^2\mathsf{coCom}$-coalgebra $\Lambda^2\mathsf{coCom}(V_A)$ gives us a $\Lambda\mathsf{Lie}_\infty$-automorphism

$$U^{\mathcal{D}} : V_A \rightsquigarrow V_A. \tag{3.9}$$

Note that, for a fixed Ger_∞-structure $Q^{\exp(\mathcal{D})}$, Ger_∞ quasi-isomorphism (3.7) is far from unique. However, the second statement of Corollary 6.4 implies that the homotopy class of (3.7) is unique. Therefore, the assignment

$$\mathcal{D} \mapsto \left[U^{\mathcal{D}}\right]$$

is a well defined map from the set of degree zero cocycles of $\mathrm{Der}'(\mathsf{Ger}_\infty)$ to homotopy classes of $\Lambda\mathsf{Lie}_\infty$-automorphisms of V_A.

This statement can be strengthened further:

Proposition 3.1 *The homotopy type of $U^{\mathcal{D}}$ does not depend on the choice of the representative $\mathcal{D}$ of the cohomology class v. Furthermore, for any pair of degree zero cocycles $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Der}'(\mathsf{Ger}_\infty)$, the composition $U^{\mathcal{D}_1} \circ U^{\mathcal{D}_2}$ is homotopy equivalent to $U^{\mathrm{CH}(\mathcal{D}_1, \mathcal{D}_2)}$, where $\mathrm{CH}(x, y)$ denotes the Campbell–Hausdorff series in symbols x, y.*

Let us postpone the technical Proof of Proposition 3.1 to Sect. 3.4 and observe that this proposition implies the following statement:

Corollary 3.2 *Let $\mathcal{D}$ be a degree zero cocycle in $\mathrm{Der}'(\mathsf{Ger}_\infty)$ representing a cohomology class $v \in \mathfrak{g}$ and let U_{Lie} be a $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism from V_A to $C^\bullet(A)$. The assignment*

$$U_{\mathsf{Lie}} \mapsto U_{\mathsf{Lie}} \circ U^{\mathcal{D}} \tag{3.10}$$

induces a right action of the group $\exp(\mathfrak{g})$ on the set $\pi_0\,(V_A \rightsquigarrow C^\bullet(A))$ of homotopy classes of $\Lambda\mathsf{Lie}_\infty$-morphisms from V_A to $C^\bullet(A)$. □

From now on, by abuse of notation, we denote by $U^{\mathcal{D}}$ any representative in the homotopy class of $\Lambda\mathsf{Lie}_\infty$-automorphism (3.9).

3.3 The theorem on GRT_1-equivariance

The following theorem is the main result of this paper:

Theorem 3.3 *Let $\pi_0\,(\mathsf{Ger}_\infty \to \mathsf{Braces})$ be the set of homotopy classes of operad maps (2.1) from the dg operad Ger_∞ governing homotopy Gerstenhaber algebras to the dg operad Braces of brace trees. Let $\pi_0\,(V_A \rightsquigarrow C^\bullet(A))$ be the set of homotopy classes of $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphisms*[10] *from the algebra V_A of polyvector fields to*

[10] We tacitly assume that operad maps (2.1) satisfies conditions (2.2) and $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphisms $V_A \rightsquigarrow C^\bullet(A)$ extend the Hochschild–Kostant–Rosenberg embedding.

the algebra $C^\bullet(A)$ of Hochschild cochains of a graded affine space. Then Tamarkin's map $\mathfrak{T}$ (2.20) commutes with the action of the group $\exp(\mathfrak{g})$ *which corresponds to Lie algebra* (3.4).

Proof Following [22, Section 3], [13], we will denote by $\mathrm{Cyl}(\mathsf{Ger}^\vee)$ the 2-colored dg operad whose algebras are pairs (V, W) with the data

1. a Ger_∞-structure on V,
2. a Ger_∞-structure on W, and
3. a Ger_∞-morphism F from V to W, i.e. a homomorphism of corresponding dg $\mathsf{Ger}^\vee$-coalgebras $\mathsf{Ger}^\vee(V) \to \mathsf{Ger}^\vee(W)$.

In fact, if we forget about the differential, then the operad $\mathrm{Cyl}(\mathsf{Ger}^\vee)$ is a free operad on a certain 2-colored collection $\mathcal{M}(\mathsf{Ger}^\vee)$ naturally associated to $\mathsf{Ger}^\vee$.

Let us denote by

$$\mathrm{Der}'(\mathrm{Cyl}(\mathsf{Ger}^\vee)) \tag{3.11}$$

the dg Lie algebra of derivations $\mathcal{D}$ of $\mathrm{Cyl}(\mathsf{Ger}^\vee)$ subject to the condition[11]

$$p \circ \mathcal{D} = 0, \tag{3.12}$$

where p is the canonical projection from $\mathrm{Cyl}(\mathsf{Ger}^\vee)$ onto $\mathcal{M}(\mathsf{Ger}^\vee)$.

The restrictions to the first color part and the second color part of $\mathrm{Cyl}(\mathsf{Ger}^\vee)$, respectively, give us natural maps of dg Lie algebras

$$\mathrm{res}_1,\ \mathrm{res}_2 : \mathrm{Der}'(\mathrm{Cyl}(\mathsf{Ger}^\vee)) \to \mathrm{Der}'(\mathsf{Ger}_\infty), \tag{3.13}$$

and, due to [22, Theorem 4.3], res_1 and res_2 are chain homotopic quasi-isomorphisms.

Therefore, for every $v \in \mathfrak{g}$ there exists a degree zero cocycle

$$\mathcal{D} \in \mathrm{Der}'(\mathrm{Cyl}(\mathsf{Ger}^\vee)) \tag{3.14}$$

such that both $\mathrm{res}_1(\mathcal{D})$ and $\mathrm{res}_2(\mathcal{D})$ represent the cohomology class v.

Let

$$U_{\mathsf{Ger}} : V_A \rightsquigarrow C^\bullet(A)^\Psi \tag{3.15}$$

be a Ger_∞-morphism from V_A to $C^\bullet(A)$ which restricts to a $\Lambda\mathsf{Lie}_\infty$-morphism

$$U_{\mathsf{Lie}} : V_A \to C^\bullet(A). \tag{3.16}$$

The triple consisting of

- the standard Gerstenhaber structure on V_A,

[11] It is condition (3.12) which guarantees that any degree zero cocycle in $\mathrm{Der}'(\mathrm{Cyl}(\mathsf{Ger}^\vee))$ can be exponentiated to an automorphism of $\mathrm{Cyl}(\mathsf{Ger}^\vee)$.

- the Ger_∞-structure on $C^\bullet(A)$ coming from a map Ψ, and
- Ger_∞-morphism (3.15)

gives us a map of dg operads

$$U_{\mathrm{Cyl}} : \mathrm{Cyl}(\mathsf{Ger}^\vee) \to \mathsf{End}_{V_A, C^\bullet(A)} \tag{3.17}$$

from $\mathrm{Cyl}(\mathsf{Ger}^\vee)$ to the 2-colored endomorphism operad $\mathsf{End}_{V_A, C^\bullet(A)}$ of the pair $(V_A, C^\bullet(A))$.

Precomposing U_{Cyl} with the endomorphism

$$\exp(\mathcal{D}) : \mathrm{Cyl}(\mathsf{Ger}^\vee) \to \mathrm{Cyl}(\mathsf{Ger}^\vee)$$

we get another operad map

$$U_{\mathrm{Cyl}} \circ \exp(\mathcal{D}) : \mathrm{Cyl}(\mathsf{Ger}^\vee) \to \mathsf{End}_{V_A, C^\bullet(A)} \tag{3.18}$$

which corresponds to the triple consisting of

- the new Ger_∞-structure $Q^{\exp(\mathrm{res}_1(\mathcal{D}))}$ on V_A,
- the Ger_∞-structure on $C^\bullet(A)$ corresponding to $\Psi \circ \exp(\mathrm{res}_2(\mathcal{D}))$, and
- a Ger_∞ quasi-isomorphism

$$\widetilde{U}_{\mathsf{Ger}} : V_A^{Q^{\exp(\mathrm{res}_1(\mathcal{D}))}} \rightsquigarrow C^\bullet(A)^{\Psi \circ \exp(\mathrm{res}_2(\mathcal{D}))} \tag{3.19}$$

Due to technical Proposition 7.1 proved in Appendix C below, the restriction of the Ger_∞ quasi-isomorphism $\widetilde{U}_{\mathsf{Ger}}$ (3.19) to $\Lambda^2\mathsf{coCom}(V_A)$ gives us the same $\Lambda\mathsf{Lie}_\infty$-morphism (3.16).

On the other hand, by Corollary 6.3 from Appendix B.1, there exists a Ger_∞ quasi-isomorphism

$$U_{\mathrm{corr}} : V_A \to V_A^{Q^{\exp(\mathrm{res}_1(\mathcal{D}))}} \tag{3.20}$$

from V_A with the standard Gerstenhaber structure to V_A with the new Ger_∞-structure $Q^{\exp(\mathrm{res}_1(\mathcal{D}))}$.

Thus, composing U_{corr} with $\widetilde{U}_{\mathsf{Ger}}$ (3.19), we get a Ger_∞ quasi-isomorphism

$$U_{\mathsf{Ger}}^{\exp(\mathcal{D})} : V_A \rightsquigarrow C^\bullet(A)^{\Psi \circ \exp(\mathrm{res}_2(\mathcal{D}))} \tag{3.21}$$

from V_A with the standard Gerstenhaber structure to $C^\bullet(A)$ with the Ger_∞-structure coming from $\Psi \circ \exp(\mathrm{res}_2(\mathcal{D}))$.

The restriction of this Ger_∞-morphism $U_{\mathsf{Ger}}^{\exp(\mathcal{D})}$ to $\Lambda^2\mathsf{coCom}(V_A)$ gives us the $\Lambda\mathsf{Lie}_\infty$-morphism

$$U_{\mathsf{Lie}} \circ U^{\mathrm{res}_1(\mathcal{D})} \tag{3.22}$$

where $U^{\mathrm{res}_1(\mathcal{D})}$ is the $\Lambda\mathsf{Lie}_\infty$-automorphism of V_A obtained by restricting (3.20) to $\Lambda^2\mathsf{coCom}(V_A)$.

Since both cocycles $\mathrm{res}_1(\mathcal{D})$ and $\mathrm{res}_2(\mathcal{D})$ of $\mathrm{Der}'(\mathsf{Ger}_\infty)$ represent the same cohomology class $v \in \mathfrak{g}$, Theorem 3.3 follows. □

3.4 The proof of Proposition 3.1

Let $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ be two cohomologous cocycles in $\mathrm{Der}'(\mathsf{Ger}_\infty)$ and let $Q^{\exp(\mathcal{D})}$, $Q^{\exp(\widetilde{\mathcal{D}})}$ be Ger_∞-structures on V_A corresponding to the operad maps

$$\mathrm{Act}_{stan} \circ \exp(\mathcal{D}) : \mathsf{Ger}_\infty \to \mathsf{End}_{V_A}, \tag{3.23}$$

$$\mathrm{Act}_{stan} \circ \exp(\widetilde{\mathcal{D}}) : \mathsf{Ger}_\infty \to \mathsf{End}_{V_A}, \tag{3.24}$$

respectively. Here Act_{stan} is the map $\mathsf{Ger}_\infty \to \mathsf{End}_{V_A}$ corresponding to the standard Gerstenhaber structure on V_A.

Due to Proposition B.2 in [22], operad maps (3.23) and (3.24) are homotopy equivalent. Hence there exists a Ger_∞-structure Q_t on $V_A \otimes \Omega^\bullet(\mathbb{K})$ such that the evaluation maps

$$\begin{aligned} p_0 &: V_A \otimes \Omega^\bullet(\mathbb{K}) \to V_A^{Q^{\exp(\mathcal{D})}}, \quad p_0(v) := v\big|_{dt=0,\ t=0}, \\ p_1 &: V_A \otimes \Omega^\bullet(\mathbb{K}) \to V_A^{Q^{\exp(\widetilde{\mathcal{D}})}}, \quad p_1(v) := v\big|_{dt=0,\ t=1}. \end{aligned} \tag{3.25}$$

are strict quasi-isomorphisms of the corresponding Ger_∞-algebras.

Furthermore, observation (3.3) implies that the restriction of a homotopy connecting the automorphisms $\exp(\mathcal{D})$ and $\exp(\widetilde{\mathcal{D}})$ of Ger_∞ to the suboperad $\Lambda\mathsf{Lie}_\infty$ coincides with the identity map on $\Lambda\mathsf{Lie}_\infty$ for every t. Therefore, the $\Lambda\mathsf{Lie}_\infty$-part of the Ger_∞-structure Q_t on $V_A \otimes \Omega^\bullet(\mathbb{K})$ coincides with the standard $\Lambda\mathsf{Lie}$-structure given by the Schouten bracket.

