J. Homotopy Relat. Struct. (2014) 9:263–271
DOI 10.1007/s40062-013-0024-9

Existence of $b_0 b_1 g_0 \widetilde{\gamma}_s$-element in the stable homotopy of spheres

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Received: 7 November 2012 / Accepted: 2 February 2013 / Published online: 23 February 2013

Abstract Let p be a prime greater than five and A be the mod p Steenrod algebra. In this paper, we show that the composite map $\beta_1 \varphi_s$ is nontrivial in the stable homotopy of spheres $\pi_{2(p-1)[(s+1)p^2+(s+1)p+s]-9}(S)$, where $4 \leq s < p$ and φ_s is represented by $b_1 g_0 \widetilde{\gamma}_s \in \mathrm{Ext}_A^{s+4,2(p-1)[(s+1)p^2+sp+s]+s-3}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the Adams spectral sequence.

Keywords Stable homotopy of spheres · Adams spectral sequence · Smith-Toda spectrum · May spectral sequence

Mathematics Subject Classification (2000) 55Q45 · 55T10

Communicated by Jim Stasheff.

The first author was supported in part by the National Natural Science Foundation of China (No. 11171161) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

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1 Introduction and statement of the main results

Let A be the mod p Steenrod algebra and S be the sphere spectrum localized at an arbitrary odd prime p. To determine the stable homotopy groups of spheres $\pi_*(S)$ is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence (see [1]):

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \Rightarrow \pi_{t-s}(S),$$

where the E_2-term is the cohomology of A.

Throughout this paper, we fix $q = 2(p-1)$. The known results on $\mathrm{Ext}_A^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ are as follows: $\mathrm{Ext}_A^{0,*}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p$ by its definition. From [6], we have that for odd prime p $\mathrm{Ext}_A^{1,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ has $\mathbb{Z}/p$-basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}/p, \mathbb{Z}/p)$ and $h_i \in \mathrm{Ext}_A^{1,p^i q}(\mathbb{Z}/p, \mathbb{Z}/p)$ for all $i \geq 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ has $\mathbb{Z}/p$-basis consisting of α_2, a_0^2, $a_0 h_i$ $(i > 0)$, g_i $(i \geq 0)$, k_i $(i \geq 0)$, b_i $(i \geq 0)$, and $h_i h_j$ $(j \geq i+2, i \geq 0)$ whose internal degrees are $2q+1$, 2, $p^i q + 1$, $p^{i+1}q + 2p^i q$, $2p^{i+1}q + p^i q$, $p^{i+1}q$ and $p^i q + p^j q$, respectively. Aikawa [2] determined $\mathrm{Ext}_A^{3,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ by virtue of the lambda algebra.

Let M be the mod p Moore spectrum given by the following cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S. \tag{1.1}$$

Let $\alpha : \Sigma^q M \longrightarrow M$ be the known Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M. \tag{1.2}$$

This spectrum which we briefly write as K is known to be the Smith-Toda spectrum $V(1)$. For $p > 3$, Smith [9] showed that there exists a periodic map

$$\beta : \Sigma^{(p+1)q} K \longrightarrow K$$

which induces multiplication by v_2 in $K(2)$-theory. Let $V(2)$ be the cofibre of $\beta : \Sigma^{(p+1)q} K \longrightarrow K$ given by the cofibration

$$\Sigma^{(p+1)q} K \xrightarrow{\beta} K \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1} K. \tag{1.3}$$

For $p \geq 7$, there exists the Smith-Toda map $\gamma : \Sigma^{q(p^2+p+1)} V(2) \longrightarrow V(2)$ (see [10]).

Definition 1.1 For $s \geq 1$, the β-element β_s is defined to be the composite map

$$j j' \beta^s i' i \in \pi_{q[sp+(s-1)]-2}(S)$$

and the γ-element γ_s is defined to be the composite map

$$jj'\bar{j}\gamma^s\bar{i}i'i \in \pi_{q[sp^2+(s-1)p+(s-2)]-3}(S).$$

Theorem 1.2 (1) ([9]) *For $p \geq 5$ and $s \geq 1$, $\beta_s \neq 0$ in $\pi_*(S)$.*
(2) ([7]) *For $p \geq 7$ and $s \geq 1$, $\gamma_s \neq 0$ in $\pi_*(S)$.*

So far, not so many families of homotopy elements in π_*S have been detected. For example, Cohen [3] detected a new element $\zeta_n \in \pi_{q(p^{n+1}+1)-3}(S)$ for $n \geq 1$, which is represented by $h_0b_n \in \mathrm{Ext}_A^{3,q(p^{n+1}+1)}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the Adams spectral sequence.

