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The rational homotopy type of the complement of the graph of a map

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Abstract We prove that the rational homotopy type of the complement of the graph of a continuous map from a simply connected closed manifold to a 2-connected closed manifold of the same dimension depends only on the rational homotopy class of the map. We give a commutative differential graded algebra model (in the sense of Sullivan) of the complement of the graph and study its formality.

Keywords Commutative differential graded algebra · Sullivan model · Minimal model · Formal space and map · Leray spectral sequence

Mathematics Subject Classification 55P62 · 55M05 · 55N30 · 55R20 · 55U25

1 Introduction

In this paper we have considered only smooth manifolds. However using the results of Milnor [14], Kister [8] and Kirby–Siebenmann [7] the arguments given for smooth manifolds can be carried over to topological manifolds.

Let $f : M \to N$ be a continuous map of closed connected oriented manifolds of dimension n. We aim to study the rational homotopy of $M \times N \backslash \Gamma(f)$, where $\Gamma(f) = \{(x, f(x)) | x \in M\}$ is the graph of f.

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We refer the reader to [3] and [5] for standard definitions and terminologies of: Free commutative differential graded algebra (CDGA), (minimal) Sullivan CDGA, (minimal) Sullivan model for a CDGA and for a space X, homotopy between morphisms of CDGAs, formality of spaces.

Definition 1.1 ([5], [22], [4], [12], [20]) A *model for a continuous map* $f : X \to Y$ between connected topological spaces is a morphism of CDGAs, $\psi_f : (A_Y, d_{A_Y}) \to (A_X, d_{A_X})$, such that there exists a homotopy commutative diagram, where $(\Lambda W, d_W)$, $(\Lambda V, d_V)$ are minimal models of Y and X respectively and the vertical arrows are quasi-isomorphisms:

$$\begin{array}{ccc} (A_Y, d_{A_Y}) & \overset{\psi_f}{\to} & (A_X, d_{A_X}) \\ \cong_q \uparrow \ \phi_W & & \cong_q \uparrow \ \phi_V \\ (\Lambda W, d_W) & \overset{\psi_{\Lambda f}}{\to} & (\Lambda V, d_V) \\ \cong_q \downarrow \ \rho_W & & \cong_q \downarrow \ \rho_V \\ (A_{PL}(Y), d_Y) & \overset{A_{PL}(f)}{\to} & (A_{PL}(X), d_X). \end{array}$$

A continuous map $f : X \to Y$ between formal spaces is said to be *formal* (also referred to as *formalizable* in the literature) if $f^* : (H^*(Y; \mathbb{Q}), 0) \to (H^*(X; \mathbb{Q}), 0)$ is a model for f.

Many authors have studied rational homotopy type of maps and their formality; see e.g. [4,13,16–18,22].

These authors have studied spaces $C(f)$ closely associated with f, like the mapping cone of f, the cofibre of f, the homotopy fibre of f and have recorded results related to problems of following types:

Problem 1.2 1. *Find out if the rational homotopy of $C(f)$ is determined by the rational homotopy of f.*
2. *Given that f is formal, determine if $C(f)$ is necessarily formal.*

We study the following problems of a similar nature:

Problem 1.3 1. *Find out if the rational homotopy of $M \times N \backslash \Gamma(f)$, where $\Gamma(f)$ is the graph of f, is determined by the rational homotopy of f.*
2. *Given that f is formal, determine if $M \times N \backslash \Gamma(f)$ is necessarily formal.*

The motivation of this paper comes from the work of Lambrechts and Stanley [9] on the rational homotopy type of configuration spaces of two points, and the observation that if f is the identity map $1_M : M \to M$ then, the graph of f, $\Gamma(f) = \Delta$, the diagonal, and $M \times M \backslash \Gamma(f) = M \times M \backslash \Delta = F(M, 2)$, the configuration space of two points. We begin with the following immediate observations:

1. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is any continuous map then $\mathbb{R}^n \times \mathbb{R}^n \backslash \Gamma(f)$ is homeomorphic to $F(\mathbb{R}^n, 2)$ under the homeomorphism $(u, v) \mapsto (u, v - f(u) + u)$.

2. For a topological space X if $f : X \to X$ is a homeomorphism then $X \times X \backslash \Gamma(f)$ is homeomorphic to $X \times X \backslash \Delta = F(X, 2)$ under the homeomorphism $(x, y) \mapsto (x, f^{-1}(y))$.
3. Suppose that $X = S^2 \vee S^3$, $f : X \to X$ is the constant map $x \mapsto (0, 0, 1)$ and $g : X \to X$ is the constant map $x \mapsto (0, 0, 0, -1)$. Then $X \times X \setminus \Gamma(f) \simeq X \times (\mathbb{R}^2 \vee S^3)$ whereas $X \times X \setminus \Gamma(g) \simeq X \times (S^2 \vee \mathbb{R}^3)$. Therefore they are not homotopy equivalent. Thus for an arbitrary space X and a continuous map $f : X \to X$, $X \times X \backslash \Gamma(f)$ need not even be homotopy equivalent to $F(X, 2)$.

Lambrechts and Stanley [9,10] proved that if M is a simply-connected closed manifold such that $H^2(M; \mathbb{Q}) = 0$ then the rational homotopy type of $F(M, 2)$ depends only on the rational homotopy type of M. They also proved in [9] that if M is a 2-connected closed manifold, then $F(M, 2)$ is formal if and only if M is formal by constructing explicitly a CDGA model for $F(M, 2)$ out of a differential Poincaré duality algebra model of M and a model of the diagonal map.

We show that starting from a CDGA model of $f : M \to N$ one can construct a CDGA model of $(1_M \times f) \circ \Delta_M : M \to M \times N$, where $\Delta_M(x) = (x, x)$ for $x \in M$. This allows us by Corollary 1.5 of [10], to conclude that if we take $M,\ N$ to be simply connected and $H^2(N; \mathbb{Q}) = 0$ then the rational homotopy type of $M \times N \setminus \Gamma(f)$ is determined by the rational homotopy class of f:

Theorem 1.4 *Let $f : M \to N$ be a continuous map of closed connected oriented manifolds of dimension n such that $H^1(M; \mathbb{Q}) = 0 = H^1(N; \mathbb{Q}) = H^2(N; \mathbb{Q})$, then a CDGA-model of $M \times N \setminus \Gamma(f)$ can be explicitly determined out of any CDGA-model of f.*

Corollary 1.5 *If $f : M \to N$ is a continuous map of closed simply connected manifolds of dimension n such that $H^2(N; \mathbb{Q}) = 0$, then the rational homotopy type of $M \times N \backslash \Gamma(f)$ depends only on the rational homotopy class of f.*

We relativize results of [9] by defining a class Δ_ψ depending on a morphism ψ of differential Poincaré duality algebras similar to the diagonal class Δ in [9]. We know from Theorem 1.1 of [11] that if $M,\ N$ are closed connected oriented manifolds with $H^1(M; \mathbb{Q}) = 0 = H^1(N; \mathbb{Q})$, then they admit differential Poincaré duality algebra models. We also know from Proposition 1 of [2] that if $f : M \to N$ is a continuous map of simply connected manifolds of dimension $n \geq 7$ (it is a conjecture that this dimensional restriction can be removed) such that $H^2(f)$ is injective, then f admits a Sullivan model $\psi_f : (A_N, d_{A_N}) \to (A_M, d_{A_M})$, where $(A_M, d_{A_M}),\ (A_N, d_{A_N})$ are Poincaré duality algebras. Thus assuming the existence of such a model of f we can construct a specific CDGA model of $M \times N \backslash \Gamma(f)$ from the model of f:

Theorem 1.6 *Let $f : M \to N$ be a continuous map of closed connected oriented manifolds of dimension n such that $H^1(M; \mathbb{Q}) = 0 = H^1(N; \mathbb{Q}) = H^2(N; \mathbb{Q})$. If $\psi_f : (A_N, d_{A_N}) \to (A_M, d_{A_M})$ is a model of f, where (A_M, d_{A_M}), (A_N, d_{A_N}) are oriented differential Poincaré duality algebras, $\{a_i\}_{1 \leq i \leq l}$ a homogeneous basis of A_N and $\{a_i^*\}_{1 \leq i \leq l}$ its Poincaré dual basis*

$$\Delta = \sum_{i=1}^{l_N} (-1)^{\deg\,(a_i)} a_i \otimes a_i^* \in (A_N \otimes A_N)^n,$$

is the diagonal class (defined in [9], p. 1030) and $\Delta_{\psi_f} := (\psi_f \otimes 1)(\Delta)$, *called the class of* ψ_f, *then the ideal* $(\Delta_{\psi_f}) = \Delta_{\psi_f}.(A_M \otimes A_N)$ *is a differential ideal of* $A_M \otimes A_N$, *and the quotient CDGA*

$$\left(\frac{A_M \otimes A_N}{(\Delta_{\psi_f})}, \ \overline{d_{A_M} \otimes d_{A_N}} \right)$$

is a CDGA model of $M \times N \backslash \Gamma(f)$.