Since tensoring with $\Omega^\bullet(\mathbb{K})$ does not change cohomology, Corollary 6.4 from Appendix B.2 implies that the canonical embedding $V_A \hookrightarrow V_A \otimes \Omega^\bullet(\mathbb{K})$ can be extended to a Ger_∞ quasi-isomorphism

$$U^{\mathfrak{H}}_{\mathrm{corr}} : V_A \rightsquigarrow V_A \otimes \Omega^\bullet(\mathbb{K}) \tag{3.26}$$

from V_A with the standard Gerstenhaber structure to $V_A \otimes \Omega^\bullet(\mathbb{K})$ with the Ger_∞-structure Q_t.

Since the $\Lambda\mathsf{Lie}_\infty$-part of the Ger_∞-structure Q_t on $V_A \otimes \Omega^\bullet(\mathbb{K})$ coincides with the standard $\Lambda\mathsf{Lie}$-structure given by the Schouten bracket, the restriction of $U^{\mathfrak{H}}_{\mathrm{corr}}$ onto $\Lambda^2\mathsf{coCom}(V_A)$ gives us a homotopy connecting the $\Lambda\mathsf{Lie}_\infty$-automorphisms

$$p_0 \circ U^{\mathfrak{H}}_{\mathrm{corr}}\Big|_{\Lambda^2\mathsf{coCom}(V_A)} : V_A \rightsquigarrow V_A \tag{3.27}$$

and

$$p_1 \circ U^{\mathfrak{H}}_{\mathrm{corr}}\Big|_{\Lambda^2\mathsf{coCom}(V_A)} : V_A \rightsquigarrow V_A. \tag{3.28}$$

Due to the second part of Corollary 6.4, $\Lambda\mathsf{Lie}_\infty$-automorphism (3.27) is homotopy equivalent to $U^{\mathcal{D}}$ and $\Lambda\mathsf{Lie}_\infty$-automorphism (3.28) is homotopy equivalent to $U^{\widetilde{\mathcal{D}}}$.

Thus the homotopy type of $U^{\mathcal{D}}$ is indeed independent of the representative $\mathcal{D}$ of the cohomology class.

To prove the second claim of Proposition 3.1, we will need to use the 2-colored dg operad $\mathrm{Cyl}(\mathsf{Ger}^\vee)$ recalled in the Proof of Theorem 3.3 above. Moreover, we need [22, Theorem 4.3] which implies that restrictions (3.13) are homotopic quasi-isomorphisms of cochain complexes.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be degree zero cocycles in $\mathrm{Der}'(\mathsf{Ger}_\infty)$ and let $Q^{\exp(\mathcal{D}_1)}$ be the Ger_∞-structure on V_A which comes from the composition

$$\mathrm{Act}_{stan} \circ \exp(\mathcal{D}_1) : \mathsf{Ger}_\infty \to \mathsf{End}_{V_A}, \tag{3.29}$$

where Act_{stan} denotes the map $\mathsf{Ger}_\infty \to \mathsf{End}_{V_A}$ corresponding to the standard Gerstenhaber structure on V_A.

Let $U_{\mathsf{Ger},1}$ be a Ger_∞-quasi-isomorphism

$$U_{\mathsf{Ger},1} : V_A \rightsquigarrow V_A^{Q^{\exp(\mathcal{D}_1)}}, \tag{3.30}$$

where the source is considered with the standard Gerstenhaber structure.

By construction, the $\Lambda\mathsf{Lie}_\infty$-automorphism

$$U^{\mathcal{D}_1} : V_A \rightsquigarrow V_A$$

is the restriction of $U_{\mathsf{Ger},1}$ onto $\Lambda^2\mathsf{coCom}(V_A)$.

Let us denote by $U^{V_A}_{\mathrm{Cyl}}$ the operad map

$$U^{V_A}_{\mathrm{Cyl}} : \mathrm{Cyl}(\mathsf{Ger}^\vee) \to \mathsf{End}_{V_A, V_A}$$

which corresponds to the triple:

- the standard Gerstenhaber structure on the first copy of V_A,
- the Ger_∞-structure $Q^{\exp(\mathcal{D}_1)}$ on the second copy of V_A, and
- the chosen Ger_∞ quasi-isomorphism in (3.30).

Due to [22, Theorem 4.3], there exists a degree zero cocycle $\mathcal{D}_{\mathrm{Cyl}}$ in $\mathrm{Der}'\big(\mathrm{Cyl}(\mathsf{Ger}^\vee)\big)$ for which the cocycles

$$\mathcal{D} := \mathrm{res}_1(\mathcal{D}_{\mathrm{Cyl}}), \quad \mathcal{D}' := \mathrm{res}_2(\mathcal{D}_{\mathrm{Cyl}}) \tag{3.31}$$

are both cohomologous to the given cocycle $\mathcal{D}_2$.

Precomposing the map $U^{V_A}_{\mathrm{Cyl}}$ with the automorphism $\exp(\mathcal{D}_{\mathrm{Cyl}})$ we get a new $\mathrm{Cyl}(\mathsf{Ger}^\vee)$-algebra structure on the pair (V_A, V_A) which corresponds to the triple

- the Ger_∞-structure $Q^{\exp(\mathcal{D})}$ on the first copy of V_A,
- the Ger_∞-structure $Q^{\exp(\mathrm{CH}(\mathcal{D}_1, \mathcal{D}'))}$ on the second copy of V_A, and

- a Ger_∞ quasi-isomorphism

$$\widetilde{U}_{\mathsf{Ger}} : V_A^{Q^{\exp(\mathcal{D})}} \rightsquigarrow V_A^{Q^{\exp(\mathrm{CH}(\mathcal{D}_1, \mathcal{D}'))}}. \tag{3.32}$$

Let us observe that, due to Proposition 7.1 from Appendix C, the restriction of $\widetilde{U}_{\mathsf{Ger}}$ onto $\Lambda^2\mathsf{coCom}(V_A)$ coincides with the restriction of (3.30) onto $\Lambda^2\mathsf{coCom}(V_A)$. Hence,

$$\widetilde{U}_{\mathsf{Ger}}\Big|_{\Lambda^2\mathsf{coCom}(V_A)} = U^{\mathcal{D}_1}, \tag{3.33}$$

where $U^{\mathcal{D}_1}$ is a $\Lambda\mathsf{Lie}_\infty$-automorphism of V_A corresponding[12] to $\mathcal{D}_1$.

Recall that there exists a Ger_∞ quasi-isomorphism

$$U_{\mathsf{Ger}} : V_A \rightsquigarrow V_A^{Q^{\exp(\mathcal{D})}}. \tag{3.34}$$

where the source is considered with the standard Gerstenhaber structure. Furthermore, since $\mathcal{D}$ is cohomologous to $\mathcal{D}_2$, the first claim of Proposition 3.1 implies that the restriction of U_{Ger} onto $\Lambda^2\mathsf{coCom}(V_A)$ gives us a $\Lambda\mathsf{Lie}_\infty$-automorphism $U^{\mathcal{D}}$ of V_A which is homotopy equivalent to $U^{\mathcal{D}_2}$.

Let us also observe that the composition $\widetilde{U}_{\mathsf{Ger}} \circ U_{\mathsf{Ger}}$ gives us a Ger_∞ quasi-isomorphism

$$\widetilde{U}_{\mathsf{Ger}} \circ U_{\mathsf{Ger}} : V_A \rightsquigarrow V_A^{Q^{\exp(\mathrm{CH}(\mathcal{D}_1, \mathcal{D}'))}} \tag{3.35}$$

Hence, the restriction of $\widetilde{U}_{\mathsf{Ger}} \circ U_{\mathsf{Ger}}$ gives us a $\Lambda\mathsf{Lie}_\infty$-automorphism of V_A corresponding to $\mathrm{CH}(\mathcal{D}_1, \mathcal{D}')$. Due to (3.33), this $\Lambda\mathsf{Lie}_\infty$-automorphism coincides with

$$U^{\mathcal{D}_1} \circ U^{\mathcal{D}}.$$

Since $\mathcal{D}$ and $\mathcal{D}'$ are both cohomologous to $\mathcal{D}_2$, the second claim of Proposition 3.1 follows. □

Remark 3.4 The second claim of Proposition 3.1 can probably be deduced from [27, Proposition 5.4] and some other statements in [27]. However, this would require a digression to "stable setting" which we avoid in this paper. For this reason, we decided to present a complete proof of Proposition 3.1 which is independent of any intermediate steps in [27].

12 Strictly speaking, only the homotopy class of the $\Lambda\mathsf{Lie}_\infty$-automorphism $U^{\mathcal{D}_1}$ is uniquely determined by $\mathcal{D}_1$.

4 Final remarks: connecting Drinfeld associators to the set of homotopy classes $\pi_0\,(V_A \rightsquigarrow C^\bullet(A))$

In this section we recall how to construct a GRT_1-equivariant map $\mathfrak{B}$ from the set $\mathrm{DrAssoc}_1$ of Drinfeld associators to the set

$$\pi_0\,(\mathsf{Ger}_\infty \to \mathsf{Braces})$$

of homotopy classes of operad morphisms (2.1) satisfying conditions (2.2).

Composing $\mathfrak{B}$ with the map $\mathfrak{T}$ (2.20), we get the desired map

$$\mathfrak{T}\circ\mathfrak{B} : \mathrm{DrAssoc}_1 \to \pi_0\left(V_A \rightsquigarrow C^\bullet(A)\right) \tag{4.1}$$

from the set $\mathrm{DrAssoc}_1$ to the set of homotopy classes of $\Lambda\mathsf{Lie}_\infty$-morphisms from V_A to $C^\bullet(A)$ whose linear term is the Hochschild–Kostant–Rosenberg embedding.

Theorem 3.3 will then imply that map (4.1) is GRT_1-equivariant.

4.1 The sets $\mathrm{DrAssoc}_\kappa$ of Drinfeld associators

In this short subsection, we briefly recall Drinfeld's associators and the Grothendieck–Teichmueller group GRT_1. For more details we refer the reader to [1,2], or [11].

Let m be an integer ≥ 2. We denote by $\mathfrak{t}_m$ the Lie algebra generated by symbols $\{t^{ij} = t^{ji}\}_{1\leq i\neq j\leq m}$ subject to the following relations:

$$\begin{aligned} [t^{ij}, t^{ik} + t^{jk}] &= 0 \quad \text{for any triple of distinct indices } i, j, k, \\ [t^{ij}, t^{kl}] &= 0 \quad \text{for any quadruple of distinct indices } i, j, k, l. \end{aligned} \tag{4.2}$$

The notation $\mathsf{A}^{\mathrm{pb}}_m$ is reserved for the associative algebra (over $\mathbb{K}$) of formal power series in noncommutative symbols $\{t^{ij} = t^{ji}\}_{1\leq i\neq j\leq m}$ subject to the same relations (4.2). Let us recall [25, Section 4] that the collection $\mathsf{A}^{\mathrm{pb}} := \{\mathsf{A}^{\mathrm{pb}}_m\}_{m\geq 1}$ with $\mathsf{A}^{\mathrm{pb}}_1 := \mathbb{K}$ forms an operad in the category of associative $\mathbb{K}$-algebras.

Let $\mathfrak{lie}(x, y)$ be the degree completion of the free Lie algebra in two symbols x and y and let κ be any element of $\mathbb{K}$.

The set $\mathrm{DrAssoc}_\kappa$ consists of elements $\Phi \in \exp\,(\mathfrak{lie}(x, y))$ which satisfy the equations

$$\Phi(y, x)\Phi(x, y) = 1, \tag{4.3}$$

$$\Phi(t^{12}, t^{23} + t^{24})\,\Phi(t^{13} + t^{23}, t^{34}) = \Phi(t^{23}, t^{34})\,\Phi(t^{12} + t^{13}, t^{24} + t^{34})\,\Phi(t^{12}, t^{23}), \tag{4.4}$$

$$e^{\kappa(t^{13}+t^{23})/2} = \Phi(t^{13}, t^{12})e^{\kappa t^{13}/2}\Phi(t^{13}, t^{23})^{-1}e^{\kappa t^{23}/2}\Phi(t^{12}, t^{23}), \tag{4.5}$$

and

$$e^{\kappa(t^{12}+t^{13})/2} = \Phi(t^{23}, t^{13})^{-1} e^{\kappa t^{13}/2} \Phi(t^{12}, t^{13}) e^{\kappa t^{12}/2} \Phi(t^{12}, t^{23})^{-1}. \qquad (4.6)$$

For $\kappa \neq 0$, elements Φ of $\mathrm{DrAssoc}_\kappa$ are called Drinfeld associators. However, for our purposes, we only need the set $\mathrm{DrAssoc}_1$ and the set $\mathrm{DrAssoc}_0$.