Liu [5] also detected an infinite family of homotopy elements in the stable homotopy groups of spheres and obtained the following

Theorem 1.3 ([5, Theorem 1.1]) *Let $p \geq 7$ and $3 \leq s < p$. Then the product element*

$$b_1g_0\widetilde{\gamma}_s \in \mathrm{Ext}_A^{s+4,*}(\mathbb{Z}/p, \mathbb{Z}/p)$$

is a permanent cycle in the Adams spectral sequence and converges nontrivially to a homotopy element $\varphi_s \in \pi_*(S)$.

In this paper, we consider the composite element $\beta_1\varphi_s$ and show its nontriviality under some conditions. The main result can be stated as follows:

Theorem 1.4 *Let $p \geq 7$ and $4 \leq s < p$. Then the composite map $\beta_1\varphi_s$ is nontrivial in $\pi_*(S)$.*

The paper is arranged as follows: after recalling some knowledge on the May spectral sequence in Sect. 2, we make use of the May spectral sequence and the Adams spectral sequence to prove Theorem 1.4 in Sect. 3.

2 Preliminaries on the May spectral sequence

The May spectral sequence is one of our main tools in this paper. For the sake of completeness, we briefly recall some knowledge on the May spectral sequence.

From [8], there is the May spectral sequence $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{m,i}|m > 0, i \geq 0) \otimes P(b_{m,i}|m > 0, i \geq 0) \otimes P(a_n|n \geq 0), \quad (2.1)$$

where E is the exterior algebra, P is the polynomial algebra, and

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}, a_n \in E_1^{1,2p^n-1,2n+1}. \quad (2.2)$$

One has

$$d_r : E_r^{s,t,u} \longrightarrow E_r^{s+1,t,u-r} \quad (2.3)$$

for $r \geq 1$ and if $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y). \tag{2.4}$$

There also exists a graded commutativity in the May spectral sequence as follows:

$$x \cdot y = (-1)^{ss'+tt'} y \cdot x$$

for $x, y = h_{i,j}, b_{k,l}$, or a_n. The first May differential d_1 is given by

$$\begin{cases} d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j} h_{k,j}, \\ d_1(a_i) = \sum_{0 \leq k < i} h_{i-k,k} a_k, \\ d_1(b_{i,j}) = 0. \end{cases} \tag{2.5}$$

For convenience, we define dim $x = s$, deg $x = t$ and $\mathrm{M}(x) = u$ for any element $x \in E_1^{s,t,u}$.

For each integer $t \geq 0$, it can be always expressed uniquely as

$$t = q(c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0) + e,$$

where $0 \leq c_i < p\,(0 \leq i \leq n), c_n > 0, 0 \leq e < q$. Suppose $g = x_1 x_2 \ldots x_m \in E_1^{\bar{s},t,*}$, where $m \leq \bar{s}$, x_i is one of $a_k, h_{l,j}$ or $b_{u,z}$, $0 \leq k \leq n+1$, $0 \leq l + j \leq n+1$, $0 \leq u + z \leq n$, $l > 0$, $j \geq 0$, $u > 0$, $z \geq 0$. We also suppose deg $x_i = q(c_{i,n} p^n + c_{i,n-1} p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then we have dim $g = \sum_{i=1}^m \dim x_i = \bar{s}$ and

$$\deg g = t = \left(\left(\sum_{i=1}^m c_{i,n}\right) p^n + \cdots + \left(\sum_{i=1}^m c_{i,2}\right) p^2 + \left(\sum_{i=1}^m c_{i,1}\right) p + \left(\sum_{i=1}^m c_{i,0}\right)\right) + \left(\sum_{i=1}^m e_i\right).$$

Lemma 2.1 *If* $\left(\sum_{i=1}^m c_{i,0}\right) - \left(\sum_{i=1}^m e_i\right) = v$, *then there exist integers* $i_v > i_{v-1} > \cdots > i_1 \geq 1$ *and an element* $\hat{g}$ *such that* $g = h_{i_v,0} h_{i_{v-1},0} \ldots h_{i_1,0} \hat{g}$ *up to sign.*

Proof By use of (2.2), there must exist v elements of the form $h_{\mu,0}$ at least among g. Note that $h_{i,0}^2 = 0$ for $i \geq 1$. Thus, there exist integers i_k $(0 < k < v)$ and an element $\hat{g}$ such that $i_v > i_{v-1} > \cdots > i_1 \geq 1$ and $g = h_{i_v,0} h_{i_{v-1},0} \ldots h_{i_1,0} \hat{g}$ up to sign. The lemma follows. □

3 Proof of Theorem 1.4

Before showing our main theorem, we first give some important results on the May E_r-term ($r \geq 1$) and the Adams E_2-term.