This allows us to prove that if f is formal then $M \times N \backslash \Gamma(f)$ is also formal.

Corollary 1.7 *Let* $f : M \to N$ *be a formal map of closed connected oriented formal manifolds of dimension n such that* $H^1(M; \mathbb{Q}) = 0 = H^1(N; \mathbb{Q}) = H^2(N; \mathbb{Q})$, *then* $M \times N \setminus \Gamma(f)$ *is a formal space.*

As the first major step in this paper we compute the cohomology algebra $H^*(M \times N \backslash \Gamma(f))$ using the Leray spectral sequence as described in §2 of Totaro [21]. We analyze at depth the cohomology class μ_f, as described in Chapter 30 of Greenberg and Harper [6].

Let $\tilde{\mu} \in H^n(M \times N, M \times N \backslash \Gamma(f); \mathbb{Z})$ be the Thom class as in Lemma 2.1, and $\tilde{\mu}' = j^*(\tilde{\mu}) \in H^n(M \times N; \mathbb{Z})$, where $j : M \times N \to (M \times N, M \times N \setminus \Gamma(f))$ is the natural injection. Let us now take cohomology with rational coefficients (or coefficients in any field). Let $\{b_i\}_{1 \le i \le l_N}$ be a homogeneous basis of $H^*(N; \mathbb{Q})$, and let $\{b_i^*\}_{1 \le i \le l_N}$ be its Poincaré dual basis, that is $\langle b_i \cup b_j^*, [N] \rangle = \delta_{ij}$, where $[N]$ is the fundamental homology class of N. Let $\mu'_N := \sum_{i=1}^{l_N} (-1)^{\deg\,(b_i)} b_i \times b_i^* \in (H^*(N \times N; \mathbb{Q}))^n$ be the diagonal class of N, and let $\mu_f := (f \times id_N)^*(\mu'_N)$, called the cohomology class of the graph of f (see p. 284 of [6]). Then there is a unique class $\Delta_f \in (H^*(M; \mathbb{Q}) \otimes H^*(N; \mathbb{Q}))^n$ which maps to μ_f under the isomorphism of the Künneth theorem $(H^*(M; \mathbb{Q}) \otimes H^*(N; \mathbb{Q}))^n \xrightarrow{\cong} (H^*(M \times N; \mathbb{Q}))^n$, we call Δ_f the *class of* f.

Theorem 1.8 *Let* $f : M \to N$ *be a continuous map of closed oriented manifolds of dimension* $n \ge 2$ *and* $\Gamma(f)$ *be the graph of* f. *Then we have the following isomorphism of rings:*

$$H^*(M \times N \backslash \Gamma(f); \mathbb{Z}) \cong \frac{H^*(M \times N; \mathbb{Z})}{(\tilde{\mu}')},$$

where $(\tilde{\mu}')$ *denotes the ideal* $H^*(M \times N; \mathbb{Z}) \cup \tilde{\mu}'$. *For coefficients in* $\mathbb{Q}$, *and* $\Delta_f = \sum_{i=1}^{l_N} (-1)^{\deg\,(b_i)} f^*(b_i) \otimes b_i^*$, *we have the following isomorphism of algebras:*

$$H^*(M \times N \backslash \Gamma(f); \mathbb{Q}) \cong \frac{H^*(M \times N; \mathbb{Q})}{(\mu_f)} \cong \frac{H^*(M; \mathbb{Q}) \otimes H^*(N; \mathbb{Q})}{(\Delta_f)}.$$

The paper is arranged as follows. In §2 we determine the cohomology algebra of $M \times N \backslash \Gamma(f)$. In §3 we adapt for a continuous map $f : M \to N$ some results of [9] and [10] to construct a CDGA model of $M \times N \backslash \Gamma(f)$, and prove Theorem 1.4 and Corollary 1.5. In §4 we prove several results about differential Poincaré duality algebra associated to a morphism ψ by adapting similar results of [9], proved in the absolute case. These results and some results of [10] are used for the proofs of Theorem 1.6 and Corollary 1.7. In the final §5 we give some examples and applications of our results.

2 The cohomology algebra of $M \times N \backslash \Gamma(f)$

In this section we prove Theorem 1.8.

We assume that $f : M \to N$ is a continuous map of closed connected oriented manifolds of dimension n with graph $\Gamma(f)$. Since $\Gamma(f) \subset M \times N$ is an embedding, one can derive the following analogue of Corollary 11.2 of Milnor-Stasheff [15] (and also of Corollary (30.2) of Greenberg-Harper [6]) in a similar fashion. We follow the notations which preceded the statement of Theorem 1.8 in §1.

Lemma 2.1 *There is a Thom class* $\tilde{\mu} \in H^n(M \times N, M \times N \backslash \Gamma(f); \mathbb{Z})$ *associated to the oriented normal bundle of the embedding* $\Gamma(f) \subset M \times N$.

We next prove the following analogue of Corollary (30.3) of [6]:

Theorem 2.2 *Let* $[N] \in H_n(N; \mathbb{Z})$ *be the fundamental homology class of* N, $j^* : H^n(M \times N, M \times N \backslash \Gamma(f); \mathbb{Z}) \to H^n(M \times N; \mathbb{Z})$ *be the homomorphism in cohomology induced by the map* $j : M \times N \hookrightarrow (M \times N, M \times N \backslash \Gamma(f))$ *and* $\tilde{\mu}' = j^*(\tilde{\mu})$, *then* $\tilde{\mu}'/[N] = 1$.

Proof For $x \in M$, consider the commutative diagram:

$$\begin{array}{ccc} (N, N\backslash f(x)) & \xrightarrow{\tilde{i_x}} & (M \times N, M \times N\backslash\Gamma(f)) \\ j_{f(x)} \uparrow & & j \uparrow \\ N & \xrightarrow{\tilde{i_x}} & M \times N \end{array}$$

where $\tilde{i_x} : (N, N\backslash f(x)) \to (M \times N, M \times N\backslash\Gamma(f))$ is defined by $\tilde{i_x}(x') = (x, x'), \forall x' \in N$, $\tilde{i_x} : N \to M \times N$ its restriction to $(N, \emptyset)$ and $j_{f(x)} : N \to (N, N \backslash f(x))$ is the natural injection. If $s : N \to N^0$ is a section of the orientation sheaf N^0 over N, then

$$\begin{aligned} 1 &= [s(f(x)), \tilde{i_x}^*(\tilde{\mu})] = [j_{f(x)*}[N], \tilde{i_x}^*(\tilde{\mu})] = [[N], j^*_{f(x)}\tilde{i_x}^*(\tilde{\mu})] \\ &= [[N], (\tilde{i_x} \circ j_{f(x)})^*(\tilde{\mu})] = [[N], (j \circ \tilde{i_x})^*(\tilde{\mu})] = [[N], \tilde{i_x}^* \circ j^*(\tilde{\mu})] \\ &= [\tilde{i_x}_*[N], j^*(\tilde{\mu})] = [\tilde{i_x}_*[N], \tilde{\mu}']. \end{aligned}$$

Since $N \cong \{x\} \times N$, if $\bar{x}$ is the homology class of the zero cycle x, $\tilde{i_x}_*([N]) = \bar{x} \times [N]$. So from the above expression we get

$1 = [\tilde{i_{x*}}[N], \tilde{\mu}'] = [\bar{x} \times [N], \tilde{\mu}'] = [\bar{x}, \tilde{\mu}'/[N]]$ for all $x \in M$, which proves the result. □

We now prove the following analogue of Lemma 11.8 of [15] (and also of Lemma (30.5) of [6]):

Theorem 2.3 *If $a \in H^*(N;\mathbb{Z})$, then $(f^*(a) \times 1) \cup \tilde{\mu}' = (1 \times a) \cup \tilde{\mu}'$.*

Proof Let $V_{\Gamma(f)}$ be a tubular neighborhood of the graph $\Gamma(f)$ in $M \times N$. So $\Gamma(f)$ is a deformation retract of $V_{\Gamma(f)}$. Let $i_{\Gamma(f)} : \Gamma(f) \to V_{\Gamma(f)}$ be the inclusion map and $r_{\Gamma(f)} : V_{\Gamma(f)} \to \Gamma(f)$ be the retraction map. Then $r_{\Gamma(f)} \circ i_{\Gamma(f)} = 1_{\Gamma(f)}$ and $i_{\Gamma(f)} \circ r_{\Gamma(f)} \simeq 1_{V_{\Gamma(f)}}$. Consider the projections $p_1 : M \times N \to M$ and $p_2 : M \times N \to N$. Since $f \circ p_1$ and p_2 coincide on $\Gamma(f)$, we have $(f \circ p_1)\,|_{V_{\Gamma(f)}} \circ i_{\Gamma(f)} \circ r_{\Gamma(f)} = p_2\,|_{V_{\Gamma(f)}} \circ i_{\Gamma(f)} \circ r_{\Gamma(f)}$. So the $(f \circ p_1)\,|_{V_{\Gamma(f)}} \simeq p_2\,|_{V_{\Gamma(f)}}$. Therefore the cohomology classes $(f \circ p_1)^*(a) = p_1^* \circ f^*(a) = f^*(a) \times 1$ and $p_2^*(a) = 1 \times a$ have the same image under the restriction homomorphism $H^i(M \times N;\mathbb{Z}) \to H^i(V_{\Gamma(f)};\mathbb{Z})$. Now using the commutative diagram