According to [11, Section 5], the set

$$\mathrm{DrAssoc}_0 \qquad (4.7)$$

forms a prounipotent group and, by [11, Proposition 5.5], this group acts simply transitively on the set of associators in $\mathrm{DrAssoc}_1$. Following [11], we denote the group $\mathrm{DrAssoc}_0$ by GRT_1.

4.2 A map $\mathfrak{B}$ from $\mathrm{DrAssoc}_1$ to $\pi_0(\mathsf{Ger}_\infty \to \mathsf{Braces})$

Let us recall [2,25] that collections of all braid groups can be assembled into the operad PaB in the category of $\mathbb{K}$-linear categories. Similarly, the collection of algebras $\{\mathsf{A}^{\mathrm{pb}}_m\}_{m\geq 1}$ can be "upgraded" to the operad PaCD also in the category of $\mathbb{K}$-linear categories. Every associator $\Phi \in \mathrm{DrAssoc}_1$ gives us an isomorphism of these operads

$$I_\Phi : \mathsf{PaB} \xrightarrow{\cong} \mathsf{PaCD}. \qquad (4.8)$$

The group GRT_1 acts on the operad PaCD in such a way that, for every pair $g \in \mathsf{GRT}_1$, $\Phi \in \mathrm{DrAssoc}_1$, the diagram

$$\begin{array}{ccc} \mathsf{PaB} & \xrightarrow{I_\Phi} & \mathsf{PaCD} \\ \downarrow{\scriptstyle \mathsf{id}} & & \downarrow{\scriptstyle g} \\ \mathsf{PaB} & \xrightarrow{I_{g(\Phi)}} & \mathsf{PaCD} \end{array} \qquad (4.9)$$

commutes.

Applying to PaB and PaCD the functor $C_{-\bullet}(\ ,\mathbb{K})$, where $C_\bullet(\ ,\mathbb{K})$ denotes the Hochschild chain complex with coefficients in $\mathbb{K}$, we get dg operads

$$C_{-\bullet}(\mathsf{PaB}, \mathbb{K}) \qquad (4.10)$$

and

$$C_{-\bullet}(\mathsf{PaCD}, \mathbb{K}). \qquad (4.11)$$

By naturality of $C_{-\bullet}(\ ,\mathbb{K})$, diagram (4.9) gives us the commutative diagram

$$\begin{array}{ccc} C_{-\bullet}(\mathsf{PaB},\mathbb{K}) & \xrightarrow{I_\Phi} & C_{-\bullet}(\mathsf{PaCD},\mathbb{K}) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} \\ C_{-\bullet}(\mathsf{PaB},\mathbb{K}) & \xrightarrow{I_{g(\Phi)}} & C_{-\bullet}(\mathsf{PaCD},\mathbb{K}), \end{array} \tag{4.12}$$

where, for simplicity, the maps corresponding to I_Φ, $I_{g(\Phi)}$ and g are denoted by the same letters, respectively.

Recall that Eq. (5) from [25] gives us the canonical quasi-isomorphism from the operad Ger to $C_{-\bullet}(\mathsf{A}^{\mathrm{pb}},\mathbb{K})$. The latter operad, in turn, receives the natural map

$$C_{-\bullet}(\mathsf{PaCD},\mathbb{K}) \to C_{-\bullet}(\mathsf{A}^{\mathrm{pb}},\mathbb{K})$$

from $C_{-\bullet}(\mathsf{PaCD},\mathbb{K})$ which is also known to be a quasi-isomorphism.

Thus, using the lifting property (see [5, Corollary 5.8]) for maps from the operad $\mathsf{Ger}_\infty = \mathrm{Cobar}(\mathsf{Ger}^\vee)$, we get the quasi-isomorphism[13]

$$\mathsf{Ger}_\infty \ \stackrel{\sim}{\longrightarrow}\ C_{-\bullet}(\mathsf{PaCD},\mathbb{K}). \tag{4.13}$$

Using this quasi-isomorphism and [5, Corollary 5.8], one can construct (see [27, Section 6.3.1]) a group homomorphism

$$\mathsf{GRT}_1 \to \exp(\mathfrak{g}), \tag{4.14}$$

where the Lie algebra $\mathfrak{g}$ is defined in (3.4). By [27, Theorem 1.2], homomorphism (4.14) is an isomorphism.

Any specific solution of Deligne's conjecture on the Hochschild complex (see, for example, [4,8], or [21]) combined with Fiedorowicz's recognition principle [12] provides us with a sequence of quasi-isomorphisms

$$\mathsf{Braces} \stackrel{\sim}{\leftarrow} \bullet \stackrel{\sim}{\rightarrow} \bullet \stackrel{\sim}{\leftarrow} \bullet \cdots \bullet \stackrel{\sim}{\rightarrow} C_{-\bullet}(\mathsf{PaB},\mathbb{K}) \tag{4.15}$$

which connects the dg operad Braces to $C_{-\bullet}(\mathsf{PaB},\mathbb{K})$.

Hence, every associator $\Phi \in \mathrm{DrAssoc}_1$ gives us a sequence of quasi-isomorphisms

$$\mathsf{Braces} \stackrel{\sim}{\leftarrow} \bullet \stackrel{\sim}{\rightarrow} \bullet \stackrel{\sim}{\leftarrow} \bullet \cdots \bullet \stackrel{\sim}{\rightarrow} C_{-\bullet}(\mathsf{PaB},\mathbb{K}) \xrightarrow{I_\Phi} C_{-\bullet}(\mathsf{PaCD},\mathbb{K}) \stackrel{\sim}{\longleftarrow} \mathsf{Ger}_\infty \tag{4.16}$$

connecting the dg operads Braces to Ger_∞.

[13] By the same lifting property (see [5, Corollary 5.8]), we know that the homotopy type of the quasi-isomorphism (4.13) is uniquely determined by the operad map $\mathsf{Ger} \to C_{-\bullet}(\mathsf{A}^{\mathrm{pb}},\mathbb{K})$ from [25, Eq. (5)].

Using [5, Corollary 5.8] once again, we conclude that the sequence of quasi-isomorphisms (4.16) determines a unique homotopy class of quasi-isomorphisms (of dg operads)

$$\Psi : \mathsf{Ger}_\infty \to \mathsf{Braces}. \tag{4.17}$$

Thus we get a well defined map

$$\mathfrak{B} : \mathrm{DrAssoc}_1 \to \pi_0 \left(\mathsf{Ger}_\infty \to \mathsf{Braces}\right). \tag{4.18}$$

In view of isomorphism (4.14), the set of homotopy classes $\pi_0 (\mathsf{Ger}_\infty \to \mathsf{Braces})$ is equipped with a natural action of GRT_1. Moreover, the commutativity of diagram (4.12) implies that the map $\mathfrak{B}$ is GRT_1-equivariant.

Thus, combining this observation with Theorem 3.3 we deduce the following corollary:

Corollary 4.1 *Let $\pi_0 (V_A \rightsquigarrow C^\bullet(A))$ be the set of homotopy classes of $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphisms which extend the Hochschild–Kostant–Rosenberg embedding of polyvector fields into Hochschild cochains. If we consider $\pi_0 (V_A \rightsquigarrow C^\bullet(A))$ as a set with the GRT_1-action induced by isomorphism (4.14) then the composition*

$$\mathfrak{T} \circ \mathfrak{B} : \mathrm{DrAssoc}_1 \to \pi_0 \left(V_A \rightsquigarrow C^\bullet(A)\right) \tag{4.19}$$

is GRT_1-equivariant. □

Remark 4.2 Any sequence of quasi-isomorphisms of dg operads (4.15) gives us an isomorphism between the objects corresponding to $C_{-\bullet}(\mathsf{PaB}, \mathbb{K})$ and Braces in the homotopy category of dg operads. However, there is no reason to expect that different solutions of the Deligne conjecture give the same isomorphisms from $C_{-\bullet}(\mathsf{PaB}, \mathbb{K})$ to Braces in the homotopy category. Hence the resulting composition in (4.19) may depend on the choice of a specific solution of Deligne's conjecture on the Hochschild complex.

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Appendix A: Filtered $\Lambda^{-1}\mathsf{Lie}_\infty$-algebras

Let L be a cochain complex with the differential ∂. Recall that a $\Lambda^{-1}\mathsf{Lie}_\infty$-structure on L is a sequence of degree 1 multi-brackets

$$\{\ ,\ ,\dots,\ \}_m : S^m(L) \to L, \quad m \ge 2 \tag{5.1}$$

satisfying the relations

$$\partial\{v_1, v_2, \ldots, v_m\} + \sum_{i=1}^{m} (-1)^{|v_1|+\cdots+|v_{i-1}|}\{v_1, \ldots, v_{i-1}, \partial v_i, v_{i+1}, \ldots, v_m\}$$
$$+ \sum_{k=2}^{m-1} \sum_{\sigma \in \mathrm{Sh}_{k,m-k}} (-1)^{\varepsilon(\sigma; v_1,\ldots,v_m)}\{\{v_{\sigma(1)}, \ldots, v_{\sigma(k)}\}, v_{\sigma(k+1)}, \ldots, v_{\sigma(m)}\} = 0, \tag{5.2}$$

where $(-1)^{\varepsilon(\sigma;v_1,\ldots,v_m)}$ is the Koszul sign factor [see Eq. (1.1)].

We say that a $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra L is *filtered* if it is equipped with a complete descending filtration

$$L = \mathcal{F}_1 L \supset \mathcal{F}_2 L \supset \mathcal{F}_3 L \supset \ldots. \tag{5.3}$$

For such filtered $\Lambda^{-1}\mathsf{Lie}_\infty$-algebras we may define a Maurer–Cartan element as a degree zero element α satisfying the equation

$$\partial\alpha + \sum_{m \geq 2} \frac{1}{m!}\{\alpha, \alpha, \ldots, \alpha\}_m = 0. \tag{5.4}$$

Note that this equation makes sense for any degree 0 element α because $L = \mathcal{F}_1 L$ and L is complete with respect to filtration (5.3). Let us denote by $\mathrm{MC}(L)$ the set of Maurer–Cartan elements of a filtered $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra L.

According to[14] [14], the set $\mathrm{MC}(L)$ can be upgraded to an ∞-groupoid $\mathfrak{MC}(L)$ (i.e. a simplicial set satisfying the Kan condition). To introduce the ∞-groupoid $\mathfrak{MC}(L)$, we denote by $\Omega^\bullet(\Delta_n)$ the dg commutative $\mathbb{K}$-algebra of polynomial forms [14, Section 3] on the n-th geometric simplex Δ_n. Next, we declare that set of n-simplices of $\mathfrak{MC}(L)$ is

$$\mathrm{MC}\left(L \,\hat{\otimes}\, \Omega^\bullet(\Delta_n)\right), \tag{5.5}$$

where L is considered with the topology coming from filtration (5.3) and $\Omega^\bullet(\Delta_n)$ is considered with the discrete topology. The structure of the simplicial set is induced from the structure of a simplicial set on the sequence $\{\Omega^\bullet(\Delta_n)\}_{n\geq 0}$.

For example, 0-cells of $\mathfrak{MC}(L)$ are precisely Maurer–Cartan elements of L and 1-cells are sums

$$\alpha' + dt\,\alpha'', \quad \alpha' \in L^0 \,\hat{\otimes}\, \mathbb{K}[t], \quad \alpha'' \in L^{-1} \,\hat{\otimes}\, \mathbb{K}[t] \tag{5.6}$$

[14] A version of the Deligne–Getzler–Hinich ∞-groupoid for pro-nilpotent $\Lambda^{-1}\mathsf{Lie}_\infty$-algebras is introduced in [6, Section 4].

satisfying the pair of equations

$$\partial\alpha' + \sum_{m\geq 2} \frac{1}{m!}\{\alpha', \alpha', \dots, \alpha'\}_m = 0, \tag{5.7}$$

$$\frac{d}{dt}\alpha' = \partial\alpha'' + \sum_{m\geq 1} \frac{1}{m!}\{\alpha', \alpha', \dots, \alpha', \alpha''\}_{m+1}. \tag{5.8}$$

Thus, two 0-cells α_0, α_1 of $\mathfrak{MC}(L)$ (i.e. Maurer–Cartan elements of L) are isomorphic if there exists an element (5.6) satisfying (5.7) and (5.8) and such that

$$\alpha_0 = \alpha'\big|_{t=0} \quad \text{and} \quad \alpha_1 = \alpha'\big|_{t=1}. \tag{5.9}$$

We say that a 1-cell (5.6) connects α_0 and α_1.

A.1: A lemma on adjusting Maurer–Cartan elements

Let α be a Maurer–Cartan element of a filtered $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra and ξ be a degree -1 element in $\mathcal{F}_n L$ for some integer $n \geq 1$.