The following Representation Theorem is due to X. Liu.

Lemma 3.1 ([4, Theorem 1.1]) *Let $p \geq 7$, $3 \leq s < p$. Then the element*

$$a_3^{s-3} h_{3,0} h_{2,1} h_{1,2} \in E_1^{s,t(s),*}$$

detects the third Greek letter element $\tilde{\gamma}_s \in \mathrm{Ext}_A^{s,t(s)}(\mathbb{Z}/p, \mathbb{Z}/p)$ in the May spectral sequence, where $t(s) = sp^2q + (s-1)pq + (s-2)q + s - 3$ and $\tilde{\gamma}_s$ detects the γ-element γ_s in the Adams spectral sequence.

The following lemma plays an important role in showing Theorem 1.4 and can be stated as follows:

Lemma 3.2 *Let $p \geq 7$, $4 \leq s < p$ and $r \geq 1$. Then the May E_1-term*

$$E_1^{s+6-r,t(s,r),*} = \begin{cases} G, & r = 1, \\ 0, & r > 1, \end{cases}$$

where $t(s,r) = q[(s+1)p^2 + (s+1)p + s] + s - r - 2$ and G is the $\mathbb{Z}/p$-module generated by $a_3^{s-3} h_{3,0} h_{2,0} h_{1,0} h_{2,1} b_{2,0} b_{1,1}$, $a_3^{s-3} h_{3,0} h_{2,0} h_{1,0} h_{1,2} b_{2,0}^2$, $a_3^{s-3} h_{3,0} h_{2,0} h_{1,0} h_{2,1} h_{1,1} h_{1,2} b_{1,1}$.

Proof When $r \geq s - 1$, it is easy to check that that the May E_1-term

$$E_1^{s+6-r,t(s,r),*} = 0. \tag{3.1}$$

Therefore, we assume $1 \leq r \leq s - 2$ in the rest of the proof.

If $s = 4$, then r may equal 1 or 2 by $1 \leq r \leq s - 2$. We can show that

$$E_1^{9,t(4,1),*} = \mathbb{Z}/p\{a_3 h_{3,0} h_{2,0} h_{1,0} h_{2,1} b_{1,1} b_{2,0},\ a_3 h_{3,0} h_{2,0} h_{1,0} h_{1,2} b_{2,0}^2,\\ a_3 h_{3,0} h_{2,0} h_{1,0} h_{2,1} h_{1,1} h_{1,2} b_{1,1}\}$$

and

$$E_1^{8,t(4,2),*} = 0$$

through easy computations. Thus, we assume $s \geq 5$ in the rest of the proof.

Case 1 $5 \leq s < p - 1$. By replacing $\bar{s}$ and t by $s + 6 - r$ and $t(s,r)$ in the argument given above Lemma 2.1 respectively, we have $\dim g = \sum_{i=1}^{m} \dim x_i = s + 6 - r$ which implies that $m \leq s + 6 - r \leq s + 5 < p + 4$. Thus, we have

$$\begin{cases} \sum_{i=1}^{m} e_i = s - r - 2, \ \sum_{i=1}^{m} c_{i,0} = s, \\ \sum_{i=1}^{m} c_{i,1} = s + 1, \quad \sum_{i=1}^{m} c_{i,2} = s + 1. \end{cases} \tag{3.2}$$

Subcase 1.1 $2 \leq r \leq s - 2$. Note that

$$\sum_{i=1}^{m} c_{i,0} - \sum_{i=1}^{m} e_i = s - (s - r - 2) = r + 2 > 2 + 1.$$

By Lemma 2.1, we have

$$E_1^{s+6-r, t(s,r),*} = 0.$$

Subcase 1.2 $r = 1$. Note that in this case $\sum_{i=1}^{m} c_{i,0} - \sum_{i=1}^{m} e_i = s - (s - 3) = 3$. From Lemma 2.1, $g = \hat{g} h_{1,0} h_{2,0} h_{3,0}$ up to sign if g exists, where $\hat{g} = x_1 \ldots x_{m-3}$. For the above $\hat{g}$, we have:

$$\begin{cases} \sum_{i=1}^{m-3} e_i = s - 3, \\ \sum_{i=1}^{m-3} c_{i,0} = s - 3, \\ \sum_{i=1}^{m-3} c_{i,1} = s - 1, \\ \sum_{i=1}^{m-3} c_{i,2} = s. \end{cases} \tag{3.3}$$

