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Pre-c-symplectic condition for the product of odd-spheres

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Abstract We say that a simply connected space X is *pre-c-symplectic* if it is the fibre of a rational fibration $X \to Y \to \mathbb{C}P^{\infty}$ where Y is cohomologically symplectic in the sense that there is a degree 2 cohomology class which cups to a top class. It is a rational homotopical property but not a cohomological one. By using Sullivan's minimal models (Félix et al. in Rational homotopy theory. Graduate Texts in Mathematics, vol. 205. Springer, Berlin, 2001), we give the necessary and sufficient condition that the product of odd-spheres $X = S^{k_1} \times \cdots \times S^{k_n}$ is pre-c-symplectic and see some related topics. Also we give a charactarization of the Hasse diagram of rational toral ranks for a space X (Yamaguchi in Bull Belg Math Soc Simon Stevin 18:493–508, 2011) as a necessary condition to be pre-c-symplectic and see some examples in the cases of finite-oddly generated rational homotopy groups.

Keywords Symplectic · c-Symplectic · Pre-c-symplectic · Sullivan model · Rational homotopy type · Almost free toral action · Rational toral rank · Hasse diagram of rational toral ranks · KS-model · Elliptic · Formal

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Our definition of *pre-c-symplectic* is completely different from usual one of *presymplectic* (cf. [12,14]).

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1 Introduction

Recall the question:"*If a symplectic manifold is a nilpotent space, what special homotopical properties are apparent? Conversely, what nilpotent spaces have symplectic or c-symplectic structures?*" [9, (4.99)]. Here a rationally Poincaré dual space Y (the graded algebra $H^*(Y;\mathbb{Q})$ is a Poincaré duality algebra [9, Def. 3.1]) with formal dimension

$$fd(Y) := \max\{i \mid H^i(Y;\mathbb{Q}) \neq 0\}$$

$= 2n$ is said to be *c-symplectic (cohomologically symplectic)* if there is a rational cohomology class $\omega \in H^2(Y;\mathbb{Q})$ such that ω^n is a top class for Y [9, Def. 4.87] [22,29], many of which are known to be realized by $2n$-dimensional smooth manifolds [9]. A lot of results on the above problem and related topics are given in rational homotopy theory (cf. [5,6,9,15,16,18–21,29]). For example, Lupton and Oprea [20] study the formalising tendency of certain symplectic manifolds using techniques of D.Sullivan's rational model [28]. Notice that it is known that the connected sum $\mathbb{C}P^2\sharp\mathbb{C}P^2$ is c-symplectic but not symplectic [4] [21, p. 263], for the n-dimensional complex projective space $\mathbb{C}P^n$. In [15,18] [22, Theorem 6.3] [30], we can see conditions that a total space with a degree 2 cohomology class admits a symplectic structure in a certain fibration. But we don't mention anything about symplectic geometry in this paper.

For a simply connected c-symplectic space Y, we have $\omega \in Hom(\pi_2(Y),\mathbb{Q})$ for the class ω of $H^2(Y;\mathbb{Q})$ from Hurewicz isomorphism. In particular, $\pi_2(Y)\otimes\mathbb{Q} \neq 0$. So there is a simply connected space X that is the fibre of a fibration

$$X \to Y \to \mathbb{C}P^\infty \tag{1}$$

where $\mathbb{C}P^\infty = \cup_{n=1}^\infty \mathbb{C}P^n (= K(\mathbb{Z},2))$, $\pi_*(X)\otimes\mathbb{Q}\oplus\mathbb{Q}\cdot t^* = \pi_*(Y)\otimes\mathbb{Q}$ for a cohomology element t with $\deg(t) = 2$ (necessarily we don't need $t = \omega$) and $H^*(\mathbb{C}P^\infty;\mathbb{Q}) = \mathbb{Q}[t]$.

Definition 1.1 We say a simply connected space X to be *pre-c-symplectic* (*pre-cohomologically symplectic*) if X is the fibre of a fibration (1) where Y is c-symplectic.