$$\begin{array}{ccc} H^i(M \times N;\mathbb{Z}) & \longrightarrow & H^i(V_{\Gamma(f)};\mathbb{Z}) \\ \downarrow \cup\tilde{\mu} & & \downarrow \cup\tilde{\mu}\,|_{(V_{\Gamma(f)}, V_{\Gamma(f)}\setminus\Gamma(f))} \\ H^{i+n}(M \times N, M \times N\setminus\Gamma(f);\mathbb{Z}) & \cong & H^{i+n}(V_{\Gamma(f)}, V_{\Gamma(f)}\setminus\Gamma(f);\mathbb{Z}) \end{array}$$

it follows that $(f^*(a) \times 1) \cup \tilde{\mu} = (1 \times a) \cup \tilde{\mu}$. So $j^*((f^*(a) \times 1) \cup \tilde{\mu}) = j^*((1 \times a) \cup \tilde{\mu})$, j as defined earlier. By the properties of mixed cup products $(f^*(a) \times 1) \cup j^*(\tilde{\mu}) = (1 \times a) \cup j^*(\tilde{\mu})$. So $(f^*(a) \times 1) \cup \tilde{\mu}' = (1 \times a) \cup \tilde{\mu}'$. □

The next theorem is an analogue of Theorem 11.11 of [15] (and also of Proposition (30.18) of [6]):

Theorem 2.4 *Let $\{b_i\}_{i=1}^{l_N}$ be a homogeneous basis of $H^*(N;\mathbb{Q})$, and let $\{b_i^*\}_{i=1}^{l_N}$ be its Poincaré dual basis, then the cohomology class $\tilde{\mu}'$ is given by*

$$\tilde{\mu}' = \sum_{i=1}^{l_N} (-1)^{\deg(b_i)} f^*(b_i) \times b_i^*,$$

where deg $(b_i) = k$ if $b_i \in H^k(N;\mathbb{Q})$.

Proof Using the Künneth formula we can write $\tilde{\mu}' \in H^n(M \times N;\mathbb{Q})$ as

$$\tilde{\mu}' = c_1 \times b_1^* + ... + c_{l_N} \times b_{l_N}^*,$$

where $c_1, \ldots, c_{l_N}$ are certain well defined cohomology classes in $H^*(M;\mathbb{Q})$ with $\deg(b_i) + \deg(c_i) = n$. We apply the homomorphism $/[N]$ on both sides of the identity $(f^*(a) \times 1) \cup \tilde{\mu}' = (1 \times a) \cup \tilde{\mu}'$, where $a \in H^*(N;\mathbb{Q})$. By Property 4 of slant product on p. 288 of Spanier [19] and by our Theorem 2.2 we get

$$((f^*(a) \times 1) \cup \tilde{\mu}')/[N] = f^*(a) \cup (\tilde{\mu}'/[N]) = f^*(a). \qquad (1)$$

Also,

$$((1 \times a) \cup \tilde{\mu}')/[N] = (1 \times a) \cup \sum_{i=1}^{l_N} c_i \times b_i^*/[N] = \sum_{i=1}^{l_N} (-1)^{\deg(a)\deg(c_i)} c_i \times (a \cup b_i^*)/[N]$$
$$= \sum_{i=1}^{l_N} (-1)^{\deg(a)\deg(c_i)} c_i \langle a \cup b_i^*, [N]\rangle \tag{2}$$

Using equations (1) and (2) above and substituting b_j for a, we get

$$f^*(b_j) = \sum_{i=1}^{l_N} (-1)^{\deg(b_j)\deg(c_i)} c_i \langle b_j \cup b_i^*, [N]\rangle = (-1)^{\deg(b_j)\deg(c_j)} c_j,$$

but $\deg(b_j) = \deg(c_j)$, so we get that

$$(-1)^{\deg(b_j)} f^*(b_j) = (-1)^{\deg(b_j)}(-1)^{\deg(b_j)\deg(b_j)} c_j = (-1)^{\deg(b_j)(1+\deg(b_j))} c_j = c_j.$$

This proves the theorem. □

Let $[\Gamma(f)] = (1_M \times f)_* \circ \Delta_{M*}([M])$. Let $\mu_N \in H^n(N \times N, N \times N \backslash \Gamma(1_N); \mathbb{Z})$ be the Thom class of the oriented normal bundle of the embedding $\Gamma(1_N) \hookrightarrow N \times N$, let $\mu'_N \in H^n(N \times N; \mathbb{Z})$ be its image under the homomorphism in cohomology induced by $j_N : N \times N \to (N \times N, N \times N \backslash \Gamma(1_N))$ and $\mu_f = (f \times 1_N)^*(\mu'_N) \in H^n(M \times N; \mathbb{Z})$.

Remark 2.5 1. When coefficients belong to $\mathbb{Q}$ (or any field) by Theorem 11.11 of [15] (or by Proposition (30.18) of [6]), if $\{a_i\}_{i=1}^{l_M}$ is a homogeneous basis of $H^*(M; \mathbb{Q})$ and $\{a_i^*\}_{i=1}^{l_M}$ its Poincaré dual basis, we have $\mu'_M = \sum_{i=1}^{l_M} (-1)^{\deg(a_i)} a_i \times a_i^*$, and $\mu'_N = \sum_{i=1}^{l_N} (-1)^{\deg(b_i)} b_i \times b_i^*$. Therefore, we see using Theorem 2.4 , that in rational cohomology, $\mu_f = \tilde{\mu}'$.

2. By Exercises (30.15) and (30.17) of [6] we have $[M \times M] \cap \mu'_M = \Delta_{M*}([M]) = [\Gamma(1_M)]$, and $[N \times N] \cap \mu'_N = \Delta_{N*}([N]) = [\Gamma(1_N)]$. Using this and Part 1 of this remark one can prove the following statement which is the assertion of the Exercise (30.22) of [6]:

$$[M] \times [N] \cap \tilde{\mu}' = [M] \times [N] \cap \mu_f = [\Gamma(f)] \tag{A}$$

We are now ready to prove Theorem 1.8.

Proof of Theorem.1.8 Let $h : M \times N \backslash \Gamma(f) \hookrightarrow M \times N$ denote the inclusion map. The Leray spectral sequence for this inclusion has the form

$$E_2^{i,j} = H^i(M \times N; R^j h_* \mathbb{Z}) \Rightarrow H^{i+j}(M \times N \setminus \Gamma(f); \mathbb{Z}).$$

Here $R^j h_*\mathbb{Z}$ is the Leray sheaf on $M \times N$ associated with the presheaf $U \to H^j(U \cap (M \times N \backslash \Gamma(f)); \mathbb{Z})$ where U runs over the family of all open subsets of $M \times N$. The stalk of $R^j h_*\mathbb{Z}$ at a point $\bar{x} = (x_1, x_2) \in M \times N$ is described below.

1. If $x_2 \neq f(x_1)$, we can choose small coordinate neighborhoods U_1 and U_2 of x_1 and x_2 in M and N respectively, such that $f(U_1) \cap U_2 = \emptyset$. If $U = U_1 \times U_2$, then $U \subset M \times N \backslash \Gamma(f)$. Therefore, for every $\bar{x} = (x_1, x_2) \in M \times N, x_2 \neq f(x_1)$, $(R^j h_*\mathbb{Z})_{\bar{x}} = H^j(U \cap (M \times N \backslash \Gamma(f)); \mathbb{Z}) = H^j(U; \mathbb{Z}) = H^j(U_1 \times U_2; \mathbb{Z}) = 0$ if $j \neq 0$, and $(R^0 h_*\mathbb{Z})_{\bar{x}} = \mathbb{Z}$.
2. If $x_2 = f(x_1)$, we can choose small coordinate neighborhoods U_1 and U_2 of x_1 and x_2 in M and N respectively, such that $f(U_1) \subset U_2$. For $U = U_1 \times U_2$, $(R^j g_*\mathbb{Z})_{\bar{x}} = H^j(U \cap (M \times N \backslash \Gamma(f)); \mathbb{Z}) = H^j(U \backslash \Gamma(f); \mathbb{Z}) = H^j(U_1 \times U_2 \backslash \Gamma(f); \mathbb{Z})$. Let $h_1 : U_1 \overset{\cong}{\to} \mathbb{R}^n$ and $h_2 : U_2 \overset{\cong}{\to} \mathbb{R}^n$, be the coordinate charts of U_1 and U_2 respectively. Consider the composite $\hat{f} = h_2 \circ f \circ h_1^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, then $\Gamma(\hat{f}) = h_1 \times h_2(\Gamma(f)|_{U_1 \times U_2})$. Therefore, $(R^j h_*\mathbb{Z})_{\bar{x}} = H^j(U_1 \times U_2 \backslash \Gamma(f); \mathbb{Z}) \cong H^j((\mathbb{R}^n \times \mathbb{R}^n) \backslash \Gamma(\hat{f}); \mathbb{Z})$. But as observed in the introduction $\mathbb{R}^n \times \mathbb{R}^n \backslash \Gamma(\hat{f})$ is homeomorphic to $F(\mathbb{R}^n, 2)$, so for every $\bar{x} = (x_1, x_2) \in M \times N, x_2 = f(x_1)$

$$(R^j h_*\mathbb{Z})_{\bar{x}} = H^j(F(\mathbb{R}^n, 2); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } j = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