Note that $m - 3 \leq s + 2$ and $\sum_{i=1}^{m-3} c_{i,2} + \sum_{i=1}^{m-3} e_i - (m-3) \geq s + (s-3) - (s+2) = s - 5$. By (2.2) and degree reason, $\hat{g}$ must be of the form $\hat{g} = a_3^{s-5} \hat{\hat{g}}$ up to sign, where $\hat{\hat{g}} = x_{s-4} \ldots x_{m-3} \in E_1^{7, q(5p^2+4p+2)+2,*}$. By (2.1) and (2.2), we easily get that

$$E_1^{7,q(5p^2+4p+2)+2,*} = \mathbb{Z}/p\{a_3^2 h_{2,1} b_{2,0} b_{1,1}, a_3^2 h_{1,2} b_{2,0}^2, a_3^2 h_{2,1} h_{1,1} h_{1,2} b_{1,1}, a_3 a_0 h_{3,0} h_{2,1} h_{1,2} b_{2,0}\}.$$

Thus, we have that up to sign $g = a_3^{s-3} h_{3,0} h_{2,0} h_{1,0} h_{2,1} b_{2,0} b_{1,1}$, $a_3^{s-3} h_{3,0} h_{2,0} h_{1,0} h_{1,2} b_{2,0}^2$, $a_3^{s-3} h_{3,0} h_{2,0} h_{1,0} h_{2,1} h_{1,1} h_{1,2} b_{1,1}$, $a_3^{s-4} a_0 h_{3,0}^2 h_{2,0} h_{1,0} h_{2,1} h_{1,2} b_{2,0}$ in which the last one is trivial by $h_{3,0}^2 = 0$.

Case 2 $s = p - 1$. By replacing $\bar{s}$ and t by $p + 5 - r$ and $t(p-1, r)$ in the argument given above Lemma 2.1 respectively, we have $\dim g = \sum_{i=1}^{m} \dim x_i = p + 5 - r$ which implies that $m \leq p + 5 - r \leq p + 4$. Thus, we have:

$$\begin{cases} \sum_{i=1}^{m} e_i = p-3-r, \\ \sum_{i=1}^{m} c_{i,0} = p-1, \\ \sum_{i=1}^{m} c_{i,1} = \lambda_1 p, \quad \lambda_1 \geq 0, \\ \sum_{i=1}^{m} c_{i,2} + \lambda_1 = 1 + \lambda_2 p, \quad \lambda_2 \geq 0, \\ \sum_{i=1}^{m} c_{i,3} + \lambda_2 = 1. \end{cases} \tag{3.4}$$

From $\sum_{i=1}^{m} c_{i,1} = \lambda_1 p$ and $m \leq p+4$, we have $\lambda_1 = 0$ or 1.

Subcase 2.1 $\lambda_1 = 0$. From $\sum_{i=1}^{m} c_{i,0} - \sum_{i=1}^{m} e_i = p-1-(p-3-r) = r+2$ and Lemma 2.1, we have that there would be $r+2$ factors of the form $h_{i,0}$ at least among g if g exists. Note that $\sum_{i=1}^{m} c_{i,1} = 0$ and $h_{i,0}^2$=0 for any $i \geq 1$. From (2.2), it follows that g is impossible to exist in this case.

Subcase 2.2 $\lambda_1 = 1$. From $\sum_{i=1}^{m} c_{i,2} = \lambda_2 p$, we have that $\lambda_2 = 0$ or 1 by $m \leq p+4$.

If $\lambda_2 = 0$, from (3.4) and (2.2) we can deduce that there must exist a factor $h_{1,3}$ or $b_{1,2}$ among g if g exists. Thus we can write g as $g = x_1x_2\ldots x_{m-1}\bar{g}$, where $\bar{g}$=$h_{1,3}$ or $b_{1,2}$. Let $g_1 = x_1x_2\ldots x_{m-1}$. By Lemma 2.1, g_1 is impossible to exist. Thus, g is impossible to exist. If $\lambda_2 = 1$, by an argument similar to that used in Case 1, we have that, g exists only when $r = 1$, and up to sign $g = a_3^{p-4}h_{3,0}h_{2,0}h_{1,0}h_{2,1}b_{2,0}b_{1,1}$, $a_3^{p-4}h_{3,0}h_{2,0}h_{1,0}h_{1,2}b_{2,0}^2$, $a_3^{p-4}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,1}h_{1,2}b_{1,1}$.