For example, $\mathbb{C}P^n$ is a symplectic manifold, whose pre-c-symplectic space must be the $2n+1$-dimensional sphere S^{2n+1}. It is induced by the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ [1, p. 95]. We know that $fd(Y) = 2n$ if and only if $fd(X) = 2n+1$ in (1) from the Gysin exact sequence of of the induced fibration $S^1 \to X \to Y$. When $\dim \pi_2(Y)\otimes\mathbb{Q} > 1$, (1) may not be rational homotopically unique for Y. For example, when Y is $S^2\times\mathbb{C}P^2$, two spaces $S^3\times\mathbb{C}P^2$ and $S^2\times S^5$ are both its pre-c-symplectic spaces (there are three pre-c-symplectic spaces in the case of [20, Example 2.12]). The being c-symplectic and the being pre-c-symplectic are complementary. If a space is c-symplectic, it is not pre-c-symplectic and moreover if a space is pre-c-symplectic, it is not c-symplectic. The being c-symplectic is preserved by product; i.e., $Y_1\times Y_2$ is pre-c-symplectic by the class $\omega_1+\omega_2$ when Y_1 and Y_2 are both c-symplectic by classes ω_1 and ω_2, respectively. But the being pre-c-symplectic can not since then the formal dimension is even.

Of course, the being pre-c-symplectic depends on the rational homotopy type of X. Recall the Sullivan's rational model theory [28]. Let the Sullivan minimal model of X be $M(X) = (\Lambda V, d)$. It is a free $\mathbb{Q}$-commutative differential graded algebra (dga) with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i\geq 2} V^i$ where $\dim V^i < \infty$ and a decomposable differential; i.e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda^+ V$ is the ideal of ΛV generated by elements of positive degree. Denote the degree of a homogeneous element f of a graded algebra as $|f|$. Then $xy = (-1)^{|x||y|} yx$ and $d(xy) = d(x)y + (-1)^{|x|} xd(y)$. Note that $M(X)$ determines the rational homotopy type of X. In particular, it is known that

$$H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q}) \quad \text{and} \quad V^i \cong Hom(\pi_i(X), \mathbb{Q}).$$

Refer [8, Sections 12–15] for detail. Especially, (1) is replaced with the relative model (KS-model) [8]

$$(\mathbb{Q}[t], 0) \to (\mathbb{Q}[t] \otimes \Lambda V, D) \to (\Lambda V, d) \tag{2}$$

where $|t| = 2$ and $\overline{D} = d$. We often say that $M(Y) = (\mathbb{Q}[t] \otimes \Lambda V, D)$ is c-symplectic when Y is so. When $\pi_*(X) \otimes \mathbb{Q} < \infty$ and $\dim H^*(X; \mathbb{Q}) < \infty$, a simply connected space X is said to be elliptic. It is known that

$$fd(X) = fd(\Lambda V, d) = \sum_i |y_i| - \sum_i (|x_i| - 1)$$

for $V^{odd} = \mathbb{Q}(y_i)_i$ and $V^{even} = \mathbb{Q}(x_i)_i$ when X is elliptic [8, Section 32]. When is a simply connected space X pre-c-symplectic? Notice that if a pure model $M(Y) = (\Lambda U, d_Y)$, which satisfies $d_Y U^{even} = 0$ and $d_Y U^{odd} \subset \Lambda U^{even}$, is c-symplectic, then $\dim U^{even} = \dim U^{odd}$ [20]. For example, any simply connected symplectic homogeneous space is a maximal rank homogeneous space [20, Corollary 2.5]. So, from (2), it may be natural to expect that $\dim V^{even} = \dim V^{odd} - 1$ if a pure model $M(X) = (\Lambda V, d)$ is pre-c-symplectic. But it is false (cf. Theorem 1.2 below). If anything, "it is relatively easy to construct c-symplectic Sullivan minimal models" (cf. [20, Example 2.9] [21, p. 263]) and furthermore *pre-c-symplectic spaces exist everywhere.* The latter is nearly true if we can suitably change the ratio of degrees of basis elements of V for $M(X) = (\Lambda V, d)$. For example, for any even dimensional simply connected compact manifold B, the product space $X = B \times S^N$ for the N-dimensional sphere S^N is pre-c-symplectic for any odd integer N with $N > \dim B$. Indeed, we can put the model of (2) as $M(Y) = (\mathbb{Q}[t] \otimes \Lambda V \otimes \Lambda v, D)$ by

$$D(v) = \alpha \cdot t^{(N+1-\dim B)/2} - t^{(N+1)/2} \quad \text{and} \quad D(b) = d_B(b)$$

for $b \in M(B) = (\Lambda V, d_B)$, the fundamental class $[\alpha]$ of $H^*(B; \mathbb{Q})$ and $M(S^N) = (\Lambda v, 0)$ with $|v| = N$. Then

$$H^*(Y; \mathbb{Q}) = H^*(B; \mathbb{Q})[t]/(\alpha \cdot t^{(N+1-\dim B)/2} - t^{(N+1)/2})$$

and $[t]^{(\dim B+N-1)/2} = [\alpha \cdot t^{(N-1)/2}] \neq 0$. Since $fd(Y) = \dim B + N - 1$, we see Y is c-symplectic, that is, X is pre-c-symplectic. In general, it seems difficult to find the smallest N such that X is pre-c-symplectic. This is a symbolic example in this paper.