This shows that the Leray sheaf $(R^{n-1} h_*\mathbb{Z})$ is supported and is locally constant along the graph $\Gamma(f)$ with stalks $\mathbb{Z}$, and $(R^0 h_*\mathbb{Z})$ is locally constant on $M \times N$ with stalks $\mathbb{Z}$. Since M, N and hence $\Gamma(f)$ are orientable, we have

$$H^i(M \times N; R^j h_*\mathbb{Z}) = \begin{cases} H^i(M \times N; \mathbb{Z}) & \text{if } j = 0, \\ H^i(\Gamma(f); \mathbb{Z}) & \text{if } j = n-1, \\ 0 & \text{otherwise} \end{cases}$$

Hence, the E_2 terms of Leray spectral sequence take the form:

$$\begin{array}{cccccc}
0 & 0 & \cdots & 0 & \cdots \\
E_2^{0,n-1} = H^0(\Gamma(f); \mathbb{Z}) & E_2^{1,n-1} = H^1(\Gamma(f); \mathbb{Z}) & \cdots & E_2^{n,n-1} = H^n(\Gamma(f); \mathbb{Z}) & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
E_2^{0,0} = H^0(M \times N; \mathbb{Z}) & E_2^{1,0} = H^1(M \times N; \mathbb{Z}) & \cdots & E_2^{n,0} = H^n(M \times N; \mathbb{Z}) & \cdots
\end{array}$$

and E_n terms coincide with E_2 terms:

$$\begin{array}{ccccc}
0 & 0 & \cdots & 0 & \cdots \\
E_n^{0,n-1} = H^0(\Gamma(f);\mathbb{Z}) & E_n^{1,n-1} = H^1(\Gamma(f);\mathbb{Z}) & \cdots & E_n^{n,n-1} = H^n(\Gamma(f);\mathbb{Z}) & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
0 & d_n & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & \cdots \\
E_n^{0,0} = H^0(M\times N;\mathbb{Z}) & E_n^{1,0} = H^1(M\times N;\mathbb{Z}) & \cdots & E_n^{n,0} = H^n(M\times N;\mathbb{Z}) & \cdots
\end{array}$$

We now determine d_n on the E_n page. Note that the differential d_n is 0 on the bottom row, since it maps each row to a lower row. Let $i:\Gamma(f)\to M\times N$ be the inclusion map and let $\pi: M\times N\to\Gamma(f)$ be defined by $\pi(x_1,x_2)=(x_1,f(x_1))$. Then $\pi\circ i=1_{\Gamma(f)}$. Therefore, by the functoriality of cohomology, the homomorphisms induced in cohomology, π^* and i^* are, respectively, injective and surjective. Hence the long cohomology exact sequence of the pair $(M\times N,\Gamma(f))$ gives rise to the following split short exact sequence:

$$0\to H^*(M\times N,\Gamma(f);\mathbb{Z})\xrightarrow{j_1^*} H^*(M\times N;\mathbb{Z})\underset{\pi^*}{\overset{i^*}{\rightleftharpoons}}(H^*(\Gamma(f);\mathbb{Z}))\to 0,$$

where $j_1: M\times N\to(M\times N,\Gamma(f))$ is the natural injection.

As $\Gamma(f)$ is a smooth submanifold with an orientable normal bundle in the smooth manifold $M\times N$, the differentials d_n in the above Leray spectral sequence originating from the $(n-1)^{th}$ row are Gysin maps. If $z\in H^i(M\times N;\mathbb{Z})$, $i^*(z)\in H^i(\Gamma(f);\mathbb{Z})$ and we have $d_n(i^*(z))=\pi^*(i^*(z))\cup\tilde{\mu}'$. From the above split exact sequence we see that $z-\pi^*(i^*(z))=j_1^*(w)$, for some element $w\in H^i(M\times N,\Gamma(f);\mathbb{Z})$. Recall that $\tilde{\mu}'=j^*(\tilde{\mu})$, the image of the Thom class $\tilde{\mu}\in H^*(M\times N, M\times N\setminus\Gamma(f);\mathbb{Z})$, where $j: M\times N\to(M\times N, M\times N\backslash\Gamma(f))$ is the natural injection. So by Property 8 on p. 251 of [19] and the fact that $w\cup\tilde{\mu}=0$ (since the mixed cup product $\cup$: $H^i(M\times N,\Gamma(f);\mathbb{Z})\otimes H^n(M\times N, M\times N\setminus\Gamma(f);\mathbb{Z})\to H^{i+n}(M\times N,\Gamma(f)\cup M\times N\setminus\Gamma(f);\mathbb{Z})=H^{i+n}(M\times N, M\times N;\mathbb{Z})=0$ has its image in a zero module) we get that $(z-\pi^*(i^*(z)))\cup\tilde{\mu}'=j_1^*(w)\cup\tilde{\mu}'=j_1^*(w)\cup j^*(\tilde{\mu})=0$. This means, in particular, that $d_n(1)=\tilde{\mu}'$.

If $k: M\to\Gamma(f)$ denotes the mapping $x\mapsto(x,f(x))$ and $\pi_M: M\times N\to M$ be the projection onto M, then k^* is an isomorphism and $\pi^*=\pi_M^*\circ k^*$. Thus $d_n(i^*(z))=\pi_M^*(k^*(i^*(z)))\cup\tilde{\mu}'=(k^*(i^*(z))\times 1)\cup\tilde{\mu}'$. But by Property 4 of slant products on p. 288 of [19], $\{(k^*(i^*(z))\times 1)\cup\tilde{\mu}'\}/[N]=(k^*(i^*(z))\cup(\tilde{\mu}'/[N])=k^*(i^*(z))$, as $\tilde{\mu}'/[N]=1$ by Theorem 2.2 above. So we have: if $i^*(z)\neq 0$ then $(k^*(i^*(z))\times 1)\cup\tilde{\mu}'\neq 0$. Therefore d_n is injective on the $(n-1)^{th}$-row of $E_n=E_2$.

This yields $E_{n+1}^{i,j}=E_\infty^{i,j}=0$ for $j\neq 0$, and $E_\infty^{i,0}=E_{n+1}^{i,0}=\frac{E_n^{i,0}}{\operatorname{im} d_n}=\frac{E_2^{i,0}}{\operatorname{im} d_n}=\frac{H^i(M\times N;\mathbb{Z})}{\operatorname{im} d_n}$ for all i. But, for $0\leq i\leq n-1$, we have $E_\infty^{i,0}=E_{n+1}^{i,0}=E_n^{i,0}=E_2^{i,0}=H^i(M\times N;\mathbb{Z})$, and for $n\leq i\leq 2n$, $E_{n+1}^{i,0}=E_\infty^{i,0}=\frac{H^i(M\times N;\mathbb{Z})}{\operatorname{im} d_n}$.

Thus

$$H^*(M\times N\backslash\Gamma(f);\mathbb{Z})\cong\frac{H^*(M\times N;\mathbb{Z})}{H^*(M\times N;\mathbb{Z})\cup\tilde{\mu}'}\cong\frac{H^*(M\times N;\mathbb{Z})}{(\tilde{\mu}')},$$

as rings, where $(\tilde{\mu}')$ denotes the ideal of $H^*(M\times N;\mathbb{Z})$ generated by $\tilde{\mu}'$.

If we take cohomology with coefficients in $\mathbb{Q}$, then by an application of the Künneth theorem $H^*(M;\mathbb{Q}) \otimes H^*(N;\mathbb{Q}) \xrightarrow{\cong} H^*(M \times N;\mathbb{Q})$. Hence there is a unique class $\Delta_f = \sum_{i=1}^{n} (-1)^{\deg(b_i)} f^*(b_i) \otimes b_i^*$ in $H^*(M;\mathbb{Q}) \otimes H^*(N;\mathbb{Q})$ which maps to $\tilde{\mu}' = \sum_{i=1}^{N} (-1)^{\deg(b_i)} f^*(b_i) \times b_i^* = \mu_f$ in $H^*(M \times N;\mathbb{Q})$ under this isomorphism, and therefore

$$H^*(M \times N \setminus \Gamma(f);\mathbb{Q}) \cong \frac{H^*(M \times N;\mathbb{Q})}{(\mu_f)} \cong \frac{H^*(M;\mathbb{Q}) \otimes H^*(N;\mathbb{Q})}{(\Delta_f)},$$

as algebras. □

A similar argument can be given when $\mathbb{Q}$ is replaced by an arbitrary field.

