

# On a cohomology of digraphs and Hochschild cohomology

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**Abstract** We use a differential form cohomology theory on transitive digraphs to give a new proof of a theorem of Gerstenhaber and Schack about isomorphism between simplicial cohomology and Hochschild cohomology of a certain algebra associated with the simplicial complex.

**Keywords** Cohomology theory of digraph · Simplicial complex · Hochschild cohomology

**Mathematics Subject Classification** 05C20 · 55U10 · 13D03

## 1 Introduction

At present time there exist different notions of homology (cohomology) of digraphs and undirected graphs, see [2, 5, 6, 8, 10, 11]. In this paper we use cohomology theory of digraphs of [2, 6], that is reminiscent of the classical de Rham cohomology theory of

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differential forms, in order to give a new proof of a theorem of Gerstenhaber and Schack [4]. In the course of the proof we obtain close relations between the cohomologies of digraphs, those of simplicial complexes, and the Hochschild cohomologies of algebras [9].

Let  $\mathbb{K}$  be a commutative unital ring and  $A$  be an associative unital algebra over a  $\mathbb{K}$ . Denote by  $C^n(A)$  the set of all  $\mathbb{K}$ -multilinear functions  $f : A^n \rightarrow A$ . Hochschild [9] introduced a differential  $D : C^n(A) \rightarrow C^{n+1}(A)$  that satisfies  $D^2 = 0$  thus making  $\{C^n(A)\}_{n=0}^\infty$  into a cochain complex. Its cohomologies are called Hochschild cohomologies of  $A$  and are denoted by  $HH^n(A)$ .

In 1983 Gerstenhaber and Schack [4] established the following beautiful relation between the Hochschild and simplicial cohomologies. Let  $S$  be a (finite) simplicial complex. Denote by  $V$  the set of all barycenters of the simplexes of  $S$ , and introduce on  $V$  a structure of a digraph (=directed graph) by defining the edges  $a \rightarrow b$  between two barycenters  $a, b \in V$  as follows: if  $s_a$  is the simplex whose barycenter is  $a$ , and  $s_b$  the simplex with the barycenter  $b$ , then by definition

$$a \rightarrow b \Leftrightarrow s_a \supset s_b. \tag{1}$$

In particular, the underlying undirected graph of  $V$  with removed loops is the 1-skeleton of the barycentric subdivision  $B_S$  of  $S$  (see Fig. 1).

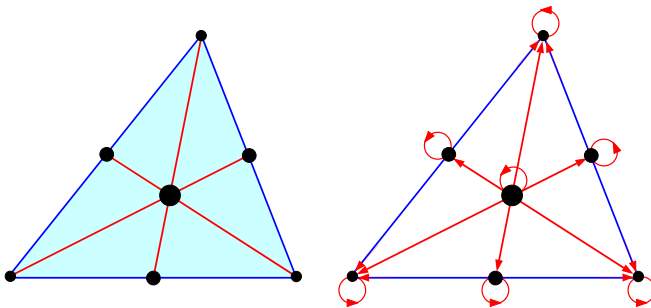
Denote by  $A$  the left  $\mathbb{K}$ -module of all formal linear combinations of the edges of  $V$  with coefficients from  $\mathbb{K}$ . Define the multiplication of the edges by

$$(a, b)(a', b') = \begin{cases} (a, b'), & \text{if } b = a' \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

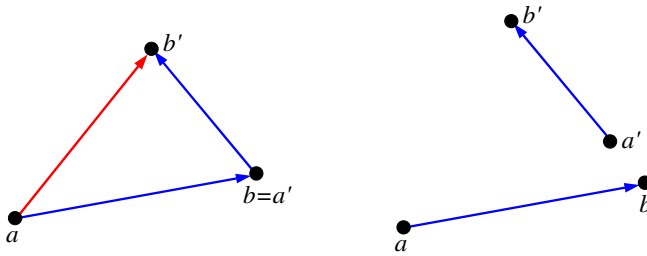
(cf. Fig. 2) and then extend it by  $\mathbb{K}$ -bilinearly to a multiplication in  $A$ , thus making  $A$  into an associative algebra over  $\mathbb{K}$ . The algebra  $A$  has a unity that is the sum of all loops in  $V$ .

Gerstenhaber and Schack proved in [4] that

$$HH^*(A) \cong H^*(S), \tag{3}$$



**Fig. 1** The barycentric subdivision  $B_S$  of a 2-simplex (left) and the associated digraph  $(V, \rightarrow)$  (right)



**Fig. 2** Illustration to (2):  $(a, b) (a', b') = (a, b')$  (left) and  $(a, b) (a', b') = 0$  (right)

where  $H^*(S)$  is the simplicial cohomology of  $S$ . Although they constructed in [4] an explicit cochain map  $\tau : C^*(S) \rightarrow C^*(A)$ , they wrote: “we are unable to show by direct calculation that  $H^*(\tau)$  is an isomorphism”. To prove the isomorphism (3) in cohomologies, the Gerstenhaber and Schack [3] used instead the cohomology comparison theorem developed in their previous paper.

Another proof of (3) was given by Cibils [1] using a cleverly chosen resolution of the path algebra of a poset associated with the simplicial complex.

The purpose of this paper is to give a direct proof of (3) by constructing a cochain map between  $C^*(S)$  and  $C^*(A)$ , although in a number of steps, and showing that the associated cohomology map is an isomorphism, thus completing the program of Gerstenhaber and Schack.

Let us briefly describe our approach. We make a crucial use of cohomologies of digraphs that are analogous to the differential forms cohomologies. The elements of the cochain complex  $\Omega^n$ , based on the digraph  $(V, \rightarrow)$ , are  $\mathbb{K}$ -valued functions on  $n$ -paths  $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n$  of the vertices of  $V$ , with appropriately defined exterior differential  $d : \Omega^n \rightarrow \Omega^{n+1}$ . We use also a regularized cochain complex  $\Omega_{reg}^*$  where the functions are defined only on the paths  $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n$  with distinct vertices.

The notions of exterior differential and exterior forms on general digraphs as well as the associated cohomology theory were developed in [2,5,6]. In particular, it was shown in [6] (based on [7]) that the cohomology groups  $H^*(\Omega_{reg}, \mathbb{Z})$  are homotopy invariant. We refer the reader to [5,6] for general overview of (co)homology theories on digraphs. In the present paper we need the aforementioned theory only in a special case of transitive digraphs. In the latter case the definitions of the cochain complex and the associated cohomologies are significantly simplified and can be easily introduced from scratch as we do in Sect. 3.

Consider the product  $\tilde{V} = V \times V$  that also becomes a digraph with the following definition of edges:

$$(i, j) \rightarrow (i', j') \quad \text{in } \tilde{V} \Leftrightarrow j \rightarrow i' \quad \text{in } V$$

(see Sect. 4). This type of product of graphs is unusual in graph theory, but is appropriate for our purpose. On the digraph  $(\tilde{V}, \rightarrow)$  we also introduce a cochain complex  $\tilde{\Omega}^*$  analogously to the cochain complex  $\Omega^*$  on  $(V, \rightarrow)$ , with an additional weight in the exterior differential, which strengthens the role of the diagonal  $\text{diag } \tilde{V} \cong V$ .

Denote by  $\overline{A}$  the algebra spanned by all pairs  $(a, b) \in V \times V$  (not only edges of  $V$ ) with multiplication (2), and consider a Hochschild cohomology  $HH^*(A, \overline{A})$  that comes from the cochain complex  $C^*(A, \overline{A})$  of multilinear functions  $f : A^n \rightarrow \overline{A}$ .

We prove (3) by means of the following chain of relations:

$$HH^*(A) \cong HH^*(A, \overline{A}) \cong H^*(\widetilde{\Omega}) \cong H^*(\Omega) \cong H^*(\Omega_{reg}) \cong H^*(B_S) \cong H^*(S), \tag{4}$$

where  $B_S$  is the barycentric subdivision of  $S$ . The corresponding cochain complexes are related as follows:

$$C^*(A) \subset C^*(A, \overline{A}) \cong \widetilde{\Omega}^* \succ \Omega^* \succ \Omega_{reg}^* \cong C^*(B_S) \supset C^*(S),$$

where  $\mathcal{A}^* \succ \mathcal{B}^*$  means that  $\mathcal{B}^*$  is a quotient cochain complex of  $\mathcal{A}^*$ . Hence, in the first step we enlarge the cochain complex in question, but in each of the following steps we reduce it while preserving cohomologies, until we arrive at  $C^*(S)$ . Let us comment on each of the relations in (4).