Let us consider the following sequence $\{\alpha'_k\}_{k\geq 0}$ of degree zero elements in $L \,\hat{\otimes}\, \mathbb{K}[t]$

$$\alpha'_0 := \alpha, \quad \alpha'_{k+1}(t) := \alpha + \int_0^t dt_1 \left(\partial\xi + \sum_{m\geq 1} \frac{1}{m!}\{\alpha'_k(t_1), \dots, \alpha'_k(t_1), \xi\}_{m+1} \right). \tag{5.10}$$

Since L is complete with respect to filtration (5.3), the sequence $\{\alpha'_k\}_{k\geq 0}$ convergences to a (degree 0) element $\alpha' \in L \,\hat{\otimes}\, \mathbb{K}[t]$ which satisfies the integral equation

$$\alpha'(t) = \alpha + \int_0^t dt_1 \left(\partial\xi + \sum_{m\geq 1} \frac{1}{m!}\{\alpha'(t_1), \dots, \alpha'(t_1), \xi\}_{m+1} \right). \tag{5.11}$$

We claim that

Lemma 5.1 *If, as above, ξ is a degree -1 element in $\mathcal{F}_n L$ and α' is an element of $L \,\hat{\otimes}\, \mathbb{K}[t]$ obtained by recursive procedure (5.10) then the sum*

$$\alpha' + dt\, \xi \tag{5.12}$$

is a 1-cell of $\mathfrak{MC}(L)$ which connects α to another Maurer–Cartan element $\widetilde{\alpha}$ of L such that

$$\alpha' - \alpha \in \mathcal{F}_n L \,\hat{\otimes}\, \mathbb{K}[t], \tag{5.13}$$

and

$$\widetilde{\alpha} - \alpha - \partial \xi \in \mathcal{F}_{n+1} L. \tag{5.14}$$

If the element ξ *satisfies the additional condition*

$$\partial \xi \in \mathcal{F}_{n+1} L \tag{5.15}$$

then

$$\alpha' - \alpha \in \mathcal{F}_{n+1} L \,\hat{\otimes}\, \mathbb{K}[t], \tag{5.16}$$

and

$$\widetilde{\alpha} - \alpha - \partial \xi - \{\alpha, \xi\} \in \mathcal{F}_{n+2} L. \tag{5.17}$$

Proof Equation (5.11) implies that α' satisfies the differential equation

$$\frac{d}{dt}\alpha' = \partial \xi + \sum_{m \geq 1} \frac{1}{m!}\{\alpha', \ldots, \alpha', \xi\}_{m+1} \tag{5.18}$$

with the initial condition

$$\alpha' \Big|_{t=0} = \alpha. \tag{5.19}$$

Let us denote by Ξ the following degree 1 element of $L \,\hat{\otimes}\, \mathbb{K}[t]$

$$\Xi := \partial \alpha' + \sum_{m \geq 2} \frac{1}{m!}\{\alpha', \alpha', \ldots, \alpha'\}_m. \tag{5.20}$$

A direct computation shows that Ξ satisfies the following differential equation

$$\frac{d}{dt}\Xi = -\sum_{m \geq 0} \frac{1}{m!}\{\alpha', \ldots, \alpha', \Xi, \xi\}_{m+2}. \tag{5.21}$$

Furthermore, since α is a Maurer–Cartan element of L, the element Ξ satisfies the condition

$$\Xi\Big|_{t=0} = 0$$

and hence Ξ satisfies the integral equation

$$\Xi(t) = -\int_0^t dt_1 \left(\sum_{m \geq 0} \frac{1}{m!}\{\alpha'(t_1), \ldots, \alpha'(t_1), \Xi(t_1), \xi\}_{m+2} \right). \tag{5.22}$$

Equation (5.22) implies that

$$\Xi \in \bigcap_{n\geq 1} \mathcal{F}_n L \,\hat{\otimes}\, \mathbb{K}[t].$$

Therefore $\Xi = 0$ and hence the limiting element α' of sequence (5.10) is a Maurer–Cartan element of $L \,\hat{\otimes}\, \mathbb{K}[t]$.

Combining this observation with differential equation (5.18), we conclude that the element $\alpha' + dt\,\xi \in L \,\hat{\otimes}\, \Omega^{\bullet}(\Delta_1)$ is indeed a 1-cell in $\mathfrak{MC}(L)$ which connects the Maurer–Cartan element α to the Maurer–Cartan element

$$\widetilde{\alpha} := \alpha + \int_0^1 dt \left(\partial\xi + \sum_{m\geq 1} \frac{1}{m!}\{\alpha'(t), \dots, \alpha'(t), \xi\}_{m+1} \right). \tag{5.23}$$

Since $\xi \in \mathcal{F}_n L$ and $L = \mathcal{F}_1 L$, equation (5.11) implies that

$$\alpha' - \alpha \in \mathcal{F}_n L \,\hat{\otimes}\, \mathbb{K}[t]$$

and equation (5.23) implies that

$$\widetilde{\alpha} - \alpha - \partial\xi \in \mathcal{F}_{n+1} L.$$

Thus, the first part of Lemma 5.1 is proved.

If $\xi \in \mathcal{F}_n L$ and $\partial\xi \in \mathcal{F}_{n+1} L$ then, again, it is clear from (5.11) that inclusion (5.16) holds.

Finally, using inclusion (5.16) and equation (5.23), it is easy to see that

$$\widetilde{\alpha} - \alpha - \partial\xi - \{\alpha, \xi\} \in \mathcal{F}_{n+2} L.$$

Lemma 5.1 is proved. □

A.2: Convolution $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra, ∞-morphisms and their homotopies

Let $\mathcal{C}$ be a coaugmented cooperad (in the category of graded vector spaces) satisfying the additional condition

$$\mathcal{C}(0) = \mathbf{0} \tag{5.24}$$

and V be a cochain complex. (In this paper, $\mathcal{C}$ is usually the cooperad $\mathsf{Ger}^\vee$).

Following [7], we say that V is a homotopy algebra of type $\mathcal{C}$ if V carries $\mathrm{Cobar}(\mathcal{C})$-algebra structure or equivalently the $\mathcal{C}$-coalgebra

$$\mathcal{C}(V)$$

has a degree 1 coderivation Q satisfying

$$Q\Big|_V = 0$$

and the Maurer–Cartan equation

$$[d_V, Q] + \frac{1}{2}[Q, Q] = 0$$

where d_V is the differential on $\mathcal{C}(V)$ induced from the one on V.

For two homotopy algebras (V, Q_V) and (W, Q_W) of type $\mathcal{C}$, we consider the graded vector space

$$\mathrm{Hom}(\mathcal{C}(V), W) \tag{5.25}$$

with the differential ∂

$$\partial(f) := d_W \circ f - (-1)^{|f|} f \circ (d_V + Q_V) \tag{5.26}$$

and the multi-brackets (of degree 1)

$$\{\ ,\ ,\dots,\}_m : S^m\left(\mathrm{Hom}(\mathcal{C}(V), W)\right) \to \mathrm{Hom}(\mathcal{C}(V), W), \quad m \geq 2$$

$$\{f_1, \dots, f_m\}(X) = p_W \circ Q_W\left(1 \otimes f_1 \otimes \cdots \otimes f_m(\Delta_m(X))\right), \tag{5.27}$$

where Δ_m is the m-th component of the comultiplication

$$\Delta_m : \mathcal{C}(V) \to \left(\mathcal{C}(m) \otimes \mathcal{C}(V)^{\otimes m}\right)^{S_m}$$

and p_W is the canonical projection

$$p_W : \mathcal{C}(W) \to W.$$

According to [7] or [10, Section 1.3], Eq. (5.27) define a $\Lambda^{-1}\mathsf{Lie}_\infty$-structure on the cochain complex $\mathrm{Hom}(\mathcal{C}(V), W)$ with the differential ∂ (5.26). The $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra

$$\mathrm{Hom}(\mathcal{C}(V), W) \tag{5.28}$$

is called the *convolution* $\Lambda^{-1}\mathsf{Lie}_\infty$*-algebra* of the pair V, W.

The convolution $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra $\mathrm{Hom}(\mathcal{C}(V), W)$ carries the obvious descending filtration "by arity"

$$\mathcal{F}_n\mathrm{Hom}(\mathcal{C}(V), W) = \left\{ f \in \mathrm{Hom}(\mathcal{C}(V), W) \mid f\big|_{\mathcal{C}(m)\otimes_{S_m} V^{\otimes m}} = 0 \ \ \forall\, m < n \right\}. \tag{5.29}$$

$\mathrm{Hom}(\mathcal{C}(V), W)$ is obviously complete with respect to this filtration and

$$\mathrm{Hom}(\mathcal{C}(V), W) = \mathcal{F}_1\mathrm{Hom}(\mathcal{C}(V), W) \tag{5.30}$$

due to condition (5.24). In other words, under our assumption on the cooperad $\mathcal{C}$, the convolution $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra $\mathrm{Hom}(\mathcal{C}(V), W)$ is pronilpotent.

According to [10, Proposition 3], ∞-morphisms from V to W are in bijection with Maurer–Cartan elements of $\mathrm{Hom}(\mathcal{C}(V), W)$ i.e. 0-cells of the Deligne–Getzler–Hinich ∞-groupoid corresponding to $\mathrm{Hom}(\mathcal{C}(V), W)$. Furthermore, due to [10, Corollary 2], two ∞-morphisms from V to W are homotopic if and only if the corresponding Maurer–Cartan elements are isomorphic 0-cells in the Deligne–Getzler–Hinich ∞-groupoid of $\mathrm{Hom}(\mathcal{C}(V), W)$.

Appendix B: Tamarkin's rigidity

Let V_A denote the Gerstenhaber algebra of polyvector fields on the graded affine space corresponding to $A = \mathbb{K}[x^1, x^2, \dots, x^d]$ with

$$|x^i| = t_i.$$

As the graded commutative algebra over $\mathbb{K}$, V_A is freely generated by variables

$$x^1, x^2, \dots, x^d, \theta_1, \theta_2, \dots, \theta_d,$$

where θ_i carries degree $1 - t_i$.

$$V_A = \mathbb{K}[x^1, x^2, \dots, x^d, \theta_1, \theta_2, \dots, \theta_d]. \tag{6.1}$$

Let us denote by $\mu_\wedge$ and $\mu_{\{\,,\,\}}$ the vectors in $\mathsf{End}_{V_A}(2)$ corresponding to the multiplication and the Schouten bracket $\{\,,\,\}$ on V_A, respectively.

The composition of the canonical quasi-isomorphism

$$\mathrm{Cobar}(\mathsf{Ger}^\vee) \to \mathsf{Ger}$$

and the map $\mathsf{Ger} \to \mathsf{End}_{V_A}$ corresponds to the following Maurer–Cartan element

$$\alpha := \mu_\wedge \otimes \{b_1, b_2\} + \mu_{\{\,,\,\}} \otimes b_1 b_2 \tag{6.2}$$

in the graded Lie algebra

$$\mathrm{Conv}^\oplus(\mathsf{Ger}^\vee, \mathsf{End}_{V_A}) := \bigoplus_{n\ge 1} \mathrm{Hom}_{S_n}\left(\mathsf{Ger}^\vee(n), \mathsf{End}_{V_A}(n)\right) \tag{6.3}$$

for which we frequently use the obvious identification[15]

$$\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \cong \bigoplus_{n \geq 1} \Big(\mathsf{End}_{V_A}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)^{S_n}. \tag{6.4}$$

In this section, we consider $\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A})$ as the cochain complex with the following differential

$$\partial := [\alpha, \ ]. \tag{6.5}$$

We observe that $\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A})$ carries the natural descending filtration "by arity":

$$\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) = \mathcal{F}_0\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \supset \mathcal{F}_1\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \supset \cdots$$
$$\mathcal{F}_m\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) := \bigoplus_{n \geq m+1} \big(\mathsf{End}_{V_A}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n) \big)^{S_n}. \tag{6.6}$$

More precisely,

$$\partial \Big(\mathsf{End}_{V_A}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)^{S_n} \subset \Big(\mathsf{End}_{V_A}(n+1) \otimes \Lambda^{-2}\mathsf{Ger}(n+1) \Big)^{S_{n+1}}. \tag{6.7}$$

In particular, every cocycle $X \in \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A})$ is a finite sum

$$X = \sum_{n \geq 1} X_n, \quad X_n \in \Big(\mathsf{End}_{V_A}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)^{S_n} \tag{6.8}$$

where each individual term X_n is a cocycle.