3 A CDGA model of $M \times N \setminus \Gamma(f)$

In this section we prove Theorem 1.4 and Corollary 1.5 by constructing a CDGA model of $(1_M \times f) \circ \Delta_M : M \to M \times M \to M \times N$ from a CDGA model of $f : M \to N$. (We refer the reader to [9], [10], [3] and [5] for necessary definitions, notations and results leading to the construction of a CDGA model of the configuration space of two points $F(M, 2)$).

Proof of Theorem 1.4 We first note that a CDGA-model $(A_N, d_N) \xrightarrow{\psi_f} (A_M, d_M)$ of f determines a CDGA-model $(A_M \otimes A_N, d_{A_M} \otimes d_{A_N}) \xrightarrow{1_{A_M} \otimes \psi_f} (A_M \otimes A_M, d_{A_M} \otimes d_{A_M})$ of $1_M \times f$, whose verification is left to the reader. This together with the fact that a CDGA-model of M determines a CDGA-model of Δ_M (see, e.g. Example 2.48 on p.73 of [5]) yields the conclusion that a CDGA-model of f determines a CDGA-model $(A_M \otimes A_N) \xrightarrow{1_{A_M} \otimes \psi_f} (A_M \otimes A_M) \xrightarrow{\nu_{A_M}} A_M$, defined by $x \otimes y \mapsto (\nu_{A_M} \circ (1_{A_M} \otimes \psi_f))(x \otimes y) = x.\psi_f(y)$, of $(1_M \times f) \circ \Delta_M$.

By our hypotheses $H^1(M;\mathbb{Q}) = 0$ and $H^1(N;\mathbb{Q}) = 0 = H^2(N;\mathbb{Q})$, it follows that for the embedding $(1_M \times f) \circ \Delta_M : M \to M \times N$, which actually embeds M as the graph $\Gamma(f)$ of f in $M \times N$, the hypotheses of Theorem 1.4 of [10], namely $H_1((1_M \times f) \circ \Delta_M;\mathbb{Q})$ is an isomorphism and $H_2((1_M \times f) \circ \Delta_M;\mathbb{Q})$ is an epimorphism are satisfied. Therefore by applying Theorem 1.4 of [10] one can determine explicitly a model of $M \times N \setminus \Gamma(f)$ from the above model of f, and our result follows. □

Proof of Corollary. 1.5 By hypothesis the dimension of N is at least three, so $\dim M = \dim N \geq 3$. Therefore the hypothesis of Corollary 1.5 of [10], namely the codimension of the embedding of $\Gamma(f)$ in $M \times N$ is ≥ 3, and $H_*((1_M \times f) \circ \Delta_M;\mathbb{Q})$ is 2-connected are satisfied. Hence, the rational homotopy type of $M \times N \setminus \Gamma(f)$ is determined by the rational homotopy class of $1_M \times f$. But the homotopy class of $1_M \times f$ is determined by the homotopy class of f; hence the corollary holds. □

4 The CDGA model of $M \times N \backslash \Gamma(f)$ based on differential-Poincaré-duality-algebra models of M and N

In this section we consider a continuous map $f : M \to N$ of closed connected oriented manifolds of dimension n with $H^1(M;\mathbb{Q}) = 0 = H^1(N;\mathbb{Q})$ having a CDGA model $(A_N, d_{A_N}) \overset{\psi_f}{\to} (A_M, d_{A_M})$ in which (A_M, d_{A_M}) and (A_N, d_{A_N}) are differential connected Poincaré duality algebras of formal dimension n (refer to the para preceding Theorem 1.6 of the introduction).

We note that all the algebras and relevant results of [9], proved by them in the absolute case, can analogously be developed for any given CDGA morphism of Poincaré duality algebras.

Definition 4.1 Let $(A_2, d_{A_2}) \overset{\psi}{\to} (A_1, d_{A_1})$ be a morphism of Poincaré duality algebras of formal dimension n. Define $\Delta_\psi := (\psi \otimes 1)(\Delta)$, where Δ is the diagonal class defined in the statement of Theorem 1.6 and the element Δ_ψ, which belongs to $(A_1 \otimes A_2)^n$, will be called the *class of* ψ.

As in Proposition 4.3 of [9] the element Δ_ψ does not depend on the choice of the basis $\{a_i\}_{1\le i\le l}$ of A_2.

Remark 4.2 If $s^{-n}A_1$ is the suspension of A_1 as defined in §2 of [9] then there is an $(A_1 \otimes A_2)$-module structure on $s^{-n}A_1$ given by

$$(x \otimes y).(s^{-n}a) = (-1)^{n\deg(x)+n\deg(y)+\deg(a)\deg(y)} s^{-n}(x.a.\psi(y))$$

for homogeneous elements $a, x \in A_1$, and $y \in A_2$ and an obvious $(A_1 \otimes A_2)$-module structure on $(A_1 \otimes A_2)$. In other words the $(A_1 \otimes A_2)$-module structure on $s^{-n}A_1$ is obtained from the obvious structure of $A_1 \otimes A_1$-module on $s^{-n}A_1$ by transporting it along the CDGA map $1 \otimes \psi : A_1 \otimes A_2 \to A_1 \otimes A_1$.

In view of the above, statements below follow from the corresponding statements of [9], proved there in the absolute case.

1. Let $\psi : (A_2, \omega_{A_2}) \to (A_1, \omega_{A_1})$ be a CDGA morphism of oriented Poincaré duality algebras of finite dimension and of formal dimension n. Then the map $\widehat{\Delta_\psi} : s^{-n}A_1 \to A_1 \otimes A_2$, defined by $s^{-n}a \mapsto \Delta_\psi.(a \otimes 1)$ is a morphism of $(A_1 \otimes A_2)$-modules.
2. Let $\psi : (A_2, d_{A_2}, \omega_{A_2}) \to (A_1, d_{A_1}, \omega_{A_1})$ be a morphism of oriented differential Poincaré algebras of formal dimension n. Then Δ_ψ is a cocycle.
3. If $\psi : (A_2, d_{A_2}, \omega_{A_2}) \to (A_1, d_{A_1}, \omega_{A_1})$ is a morphism of connected differential oriented Poincaré duality algebras of formal dimension n and $\widehat{\Delta_\psi} : s^{-n}A_1 \to A_1 \otimes A_2$ is defined by $s^{-n}a \mapsto \Delta_\psi.(a \otimes 1)$, then
   (a) $\widehat{\Delta_\psi}$ is a morphism of $(A_1 \otimes A_2)$-dgmodules and hence its mapping cone

$$C(\widehat{\Delta_\psi}) := A_1 \otimes A_2 \oplus_{\widehat{\Delta_\psi}} ss^{-n}A_1$$

is an $(A_1 \otimes A_2)$-dgmodule.

(b) $C(\widehat{\Delta_\psi})$ is also a $CDGA$ under the following multiplication rules: If $a, a', x, y \in A_1$, and $b, b' \in A_2$,

$$
\begin{aligned}
(a \otimes b).(a' \otimes b') &= (-1)^{\deg(b)\deg(a')}(a.a' \otimes b.b'),\\
(a \otimes b).(ss^{-n}x) &= (-1)^{(n-1)(\deg(a)+\deg(b))}ss^{-n}(a.\psi(b).x),\\
(ss^{-n}x).(a \otimes b) &= ss^{-n}(x.a.\psi(b)),\\
(ss^{-n}x).(ss^{-n}y) &= 0.
\end{aligned}
$$

Definition 4.3 Let (A, ω) be a connected oriented Poincaré duality algebra of formal dimension n with orientation class ω (refer to Definition 4.1 of [9]). Since $A^n \cong \mathbb{Q}$, there exists a unique element $\mu \in A^n$ such that $\omega(\mu) = 1$ which called the *fundamental class* of A.

Our goal in this section is to prove Theorem 1.6. Throughout we consider morphisms $\psi : (A_2, d_{A_2}, \omega_{A_2}) \to (A_1, d_{A_1}, \omega_{A_1})$ of connected differential oriented Poincaré duality algebras of formal dimension n.