1. The isomorphism  $H^*(\widetilde{\Omega}) \cong H^*(\Omega)$  is key result of this paper and is proved in Lemma 4.2. We establish first a map between the corresponding cochain complexes and then prove that the cohomology of the kernel of this map is trivial. This step is crucial in the entire proof as it allows to drastically reduce the cochain complex  $\widetilde{\Omega}^*$  to  $\Omega^*$ .
2. The isomorphism  $HH^*(A, \overline{A}) \cong H^*(\widetilde{\Omega})$  follows from the isomorphism of the corresponding cochain complexes  $C^*(A, \overline{A})$  and  $\widetilde{\Omega}^*$ . The proof of the latter contains a trick of changing the tensor notation, which in an instant transforms the elements of  $C^n(A, \overline{A})$  into those of  $\widetilde{\Omega}^n$  (Lemma 5.1).
3. The isomorphism  $HH^*(A) \cong HH^*(A, \overline{A})$  is proved in Lemma 5.2 using an argument from the proof of Lemma 4.2.
4. The isomorphism  $H^*(\Omega) \cong H^*(\Omega_{reg})$  is proved in Lemma 6.1, also similarly to Lemma 4.2.
5. The isomorphism  $H^*(\Omega_{reg}) \cong H^*(B_S)$  is rather straightforward and follows from the isomorphism of the cochain complexes  $\Omega_{reg}^*$  and  $C^*(B_S)$  (Lemma 7.1).
5. The isomorphism  $H^*(B_S) \cong H^*(S)$  is classical.

The structure of the paper is determined by the above sequence of lemmas. The main Theorem 7.2 is stated and proved at the end of the paper.

## 2 Hochschild cohomology

Let  $\mathbb{K}$  be a commutative unital ring and  $A$  be an associative unital algebra over  $\mathbb{K}$ . Let  $M$  be a left  $\mathbb{K}$ -module that has a compatible structure of a  $A$ -bimodule (in particular,  $M$  can be an algebra over  $\mathbb{K}$  containing  $A$  as a subalgebra). For any non-negative integer  $n$ , denote by  $C^n(A, M)$  the set of all  $\mathbb{K}$ -multilinear functions  $f : A^n \rightarrow M$ . Clearly,  $C^n(A, M)$  is a left  $\mathbb{K}$ -module. By definition, we take  $C^0(A, M) = M$ .

Consider the Hochschild differential  $D : C^n(A, M) \rightarrow C^{n+1}(A, M)$  defined on any function  $f \in C^n(A, M)$  as follows:

$$\begin{aligned}
 Df(a_1, a_2, \dots, a_{n+1}) &= a_1 f(a_2, \dots, a_{n+1}) \\
 &\quad + \sum_{k=1}^n (-1)^k f(a_1, \dots, a_k a_{k+1}, \dots, a_{n+1}) \\
 &\quad + (-1)^{n+1} f(a_1, \dots, a_n) a_{n+1},
 \end{aligned}$$

for all  $a_i \in A$ . In particular, for  $n = 0$ , the differential of a constant  $f = m \in M$  is a function  $Df : A \rightarrow M$  given by

$$Df(a) = am - ma.$$

It is known [9] that  $D^2 = 0$  so that we have a cochain complex

$$0 \rightarrow C^0(A, M) \xrightarrow{D} \dots \xrightarrow{D} C^n(A, M) \xrightarrow{D} C^{n+1}(A, M) \xrightarrow{D} \dots$$

whose cohomologies are called the Hochschild cohomologies and are denoted by  $HH^*(A, M)$ .

### 3 Cohomology of transitive digraphs

In this section we introduce a cohomology theory on transitive digraphs, similarly to the cohomology theory of digraph of [2,6]. However, unlike the cited works, we allow the digraphs to have loop edges, which is motivated by application to Hochschild cohomology.

Let  $V$  be a finite set, whose elements will be called vertices. Denote by  $\Lambda^n = \Lambda^n(V, \mathbb{K})$  the left  $\mathbb{K}$ -module of all functions  $f : V^{n+1} \rightarrow \mathbb{K}$ . We use the following notation

$$f_{i_0 \dots i_n} := f(i_0, \dots, i_n)$$

for all  $i_k \in V$ . Any sequence  $i_0 \dots i_n$  of  $n + 1$  vertices of  $V$  will be called an  $n$ -path. Hence, any  $f \in \Lambda^n$  is a  $\mathbb{K}$ -valued function on  $n$ -paths.

Fix a function  $\delta : V \rightarrow \mathbb{K}$  and define the exterior differential  $d : \Lambda^n \rightarrow \Lambda^{n+1}$  by

$$(df)_{i_0 \dots i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k \delta_{i_k} f_{i_0 \dots \widehat{i_k} \dots i_{n+1}}, \tag{5}$$

where the hat  $\widehat{i_k}$  means omission of the index  $i_k$ . Strictly speaking, the operator  $d$  should have been denoted by  $d_\delta$  but we omit the index  $\delta$  as usually this causes no confusion.

Claim  $d^2 = 0$ .

Proof For any  $f \in \Lambda^n$ , we have

$$\begin{aligned} (d^2 f)_{i_0 \dots i_{n+2}} &= \sum_{k=0}^{n+2} (-1)^k \delta_{i_k} (df)_{i_0 \dots \widehat{i_k} \dots i_{n+2}} \\ &= \sum_{k=0}^{n+2} (-1)^k \delta_{i_k} \left( \sum_{l=0}^{k-1} (-1)^l \delta_{i_l} f_{i_0 \dots \widehat{i_l} \dots \widehat{i_k} \dots i_{n+2}} \right. \\ &\quad \left. + \sum_{l=k+1}^{n+2} (-1)^{l-1} \delta_{i_l} f_{i_0 \dots \widehat{i_k} \dots \widehat{i_l} \dots i_{n+2}} \right) \\ &= \sum_{0 \leq l < k \leq n+2} (-1)^{k+l} \delta_{i_k} \delta_{i_l} f_{i_0 \dots \widehat{i_l} \dots \widehat{i_k} \dots i_{n+2}} \\ &\quad - \sum_{0 \leq k < l \leq n+2} (-1)^{k+l} \delta_{i_k} \delta_{i_l} f_{i_0 \dots \widehat{i_k} \dots \widehat{i_l} \dots i_{n+2}}. \end{aligned}$$

After switching  $k$  and  $l$  in the last sum we see that it is equal to the previous one, whence  $d^2 f = 0$  follows. □

Hence, we obtain a cochain complex

$$0 \rightarrow \Lambda^0 \xrightarrow{d} \dots \xrightarrow{d} \Lambda^n \xrightarrow{d} \Lambda^{n+1} \xrightarrow{d} \dots$$

So far  $V$  was just a finite set. Let us now introduce a digraph structure on  $V$ .

**Definition 3.1** A couple  $(V, E)$  is called a (finite) digraph if  $V$  is any finite set and  $E \subset V \times V$  is a relation on  $V$ .

Any couple  $(i, j) \in E$  is called a directed edge or arrow and will be denoted by  $i \rightarrow j$ . A digraph  $(V, E)$  will also be denoted by  $(V, \rightarrow)$ .

**Definition 3.2** A  $n$ -path  $i_0 \dots i_n$  of vertices of a digraph  $(V, \rightarrow)$  is called *allowed* if

$$i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n,$$

and *non-allowed* otherwise.

On any digraph  $(V, \rightarrow)$  there is a natural way of defining a cochain complex  $\Omega^*$  with the exterior differential  $d$  that takes into account allowed paths (see [6,8]). However, we need this construction only for a specific class of *transitive* digraphs, that can be done easily as below.