In this paper, we need the following version of Tamarkin's rigidity

Theorem 6.1 *If n is an integer ≥ 2 then for every cocycle*

$$X \in \Big(\mathsf{End}_{V_A}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)^{S_n} \subset \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A})$$

there exists a cochain $Y \in \big(\mathsf{End}_{V_A}(n-1) \otimes \Lambda^{-2}\mathsf{Ger}(n-1) \big)^{S_{n-1}}$ such that

$$X = \partial Y.$$

Remark 6.2 Note that the above statement is different from Tamarkin's rigidity in the "stable setting" [5, Section 12]. According to [5, Corollary 12.2], one may think that

[15] Recall that the cooperad $\mathsf{Ger}^{\vee}$ is the linear dual of the operad $\Lambda^{-2}\mathsf{Ger}$.

the vector

$$\mu_{\{\ ,\ \}} \otimes b_1 b_2$$

is a non-trivial cocycle in (6.3). In fact,

$$\mu_{\{\ ,\ \}} \otimes b_1 b_2 = [\alpha, P \otimes b_1],$$

where P is the following version of the "Euler derivation" of V_A.

$$P(v) := \sum_{i=1}^{d} \theta_i \frac{\partial}{\partial \theta_i}.$$

Proof of Theorem 6.1 Theorem 6.1 is only a slight generalization of the statement proved in Section 5.4 of [15] and, in the proof given here, we pretty much follow the same line of arguments as in [15, Section 5.4].

First, we introduce an additional set of auxiliary variables

$$\check{x}_1, \check{x}_2, \dots, \check{x}_d,\ \check{\theta}^1, \check{\theta}^2, \dots, \check{\theta}^d \tag{6.9}$$

of degrees

$$|\check{x}_i| = 2 - t_i, \quad |\check{\theta}^i| = t_i + 1.$$

Second, we consider the de Rham complex of V_A:

$$\Omega^{\bullet}_{\mathbb{K}} V_A := V_A[\check{x}_1, \check{x}_2, \dots, \check{x}_d, \check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_d] \tag{6.10}$$

with the differential

$$D = \sum_{i=1}^{d} \check{x}_i \frac{\partial}{\partial \theta_i} + \sum_{i=1}^{d} \check{\theta}^i \frac{\partial}{\partial x^i} \tag{6.11}$$

and equip it with the following descending filtration:

$$\mathcal{F}_m \Omega^{\bullet}_{\mathbb{K}} V_A := \Big\{ P \in V_A[\check{x}_1, \check{x}_2, \dots, \check{x}_d, \check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_d] \ \Big|\ \text{the total degree of } P \text{ in } \check{x}_1, \dots, \check{x}_d, \check{\theta}_1, \dots, \check{\theta}_d \text{ is } \ge m+1 \Big\}. \tag{6.12}$$

Next, we observe that every homogeneous vector[16]

$$P = P^{i_1 i_2 \dots i_k}_{j_1 j_2 \dots j_q}\, \check{x}_{i_1} \dots \check{x}_{i_k} \check{\theta}^{j_1} \dots \check{\theta}^{j_q} \in V_A[\check{x}_1, \check{x}_2, \dots, \check{x}_d, \check{\theta}_1, \check{\theta}_2, \dots, \check{\theta}_d]$$

[16] Summation over repeated indices is assumed.

defines an element $P^{\mathsf{End}} \in \mathsf{End}_{V_A}(k+q)$:

$$P^{\mathsf{End}}(v_1, v_2, \dots, v_{k+q}) := \sum_{\sigma \in S_{k+q}} \pm P^{i_1 i_2 \dots i_k}_{j_1 j_2 \dots j_q} \partial_{x^{i_1}} v_{\sigma(1)} \, \partial_{x^{i_2}} v_{\sigma(2)} \dots \partial_{x^{i_k}} v_{\sigma(k)} \times \partial_{\theta_{j_1}} v_{\sigma(k+1)} \, \partial_{\theta_{j_2}} v_{\sigma(k+2)} \dots \partial_{\theta_{j_q}} v_{\sigma(k+q)}, \quad (6.13)$$

where the sign factors $\pm$ are determined by the usual Koszul rule.

Finally, we claim that the formula

$$\mathrm{VH}(P) := P^{\mathsf{End}} \otimes b_1 b_2 \dots b_{k+q} \quad (6.14)$$

defines a degree zero injective map

$$\mathrm{VH} : \mathbf{s}^{-2} \mathcal{F}_0 \Omega^{\bullet}_{\mathbb{K}} V_A \to \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \quad (6.15)$$

which is compatible with filtrations (6.6) and (6.12).

A direct computation shows that VH intertwines differentials (6.5) and (6.11).

Let m be an integer and

$$\mathcal{G}^m \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \quad (6.16)$$

be the subspace of $\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A})$ of sums

$$\sum_i M_i \otimes q_i \in \bigoplus_{n \ge 1} \Big(\mathsf{End}_{V_A}(n) \otimes \Lambda^{-2} \mathsf{Ger}(n) \Big)^{S_n} \quad (6.17)$$

satisfying the condition

$$\text{the number of Lie brackets in } q_i - |M_i \otimes q_i| \le m. \quad (6.18)$$

It is easy to see that the sequence of subspaces (6.16)

$$\dots \subset \mathcal{G}^{-1} \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \subset \mathcal{G}^{0} \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \subset \mathcal{G}^{1} \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \subset \dots$$

form an ascending filtration on the cochain complex $\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A})$ and the associated graded cochain complex

$$\mathrm{Gr}_{\mathcal{G}} \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \quad (6.19)$$

is isomorphic to

$$\bigoplus_{n \ge 1} \Big(\mathsf{End}_{V_A}(n) \otimes \Lambda^{-2} \mathsf{Ger}(n) \Big)^{S_n}$$

with the differential

$$\partial^{\mathrm{Gr}} = [\mu_{\wedge} \otimes \{b_1, b_2\}, \;\;], \tag{6.20}$$

where $\mu_{\wedge}$ is the vector in $\mathsf{End}_{V_A}(2)$ which corresponds to the multiplication on V_A.

Let us observe that (6.19) is naturally a V_A-module (where V_A is viewed as the graded commutative algebra), differential (6.20) is V_A-linear, and since

$$\mathsf{Ger}^{\vee}(V_A) = \Lambda^2 \mathsf{coCom}(\Lambda \mathsf{coLie}(V_A)),$$

cochain complex (6.19) is isomorphic to

$$\mathrm{Hom}_{V_A}\Big(\mathbf{s}^2 \underline{S}_{V_A}(\mathbf{s}^{-1}\, V_A \otimes_{\mathbb{K}} \mathsf{coLie}(\mathbf{s}^{-1}\, V_A)),\, V_A\Big) \tag{6.21}$$

with the differential coming from the one on the Harrison homological[17] complex [19, Section 4.2.10]

$$V_A \otimes_{\mathbb{K}} \mathsf{coLie}(\mathbf{s}^{-1}\, V_A) \tag{6.22}$$

of the graded commutative algebra V_A with coefficients in V_A.

Since V_A is freely generated by elements $x^1, \ldots, x^d, \theta_1, \ldots, \theta_d$, Theorem 3.5.6 and Proposition 4.2.11 from [19] imply that the embedding

$$\begin{aligned} &I_{\mathrm{Harr}} : \bigoplus_{i=1}^{d} V_A e^i \;\oplus\; \bigoplus_{i=1}^{d} V_A f_i \;\to\; V_A \otimes \mathsf{coLie}(\mathbf{s}^{-1}\, V_A) \\ &I_{\mathrm{Harr}}(e^i) := 1 \otimes \mathbf{s}^{-1}\, x^i, \quad I_{\mathrm{Harr}}(f_i) := 1 \otimes \mathbf{s}^{-1}\, \theta_i \end{aligned} \tag{6.23}$$

from the free V_A-module

$$\bigoplus_{i=1}^{d} V_A e^i \;\oplus\; \bigoplus_{i=1}^{d} V_A f_i, \quad |e^i| := t_i - 1, \quad |f_i| := -t_i \tag{6.24}$$

is a quasi-isomorphism of cochain complexes of V_A-modules from (6.24) with the zero differential to (6.22) with the Harrison differential.

Since (6.23) is a quasi-isomorphism of cochain complexes of free V_A-modules, it induces a quasi-isomorphism of cochain complexes of (free) V_A-modules:

$$\mathbf{s}^2 V_A[\mathbf{s}^{-1}\, e^1, \ldots, \mathbf{s}^{-1}\, e^d, \mathbf{s}^{-1}\, f_1, \ldots, \mathbf{s}^{-1}\, f_d] \to \mathbf{s}^2 S_{V_A}(\mathbf{s}^{-1}\, V_A \otimes_{\mathbb{K}} \mathsf{coLie}(\mathbf{s}^{-1}\, V_A)), \tag{6.25}$$

where the source carries the zero differential.

[17] The cochain complex in (6.22) is obtained from the conventional Harrison homological complex from [19, Section 4.2.10] by reversing the grading.

Therefore, map (6.15) induces a quasi-isomorphism of cochain complexes

$$\mathbf{s}^{-2}\,\mathcal{F}_0\Omega^{\bullet}_{\mathbb{K}} V_A \;\to\; \mathrm{Gr}_{\mathcal{G}}\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}),$$

where the source is considered with the zero differential.

Thus, by Lemma A.3 from [5], map (6.15) is a quasi-isomorphism of cochain complexes.

Let $n \geq 2$ and

$$X \;\in\; \left(\mathsf{End}_{V_A}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\right)^{S_n} \;\subset\; \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \qquad (6.26)$$

be a cocycle.

Since (6.15) is a quasi-isomorphism of cochain complexes, there exists a cocycle

$$\widetilde{X} \in \mathbf{s}^{-2}\,\mathcal{F}_0\Omega^{\bullet}_{\mathbb{K}} V_A \qquad (6.27)$$

such that X is cohomologous to $\mathrm{VH}(\widetilde{X})$.

Let us observe that de Rham differential D (6.11) satisfies the property

$$D\left(\mathcal{F}_0\Omega^{\bullet}_{\mathbb{K}} V_A\right) \subset \mathcal{F}_1\Omega^{\bullet}_{\mathbb{K}} V_A.$$

Hence, since VH is injective, we conclude that

$$\widetilde{X} \in \mathbf{s}^{-2}\,\mathcal{F}_1\Omega^{\bullet}_{\mathbb{K}} V_A. \qquad (6.28)$$

It is obvious that every cocycle in $\mathcal{F}_1\Omega^{\bullet}_{\mathbb{K}} V_A$ is exact in $\mathcal{F}_0\Omega^{\bullet}_{\mathbb{K}} V_A$. Therefore $\widetilde{X}$ is exact and so is cocycle (6.26).

Combining this statement with property (6.7) we easily deduce Theorem 6.1. □

B.1: The standard Gerstenhaber structure on V_A is "rigid"

The first consequence of Theorem 6.1 is the following corollary:

Corollary 6.3 *Let V_A be, as above, the algebra of polyvector fields on a graded affine space and Q be a Ger_∞-structure on V_A whose binary operations are the Schouten bracket and the usual multiplication. Then the identity map $\mathsf{id} : V_A \to V_A$ can be extended to a Ger_∞ morphism*

$$U_{\mathrm{corr}} : V_A \rightsquigarrow V_A^{Q} \qquad (6.29)$$

from V_A with the standard Gerstenhaber structure to V_A with the Ger_∞-structure Q.

Proof To prove this statement, we consider the graded space

$$\mathrm{Hom}(\mathsf{Ger}^{\vee}(V_A), V_A) \qquad (6.30)$$

with two different algebraic structures. First, (6.30) is identified with the convolution Lie algebra[18]

$$\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{End}_{V_A}) \tag{6.31}$$

with the Lie bracket [,] defined in terms of the binary (degree zero) operation • from [5, Section 4, Eq. (4.2)].

To introduce the second algebraic structure on (6.30), we recall that a Ger_∞-structure on V_A is precisely a degree 1 element

$$Q = Q_2 + \sum_{n \ge 3} Q_n \quad Q_n \in \mathrm{Hom}_{S_n}(\mathsf{Ger}^{\vee}(n) \otimes V_A^{\otimes n}, V_A) \tag{6.32}$$

in (6.31) satisfying the Maurer–Cartan equation

$$[Q, Q] = 0 \tag{6.33}$$

and the above condition on the binary operations is equivalent to the requirement

$$Q_2 = \alpha, \tag{6.34}$$

where α is Maurer–Cartan element (6.2) of (6.31).

Given such a Ger_∞-structure Q on V_A, we get the convolution $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra

$$\mathrm{Hom}(\mathsf{Ger}^{\vee}(V_A), V_A^{Q}) \tag{6.35}$$

corresponding to the pair (V_A, V_A^Q), where the first entry V_A is considered with the standard Gerstenhaber structure and the second entry is considered with the above Ger_∞-structure Q.

As a graded vector space, $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra (6.35) coincides with (6.30). However, it carries a non-zero differential d_α given by the formula

$$d_\alpha(P) = -(-1)^{|P|} P \bullet \alpha, \tag{6.36}$$

and the corresponding (degree 1) brackets

$$\{\, ,\, ,\dots,\, \}_k : S^k\Big(\mathrm{Hom}(\mathsf{Ger}^{\vee}(V_A), V_A^{Q})\Big) \to \mathrm{Hom}(\mathsf{Ger}^{\vee}(V_A), V_A^{Q})$$

are defined by general formula (5.27) in terms of the $\mathsf{Ger}^{\vee}$-coalgebra structure on $\mathsf{Ger}^{\vee}(V_A)$ and the Ger_∞-structure Q on V_A.