Lemma 4.4 *The ideal* $(\Delta_\psi) := \Delta_\psi.(A_1 \otimes A_2)$ *generated by* Δ_ψ *in* $A_1 \otimes A_2$ *is a differential ideal and the quotient* $(A_1 \otimes A_2)/(\Delta_\psi)$ *is a CDGA.*

Proof By Remark 4.2 (2), $d_{A_1 \otimes A_2}(\Delta_\psi) = 0$; hence the ideal (Δ_ψ) is a differential ideal. This implies immediately that the quotient $(A_1 \otimes A_2)/(\Delta_\psi)$ inherits a $CDGA$ structure. □

Lemma 4.5 *The map* $\widehat{\Delta_\psi}$ *induces an isomorphism*

$$\widehat{\Delta_\psi} : s^{-n}A_1 \to (\Delta_\psi).$$

Proof We first show that im $(\widehat{\Delta_\psi}) = (\Delta_\psi)$. Clearly im$(\widehat{\Delta_\psi}) \subseteq (\Delta_\psi)$. For the reverse inclusion we have

$$
\begin{aligned}
(\Delta_\psi) &= \{\Delta_\psi.(x \otimes y) \mid x \in A_1, y \in A_2\}\\
&= \{\Delta_\psi.(-1)^{\deg(x)\deg(y)}(1 \otimes y).(x \otimes 1) \mid x \in A_1, y \in A_2\}\\
&= \{\widehat{\Delta_\psi}((-1)^{\deg(x)\deg(y)}s^{-n}(\psi(y).x)) \mid x \in A_1, y \in A_2\}\\
&\subseteq \mathrm{im}(\widehat{\Delta_\psi}).
\end{aligned}
$$

Next we show that $\widehat{\Delta_\psi}$ is injective. Let ω_{A_1} and ω_{A_2} be the orientation classes of the Poincaré duality algebras A_1 and A_2 respectively. So there exist unique elements $\mu_{A_1} \in A_1^n$ and $\mu_{A_2} \in A_2^n$, fundamental classes of A_1 and A_2, such that $\omega_{A_1}(\mu_{A_1}) = 1$ and $\omega_{A_2}(\mu_{A_2}) = 1$. Fix a basis $\{a_i\}_{i=1}^l$ of A_2 and its Poincaré dual basis $\{a_i^*\}_{i=1}^l$. Since $A_2^0 = \mathbb{Q}$ we can assume that $1 = a_1 \in A_2^0$ and remaining a_i,'s are of degree > 0 so that $a_i\mu_{A_2} \in A_2^{>n} = 0$ for $i > 1$. Also $\omega_{A_2}(a_1.a_1^*) = 1$ implies that $1.a_i^* = a_1.a_1^* = \mu_{A_2}$ so that $a_1^* = a_1^{-1}.\mu_{A_2} = \mu_{A_2}$. This yields from the definition of Δ_ψ that

$$\begin{aligned}\Delta_\psi.(\mu_{A_1}\otimes 1) &= \sum_{i=1}^{l}(-1)^{\deg(a_i)}(\psi(a_i)\otimes a_i^*).(\mu_{A_1}\otimes 1)\\ &= \sum_{i=1}^{l}(-1)^{\deg(a_i^*)}(-1)^{n\deg(a_i^*)}(\psi(a_i).\mu_{A_1}\otimes a_i^*) = \psi(a_1).\mu_{A_1}\otimes a_1^*\\ &= \psi(a_1).\mu_{A_1}\otimes a_1^{-1}.\mu_{A_2} = \mu_{A_1}\otimes\mu_{A_2}\neq 0.\end{aligned}$$

Let a be a non-zero element of A_1^i. By Poincaré duality there exists an element $b \in A_1^{n-i}$ such that $a.b = \mu_{A_1}$. We have

$$\begin{aligned}\widehat{\Delta_\psi}(s^{-n}a).(b\otimes 1) = \Delta_\psi.(a\otimes 1).(b\otimes 1) = \Delta_\psi.(a.b\otimes 1) = \Delta_\psi.(\mu_{A_1}\otimes 1)\\ = \mu_{A_1}\otimes\mu_{A_2}\neq 0.\end{aligned}$$

□

Consider the projection $A_1\otimes A_2 \xrightarrow{\pi} (A_1\otimes A_2)/(\Delta_\psi)$. We extend π to a map $\hat{\pi}$: $A_1\otimes A_2\oplus_{\widehat{\Delta_\psi}} ss^{-n}A_1 \to (A_1\otimes A_2)/(\Delta_\psi)$ by setting $\hat{\pi}(ss^{-n}A_1) = 0$.

Lemma 4.6 *The map* $\hat{\pi}$: $A_1\otimes A_2\oplus_{\widehat{\Delta_\psi}} ss^{-n}A_1 \to (A_1\otimes A_2)/(\Delta_\psi)$ *defined above is a CDGA quasi-isomorphism.*

Proof The map $\hat{\pi}$ can be seen to be a $CDGA$ morphism by a straightforward computation. Since $\widehat{\Delta_\psi}$ is injective , we have a short exact sequence

$$0\to s^{-n}A_1 \xrightarrow{\widehat{\Delta_\psi}} A_1\otimes A_2 \xrightarrow{\pi} (A_1\otimes A_2)/\mathrm{im}(\widehat{\Delta_\psi})\to 0.$$

Comparing the long cohomology exact sequence corresponding to this exact sequence and that of the mapping cone $s^{-n}A_1 \xrightarrow{\widehat{\Delta_\psi}} A_1\otimes A_2 \hookrightarrow C(\widehat{\Delta_\psi}) := A_1\otimes A_2\oplus_{\widehat{\Delta_\psi}} ss^{-n}A_1$ we get the following commutative diagram:

$$\begin{array}{ccccccccccc}\to & H^q(s^{-n}A_1) & \to & H^q(A_1\otimes A_2) & \to & H^q(C(\widehat{\Delta_\psi})) & \to & H^{q+1}(s^{-n}A_1) & \to & H^{q+1}(A_1\otimes A_2) & \to\\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\ \to & H^q(s^{-n}A_1) & \to & H^q(A_1\otimes A_2) & \to & H^q\left(\frac{A_1\otimes A_2}{\mathrm{im}\,(\widehat{\Delta_\psi})}\right) & \to & H^{q+1}(s^{-n}A_1) & \to & H^{q+1}(A_1\otimes A_2) & \to\end{array}$$

Applying five lemma and using Lemma 4.5 we get that the map

$$A_1\otimes A_2\oplus_{\widehat{\Delta_\psi}} ss^{-n}A_1 \xrightarrow{\hat{\pi}=\pi\oplus 0} (A_1\otimes A_2)/\mathrm{im}(\widehat{\Delta_\psi}) = (A_1\otimes A_2)/(\Delta_\psi)$$

is a quasi isomorphism. □

We are now ready to prove Theorem 1.6.

Proof of Theorem. 1.6 We have proved in Lemma 4.4 that (Δ_{ψ_f}) is a differential ideal. Since (A_M, d_{A_M}) and (A_N, d_{A_N}) are connected differential Poincaré duality algebras of formal dimension n and since $H^n(A_M, d_{A_M}) = H^n(M; \mathbb{Q}) \neq 0$ and $H^n(A_N, d_{A_N}) = H^n(N; \mathbb{Q}) \neq 0$, Proposition 4.8 of [9] implies that $(A_M, d_{A_M}, \omega_{A_M})$ and $(A_N, d_{A_N}, \omega_{A_N})$ are oriented differential Poincaré duality algebras in the sense of Definition 4.6 of [9] for orientations $\omega_{A_M} \in \#A_M^n$ and $\omega_{A_N} \in \#A_N^n$.

We have proved in §3 that a CDGA model $\psi_f : (A_N, d_{A_N}) \to (A_M, d_{A_M})$ of f gives a CDGA model

$$\Phi := \nu_{A_M} \circ (1_{A_M} \otimes \psi_f) : (A_M \otimes A_N, d_{A_N} \otimes d_{A_M}) \to (A_M, d_{A_M})$$

of $(1_M \times f) \circ \Delta : M \to M \times N$, and is given by $x \otimes y \mapsto x.\psi_f(y)$. Set $n = 2m$. The morphism Φ induces an obvious $(A_M \otimes A_N)$-dgmodule structure on A_M, hence on $s^{-n}\#A_M$. By Proposition 4.7 of [9] $s^{-n}\#A_M = s^{-m}(s^{-m}\#A_M)$ is isomorphic to $s^{-m}A_M$ as an A_M-dgmodule, therefore also an isomorphism as an $(A_M \otimes A_N)$-dgmodule.

By Remark 4.2a the map $\widehat{\Delta_\psi} : s^{-n}A_M \to A_M \otimes A_N$ is a map of $(A_M \otimes A_N)$-dgmodules. Moreover, it induces an isomorphism in the cohomology in the top degree, because $\widehat{\Delta_\psi}(s^{-n}\mu) = \mu_{A_M} \otimes \mu_{A_N}$. Thus $\widehat{\Delta_\psi}$ is a shriek map (or, a top-degree map) in the sense of Definition 5.1 of [10].

Let I_0 be a complement of the cocycles in $(A_M \otimes A_N)^{n-3}$ and set $I = I_0 \oplus (A_M \otimes A_N)^{>n-3}$. Let K_0 be a complement of the cocycles in $(s^{-n}A_M)^{n-2}$ and set $K = K_0 \oplus (s^{-n}A_M)^{>n-2}$. Consider the quotient

$$\frac{A_M \otimes A_N \oplus_{\widehat{\Delta_\psi}} ss^{-n}A_M}{I \oplus sK}.$$

By the hypothesis of the theorem concerning f, it follows that the embedding $(1_M \times f) \circ \Delta_M : M \hookrightarrow M \times N$, which embeds M as $\Gamma(f)$ in $M \times N$, satisfies the hypothesis of Theorem 1.4 of [10], namely that $H_1((1_M \times f) \circ \Delta_M; \mathbb{Q})$ is an isomorphism and $H_2((1_M \times f) \circ \Delta_M; \mathbb{Q})$ is an epimorphism, hence by Theorem 1.4 of [10] we get that

$$\left(\frac{A_M \otimes A_N \oplus_{\widehat{\Delta_\psi}} ss^{-n}A_M}{I \oplus sK}, \bar{d}\right)$$

is a CDGA model of $M \times N \setminus \Gamma(f)$.