**Definition 3.3** Let  $(V, \rightarrow)$  be a digraph and let  $\delta$  be a  $\mathbb{K}$ -valued function on  $V$ . We say that  $V$  is a  $\delta$ -*transitive* if

$$i \rightarrow j \quad \text{and} \quad j \rightarrow k \Rightarrow i \rightarrow k, \quad \text{provided} \quad \delta_j \neq 0,$$

for all  $i, j, k \in V$ .

Let  $(V, \rightarrow)$  be a  $\delta$ -transitive digraph. Denote by  $\Omega^n = \Omega^n((V, \rightarrow), \mathbb{K})$  the set of  $\mathbb{K}$ -valued functions on all allowed  $n$ -paths on  $V$ . Let us use the same function  $\delta$  in the definition (5) of  $d$  and show that the operator  $d = d_\delta$  can be considered as an operator from  $\Omega^n$  to  $\Omega^{n+1}$ . Indeed, given  $f \in \Omega^n$ , let us first extend  $f$  arbitrarily to a function on all  $n$ -paths, that is, to an element of  $\Lambda^n$ . Then (5) defines  $df$  as an element of  $\Lambda^{n+1}$ , that is, a function on all  $(n + 1)$ -paths. Then the restriction of  $df$  to the allowed  $(n + 1)$ -paths yields an element of  $\Omega^{n+1}$ , that will also be denoted by  $df$ .

*Claim* If  $V$  is  $\delta$ -transitive then, for any  $f \in \Omega^n$ , the differential  $df$  is well defined as an element of  $\Omega^{n+1}$ .

*Proof* We need to show that  $df$  on allowed  $(n + 1)$ -paths does not depend on how  $f$  was extended. Indeed, let  $i_0 \dots i_{n+1}$  be an allowed path. We need to verify that in the right hand side of (5) the contribution of the terms  $\delta_{i_k} f_{i_0 \dots \widehat{i_k} \dots i_{n+1}}$  with non-allowed paths  $i_0 \dots \widehat{i_k} \dots i_{n+1}$  vanishes. Indeed, if  $\delta_{i_k} = 0$  then this term does not give any contribution. If  $\delta_{i_k} \neq 0$  then  $i_{k-1} \rightarrow i_k \rightarrow i_{k+1}$  imply  $i_{k-1} \rightarrow i_{k+1}$  so that the  $n$ -path

$$i_0 \dots \widehat{i_k} \dots i_{n+1} = i_0 \dots i_{k-1} i_{k+1} \dots i_{n+1}$$

is allowed, which finishes the proof. □

Hence, we obtain a cochain complex

$$0 \rightarrow \Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n \xrightarrow{d} \Omega^{n+1} \xrightarrow{d} \dots$$

of the  $\delta$ -transitive digraph  $(V, \rightarrow)$  that will be denoted by  $\Omega^* = \Omega^*_\delta((V, \rightarrow), \mathbb{K})$ . The cohomologies of this complex

$$H^n(\Omega) = H^n_\delta(\Omega(V, \rightarrow), \mathbb{K}) = \ker d|_{\Omega^n} / \text{Im} d|_{\Omega^{n-1}}$$

(where  $n \geq 0$ ) are of primary interest for us.

Note that a digraph can be  $\delta$ -transitive with many choices of function  $\delta$  since in Definition 3.3 only the support of  $\delta$  matters. However, since we use the same function  $\delta$  also in the definition (5) of  $d$ , different choices of  $\delta$  will determine different complexes  $\Omega^*$ . Therefore, in applications one should carefully specify the choice of  $\delta$ .

A digraph  $(V, \rightarrow)$  is called *transitive* if

$$i \rightarrow j \rightarrow k \Rightarrow i \rightarrow k.$$

For example, any partially ordered set (shortly, a poset) is a transitive digraph. Of course, a digraph is transitive if and only if it is  $\delta$ -transitive with  $\delta \equiv 1$ . For transitive digraphs we will always use  $\delta \equiv 1$  in the definition (5) of  $d$ .

The cochain complex  $H^*(\Omega)$  for transitive graphs is a particular case of the construction of [6, 8] of cochain complexes for arbitrary digraphs. Note also that the ring  $\mathbb{K}$  in the definition of the cochain complex can be replaced by any abelian additive group provided the function  $\delta$  is  $\mathbb{Z}$ -valued.

### 4 Self-product of a transitive digraph

Let  $(V, \rightarrow)$  be a transitive digraph, and  $\Omega^*$  the associated cochain complex (with  $\delta \equiv 1$  as we have agreed above). Set  $\tilde{V} = V \times V$  and define the edges on  $\tilde{V}$  as follows

$$(i, j) \rightarrow (i', j') \Leftrightarrow j \rightarrow i' \tag{6}$$

for all  $i, j, i', j' \in V$  (cf. Fig. 3).

Define a function  $\delta : \tilde{V} \rightarrow \mathbb{K}$  by

$$\delta_{(i,j)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{7}$$

*Claim* If  $(V, \rightarrow)$  is transitive then  $(\tilde{V}, \rightarrow)$  is  $\delta$ -transitive with the function  $\delta$  defined by (7).

*Proof* Indeed, if

$$(i, j) \rightarrow (i', j') \rightarrow (i'', j'')$$

and  $\delta_{(i',j')} \neq 0$  then  $i' = j'$  and

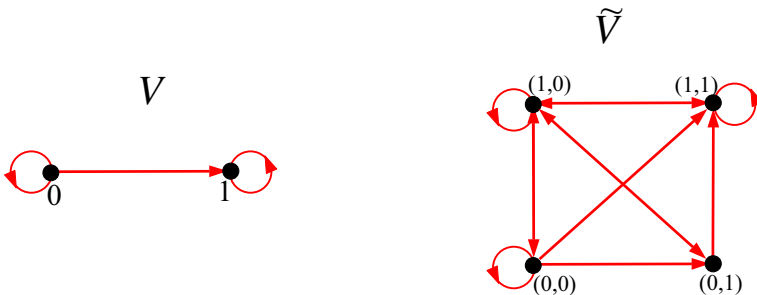
$$j \rightarrow i' = j' \rightarrow i''$$

whence  $j \rightarrow i''$  and, hence,  $(i, j) \rightarrow (i'', j'')$ . □

**Definition 4.1** The  $\delta$ -transitive digraph  $\tilde{V}$  is called a *self-product* of  $V$ .

As before, let  $\Omega^*$  be the cochain complex of the transitive digraph  $V$ , and denote by  $\tilde{\Omega}^*$  the cochain complex of the  $\delta$ -transitive digraph  $\tilde{V}$ .

We say that a digraph  $(V, \rightarrow)$  is *reflexive* if  $i \rightarrow i$  for all  $i \in V$ . For example, any poset is transitive and reflexive. Now we can state and prove our main technical lemma.



**Fig. 3** An example of digraphs  $V$  and  $\tilde{V}$



**Lemma 4.2** *If  $V$  is transitive and reflexive, then we have the isomorphism of cohomologies:*

$$H^*(\tilde{\Omega}) \cong H^*(\Omega). \tag{8}$$

*Proof* Let us identify  $V$  with  $\text{diag } \tilde{V}$ . Note that  $(i, i) \rightarrow (j, j)$  is equivalent to  $i \rightarrow j$  so that  $\text{diag } \tilde{V}$  as a subgraph of  $\tilde{V}$  is isomorphic to a digraph  $V$ . For any function  $f$  on the allowed  $n$ -paths in  $\tilde{V}$ , let  $\Phi f$  be a restriction of  $f$  to the allowed  $n$ -paths on  $V$ . In tensor notation, any function  $f \in \tilde{\Omega}^n$  has the components  $f_{(i_0, j_0) \dots (i_n, j_n)}$ , and then  $\Phi f$  is defined by

$$(\Phi f)_{i_0 \dots i_n} = f_{(i_0, i_0) \dots (i_n, i_n)}$$

for all allowed  $n$ -paths  $i_0 \dots i_n$  on  $V$ . Hence, we obtain an epimorphism

$$\Phi : \tilde{\Omega}^n \rightarrow \Omega^n. \tag{9}$$

We start with verifying that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{\Omega}^n & \xrightarrow{d} & \tilde{\Omega}^{n+1} \\ \downarrow \Phi & & \downarrow \Phi \\ \Omega^n & \xrightarrow{d} & \Omega^{n+1} \end{array} \tag{10}$$

For any  $f \in \tilde{\Omega}^n$  and any allowed  $(n + 1)$ -path  $i_0 \dots i_{n+1}$  on  $V$  we have

$$\begin{aligned} (d(\Phi f))_{i_0 \dots i_{n+1}} &= \sum_{k=0}^{n+1} (-1)^k (\Phi f)_{i_0 \dots \widehat{i_k} \dots i_{n+1}} \\ &= \sum_{k=0}^{n+1} (-1)^k \delta_{(i_k, i_k)} f_{(i_0, i_0) \dots \widehat{(i_k, i_k)} \dots (i_{n+1}, i_{n+1})} \\ &= (df)_{(i_0, i_0) \dots (i_{n+1}, i_{n+1})} \\ &= (\Phi(df))_{i_0 \dots i_{n+1}} \end{aligned}$$

whence  $d\Phi = \Phi d$  follows.