18 In our case, Lie algebra (6.31) carries the zero differential.

Let us recall [7,10] that Ger_∞-morphisms from V_A to V_A^Q are in bijection with Maurer–Cartan elements[19]

$$\beta = \sum_{n \ge 1} \beta_n, \quad \beta_n \in \mathrm{Hom}_{S_n}(\mathsf{Ger}^\vee(n) \otimes V_A^{\otimes n}, V_A) \tag{6.37}$$

of $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra (6.35) such that β_1 corresponds to the linear term of the corresponding Ger_∞-morphism.

Thus our goal is to prove that, for every Maurer–Cartan element Q (6.32) of Lie algebra (6.31) satisfying condition (6.34), there exists a Maurer–Cartan element β (see (6.37)) of $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra (6.35) such that

$$\beta_1 = \mathsf{id} : V_A \to V_A. \tag{6.38}$$

Condition (6.34) implies that the element

$$\beta^{(1)} := \mathsf{id} \in \mathrm{Hom}(\mathsf{Ger}^\vee(V_A), V_A^Q)$$

satisfies the equation (in the $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra $\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), V_A^Q)$)

$$\left(d_\alpha(\beta^{(1)}) + \sum_{k \ge 2} \frac{1}{k!} \{\beta^{(1)}, \dots, \beta^{(1)}\}_k \right)(X) = 0 \tag{6.39}$$

for every $X \in (\mathsf{Ger}^\vee(m) \otimes V_A^{\otimes m})_{S_m}$ with $m \le 2$.

Let us assume that we constructed (by induction) a degree zero element

$$\beta^{(n-1)} = \mathsf{id} + \beta_2 + \beta_3 + \dots + \beta_{n-1}, \quad \beta_j \in \mathrm{Hom}_{S_j}(\mathsf{Ger}^\vee(j) \otimes V_A^{\otimes j}, V_A) \tag{6.40}$$

such that

$$\left(d_\alpha(\beta^{(n-1)}) + \sum_{k \ge 2} \frac{1}{k!} \{\beta^{(n-1)}, \dots, \beta^{(n-1)}\}_k \right)(X) = 0 \tag{6.41}$$

for every $X \in (\mathsf{Ger}^\vee(m) \otimes V_A^{\otimes m})_{S_m}$ with $m \le n$.

We will try to find an element

$$\beta_n \in \mathrm{Hom}_{S_n}(\mathsf{Ger}^\vee(n) \otimes V_A^{\otimes n}, V_A) \tag{6.42}$$

such that the sum

$$\beta^{(n)} := \mathsf{id} + \beta_2 + \beta_3 + \dots + \beta_{n-1} + \beta_n \tag{6.43}$$

[19] Recall that Maurer–Cartan elements of a $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra have degree 0.

satisfies the equation

$$\left(d_\alpha(\beta^{(n)}) + \sum_{k\geq 2} \frac{1}{k!}\{\beta^{(n)}, \dots, \beta^{(n)}\}_k \right)(X) = 0 \tag{6.44}$$

for every $X \in (\mathsf{Ger}^\vee(m) \otimes V_A^{\otimes m})_{S_m}$ with $m \leq n+1$.

Since $\beta_n \in \mathrm{Hom}_{S_n}(\mathsf{Ger}^\vee(n) \otimes V_A^{\otimes n}, V_A)$ and (6.41) is satisfied for every $X \in (\mathsf{Ger}^\vee(m) \otimes V_A^{\otimes m})_{S_m}$ with $m \leq n$, equation (6.44) is also satisfied for every $X \in (\mathsf{Ger}^\vee(m) \otimes V_A^{\otimes m})_{S_m}$ with $m \leq n$.

For $X \in (\mathsf{Ger}^\vee(n+1) \otimes V_A^{\otimes (n+1)})_{S_{n+1}}$, Eq. (6.44) can be rewritten as

$$-\beta_n \bullet \alpha(X) + \alpha \bullet \beta_n(X) = -\sum_{k\geq 2} \frac{1}{k!}\{\beta^{(n-1)}, \dots, \beta^{(n-1)}\}_k(X). \tag{6.45}$$

Let us denote by γ the element in $\mathrm{Hom}_{S_{n+1}}(\mathsf{Ger}^\vee(n+1) \otimes V_A^{\otimes (n+1)}, V_A)$ defined as

$$\gamma := \sum_{k\geq 2} \frac{1}{k!}\{\beta^{(n-1)}, \dots, \beta^{(n-1)}\}_k \Big|_{\mathsf{Ger}^\vee(n+1)\otimes V_A^{\otimes (n+1)}} \tag{6.46}$$

Evaluating the Bianchi type identity [14, Lemma 4.5]

$$\sum_{k\geq 2} \frac{1}{k!} d_\alpha\{\beta^{(n-1)}, \dots, \beta^{(n-1)}\}_k + \sum_{k\geq 1} \frac{1}{k!}\{\beta^{(n-1)}, \dots, \beta^{(n-1)}, d_\alpha\beta^{(n-1)}\}_{k+1} + \sum_{\substack{k\geq 2\\ t\geq 1}} \frac{1}{k!t!}\{\beta^{(n-1)}, \dots, \beta^{(n-1)}, \{\beta^{(n-1)}, \dots, \beta^{(n-1)}\}_k\}_{t+1} = 0 \tag{6.47}$$

on an arbitrary element

$$Y \in (\mathsf{Ger}^\vee(n+2) \otimes V_A^{\otimes (n+2)})_{S_{n+2}}$$

and using the fact that

$$\beta^{(n-1)}(X) = 0, \quad \forall\ X \in (\mathsf{Ger}^\vee(m) \otimes V_A^{\otimes m})_{S_m} \quad \text{with} \quad m \geq n$$

we deduce that element γ (6.46) is a cocycle in cochain complex (6.3) with differential (6.5).

Thus Theorem 6.1 implies that Eq. (6.45) can always be solved for β_n.

This inductive argument concludes the proof of Corollary 6.3. □

B.2 The Gerstenhaber algebra V_A is intrinsically formal

Let $(C^\bullet, \mathfrak{d})$ be an arbitrary cochain complex whose cohomology is isomorphic to V_A

$$H^\bullet(C^\bullet) \cong V_A. \tag{6.48}$$

Let us consider V_A as the cochain complex with the zero differential and choose[20] a quasi-isomorphism of cochain complexes

$$I : V_A \to C^\bullet. \tag{6.49}$$

Let us assume that $C^\bullet$ carries a Ger_∞-structure such that the map I induces an isomorphism of Gerstenhaber algebras $V_A \cong H^\bullet(C^\bullet)$.

Then Theorem 6.1 gives us the following remarkable corollary:

Corollary 6.4 *There exists a* Ger_∞*-morphism*

$$U : V_A \rightsquigarrow C^\bullet \tag{6.50}$$

whose linear term coincides with I (6.49)*. Moreover, any two such* Ger_∞*-morphisms*

$$U, \ \widetilde{U} \ : \ V_A \rightsquigarrow C^\bullet \tag{6.51}$$

are homotopy equivalent.

Remark 6.5 The above statement is a slight refinement of one proved in [15, Section 5]. Following Hinich, we say that the Gerstenhaber algebra V_A is intrinsically formal.

Proof of Corollary 6.4 By the Homotopy Transfer Theorem [7, Section 5], [20, Section 10.3], there exists a Ger_∞-structure Q on V_A and a Ger_∞-quasi-isomorphism

$$U' : V_A^Q \rightsquigarrow C^\bullet, \tag{6.52}$$

such that

- the binary operations of the Ger_∞-structure Q on V_A are the Schouten bracket and the usual multiplication of polyvector fields,
- the linear term of U' coincides with I.

Corollary 6.3 implies that there exists a Ger_∞-morphism

$$U_{\mathrm{corr}} : V_A \rightsquigarrow V_A^Q, \tag{6.53}$$

whose linear term is the identity map $\mathsf{id} : V_A \to V_A$.

[20] Such a quasi-isomorphism exists since we are dealing with cochain complexes of vector spaces over a field.

Hence the composition

$$U = U' \circ U_{\text{corr}} : V_A \rightsquigarrow C^\bullet \tag{6.54}$$

is a desired Ger_∞-morphism.

To prove the second claim, we need the $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra

$$\text{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet) \tag{6.55}$$

corresponding to the Gerstenhaber algebra V_A and the Ger_∞-algebra $C^\bullet$. The differential $\mathcal{D}$ on (6.55) is given by the formula

$$\mathcal{D}(\Psi) := \mathfrak{d} \circ \Psi - (-1)^{|\Psi|} \Psi \circ Q_{\wedge, \{\,,\,\}}, \quad \Psi \in \text{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet), \tag{6.56}$$

where $\mathfrak{d}$ is the differential on $C^\bullet$ and $Q_{\wedge,\{\,,\,\}}$ is the differential on the $\mathsf{Ger}^\vee$-coalgebra $\mathsf{Ger}^\vee(V_A)$ corresponding to the standard Gerstenhaber structure on V_A.

The multi-brackets $\{\,,\,,\dots,\,\}_m$ are defined by the general formula [see Eq. (5.27)] in terms of the $\mathsf{Ger}^\vee$-coalgebra structure on $\mathsf{Ger}^\vee(V_A)$ and the Ger_∞-structure on $C^\bullet$.

Let us recall (see Appendix A.2 for more details) that Ger_∞-morphisms from V_A to $C^\bullet$ are in bijection with Maurer–Cartan elements of $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra (6.55) and Ger_∞-morphisms (6.51) are homotopy equivalent if and only if the corresponding Maurer–Cartan elements P and $\widetilde{P}$ in (6.55) are isomorphic 0-cells in the Deligne–Getzler–Hinich ∞-groupoid [14] of (6.55).

So our goal is to prove that any two Maurer–Cartan elements P and $\widetilde{P}$ in (6.55) satisfying

$$P\Big|_{V_A} = \widetilde{P}\Big|_{V_A} = I : V_A \to C^\bullet \tag{6.57}$$

are isomorphic.

Condition (6.57) implies that

$$\widetilde{P} - P \in \mathcal{F}_2\text{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet),$$

where $\mathcal{F}_\bullet\text{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)$ is the arity filtration (5.29) on $\text{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)$.

Let us assume that we constructed a sequence of Maurer–Cartan elements

$$P = P_2, P_3, P_4, \dots, P_{n+1} \tag{6.58}$$

such that for every $2 \le m \le n+1$

$$\widetilde{P} - P_m \in \mathcal{F}_m\text{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet) \tag{6.59}$$

and for every $2 \le m \le n$ there exists 1-cell

$$P'_m(t) + dt\, \xi_{m-1} \in \text{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)\, \hat{\otimes}\, \Omega^\bullet(\Delta_1)$$

which connects P_m to P_{m+1} and such that

$$\xi_{m-1} \in \mathcal{F}_{m-1}\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet), \tag{6.60}$$

and

$$P'_m(t) - P_m \in \mathcal{F}_m\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)\,\hat{\otimes}\,\mathbb{K}[t]. \tag{6.61}$$

Let us now prove that one can construct a 1-cell

$$P'_{n+1}(t) + dt\,\xi_n \in \mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)\,\hat{\otimes}\,\Omega^\bullet(\Delta_1) \tag{6.62}$$

such that

$$P'_{n+1}(t)\Big|_{t=0} = P_{n+1},$$

$$\xi_n \in \mathcal{F}_n\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet), \tag{6.63}$$

$$P'_{n+1}(t) - P_{n+1} \in \mathcal{F}_{n+1}\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)\,\hat{\otimes}\,\mathbb{K}[t], \tag{6.64}$$

and the Maurer–Cartan element

$$P_{n+2} := P'_{n+1}(t)\Big|_{t=1} \tag{6.65}$$

satisfies the condition

$$\widetilde{P} - P_{n+2} \in \mathcal{F}_{n+2}\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet). \tag{6.66}$$

Let us denote the difference $\widetilde{P} - P_{n+1}$ by K. Since $\widetilde{P} - P_{n+1} \in \mathcal{F}_{n+1}\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)$,

$$K = \sum_{m \ge n+1} K_m, \quad K_m \in \mathrm{Hom}_{S_m}(\mathsf{Ger}^\vee(m) \otimes V_A^{\otimes m}, C^\bullet). \tag{6.67}$$

Subtracting the left hand side of the Maurer–Cartan equation

$$\mathcal{D}(P_{n+1}) + \sum_{m \ge 2} \frac{1}{m!}\{P_{n+1}, P_{n+1}, \dots, P_{n+1}\}_m = 0 \tag{6.68}$$

from the left hand side of the Maurer–Cartan equation

$$\mathcal{D}(\widetilde{P}) + \sum_{m \ge 2} \frac{1}{m!}\{\widetilde{P}, \widetilde{P}, \dots, \widetilde{P}\}_m = 0 \tag{6.69}$$

we see that element (6.67) satisfies the equation

$$\mathcal{D}(K) + \sum_{m\geq 1} \frac{1}{m!}\{P_{n+1}, \dots, P_{n+1}, K\}_{m+1} + \sum_{m\geq 2} \frac{1}{m!}\{K, K, \dots, K\}_m^{P_{n+1}} = 0, \tag{6.70}$$

where the multi-bracket $\{K, K, \dots, K\}_m^{P_{n+1}}$ is defined by the formula

$$\{X_1, X_2, \dots, X_m\}_m^{P_{n+1}} := \sum_{q\geq 0} \frac{1}{q!}\{P_{n+1}, \dots, P_{n+1}, X_1, X_2, \dots, X_m\}_{q+m} \tag{6.71}$$

Evaluating (6.70) on $\mathsf{Ger}^\vee(n+1) \otimes V_A^{\otimes (n+1)}$ and using the fact that

$$K \in \mathcal{F}_{n+1}\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet), \tag{6.72}$$

we conclude that

$$\mathfrak{d} \circ K_{n+1} = 0, \tag{6.73}$$

where $\mathfrak{d}$ is the differential on $C^\bullet$.