Comparing the semi-trivial CDGA structure as defined in Definition 4.1 of [10] on $(A_M \otimes A_N \oplus_{\widehat{\Delta_\psi}} ss^{-n}A_M)/(I \oplus sK)$ with the multiplication in the mapping cone given in this section, we conclude that the projection

$$(A_M \otimes A_N \oplus_{\widehat{\Delta_\psi}} ss^{-n}A_M, d) \to \left(\frac{A_M \otimes A_N \oplus_{\widehat{\Delta_\psi}} ss^{-n}A_M}{I \oplus sK}, \bar{d}\right)$$

is a CDGA map. Moreover it is a quasi-isomorphism by Lemma 8.6 of [10].

Therefore $(A_M \otimes A_N \oplus_{\widehat{\Delta_\psi}} ss^{-n}A_M, d)$ is a CDGA model of $M \times N \setminus \Gamma(f)$. Hence by Lemma 4.6 $\left((A_M \otimes A_N)/(\Delta_\psi), \overline{d_{A_M} \otimes d_{A_N}}\right)$ is a CDGA model of $M \times N \setminus \Gamma(f)$. □

We finally prove Corollary 1.7 on formality of the complement of the graph of a map.

Proof of corollary. 1.7 Since $f : M \to N$ is formal, $(H^*(N;\mathbb{Q}), 0) \xrightarrow{f^*} .(H^*(M;\mathbb{Q}), 0)$ is a model of f.

Moreover, since $(H^*(N;\mathbb{Q}), 0)$ and $(H^*(M;\mathbb{Q}), 0)$ are oriented differential Poincaré duality algebras, by Theorems 1.6 and 1.8,

$$\left(\frac{H^*(M;\mathbb{Q}) \otimes H^*(N;\mathbb{Q})}{(\Delta_f)}, 0\right) \cong (H^*(M \times N \setminus \Gamma(f);\mathbb{Q}), 0) \ \ \text{(as CDGAs)}$$

is a CDGA-model of $M \times N \setminus \Gamma(f)$, and hence it is a formal space. □

5 Examples and applications

In the last section we saw that if $f : M \to N$ is a formal map of closed connected oriented formal manifolds of dimension n such that $H^1(M;\mathbb{Q}) = 0 = H^1(N;\mathbb{Q}) = H^2(N;\mathbb{Q})$, then $M \times N \setminus \Gamma(f)$ is formal (Corollary 1.7). In Proposition 6.6 of [9] Lambrechts and Stanley proved that if M is a closed connected orientable manifold of dimension n such that $M \times M \setminus \Gamma(1_M) = F(M, 2)$ is formal then M is formal. We ask a similar question (this is the converse of Corollary 1.7):

Question 5.1 *Given closed connected oriented formal manifolds M and N of dimension n and a continuous map $f : M \to N$ such that $M \times N \setminus \Gamma(f)$ is formal, is f necessarily a formal map?*

The answer is in the negative. Here we construct such an example.

Example 5.2 Consider $S^7 \xrightarrow{f=(h,c)} S^4 \times S^3$, where $h : S^7 \to S^4$ is the Hopf map and $c : S^7 \to S^3$ is a constant map. It follows from Proposition 2.5 of [1] that any formal map $S^7 \to S^4 \times S^3$ must be nullhomotopic; but f is not nullhomotopic. Therefore f is *not* formal.

Now we show that $S^7 \times (S^4 \times S^3) \setminus \Gamma(f)$ is formal. Since $S^3 \xrightarrow{j} S^7 \xrightarrow{h} S^4$ is a Hopf fibration we have the following commutative diagram [see example (2.68) page 82 of [5] or Chapter 15 of [3]

$$\begin{array}{ccccc}
(\Lambda(x,y), d') & \xrightarrow{i} & (\Lambda(x,y)\otimes\Lambda(z), d) & \xrightarrow{\eta} & (\Lambda(z), d'') \\
\rho\downarrow & & \sigma\downarrow & & \xi\downarrow \\
A_{PL}(S^4) & \xrightarrow{A_{PL}(h)} & A_{PL}(S^7) & \xrightarrow{A_{PL}(j)} & A_{PL}(S^3).
\end{array}$$

Here deg $(x) = 4$, deg $(y) = 7$, and deg $(z) = 3$; $d'x = 0$, $d'y = x^2$, $dx = 0$, $dy = x^2$, $dz = x$; and $d''z = 0$; the morphism $\rho : (\Lambda(x,y), d') \to A_{PL}(S^4)$ is the minimal model of S^4, $\sigma : (\Lambda(x, y) \otimes \Lambda(z), d) \to A_{PL}(S^7)$ is a quasi-isomorphism, $\xi : (\Lambda(z), d'') \to A_{PL}(S^3)$ is the minimal model of S^3, $i : (\Lambda(x, y), d') \to (\Lambda(x, y) \otimes \Lambda(z), d)$ is a relative minimal CDGA and $\eta : (\Lambda(x, y) \otimes \Lambda(z), d) \to (\Lambda(z), d'')$ is the projection.

From the above diagram we get the following commutative diagram

$$\begin{array}{ccc} (\Lambda(x, y) \otimes \Lambda(z), d' \otimes d'') & \xrightarrow{i} & (\Lambda(x, y) \otimes \Lambda(z), d) \\ \rho\otimes\xi \big\downarrow & & \sigma \big\downarrow \\ A_{PL}(S^4) \otimes A_{PL}(S^3) & \xrightarrow{A_{PL}(h).A_{PL}(c)} & A_{PL}(S^7), \end{array}$$

where $i(x) = x,\ i(y) = y$ and $i(z) = 0$.

Let us define CDGAs

$$(B, 0) := (\Lambda(e_4, e_3)/I_B, 0),$$

where deg $(e_4) = 4$; deg$(e_3) = 3$, I_B is the ideal generated by e_4^2, and

$$(A, d_A) := (\Lambda(u_4, u_3)/I_A, d_A),$$

where deg $(u_4) = 4$; deg$(u_3) = 3$, I_A is the ideal generated by u_4^2, $d_A\overline{u_3} = \overline{u_4}$, and $d_A\overline{u_4} = 0$. It is easily checked that $(B, 0)$, and (A, d_A) are oriented differential Poincaré duality algebras.

We define morphisms

$$\rho_B : (\Lambda(x, y) \otimes \Lambda(z), d' \otimes d'') \to (B, 0),$$

where $\rho_B(x) = \overline{e_4},\ \rho_B(y) = 0,\ \rho_B(z) = \overline{e_3}$ and

$$\rho_A : (\Lambda(x, y) \otimes \Lambda(z), d) \to (A, d_A),$$

where $\rho_A(x) = \overline{u_4},\ \rho_A(y) = 0,\ \rho_A(z) = \overline{u_3}$.

Clearly ρ_B and ρ_A are quasi-isomorphisms.

Define $\psi_f : (B, 0) \to (A, d_A)$ by $\psi_f(\overline{e_4}) = \overline{u_4},\ \psi_f(\overline{e_3}) = 0$ so that the following diagram is commutative.

$$\begin{array}{ccc} (B, 0) = (\Lambda(e_4, e_3)/I_B, 0) & \xrightarrow{\psi_f} & (\Lambda(u_4, u_3)/I_A, d_A) = (A, d_A) \\ \rho_B \big\uparrow & & \rho_A \big\uparrow \\ (\Lambda(x, y) \otimes \Lambda(z), d' \otimes d'') & \xrightarrow{i} & (\Lambda(x, y) \otimes \Lambda(z), d). \end{array}$$

Using the fact that if $X \xrightarrow{(k,l)} Y \times Z$ is a continuous map then $A_{PL}(k) \cdot A_{PL}(l) : A_{PL}(Y) \otimes A_{PL}(Z) \to A_{PL}(X)$ is its model, we get the following commutative diagram

$$\begin{array}{ccc}
(B,0) = (\Lambda(e_4, e_3)/I_B, 0) & \xrightarrow{\psi_f} & (\Lambda(u_4, u_3)/I_A, d_A) = (A, d_A) \\
{\scriptstyle \rho_B}\uparrow & & {\scriptstyle \rho_A}\uparrow \\
(\Lambda(x, y) \otimes \Lambda(z), d' \otimes d'') & \xrightarrow{i} & (\Lambda(x, y) \otimes \Lambda(z), d) \\
{\scriptstyle \rho\otimes\xi}\downarrow & & {\scriptstyle \sigma}\downarrow \\
A_{PL}(S^4) \otimes A_{PL}(S^3) & \xrightarrow{A_{PL}(h).A_{PL}(c)} & A_{PL}(S^7) \\
{\scriptstyle A_{PL}(\pi_{S^4})\otimes A_{PL}(\pi_{S^4})}\downarrow & & {\scriptstyle =}\downarrow \\
A_{PL}(S^4 \times S^3) & \xrightarrow{A_{PL}(h,c)} & A_{PL}(S^7).
\end{array}$$

Therefore ψ_f is a model of $f = (h, c)$. So, by Theorem 1.6

$$\left(\frac{A \otimes B}{(\triangle_{\psi_f})}, \overline{d_A \otimes 0}\right)$$

is a CGDA model of $S^7 \times (S^4 \times S^3) \setminus \Gamma(f)$.