Denote by  $K^n$  the kernel of the mapping (9). Since  $\Phi$  is a cochain map from  $\tilde{\Omega}^*$  to  $\Omega^*$ , we see that  $K^*$  is a sub cochain complex. The short exact sequence

$$0 \rightarrow K^* \rightarrow \tilde{\Omega}^* \rightarrow \Omega^* \rightarrow 0$$

of cochain complexes yields by zigzag Lemma the long exact sequence in cohomologies

$$\dots \rightarrow H^n(K) \rightarrow H^n(\tilde{\Omega}) \rightarrow H^n(\Omega) \rightarrow H^{n+1}(K) \rightarrow \dots$$

Hence, in order to prove the isomorphism (8) it suffices to show that, for all non-negative integers  $n$ ,

$$H^n(K) \cong \{0\},$$

that is, if  $f \in K^n$  and  $df = 0$  then there is  $g \in K^{n-1}$  such that  $f = dg$ .

Let us denote the elements of  $\tilde{V}$  by  $\mathbf{i}_1, \mathbf{i}_2$  etc. If  $\mathbf{i} = (i, j)$  then set

$$\mathbf{i}' = (j, j) \in \text{diag} \tilde{V}$$

so that  $\mathbf{i}'$  can be regarded as “projection” of  $\mathbf{i}$  onto  $V$ . Note that always  $\mathbf{i} \rightarrow \mathbf{i}'$ , which follows from the reflexivity of  $V$ , and

$$\mathbf{i} \rightarrow \mathbf{j} \Leftrightarrow \mathbf{i}' \rightarrow \mathbf{j}$$

for all  $\mathbf{i}, \mathbf{j} \in \tilde{V}$ , since  $\mathbf{i}$  and  $\mathbf{i}'$  have the same second component.

The following claim is a key to the proof.

*Claim* For any  $f \in \tilde{\Omega}^n$  there exists  $g \in \tilde{\Omega}^{n-1}$  such that the following holds:

(i) for any allowed  $(n - 1)$ -path  $\mathbf{i}_1 \dots \mathbf{i}_n$  that lies on  $\text{diag} \tilde{V}$ ,

$$g_{\mathbf{i}_1 \dots \mathbf{i}_n} = 0; \tag{11}$$

(ii) for any allowed  $n$ -path  $\mathbf{i}_1 \dots \mathbf{i}_{n+1}$  on  $\tilde{V}$  that does not lie entirely on  $\text{diag} \tilde{V}$ ,

$$(-1)^k (df)_{\mathbf{i}_1 \dots \mathbf{i}_k \mathbf{i}'_k \dots \mathbf{i}_{n+1}} = f_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}} - (dg)_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}}, \tag{12}$$

where  $k$  is the smallest index such that  $\mathbf{i}_k \notin \text{diag} \tilde{V}$ .

Let us define  $g$  as follows: for any allowed  $(n - 1)$ -paths  $\mathbf{i}_1 \dots \mathbf{i}_n$  on  $\tilde{V}$ , set  $g_{\mathbf{i}_1 \dots \mathbf{i}_n} = 0$  if all  $\mathbf{i}_k \in \text{diag} \tilde{V}$ , whereas otherwise set

$$g_{\mathbf{i}_1 \dots \mathbf{i}_n} = (-1)^k f_{\mathbf{i}_1 \dots \mathbf{i}_{k-1} \mathbf{i}_k \mathbf{i}'_k \mathbf{i}_{k+1} \dots \mathbf{i}_n}, \tag{13}$$

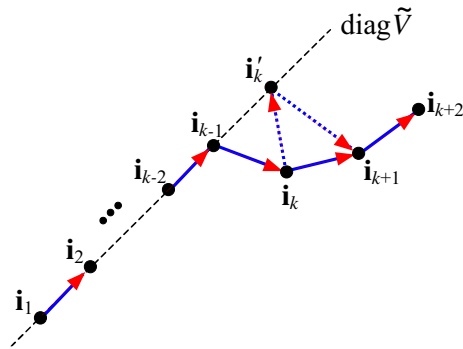
where  $k$  is the minimal index such that  $\mathbf{i}_k \notin \text{diag} \tilde{V}$ . Observe that the right hand side of (13) is well defined, since the  $n$ -path  $\mathbf{i}_1 \dots \mathbf{i}_{k-1} \mathbf{i}_k \mathbf{i}'_k \mathbf{i}_{k+1} \dots \mathbf{i}_n$  is allowed whenever  $\mathbf{i}_1 \dots \mathbf{i}_n$  is allowed, because  $\mathbf{i}_k \rightarrow \mathbf{i}'_k \rightarrow \mathbf{i}_{k+1}$  (see Fig. 4).

We have by (5)

$$(dg)_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}} = \sum_{l=1}^{n+1} (-1)^{l-1} \delta_{\mathbf{i}_l} g_{\mathbf{i}_1 \dots \widehat{\mathbf{i}}_l \dots \mathbf{i}_{n+1}}. \tag{14}$$

The term  $\delta_{\mathbf{i}_l} g_{\mathbf{i}_1 \dots \widehat{\mathbf{i}}_l \dots \mathbf{i}_{n+1}}$  in (14) vanishes whenever  $\delta_{\mathbf{i}_l} = 0$ , which, in particular, is the case when  $l = k$  as  $\mathbf{i}_k \notin \text{diag} \tilde{V}$ . Assume that  $\delta_{\mathbf{i}_l} \neq 0$ , in particular,  $l \neq k$ . Then the  $(n - 1)$ -path  $\mathbf{i}_1 \dots \widehat{\mathbf{i}}_l \dots \mathbf{i}_{n+1}$  is allowed. The first term in this path outside  $\text{diag} \tilde{V}$  is

**Fig. 4** The path  $i_1 \dots i_k i'_k i_{k+1} \dots$  is allowed



again  $i_k$ , and it has the ordinal number  $k - 1$  in this path, if  $l < k$ , and the ordinal number  $k$  if  $l > k$ . Hence, if  $l < k$ , then we obtain by (13)

$$g_{i_1 \dots \widehat{i_l} \dots i_{n+1}} = (-1)^{k-1} f_{i_1 \dots \widehat{i_l} \dots i_k i'_k \dots i_{n+1}},$$

and if  $l > k$  then

$$g_{i_1 \dots \widehat{i_l} \dots i_{n+1}} = (-1)^k f_{i_1 \dots i_k i'_k \dots \widehat{i_l} \dots i_{n+1}}.$$

Combining all the terms together, we obtain

$$\begin{aligned} (dg)_{i_1 \dots i_{n+1}} &= \sum_{l=1}^{k-1} (-1)^{k+l} \delta_{i_l} f_{i_1 \dots \widehat{i_l} \dots i_k i'_k \dots i_{n+1}} \\ &+ \sum_{l=k+1}^{n+1} (-1)^{k+l-1} \delta_{i_l} f_{i_1 \dots i_k i'_k \dots \widehat{i_l} \dots i_{n+1}}. \end{aligned} \tag{15}$$

By (5) we have also

$$\begin{aligned} (df)_{i_1 \dots i_k i'_k \dots i_{n+1}} &= \sum_{l=1}^{k-1} (-1)^{l-1} \delta_{i_l} f_{i_1 \dots \widehat{i_l} \dots i_k i'_k \dots i_{n+1}} \\ &+ (-1)^{k-1} \delta_{i_k} f_{i_1 \dots \widehat{i_k} i'_k \dots i_{n+1}} \\ &+ (-1)^k \delta_{i'_k} f_{i_1 \dots i_k \widehat{i'_k} \dots i_{n+1}} \\ &+ \sum_{l=k+1}^{n+1} (-1)^l \delta_{i_l} f_{i_1 \dots i_k i'_k \dots \widehat{i_l} \dots i_{n+1}} \end{aligned}$$

Noticing that  $\delta_{\mathbf{i}_k} = 0, \delta_{\mathbf{i}'_k} = 1, f_{\mathbf{i}_1 \dots \mathbf{i}_k \widehat{\mathbf{i}_k} \dots \mathbf{i}_{n+1}} = f_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}}$ , substituting here (15) and multiplying by  $(-1)^k$ , we obtain (12).