The Proof of Lemma 3.2 is completed. □

From the above lemma, we easily have the following:

Theorem 3.3 *Let $p \geq 7$, $4 \leq s < p$, $r \geq 2$, then we have*

$$\mathrm{Ext}_A^{s+6-r,t(s,r)}(\mathbb{Z}/p, \mathbb{Z}/p) = 0,$$

where $t(s,r) = q[(s+1)p^2 + (s+1)p + s] + s - r - 2$.

Theorem 3.4 *Let $p \geq 7$, $4 \leq s < p$. Then the product element $b_0b_1g_0\widetilde{\gamma}_s$ is nontrivial in the Adams E_2-term.*

Proof It is known that $b_{1,i}$ and $h_{2,0}h_{1,0}$ are permanent cocycles in the May spectral sequence and converge to b_i and g_0 for $i \geq 1$, respectively. From Lemma 3.1, $\widetilde{\gamma}_s$ is represented by $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}$ in the May spectral sequence. Thus,

$$b_0b_1g_0\widetilde{\gamma}_s \in \mathrm{Ext}_A^{s+6,*}(\mathbb{Z}/p, \mathbb{Z}/p)$$

is represented by

$$b_{1,0}b_{1,1}h_{2,0}h_{1,0}a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}$$

for $s \geq 4$ in the May spectral sequence. Now we show that nothing can hit the permanent cocycle $b_{1,0}b_{1,1}h_{2,0}h_{1,0}a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}$ under the May differential d_r for $r \geq 1$.

Denote the generators $b_{1,0}b_{1,1}h_{2,0}h_{1,0}a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}$, $a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}b_{2,0}b_{1,1}$, $a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{1,2}b_{2,0}^2$ and $a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,1}h_{1,2}b_{1,1}$ by $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$, respectively. Since $\mathrm{M}(\mathcal{G}) = 2p + 7s - 8 > p + 7s - 7 = \mathrm{M}(\mathcal{G}_3)$, we have that in the May spectral sequence $d_r(\mathcal{G}_3) \neq \mathcal{G}$ for any $r \geq 1$ up to a nonzero scalar. For the others, the possibilities are $d_{2p-1}(\mathcal{G}_1) = \mathcal{G}$ and $d_{4p-3}(\mathcal{G}_2) = \mathcal{G}$ up to a nonzero scalar. Note that up to sign $d_1(\mathcal{G}_1) = a_3^{s-3}h_{3,0}h_{2,0}h_{1,0}h_{1,1}h_{1,2}b_{2,0}b_{1,1} + \cdots \neq 0$ and $d_1(\mathcal{G}_2) = a_3^{s-4}a_1h_{3,0}h_{2,0}h_{1,0}h_{2,1}h_{1,2}b_{2,0}^2 + \cdots \neq 0$. Thus, $\mathcal{G}_1$ and $\mathcal{G}_2$ both die at the May E_2-term. From the above discussion, we have that nothing can hit $\mathcal{G}$ under the May differential d_r for $r \geq 1$. This completes the Proof of Theorem 3.4. □

Now we are in a position to complete the Proof of Theorem 1.4.

Proof of Theorem 1.4 It is known that $b_0 \in \mathrm{Ext}_A^{2,pq}(\mathbb{Z}/p, \mathbb{Z}/p)$ is a permanent cycle in the spectral sequence and converges nontrivially to the β-family $\beta_1 \in \pi_{pq-2}(S)$. Moreover, from Theorem 1.3 we have that $b_1 g_0 \widetilde{\gamma}_s \in \mathrm{Ext}_A^{s+4,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ detects the homotopy element $\varphi_s \in \pi_*(S)$ in the Adams spectral sequence. Thus, the composite map

$$\beta_1 \circ \varphi_s$$

is represented up to a nonzero scalar by

$$b_0 b_1 g_0 \widetilde{\gamma}_s$$

in the Adams spectral sequence. From Theorem 3.4, the product element $b_0 b_1 g_0 \widetilde{\gamma}_s$ is nontrivial in the Adams E_2-term. Meanwhile, from Theorem 3.3, $b_0 b_1 g_0 \widetilde{\gamma}_s$ does not bound in the Adams spectral sequence. Consequently, $b_0 b_1 g_0 \widetilde{\gamma}_s$ is a permanent cycle in the Adams spectral sequence and converges nontrivially to the homotopy element $\beta_1 \circ \varphi_s$. This finishes the Proof of Theorem 1.4. □

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