Note that $\{a_1 = 1,\ a_2 = \overline{e_3},\ a_3 = \overline{e_4},\ a_4 = \overline{e_4} \cdot \overline{e_3}\}$ is a homogeneous basis of $(B, 0)$ with Poincaré dual basis $\{a_1^* = \overline{e_4} \cdot \overline{e_3},\ a_2^* = \overline{e_4},\ a_3^* = \overline{e_3},\ a_4^* = 1\}$, therefore by definition

$$\begin{aligned}
\triangle_{\psi_f} &= \psi_f(1) \otimes \overline{e_4} \cdot \overline{e_3} - \psi_f(\overline{e_3}) \otimes \overline{e_4} + \psi_f(\overline{e_4}) \otimes \overline{e_3} - \psi_f(\overline{e_4} \cdot \overline{e_3}) \otimes 1 \\
&= 1 \otimes \overline{e_4} \cdot \overline{e_3} - \overline{u_4} \otimes \overline{e_3}.
\end{aligned}$$

Therefore as a CDGA

$$\left(\frac{A \otimes B}{(\triangle_{\psi_f})}, \overline{d_A \otimes 0} = \overline{d}\right) = \quad \mathbb{Q}(\overline{1 \otimes 1}) \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Q}(\overline{1 \otimes \overline{e_3}},\ \overline{\overline{u_3} \otimes 1})$$

$$\xrightarrow{\overline{d}} \mathbb{Q}(\overline{1 \otimes \overline{e_4}},\ \overline{\overline{u_4} \otimes 1}) \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Q}(\overline{\overline{u_3} \otimes \overline{e_3}}) \xrightarrow{\overline{d}} \mathbb{Q}(\overline{\overline{u_3} \otimes \overline{e_4}},\ \overline{\overline{u_4} \otimes \overline{e_3}},\ \overline{\overline{u_4}.\overline{u_3} \otimes 1})$$

$$\xrightarrow{\overline{d}} \mathbb{Q}(\overline{\overline{u_4} \otimes \overline{e_4}}) \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Q}(\overline{\overline{u_4}.\overline{u_3} \otimes \overline{e_3}}) \xrightarrow{0} \mathbb{Q}(\overline{\overline{u_4}.\overline{u_3} \otimes \overline{e_4}}) \xrightarrow{0} 0 \cdots$$

Let $c_3 \in H^3(S^3; \mathbb{Q})$, $c_4 \in H^4(S^4; \mathbb{Q})$ and $c_7 \in H^7(S^7; \mathbb{Q})$ be bases of the respective vector spaces. Then $\{b_1 = 1 \otimes 1,\ b_2 = 1 \otimes c_3,\ b_3 = c_4 \otimes 1,\ b_4 = c_4 \otimes c_3\}$ form a homogeneous basis of $H^*(S^4 \times S^3; \mathbb{Q}) = H^*(S^4; \mathbb{Q}) \otimes H^*(S^3; \mathbb{Q})$ with Poincaré dual basis $\{b_1^* = c_4 \otimes c_3,\ b_2^* = c_4 \otimes 1,\ b_3^* = 1 \otimes c_3,\ b_4^* = 1 \otimes 1\}$. We know by

Theorem 1.8 that

$$H^*(S^7 \times (S^4 \times S^3)\backslash\Gamma(f);\mathbb{Q}) = \frac{H^*(S^7;\mathbb{Q}) \otimes H^*(S^4 \times S^3;\mathbb{Q})}{(\triangle_f)},$$

where

$$\begin{aligned}\triangle_f &= f^*(1\otimes 1)\otimes c_4 \otimes c_3 - f^*(1\otimes c_3)\otimes c_4 \otimes 1 + f^*(c_4 \otimes 1)\otimes 1 \otimes c_3 \\ &\quad - f^*(c_4\otimes c_3)\otimes 1 \otimes 1 \\ &= 1 \otimes c_4 \otimes c_3.\end{aligned}$$

Therefore as a CDGA

$$\begin{aligned}&(H^*(S^7 \times (S^4 \times S^3)\backslash\Gamma(f);\mathbb{Q}),\ 0) \\ &= \ \mathbb{Q}(\overline{1\otimes 1\otimes 1}) \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \\ &\mathbb{Q}(\overline{1\otimes 1\otimes c_3}) \\ &\quad \xrightarrow{0} \mathbb{Q}(\overline{1\otimes c_4\otimes 1}) \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Q}(\overline{c_7\otimes 1\otimes 1}) \xrightarrow{0} 0 \xrightarrow{0} 0 \\ &\quad \xrightarrow{0} \mathbb{Q}(\overline{c_7\otimes 1\otimes c_3}) \xrightarrow{0} \mathbb{Q}(\overline{c_7\otimes c_4\otimes 1}) \xrightarrow{0} 0\ldots\end{aligned}$$

Now we define a CDGA morphism

$$\eta : (H^*(S^7 \times (S^4 \times S^3)\backslash\Gamma(f);\mathbb{Q}),\ 0) \to \left(\frac{A\otimes B}{(\triangle\psi_f)}, d_A \otimes 0\right),$$

by defining it on generators as follows:

$$\begin{aligned}&\eta(\overline{1\otimes 1\otimes 1}) = \overline{1\otimes 1}, \eta(\overline{1\otimes 1\otimes c_3}) = \overline{1\otimes \overline{e_3}}, \eta(\overline{1\otimes c_4\otimes 1}) = \overline{1\otimes \overline{e_4}} + \overline{\overline{u_4}\otimes 1}, \\ &\eta(\overline{c_7\otimes 1\otimes 1}) = \overline{\overline{u_4}.\overline{u_3}\otimes 1},\ \eta(\overline{c_7\otimes 1\otimes c_3}) = \overline{\overline{u_4}.\overline{u_3}\otimes \overline{e_3}},\ \eta(\overline{c_7\otimes c_4\otimes 1}) \\ &\qquad = \overline{\overline{u_4}.\overline{u_3}\otimes \overline{e_4}}.\end{aligned}$$

It is checked easily that η is a quasi-isomorphism. Therefore $(H^*(S^7 \times (S^4 \times S^3)\backslash \Gamma(f);\mathbb{Q}),\ 0)$ is a CDGA model of $S^7 \times (S^4 \times S^3)\backslash\Gamma(f)$. Hence $S^7 \times (S^4 \times S^3)\backslash\Gamma(f)$ is a formal space.

Remark 5.3 It is known (see e.g. Lemma 6.3 of [9]) that if M is a simply connected closed manifold, if $x \in M$, and if $M\backslash\{x\}$ is formal, then M is formal.

Against this background we may ask the following question.

Question 5.4 *Let $f : M \to N$ be a continuous map between simply connected formal manifolds and let $x \in M$. Suppose that $f|_{M\backslash\{x\}} : M\backslash\{x\} \longrightarrow N\backslash\{f(x)\}$ is formal. Is it necessarily true that f is formal ?*

The following example shows that the answer to this question is in the negative.

Example 5.5 Take $M = S^3$, $N = S^2$ and f the Hopf map $h : S^3 \to S^2$. Now $S^3 \backslash \{(0,0,0,1)\} \cong \mathbb{R}^3$ and $S^2 \backslash \{(0,0,1)\} \cong \mathbb{R}^2$. Since any continuous map form $\mathbb{R}^3 \to \mathbb{R}^2$ is formal, $h|_{S^3 \backslash \{(0,0,0,1)\}} : S^3 \backslash \{(0,0,0,1)\} \longrightarrow S^2 \backslash \{(0,0,1)\}$ is formal. But h is not formal.

We end the paper by recording a few simple applications of Theorem 1.6 and Corollary 1.7:

Application 5.6 1. Let M and N be two simply connected closed formal manifolds of dimension n. For any element $y \in N$, since the constant map $f_y : M \to N$ defined by $f_y(x) = y$ for all $x \in X$ is formal, by Corollary 1.7, $M \times N \setminus \Gamma(f)$ is formal. But $M \times N \setminus \Gamma(f) = M \times (N \backslash \{y\})$. Therefore $M \times (N \backslash \{y\})$ is formal.

2. Let M be a simply connected closed formal manifold of dimension n. Since the identity map $f = 1_M : M \to M$ is formal, by Corollary 1.7, $M \times M \backslash \Gamma(f)$ is formal. But $M \times M \setminus \Gamma(f) = F(M, 2)$. Therefore $F(M, 2)$ is formal, which is one of the main results (Corollary 6.1) in [9].
3. Let M and N be two simply connected Lie groups of dimension n. It is known that the minimal model of a Lie group is of the form $(\Lambda V, 0)$. Hence Lie groups are formal and continuous maps between Lie groups are also formal. Therefore, by Corollary 1.7, for a continuous map $f : M \to N$, $M \times N \backslash \Gamma(f)$ is formal.

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