Now assuming that  $f \in K^n$  and  $df = 0$ , let us show that the function  $g$  defined by (13) belongs to  $K^{n-1}$  and satisfies  $dg = f$ . That  $g \in K^{n-1}$  is true by (11). Let us show that  $dg = f$ , that is, for any allowed  $n$ -path  $\mathbf{i}_1 \dots \mathbf{i}_{n+1}$ ,

$$(dg)_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}} = f_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}}. \tag{16}$$

If all  $\mathbf{i}_1, \dots, \mathbf{i}_{n+1} \in \text{diag } \widetilde{V}$  then we have by (11) that  $(dg)_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}} = 0$ , which matches  $f_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}} = 0$  that is true by  $f \in K^n$ . Otherwise, let  $k$  be the minimal value of the index such that  $\mathbf{i}_k \notin \text{diag } \widetilde{V}$ . As it was observed above, the  $(n + 1)$ -path  $\mathbf{i}_1 \dots \mathbf{i}_k \mathbf{i}'_k \dots \mathbf{i}_{n+1}$  is allowed. Since  $df = 0$ , that is,  $df$  vanishes on all allowed  $(n + 1)$ -paths, we obtain

$$(df)_{\mathbf{i}_1 \dots \mathbf{i}_k \mathbf{i}'_k \dots \mathbf{i}_{n+1}} = 0.$$

Substituting this into (12), we obtain (16). □

### 5 Algebra associated with poset

A digraph  $(V, \rightarrow)$  is called a poset (partially ordered set) if it is reflexive, transitive, and antisymmetric, that is,  $i \rightarrow j \rightarrow i$  implies  $i = j$ . In this section  $(V, \rightarrow)$  is always a finite poset.

Denote by  $\overline{A}$  the left  $\mathbb{K}$ -module of all formal  $\mathbb{K}$ -linear combinations of pairs  $(i, j)$  of vertices of  $V$ . The pair  $(i, j)$  as an element of  $\overline{A}$  will be denoted by  $e_{ij}$  so that any element  $a \in \overline{A}$  can be uniquely represented in the form

$$a = \sum_{i, j \in V} a^{ij} e_{ij}$$

where  $a^{ij}$  are the coefficients from  $\mathbb{K}$ . Let  $A$  be a submodule of  $\overline{A}$  spanned by all  $e_{ij}$  with  $i \rightarrow j$ ; that is, any  $a \in A$  has the form

$$a = \sum_{\{i, j \in V: i \rightarrow j\}} a^{ij} e_{ij}.$$

Define in  $\overline{A}$  multiplication on the basis elements

$$e_{ij} \cdot e_{i'j'} = \delta_{ji'} e_{ij'} = \begin{cases} e_{ij'}, & \text{if } i' = j, \\ 0, & \text{otherwise,} \end{cases} \tag{17}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{18}$$

and then extend the multiplication to all the elements of  $\overline{A}$  by  $\mathbb{K}$ -bilinearity. It follows from (17) that, for all  $a, b \in \overline{A}$  and  $i, j \in V$ ,

$$(ab)^{ij} = \sum_{l \in V} a^{il} b^{lj}. \tag{19}$$

It is easy to see that  $\overline{A}$  is an associative algebra over  $\mathbb{K}$  with a unity  $1_{\overline{A}} = \sum_{i \in V} e_{ii}$ .

By a poset property we see that if  $e_{ij}, e_{kl} \in A$  then  $e_{ij}e_{kl} \in A$  so that  $A$  is a subalgebra of  $\overline{A}$  (and  $1_A = 1_{\overline{A}}$ ). Hence, the Hochschild cochain complex  $C^*(A, \overline{A})$  is well defined as in Sect. 2.

On the other hand, we have a cochain complex  $\Omega^*$  of the poset  $V$  as defined in Sect. 3, and a cochain complex  $\tilde{\Omega}^*$  of the self-product  $\tilde{V}$  of  $V$  as defined in Sect. 4.

**Lemma 5.1** *The cochain complexes  $C^*(A, \overline{A})$  and  $\tilde{\Omega}^*$  are isomorphic.*

*Proof* For any function  $f \in C^n(\overline{A}, \overline{A})$  set

$$f_{i_1 j_1 i_2 j_2 \dots i_n j_n} = f(e_{i_1 j_1}, \dots, e_{i_n j_n})$$

for all  $i_k, j_k \in V$ . Since  $f$  takes values in  $\overline{A}$ ,  $f_{i_1 j_1 i_2 j_2 \dots i_n j_n}$  is linear combination of the terms  $e_{ij}$  as follows:

$$f_{i_1 j_1 i_2 j_2 \dots i_n j_n} = \sum_{i, j \in V} f_{i_1 j_1 i_2 j_2 \dots i_n j_n}^{ij} e_{ij}$$

with the coefficients  $f_{i_1 j_1 i_2 j_2 \dots i_n j_n}^{ij} \in \mathbb{K}$ . It follows that, for all  $a_1, \dots, a_n \in A$ ,

$$f(a_1, \dots, a_n) = \sum f_{i_1 j_1 \dots i_n j_n}^{ij} a_1^{i_1 j_1} \dots a_n^{i_n j_n} e_{ij}$$

where the summation is performed over all  $i, j, i_k, j_k \in V$ .

By definition of the Hochschild differential  $D$  on  $C^*(\overline{A}, \overline{A})$  we have

$$\begin{aligned} (Df)_{i_1 j_1 \dots i_{n+1} j_{n+1}} &= (Df)(e_{i_1 j_1}, \dots, e_{i_{n+1} j_{n+1}}) \\ &= e_{i_1 j_1} f(e_{i_2 j_2}, \dots, e_{i_{n+1} j_{n+1}}) \\ &\quad + \sum_{k=1}^n (-1)^k f(e_{i_1 j_1}, \dots, e_{i_k j_k} \cdot e_{i_{k+1} j_{k+1}}, \dots, e_{i_{n+1} j_{n+1}}) \\ &\quad + (-1)^{n+1} f(e_{i_1 j_1}, \dots, e_{i_n j_n}) e_{i_{n+1} j_{n+1}}. \end{aligned}$$

Using (17) and (19) we obtain

$$\begin{aligned}
 (Df)_{i_1 j_1 \dots i_{n+1} j_{n+1}}^{ij} &= \sum_{l \in V} e_{i_1 j_1}^{il} f_{i_2 j_2 \dots i_{n+1} j_{n+1}}^{lj} \\
 &+ \sum_{k=1}^n (-1)^k \delta_{j_k i_{k+1}} f_{i_1 j_1 \dots i_{k-1} j_{k-1} i_k j_{k+1} i_{k+2} j_{k+2} \dots i_{n+1} j_{n+1}}^{ij} \\
 &+ (-1)^{n+1} \sum_{l \in V} f_{i_1 j_1 \dots i_n j_n}^{il} e_{i_{n+1} j_{n+1}}^{lj}
 \end{aligned}$$

whence

$$\begin{aligned}
 (Df)_{i_1 j_1 \dots i_{n+1} j_{n+1}}^{ij} &= \delta_{i i_1} f_{i_2 j_2 \dots i_{n+1} j_{n+1}}^{j_1 j} \\
 &+ \sum_{k=1}^n (-1)^k \delta_{j_k i_{k+1}} f_{i_1 j_1 \dots i_{k-1} j_{k-1} i_k j_{k+1} i_{k+2} j_{k+2} \dots i_{n+1} j_{n+1}}^{ij} \\
 &+ (-1)^{n+1} \delta_{j_{n+1} j} f_{i_1 j_1 \dots i_n j_n}^{i_{n+1}}.
 \end{aligned}$$