Hence there exist elements

$$K_{n+1}^{V_A} \in \mathrm{Hom}_{S_{n+1}}(\mathsf{Ger}^\vee(n+1) \otimes V_A^{\otimes (n+1)}, V_A)$$

and

$$K'_{n+1} \in \mathrm{Hom}_{S_{n+1}}(\mathsf{Ger}^\vee(n+1) \otimes V_A^{\otimes (n+1)}, C^\bullet)$$

such that

$$K_{n+1} = I \circ K_{n+1}^{V_A} + \mathfrak{d} \circ K'_{n+1}. \tag{6.74}$$

Next, evaluating (6.70) on $Y \in \mathsf{Ger}^\vee(n+2) \otimes V_A^{\otimes (n+2)}$ and using inclusion (6.72) again, we get the following identity

$$\mathfrak{d} \circ K_{n+2}(Y) - K_{n+1} \circ Q_{\wedge,\{\,,\,\}}(Y) + \{P_{n+1}, K_{n+1}\}_2(Y) = 0. \tag{6.75}$$

Unfolding $\{P_{n+1}, K_{n+1}\}_2(Y)$ we get

$$\{P_{n+1}, K_{n+1}\}_2(Y) = \sum_{i=1}^{n+2} Q_{C^\bullet}\left((\mathsf{id}_{\mathsf{Ger}^\vee(2)} \otimes K_{n+1} \otimes I) \circ \left(\Delta_{\mathbf{t}_i} \otimes \mathsf{id}^{\otimes (n+2)} \right)(Y) \right), \tag{6.76}$$

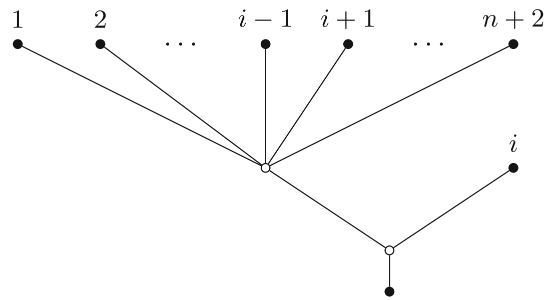

Fig. 6 The $(n+2)$-labeled planar tree $\mathbf{t}_i$

where $Q_{C^\bullet}$ is the Ger_∞-structure on $C^\bullet$, $\mathbf{t}_i$ is the $(n+2)$-labeled planar tree shown on Fig. 6, and $\Delta_{\mathbf{t}_i}$ is the corresponding component of the comultiplication

$$\Delta_{\mathbf{t}_i} : \mathsf{Ger}^\vee(n+2) \to \mathsf{Ger}^\vee(2) \otimes \mathsf{Ger}^\vee(n+1). \tag{6.77}$$

Now using (6.74) and (6.76), we rewrite (6.75) as follows

$$\begin{aligned} &\mathfrak{d} \circ K_{n+2}(Y) - I \circ (K^{V_A}_{n+1} \bullet \alpha)(Y) \\ &+ \sum_{i=1}^{n+2} Q_{C^\bullet} \Big((\mathsf{id}_{\mathsf{Ger}^\vee(2)} \otimes (\mathfrak{d} \circ K'_{n+1}) \otimes I) \circ \Big(\Delta_{\mathbf{t}_i} \otimes \mathsf{id}^{\otimes (n+2)} \Big) (Y) \Big) \\ &+ \sum_{i=1}^{n+2} Q_{C^\bullet} \Big((\mathsf{id}_{\mathsf{Ger}^\vee(2)} \otimes (I \circ K^{V_A}_{n+1}) \otimes I) \circ \Big(\Delta_{\mathbf{t}_i} \otimes \mathsf{id}^{\otimes (n+2)} \Big) (Y) \Big) = 0, \end{aligned} \tag{6.78}$$

where α is defined in (6.2).

Since the last two sums in (6.78) involve only binary Ger_∞-operations on $C^\bullet$ and these binary operations induce the usual multiplication and the Schouten bracket on V_A, we conclude that each term in the first sum in (6.78) is $\mathfrak{d}$-exact and the second sum in (6.78) is cohomologous to

$$I \circ (\alpha \bullet K^{V_A}_{n+1})(Y)$$

Therefore, identity (6.78) implies that for every $Y \in \mathsf{Ger}^\vee(n+2) \otimes V_A^{\otimes (n+2)}$ the expression

$$I \circ (\alpha \bullet K^{V_A}_{n+1} - K^{V_A}_{n+1} \bullet \alpha)(Y)$$

is $\mathfrak{d}$-exact. Thus

$$\alpha \bullet K^{V_A}_{n+1} - K^{V_A}_{n+1} \bullet \alpha = 0$$

or, in other words, the element $K^{V_A}_{n+1}$ is a cocycle in complex (6.3) with differential (6.5).

Hence, by Theorem 6.1, there exists a degree -1 element

$$\widetilde{K}^{V_A}_n \in \mathrm{Hom}_{S_n}(\mathsf{Ger}^\vee(n) \otimes V_A^{\otimes\,(n)}, V_A) \tag{6.79}$$

such that

$$K^{V_A}_{n+1} = [\alpha, \widetilde{K}^{V_A}_n]. \tag{6.80}$$

Let us now consider the degree -1 element

$$\xi_n = I \circ \widetilde{K}^{V_A}_n + K''_{n+1} \in \mathcal{F}_n\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet), \tag{6.81}$$

where $\widetilde{K}^{V_A}_n$ is element (6.79) entering Eq. (6.80) and K''_{n+1} is an element in

$$\mathrm{Hom}_{S_{n+1}}\left(\mathsf{Ger}^\vee(n+1) \otimes V_A^{\otimes\,(n+1)}, C^\bullet\right)$$

which will be determined later.

Using ξ_n, we define $P'_{n+1}(t) \in \mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)\,\hat{\otimes}\,\mathbb{K}[t]$ as the limiting element of the recursive procedure

$$\begin{aligned}(P')^{(0)} &:= P_{n+1},\\ (P')^{(k+1)}(t) &:= P_{n+1} + \int_0^t dt_1 \left(\mathcal{D}(\xi_n) + \sum_{m\geq 1} \frac{1}{m!}\{(P')^{(k)}(t_1), \ldots, (P')^{(k)}(t_1), \xi_n\}_{m+1}\right).\end{aligned} \tag{6.82}$$

Since

$$\mathfrak{d}\left(I \circ \widetilde{K}^{V_A}_n\right) = 0$$

the element ξ_n satisfies the condition

$$\mathcal{D}(\xi_n) \in \mathcal{F}_{n+1}\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet).$$

Hence, by Lemma 5.1, the sum

$$P'_{n+1}(t) + dt\xi_n \in \mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)\,\hat{\otimes}\,\Omega^\bullet(\Delta_1) \tag{6.83}$$

is a 1-cell in the ∞-groupoid corresponding to $\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)$ satisfying (6.64) and such that the Maurer–Cartan element P_{n+2} (6.65) satisfies the condition

$$P_{n+2} - P_{n+1} - \mathcal{D}(\xi_n) - \{P_{n+1}, \xi_n\}_2 \in \mathcal{F}_{n+2}\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet). \tag{6.84}$$

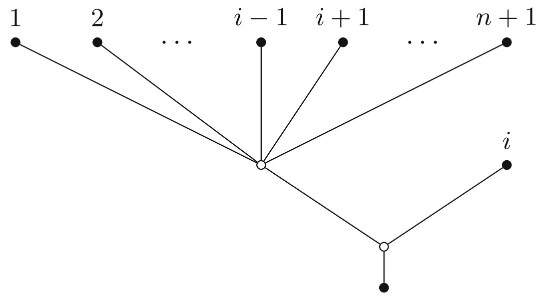

Fig. 7 The $(n+1)$-labeled planar tree $\mathbf{t}_i'$

Let us now show that, by choosing the element K''_{n+1} in (6.81) appropriately, we can get desired inclusion (6.66).

For this purpose we unfold $\{P_{n+1}, \xi_n\}_2(Y)$ for an arbitrary $Y \in \mathsf{Ger}^\vee(n+1) \otimes V_A^{\otimes (n+1)}$ and get

$$\{P_{n+1}, \xi_n\}_2(Y) = \sum_{i=1}^{n+1} Q_{C^\bullet}\left((\mathrm{id}_{\mathsf{Ger}^\vee(2)} \otimes (I \circ \widetilde{K}_n^{V_A}) \otimes I) \circ \left(\Delta_{\mathbf{t}_i'} \otimes \mathrm{id}^{\otimes (n+1)} \right)(Y) \right), \tag{6.85}$$

where $Q_{C^\bullet}$ is the Ger_∞-structure on $C^\bullet$, $\mathbf{t}_i'$ is the $(n+1)$-labeled planar tree shown on Fig. 7, and $\Delta_{\mathbf{t}_i'}$ is the corresponding component of the comultiplication

$$\Delta_{\mathbf{t}_i'} : \mathsf{Ger}^\vee(n+1) \to \mathsf{Ger}^\vee(2) \otimes \mathsf{Ger}^\vee(n). \tag{6.86}$$

Since the right hand side of (6.85) involves only binary Ger_∞-operations on $C^\bullet$ and these binary operations induce the usual multiplication and the Schouten bracket on V_A, we conclude that $\{P_{n+1}, \xi_n\}_2(Y)$ is cohomologous (in $C^\bullet$) to

$$I \circ (\alpha \bullet \widetilde{K}_n^{V_A})(Y),$$

where α is defined in (6.2).

In other words, there exists an element

$$\phi \in \mathrm{Hom}_{S_{n+1}}\left(\mathsf{Ger}^\vee(n+1) \otimes V_A^{\otimes (n+1)}, C^\bullet \right) \tag{6.87}$$

such that

$$\{P_{n+1}, \xi_n\}_2(Y) = I \circ (\alpha \bullet \widetilde{K}_n^{V_A})(Y) + \mathfrak{d} \circ \phi(Y).$$

Hence the expression $(\mathcal{D}(\xi_n) + \{P_{n+1}, \xi_n\}_2)\,(Y)$ can be rewritten as

$$(\mathcal{D}(\xi_n) + \{P_{n+1}, \xi_n\}_2)\,(Y) = \mathfrak{d} \circ K''_{n+1}(Y) + \mathfrak{d} \circ \phi(Y) + I \circ [\alpha, \widetilde{K}_n^{V_A}](Y). \tag{6.88}$$

Thus if

$$K''_{n+1} = K'_{n+1} - \phi$$

then Eqs. (6.74), (6.80), and inclusion (6.84) imply that (6.66) holds, as desired.

Thus we showed that one can construct an infinite sequence of Maurer–Cartan elements

$$P = P_2, P_3, P_4, \dots$$

and an infinite sequence of 1-cells ($m \geq 2$)

$$P'_m(t) + dt\, \xi_{m-1} \in \mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet) \,\hat{\otimes}\, \Omega^\bullet(\Delta_1) \tag{6.89}$$

such that for every $m \geq 2$

$$\widetilde{P} - P_m \in \mathcal{F}_m \mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet),$$

the 1-cell $P'_m(t) + dt\, \xi_{m-1}$ connects P_m to P_{m+1}

$$\xi_{m-1} \in \mathcal{F}_{m-1} \mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet), \tag{6.90}$$

and

$$P'_m(t) - P_m \in \mathcal{F}_m \mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet) \,\hat{\otimes}\, \mathbb{K}[t]. \tag{6.91}$$

Since the $\Lambda^{-1}\mathsf{Lie}_\infty$-algebra $\mathrm{Hom}(\mathsf{Ger}^\vee(V_A), C^\bullet)$ is complete with respect to "arity" filtration (5.29), inclusions (6.90) and (6.91) imply that we can form the infinite composition[21] of all 1-cells (6.89) and get a 1-cell which connects the Maurer–Cartan element $P = P_2$ to the Maurer–Cartan element $\widetilde{P}$.