Now we change notation, namely, we rename  $f_{i_1 j_1 \dots i_n j_n}^{ij}$  to  $f_{i i_1 j_1 \dots i_n j_n j}$  and then rename in the latter the indices as follows:  $f_{i_0 j_0 i_1 j_1 \dots i_n j_n}$ . Effectively this means the following change of notation:

$$i \rightsquigarrow i_0, \quad i_k \rightsquigarrow j_{k-1}, \quad j_k \rightsquigarrow i_k, \quad j \rightsquigarrow j_n.$$

In the notation for  $Df$  we should use  $j \rightsquigarrow j_{n+1}$  instead of  $j \rightsquigarrow j_n$ , so that the above formula for  $Df$  becomes:

$$\begin{aligned}
 (Df)_{i_0 j_0 \dots i_{n+1} j_{n+1}} &= \delta_{i_0 j_0} f_{i_1 j_1 \dots i_{n+1} j_{n+1}} \\
 &+ \sum_{k=1}^n (-1)^k \delta_{i_k j_k} f_{i_0 j_0 \dots i_{k-1} j_{k-1} i_{k+1} j_{k+1} \dots i_{n+1} j_{n+1}} \\
 &+ (-1)^{n+1} \delta_{i_{n+1} j_{n+1}} f_{i_0 j_0 \dots i_n j_n}.
 \end{aligned}$$

All pairs  $(i_k, j_k)$  can be regarded as elements of  $\tilde{V} = V \times V$ . Using notation

$$\mathbf{i}_k = (i_k, j_k)$$

we see that the components of  $f$  can be denoted by  $f_{\mathbf{i}_0 \dots \mathbf{i}_n}$ . Hence, the above formula for  $Df$  can be rewritten as follows

$$(Df)_{\mathbf{i}_0 \dots \mathbf{i}_{n+1}} = \sum_{k=0}^{n+1} (-1)^k \delta_{\mathbf{i}_k} f_{\mathbf{i}_0 \dots \widehat{\mathbf{i}}_k \dots \mathbf{i}_{n+1}} \tag{20}$$

for all  $\mathbf{i}_k \in \tilde{V}$ . Consequently, any function  $f \in C^n(\bar{A}, \bar{A})$  can be identified with a function from  $\Lambda^n(\tilde{V})$ , this identification is clearly a linear isomorphism, and it

transforms the operator  $D$  to  $d$  as one can see from comparison of (20) and (5) [note that we use the same the function  $\delta$  in (20) and (5)].

Consider now the set  $C^n(A, \bar{A})$ . The functions from  $C^n(A, \bar{A})$  can be regarded as restrictions of the functions from  $C^n(\bar{A}, \bar{A})$  to the domain  $A^n$ . In particular, for any  $f \in C^n(A, \bar{A})$ , the value  $f(e_{i_1 j_1}, \dots, e_{i_n j_n})$  is defined only if  $i_k \rightarrow j_k$ , for all  $k = 1, \dots, n$ . Hence, the component  $f_{i_1 j_1 \dots i_n j_n}^{ij}$  is defined also under the same condition, that is, for the sequences of indices such that

$$i i_1 \rightarrow j_1 i_2 \rightarrow j_2 \dots i_n \rightarrow j_n j.$$

In the notation  $f_{i_0 j_0 \dots i_n j_n} = f_{i_0 \dots i_n}$  this condition becomes  $j_{k-1} \rightarrow i_k$ , that is,

$$i_0 j_0 \rightarrow i_1 j_1 \rightarrow \dots \rightarrow i_n j_n,$$

which is equivalent to

$$i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_n,$$

given that the edges in  $\tilde{V}$  are defined by (6). Hence, any function  $f \in C^n(A, \bar{A})$  is identified with a  $\mathbb{K}$ -valued function on allowed  $n$ -paths on  $\tilde{V}$ , and such functions are exactly the elements of  $\tilde{\Omega}^n$ , which proves the isomorphism of the cochain complexes  $C^*(A, \bar{A})$  and  $\tilde{\Omega}^*$ . □

**Lemma 5.2** *We have  $HH^*(A, A) \cong HH^*(A, \bar{A})$ .*

*Proof* In the previous proof we have identified the set  $C^n(A, \bar{A})$  with the set  $\tilde{\Omega}^n$  of functions on allowed  $n$ -paths on  $\tilde{V}$ . The set  $C^n(A, A)$  is a subset of  $C^n(A, \bar{A})$ , and a function  $f \in C^n(A, \bar{A})$  belongs to  $C^n(A, A)$  if and only if, for all  $i_1 \rightarrow j_1, \dots, i_n \rightarrow j_n$ ,

$$i \rightarrow j \Rightarrow f_{i_1 j_1 \dots i_n j_n}^{ij} = 0 \tag{21}$$

(in other words, the component  $f^{ij}$  vanishes if  $ij$  is not an edge). Let  $\tilde{\Omega}_0^n$  be the image of  $C^n(A, A)$  in  $\tilde{\Omega}^n$ . Then we have the isomorphism of the cochain complexes

$$C^*(A, \bar{A}) \cong \tilde{\Omega}^* \quad \text{and} \quad C^*(A, A) \cong \tilde{\Omega}_0^*.$$

It suffices to prove that

$$H^n(\tilde{\Omega}) \cong H^n(\tilde{\Omega}_0),$$

which is equivalent to

$$H^n(\tilde{\Omega}/\tilde{\Omega}_0) = \{0\}. \tag{22}$$

Before we can prove (22), let us reformulate the property (21) in terms of the notation  $f_{\mathbf{i}_0 \dots \mathbf{i}_n}$  that we use for the elements of  $\widetilde{\Omega}^n$ . Let us introduce one more relation  $\rightarrow$  on  $\widetilde{V}$ :

$$(i, j) \rightarrow (i', j') \Leftrightarrow i \rightarrow j'$$

[observe the difference with the definition (6) of  $\rightarrow$  on  $\widetilde{V}$ ]. Then the relation  $i \rightarrow j$  in the notation  $f_{i_1 j_1 \dots i_n j_n}^{ij}$  is equivalent to  $\mathbf{i}_0 \rightarrow \mathbf{i}_n$  in the notation  $f_{\mathbf{i}_0 \dots \mathbf{i}_n}$ .

We say that an allowed  $n$ -path  $\mathbf{i}_0 \dots \mathbf{i}_n$  on  $\widetilde{V}$  is *proper* if  $\mathbf{i}_0 \rightarrow \mathbf{i}_n$ , and *improper* otherwise. Then it follows from (21) that a function  $f \in \widetilde{\Omega}^n$  belongs to  $\widetilde{\Omega}_0^n$  if and only if

$$f_{\mathbf{i}_0 \dots \mathbf{i}_n} = 0 \quad \text{for any allowed improper } n\text{-path } \mathbf{i}_0 \dots \mathbf{i}_n. \tag{23}$$

To prove (22) we need to show that if  $f \in \widetilde{\Omega}^n$  and  $df \in \widetilde{\Omega}_0^{n+1}$  then there exists  $g \in \widetilde{\Omega}^{n-1}$  such that  $dg - f \in \widetilde{\Omega}_0^n$ . Define  $g \in \widetilde{\Omega}^{n-1}$  as in Claim in the proof of Lemma 4.2. Then we must show that, for any allowed improper  $n$ -path  $\mathbf{i}_1 \dots \mathbf{i}_{n+1}$  on  $\widetilde{V}$ ,

$$(dg)_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}} = f_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}}. \tag{24}$$