Corollary 6.4 is proved. □

Appendix C: On derivations of Cyl(Λ^2coCom)

Let $\mathcal{C}$ be a coaugmented cooperad in the category of graded vector spaces and $\mathcal{C}_\circ$ be the cokernel of the coaugmentation. As above, we assume that $\mathcal{C}(0) = \mathbf{0}$ and $\mathcal{C}(1) = \mathbb{K}$.

Following [22, Section 3], [13], we will denote by Cyl($\mathcal{C}$) the 2-colored dg operad whose algebras are pairs (V, W) with the data

1. a Cobar($\mathcal{C}$)-algebra structure on V,
2. a Cobar($\mathcal{C}$)-algebra structure on W, and
3. an ∞-morphism F from V to W, i.e. a homomorphism of corresponding dg $\mathcal{C}$-coalgebras $\mathcal{C}(V) \to \mathcal{C}(W)$.

[21] Note that the composition of 1-cells in an infinity groupoid is not unique but this does not create a problem.

In fact, if we forget about the differential, then $\mathrm{Cyl}(\mathcal{C})$ is a free operad on a certain 2-colored collection $\mathcal{M}(\mathcal{C})$ naturally associated to $\mathcal{C}$.

Following the conventions of Sect. 3, we denote by

$$\mathrm{Der}'\,(\mathrm{Cyl}(\mathcal{C})) \tag{7.1}$$

the dg Lie algebra of derivations $\mathcal{D}$ of $\mathrm{Cyl}(\mathcal{C})$ subject to the condition

$$p \circ \mathcal{D} = 0, \tag{7.2}$$

where p is the canonical projection from $\mathrm{Cyl}(\mathcal{C})$ onto $\mathcal{M}(\mathcal{C})$.

We have the following generalization of (3.3):

Proposition 7.1 *The dg Lie algebra* $\mathrm{Der}'\left(\mathrm{Cyl}(\Lambda^2\mathsf{coCom})\right)$ *does not have non-zero elements in degrees* ≤ 0, *i.e.*

$$\mathrm{Der}'\left(\mathrm{Cyl}(\Lambda^2\mathsf{coCom})\right)^{\leq 0} = \mathbf{0}.$$

Proof Let us denote by α and β, respectively, the first and the second color for the collection $\mathcal{M}(\Lambda^2\mathsf{coCom})$ and the operad $\mathrm{Cyl}(\Lambda^2\mathsf{coCom})$.

Recall from [22] that $\mathrm{Cyl}(\Lambda^2\mathsf{coCom})$ is generated by the collection $\mathcal{M} = \mathcal{M}(\Lambda^2\mathsf{coCom})$ with

$$\begin{aligned}
\mathcal{M}(n, 0; \alpha) &= \mathbf{s}\,\Lambda^2\mathsf{coCom}_{\circ}(n) = \mathbf{s}^{3-2n}\mathbb{K},\\
\mathcal{M}(0, n; \beta) &= \mathbf{s}\,\Lambda^2\mathsf{coCom}_{\circ}(n) = \mathbf{s}^{3-2n}\mathbb{K},\\
\mathcal{M}(n, 0; \beta) &= \Lambda^2\mathsf{coCom}(n) = \mathbf{s}^{2-2n}\mathbb{K},
\end{aligned}$$

and with all the remaining spaces being zero. Let $\mathcal{D}$ be a derivation of $\mathrm{Cyl}(\Lambda^2\mathsf{coCom})$ of degree ≤ 0.

Since

$$\mathrm{Cyl}\left(\Lambda^2\mathsf{coCom}\right)(n, 0, \alpha) = \Lambda\mathsf{Lie}_{\infty}(n) \quad \text{and} \quad \mathrm{Cyl}\left(\Lambda^2\mathsf{coCom}\right)(0, n, \beta) = \Lambda\mathsf{Lie}_{\infty}(n),$$

observation (3.3) implies that

$$\mathcal{D}\Big|_{\mathcal{M}(n,0;\alpha)} = \mathcal{D}\Big|_{\mathcal{M}(0,n;\beta)} = 0.$$

Hence, it suffices to show that

$$\mathcal{D}\Big|_{\mathcal{M}(n,0;\beta)} = 0. \tag{7.3}$$

Let us denote by $\pi_0(\mathsf{Tree}_k(n))$ the set of isomorphism classes of labeled 2-colored planar trees corresponding to corolla $(n, 0; \beta)$ with k internal vertices. Figure 8 show

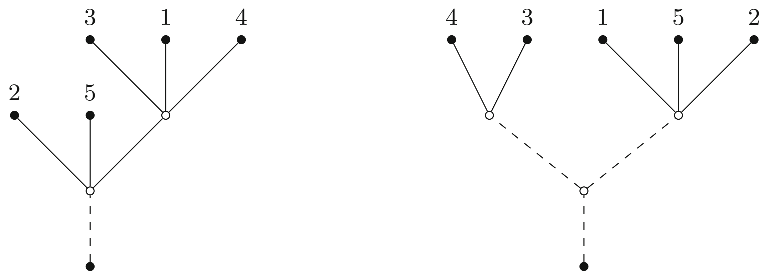

Fig. 8 *Solid edges* carry the color α and *dashed edges* carry the color β; internal vertices are denoted by *small white circles*; leaves and the root vertex are denoted by *small black circles*

two examples of such trees with $n = 5$ leaves. The left tree has $k = 2$ internal vertices and the right tree has $k = 3$ internal vertices.

For a generator $X \in \mathcal{M}(n, 0; \beta) = \mathbf{s}^{2-2n}\,\mathbb{K}$, the element $\mathcal{D}(X) \in \mathrm{Cyl}(\Lambda^2\mathsf{coCom})$ takes the form

$$\mathcal{D}(X) \;=\; \sum_{k\geq 2}\; \sum_{z\in\pi_0(\mathsf{Tree}_k(n))} (\mathbf{t}_z; X_1, \dots, X_k) \tag{7.4}$$

where $\mathbf{t}_z$ is a representative of an isomorphism class $z \in \pi_0(\mathsf{Tree}_k(n))$ and X_i are the corresponding elements of $\mathcal{M}$.

For every term in sum (7.4), we have k_1 X_i's in $\mathbf{s}\,\Lambda^2\mathsf{coCom}_\circ$ (call them X_{i_a}), and k_2 X_i's in $\Lambda^2\mathsf{coCom}$ (call them X_{j_b}).

We obviously have that $k = k_1 + k_2$ and

$$|\mathcal{D}| = \sum_{a=1}^{k_1} |X_{i_a}| + \sum_{b=1}^{k_2} |X_{j_b}| - |X| \tag{7.5}$$

or equivalently

$$|\mathcal{D}| = 2(n-1) + \sum_{a=1}^{k_1}(3 - 2n_{i_a}) + \sum_{b=1}^{k_2}(2 - 2n_{j_b}),$$

where n_{i_a} (resp. n_{j_b}) is the number of incoming edges of the vertex corresponding to X_{i_a} (resp. X_{j_b}).

On the other hand, a simple combinatorics of trees shows that

$$n - 1 = \sum_{a=1}^{k_1}(n_{i_a} - 1) + \sum_{b=1}^{k_2}(n_{j_b} - 1)$$

and hence

$$|\mathcal{D}| = k_1.$$

Since $|\mathcal{D}| \leq 0$ the latter is possible only if $k_1 = 0 = |\mathcal{D}|$, i.e. every tree in the sum $\mathcal{D}(X)$ is assembled exclusively from mixed colored corollas. That would force every tree **t** to have only one internal vertex which contradicts to the fact that the summation in (7.4) starts at $k = 2$.

Therefore (7.3) holds and the proposition follows. □

References

1. Alekseev, A., Torossian, C.: The Kashiwara–Vergne conjecture and Drinfeld's associators. Ann. Math. (2) **175**(2), 415–463 (2012). arXiv:0802.4300
2. Bar-Natan, D.: On associators and the Grothendieck-Teichmueller group. I Selecta Math. (N.S.) **4**(2), 183–212 (1998)
3. Batanin, M., Markl, M.: Crossed interval groups and operations on the Hochschild cohomology. J. Noncommut. Geom. **8**(3), 655–693 (2014). arXiv:0803.2249
4. Berger, C., Fresse, B.: Combinatorial operad actions on cochains. Math. Proc. Camb. Philos. Soc. **137**(1), 135–174 (2004)
5. Dolgushev, V.A., Rogers, C.L.: Notes on algebraic operads, graph complexes, and Willwacher's construction, mathematical aspects of quantization. In: Contemp. Math., vol. 583, pp. 25–145. AMS, Providence (2012). arXiv:1202.2937
6. Dolgushev, V.A., Rogers, C.L.: On an enhancement of the category of shifted L-infinity algebras. arXiv:1406.1744
7. Dolgushev, V.A., Hoffnung, A.E., Rogers, C.L.: What do homotopy algebras form? Adv. Math. **274**, 562–605 (2015). arXiv:1406.1751
8. Dolgushev, V.A., Tamarkin, D.E., Tsygan, B.L.: Proof of swiss cheese version of Deligne's conjecture. Int. Math. Res. Notices **2011**, 4666–4746 (2011)
9. Dolgushev, V.A., Willwacher, T.H.: Operadic Twisting—with an application to Deligne's conjecture. J. Pure Appl. Algebra **219**(5), 1349–1428 (2015). arXiv:1207.2180
10. Dotsenko, V., Poncin, N.: A tale of three homotopies. arXiv:1208.4695, Version 2
11. Drinfeld, V.G.: On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Algebra i Analiz. **2**(4), 149–181 (1990) (Russian) (translation in Leningrad Math. J. **2**(4), 829–860, 1991)
12. Fiedorowicz, Z.: The symmetric bar construction. https://people.math.osu.edu/fiedorowicz.1/. Accessed 21 June 2015
13. Fresse, B.: Operadic cobar constructions, cylinder objects and homotopy morphisms of algebras over operads, Alpine perspectives on algebraic topology. In: Contemp. Math., vol. 504, pp. 125–188. Am. Math. Soc., Providence (2009)
14. Getzler, E.: Lie theory for nilpotent L_∞-algebras. Ann. Math. (2) **170**(1), 271–301 (2009). arXiv:math/0404003
15. Hinich, V.: Tamarkin's proof of Kontsevich formality theorem. Forum Math. **15**(4), 591–614 (2003). arXiv:math.QA/0003052
16. Hochschild, G., Kostant, B., Rosenberg, A.: Differential forms on regular affine algebras. Trans. Am. Math. Soc. **102**, 383–408 (1962)
17. Kontsevich, M., Soibelman, Y.: Deformations of algebras over operads and the Deligne conjecture. In: Proceedings of the Moshé Flato Conference Math. Phys. Stud. vol. 21, pp. 255–307, Kluwer, Dordrecht (2000)
18. Kontsevich, M.: Operads and motives in deformation quantization. Lett. Math. Phys. **48**(1), 35–72 (1999)
19. Loday, J.-L.: Cyclic homology. In: Grundlehren der Mathematischen Wissenschaften (Appendix E by Maria O. Ronco), vol. 301. Springer, Berlin (1992)
20. Loday, J.-L., Vallette, B.: Algebraic operads. In: Grundlehren der Mathematischen Wissenschaften, vol. 346, pp. xxiv+634. Springer, Heidelberg (2012)
21. McClure, J.E., Smith, J.H.: A solution of Deligne's Hochschild cohomology conjecture. Contemp. Math. **293**, 153–193 (2002) (Am. Math. Soc., Providence). arXiv:math.QA/9910126
22. Paljug, B.: Action of derived automorphisms on infinity morphisms. JHRS (to appear) (2015). arXiv:1305.4699

23. Tamarkin, D.: What do DG categories form? Compos. Math. **143**(5), 1335–1358 (2007). arXiv:math.CT/0606553
24. Tamarkin, D.E.: Another proof of M. Kontsevich formality theorem (1998). arXiv:math/9803025
25. Tamarkin, D.: Formality of chain operad of little discs. Lett. Math. Phys. **66**(1–2), 65–72 (2003)
26. Voronov, A.A.: Homotopy Gerstenhaber algebras. In: Proceedings of the Moshé Flato Conference Math. Phys. Stud., vol. 22, pp. 307–331. Kluwer, Dordrecht (2000)
27. Willwacher, T.: M. Kontsevich's graph complex and the Grothendieck–Teichmueller Lie algebra, Invent. Math. **200**(3), 671–760 (2015)