An improper path cannot lie entirely on  $\text{diag } \widetilde{V}$  as otherwise  $\mathbf{i}_1 \rightarrow \dots \rightarrow \mathbf{i}_{n+1}$  implies  $\mathbf{i}_1 \rightarrow \mathbf{i}_{n+1}$  and  $\mathbf{i}_1 \rightarrow \mathbf{i}_{n+1}$ . Let  $k$  be the smallest index with  $\mathbf{i}_k \notin \text{diag } \widetilde{V}$ . Then  $g$  satisfies (12), and the identity (24) will follow from (12) if we show that

$$(df)_{\mathbf{i}_1 \dots \mathbf{i}_k \mathbf{i}'_k \dots \mathbf{i}_{n+1}} = 0. \tag{25}$$

The path  $\mathbf{i}_1 \dots \mathbf{i}_k \mathbf{i}'_k \dots \mathbf{i}_{n+1}$  is allowed, so that we obtain (25) from  $df \in \widetilde{\Omega}_0^{n+1}$  if we show that this path is improper. If  $k < n + 1$  then it is improper because its first and last vertices are the same as those of the improper path  $\mathbf{i}_1 \dots \mathbf{i}_{n+1}$ . In the case  $k = n + 1$  we need to verify that the path  $\mathbf{i}_1 \dots \mathbf{i}_{n+1} \mathbf{i}'_{n+1}$  is improper, that is,  $\mathbf{i}_1 \not\rightarrow \mathbf{i}'_{n+1}$ . Indeed, since the last components of  $\mathbf{i}_{n+1}$  and  $\mathbf{i}'_{n+1}$  are the same, the condition  $\mathbf{i}_1 \not\rightarrow \mathbf{i}'_{n+1}$  follows from  $\mathbf{i}_1 \not\rightarrow \mathbf{i}_{n+1}$ .  $\square$

### 6 Regularized cohomologies of posets

A  $n$ -path  $i_0 \dots i_n$  of the vertices of a digraph  $(V, \rightarrow)$  is called *regular* if  $i_{k-1} \neq i_k$  for all  $k = 1, \dots, n$ , and *irregular* otherwise. Let from now on  $(V, \rightarrow)$  be a poset. Any the regular allowed  $n$ -path  $i_0 \dots i_n$  on a poset has an additional property that all the vertices  $i_k$  in this path are distinct. Indeed, if  $i_l = i_k$  for some  $l < k - 1$  then  $i_l \rightarrow i_{l+1} \rightarrow \dots \rightarrow i_{k-1} \rightarrow i_k$  implies by transitivity that  $i_l \rightarrow i_{k-1} \rightarrow i_k$  whence by a poset property  $i_{k-1} = i_k$ , which contradicts the regularity.

Alongside the set  $\Omega^n$  of  $\mathbb{K}$ -valued functions on all allowed  $n$ -paths on  $V$ , consider the set  $\Omega_{reg}^n$  of  $\mathbb{K}$ -valued functions on all regular allowed  $n$ -paths. Observe that  $d\Omega_{reg}^n \subset \Omega_{reg}^{n+1}$  in the following sense: any function  $f \in \Omega_{reg}^n$  can be extended to a function from  $\Omega^n$ , also denoted by  $f$ , then  $df$  on the regular allowed  $(n + 1)$ -paths



does not depend on the choice of extension. Indeed, if  $i_0 \dots i_{n+1}$  is a regular allowed  $(n + 1)$ -path then

$$(df)_{i_0 \dots i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k f_{i_0 \dots \widehat{i}_k \dots i_{n+1}}, \tag{26}$$

and each path  $i_0 \dots \widehat{i}_k \dots i_{n+1}$  is regular and allowed. Hence,  $\Omega_{reg}^*$  is a cochain complex, that will be referred to as the regularized cochain complex of the poset  $V$ .

**Lemma 6.1** *If  $V$  is a poset then*

$$H^*(\Omega) \cong H^*(\Omega_{reg}). \tag{27}$$

*Proof* Consider the mapping  $\Omega^n \rightarrow \Omega_{reg}^n$  that is given by restriction of the functions from  $\Omega^n$  to regular allowed paths. Let  $K^n$  be the kernel of this mapping, so that we have a short exact sequence

$$0 \rightarrow K^* \rightarrow \Omega^* \rightarrow \Omega_{reg}^* \rightarrow 0.$$

Hence, the isomorphism (27) will follow if we prove that

$$H^n(K) \cong \{0\},$$

that is, for any  $f \in K^n$  with  $df = 0$  there is  $g \in K^{n-1}$  such that  $dg = f$ .

If  $n = 0$  then  $K^0 = \{0\}$  so the claim is trivial. For  $n = 1$ , a function  $f \in K^1$  may have non-zero components only of the form  $f_{ii}$ . Since  $(df)_{iii} = f_{ii}$ , it follows from  $df = 0$  that also  $f_{ii} = 0$  so that  $f = 0$ . Hence,  $g = 0$  is a solution.

Assume  $n \geq 2$ . We will construct explicitly a solution  $g \in K^{n-1}$  of the equation  $df = g$ . Let us first illustrate this construction in the case  $n = 2$ . For any allowed 1-path  $i \rightarrow j$ , set

$$g_{ij} = \begin{cases} 0, & i \neq j \\ f_{iii}, & i = j. \end{cases}$$

Then  $g \in K^1$  and let us show that  $dg = f$ , that is,

$$(dg)_{ijk} = f_{ijk} \tag{28}$$

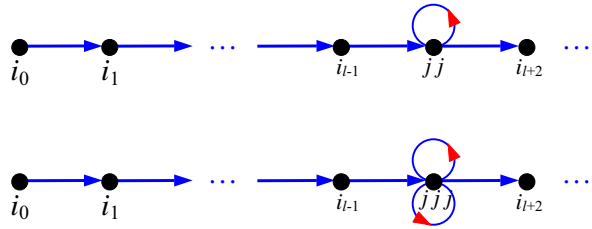
for all allowed paths  $ijk$ . If the path  $ijk$  is regular then all  $i, j, k$  are distinct, and

$$(dg)_{ijk} = g_{jk} - g_{ik} + g_{ij} = 0 = f_{ijk}.$$

If  $i = j$  then

$$(dg)_{iik} = g_{ii} = f_{iii}.$$

**Fig. 5** Paths  $i_0 \dots jj \dots i_{n-1}$  and  $i_0 \dots jjj \dots i_{n-1}$



Since  $df = 0$ , we have

$$0 = (df)_{iik} = f_{iik} - f_{iii}$$

whence

$$(dg)_{iik} = f_{iik}.$$

Similarly one treats the case  $j = k$ , which settles (28).

Now consider the general case  $n \geq 2$ . If  $i_0 \dots i_{n-1}$  is a regular allowed path then set

$$g_{i_0 \dots i_{n-1}} = 0.$$

If  $i_0 \dots i_{n-1}$  is an irregular allowed path then let  $k$  be the smallest index with the property that  $i_k = i_{k+1}$ . Denoting by  $j$  the common value of  $i_k$  and  $i_{k+1}$ , we can write

$$i_0 \dots i_{n-1} = i_0 \dots i_{k-1} jj i_{k+2} \dots i_{n-1}.$$

Then set

$$g_{i_0 \dots i_{n-1}} = (-1)^k f_{i_0 \dots i_{k-1} jj i_{k+2} \dots i_{n-1}}. \tag{29}$$

Note that the  $n$ -path  $i_0 \dots i_{k-1} jj i_{k+2} \dots i_{n-1}$  on the right hand side is obtained by inserting one more vertex  $j$  between already existing in  $i_0 \dots i_{n-1}$  two vertices  $j$  at positions  $k$  and  $k + 1$  (see Fig. 5).

By definition, we have  $g \in K^{n-1}$ . Let us show that  $dg = f$  in  $\Omega^n$ , that is,

$$(dg)_{i_0 \dots i_n} = f_{i_0 \dots i_n} \tag{30}$$

for any allowed path  $i_0 \dots i_n$ . If the path  $i_0 \dots i_n$  is regular then

$$(dg)_{i_0 \dots i_n} = \sum_{l=0}^n (-1)^l g_{i_0 \dots \hat{i}_l \dots i_n} = 0,$$

because all the paths  $i_0 \dots \hat{i}_l \dots i_n$  are also regular. Since also  $f_{i_0 \dots i_n} = 0$  by  $f \in K^n$ , we obtain the identity (30) in this case.

Let now the path  $i_0 \dots i_n$  be irregular. Let  $k$  be the smallest index with the property  $i_k = i_{k+1}$ , and denote by  $j$  the common value of  $i_k$  and  $i_{k+1}$ , that is,

$$i_0 \dots i_n = i_0 \dots i_{k-1} j j i_{k+2} \dots i_n.$$

Observe that the  $k$ -path  $i_0 \dots i_k$  is regular and allowed, which implies that all the vertices in this path are distinct. We have by (26)

$$(dg)_{i_0 \dots i_n} = \sum_{l=0}^{k-1} (-1)^l g_{i_0 \dots \widehat{i}_l \dots j j \dots i_n} \tag{31}$$

$$+ (-1)^k g_{i_0 \dots \widehat{j} j \dots i_n} + (-1)^{k+1} g_{i_0 \dots j \widehat{j} \dots i_n} \tag{32}$$

$$+ \sum_{l=k+2}^n (-1)^l g_{i_0 \dots j j \dots \widehat{i}_l \dots i_n} \tag{33}$$

Clearly, the two terms in (32) cancel out. Observe that in the paths  $i_0 \dots \widehat{i}_l \dots j j \dots i_n$  and  $i_0 \dots j j \dots \widehat{i}_l \dots i_n$  the first couple of equal consecutive vertices is  $jj$ , the same as in the full paths  $i_0 \dots j j \dots i_n$ . By (29) we obtain

$$g_{i_0 \dots \widehat{i}_l \dots j j \dots i_n} = (-1)^{k-1} f_{i_0 \dots \widehat{i}_l \dots j j j \dots i_n}$$

and

$$g_{i_0 \dots j j \dots \widehat{i}_l \dots i_n} = (-1)^k f_{i_0 \dots j j j \dots \widehat{i}_l \dots i_n}$$

whence

$$(dg)_{i_0 \dots i_n} = \sum_{l=0}^{k-1} (-1)^{l+k-1} f_{i_0 \dots \widehat{i}_l \dots j j j \dots i_n} + \sum_{l=k+2}^n (-1)^{l+k} f_{i_0 \dots j j j \dots \widehat{i}_l \dots i_n}. \tag{34}$$

On the other hand, since the  $(n + 1)$ -path  $i_0 \dots i_{k-1} j j j i_{k+2} \dots i_n$  is allowed, we have

$$(df)_{i_0 \dots i_{k-1} j j j i_{k+2} \dots i_n} = 0. \tag{35}$$

Computing this component by (26), we obtain

$$(df)_{i_0 \dots i_{k-1} j j j i_{k+2} \dots i_n} = \sum_{l=0}^{k-1} (-1)^l f_{i_0 \dots \widehat{i}_l \dots j j j \dots i_n} \tag{36}$$

$$+ (-1)^k f_{i_0 \dots \widehat{j} j j \dots i_n} + (-1)^{k+1} f_{i_0 \dots j \widehat{j} j \dots i_n} \tag{37}$$

$$+ (-1)^{k+2} f_{i_0 \dots j j \widehat{j} \dots i_n} + \sum_{l=k+2}^n (-1)^{l+1} f_{i_0 \dots j j j \dots \widehat{i}_l \dots i_n} \tag{38}$$

The three terms in (37) sum up to  $(-1)^k f_{i_0 \dots j j \dots i_n} = (-1)^k f_{i_0 \dots i_n}$ . Combining (36)–(38) with (34), we obtain

$$(df)_{i_0 \dots i_{k-1} j j j i_{k+2} \dots i_n} = (-1)^{k-1} (dg)_{i_0 \dots i_n} + (-1)^k f_{i_0 \dots i_n}.$$

Substituting here (35), we obtain (30). □

### 7 Poset associated with a simplicial complex

Let  $S$  be a finite simplicial complex. By definition,  $S$  can be regarded as a family of subsets of a finite set  $M$  such that if  $s \in S$  then any non-empty subset  $t$  of  $s$  also belongs to  $S$  (we do not allow the empty set to be in  $S$ ). Then we obtain immediately a poset structure on  $S$  as follows:

$$s \rightarrow t \Leftrightarrow s \supset t.$$

Let  $B_S$  be the barycentric subdivision of  $S$  and let  $V$  be the set of vertices of  $B_S$ , that is,  $V$  is the set of the barycenters of all simplexes of  $S$ . For any vertex  $i \in V$  denote by  $s_i$  the simplex from  $S$  whose barycenter is  $i$ . Clearly, the set  $V$  has the same poset structure as  $S$ : if  $i$  and  $j$  are two vertices from  $V$  then we set

$$i \rightarrow j \Leftrightarrow s_i \supset s_j$$

(cf. Fig. 1). As in Sect. 6, let  $\Omega_{reg}^*$  be a regularized cochain complex of the poset  $V$ .

**Lemma 7.1** *We have isomorphism*

$$H^* (\Omega_{reg}) \cong H^* (S). \tag{39}$$

*Proof* We will show that the chain complexes  $\Omega_*^{reg}$  and  $C_*(B_S)$  are isomorphic. This will imply the isomorphism of the corresponding cochain complexes and, hence,

$$H^* (\Omega_{reg}) \cong H^* (B_S).$$

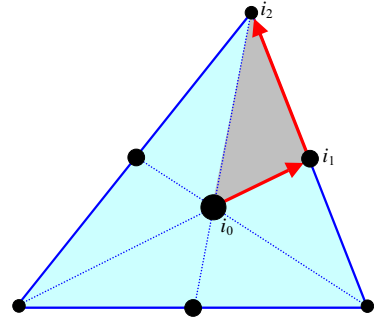
Since  $H^*(B_S) \cong H^*(S)$ , the isomorphism (39) will follow.

The set  $\Omega_n^{reg}$  consists of all formal  $\mathbb{K}$ -linear combination of the regular allowed  $n$ -paths  $i_0 \dots i_n$  on  $V$ . The  $n$ -path  $i_0 \dots i_n$  as an element of  $\Omega_n^{reg}$  will be denoted by  $e_{i_0 \dots i_n}$ . The boundary operator  $\partial : \Omega_n^{reg} \rightarrow \Omega_{n-1}^{reg}$  is defined by

$$\partial e_{i_0 \dots i_n} = \sum_{k=0}^n (-1)^k e_{i_0 \dots \hat{i}_k \dots i_n} \tag{40}$$

(observe that the  $(n - 1)$ -path  $i_0 \dots \hat{i}_k \dots i_n$  is regular allowed). It is clear that  $\Omega_{reg}^n$  can be regarded as the set of all  $\mathbb{K}$ -linear functionals on  $\Omega_n^{reg}$ , and it is easy to check that  $\partial$  is dual to  $d : \Omega_{reg}^{n-1} \rightarrow \Omega_{reg}^n$ .

**Fig. 6** A regular allowed path  $i_0i_1i_2$  determines a 2-simplex  $[i_0, i_1, i_2]$  of  $B_S$



By construction of the barycentric subdivision  $B_S$  of  $S$ , each regular allowed  $n$ -path  $i_0 \dots i_n$  in  $V$  gives rise to a  $n$ -simplex of  $B_S$  with the vertices  $i_0 \dots i_n$ , and vice versa (see Fig. 6).

Since the boundary operator on simplexes is defined by

$$\partial [i_0, \dots, i_n] = \sum_{k=0}^n (-1)^k [i_0, \dots, \widehat{i_k}, \dots, i_n]$$

that matches (40), we see that the chain complexes  $\Omega_*^{reg}$  and  $C_*(B_S)$  are isomorphic, which was to be proved.  $\square$

As in Sect. 2, define the algebras  $A$  and  $\bar{A}$  based on a poset  $V$  and the associated Hochschild cohomologies  $HH^n(A, A)$ . Now we can state and prove the main result.

**Theorem 7.2** *We have the following isomorphism of cohomologies*

$$HH^*(A, A) \cong H^*(S),$$

where  $H^*(S)$  is the simplicial cohomologies of the simplicial complex  $S$ .

*Proof* This follows from Lemmas 4.2, 5.1, 5.2, 6.1, 7.1 as in (4).  $\square$

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