We will study the conditions of spaces to be pre-c-symplectic, especially in the most rational homotopically simple case, that is, we suppose that a finite simply connected complex X has the rational cohomology structure of the exterior algebra over $\mathbb{Q}$:

$$H^*(X;\mathbb{Q}) \cong \Lambda(v_1, v_2, \dots, v_n)$$

with $1 < |v_1| = k_1 \leq |v_2| = k_2 \leq \cdots \leq |v_n| = k_n$ all odd. Then X has the rational homotopy type of the n-product of simply connected odd-spheres:

$$X \simeq_{\mathbb{Q}} S^{k_1} \times S^{k_2} \times \cdots \times S^{k_n} \quad k_i;\ \text{odd}$$

($\simeq_{\mathbb{Q}}$ means "is rational homotopy equivalent to") and the Sullivan minimal model is given by

$$M(X) \cong (\Lambda(v_1, v_2, \dots, v_n), 0).$$

For example, simply connected compact Lie groups of rank n satisfy the condition (H.Hopf). In this case, (2) is written as

$$(\mathbb{Q}[t], 0) \to (\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \dots, v_n), D) \to (\Lambda(v_1, v_2, \dots, v_n), 0).$$

In this paper, we show

Theorem 1.2 *When $H^*(X;\mathbb{Q}) \cong \Lambda(v_1, v_2, \dots, v_n)$ with all $|v_i|$ odd and $1 < |v_1| \leq |v_2| \leq \cdots \leq |v_n|$, then X is pre-c-symplectic if and only if n is odd and $|v_1| + |v_{n-1}| < |v_n|$, $|v_2| + |v_{n-2}| < |v_n|, \dots, |v_{(n-1)/2}| + |v_{(n+1)/2}| < |v_n|$.*

Remark 1.3 The "*if*" part of Theorem 1.2 does not follow when $H^*(X;\mathbb{Q})$ is not free; i.e., $d \neq 0$ for $M(X) = (\Lambda(v_1, \dots, v_n), d)$. For example, when $M(X) = (\Lambda(v_1, v_2, v_3, v_4, v_5), d)$ with $|v_1| = 3$, $|v_2| = |v_3| = 5$, $|v_4| = 9$, $|v_5| = 13$, $dv_1 = dv_2 = dv_3 = dv_5 = 0$ and $dv_4 = v_2 v_3$, any model $(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3, v_4, v_5), D)$ of (2) is not pre-c-symplectic. Indeed, the element $v_1 v_4$ can not be a D-cocycle and Dv_5 can not contain the cocycle $v_i v_4 t$ for $i = 2, 3$ from degree reasons. So we can not construct the form $Dv_5 = v_a v_b t^* + v_c v_d t^* + t^7$ with $\{a, b, c, d\} = \{1, 2, 3, 4\}$. Also the "*only if*" part of Theorem 1.2 does not follow when $H^*(X;\mathbb{Q})$ is not free. For example, when $n = 3$, $|v_1| = |v_2| = 3$, $|v_3| = 5$, $dv_1 = dv_2 = 0$, $dv_3 = v_1 v_2$, the model $(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3), D)$ of (2) with $Dv_1 = Dv_2 = 0$ and $Dv_3 = v_1 v_2 + t^3$ is c-symplectic by $[t^5] \neq 0$ but $|v_1| + |v_2| > |v_3|$ (see Theorem 2.6).

Corollary 1.4 *Let X be a compact connected simple Lie group G of rank $G > 1$. Then X is pre-c-symplectic if and only if G is B_n or C_n with n odd, or E_7.*

For example, for the 5-th symplectic group $Sp(5)$, the rational cohomology is given as $H^*(Sp(5);\mathbb{Q}) = \Lambda(v_1, v_2, v_3, v_4, v_5)$ with the degrees $|v_1| = 3$, $|v_2| = 7$, $|v_3| = 11$,

$|v_4| = 15$ and $|v_5| = 19$. From Corollary 1.4, it is pre-c-symplectic. There are at least the four rational homotopy types of c-symplectic models:

(i) $Dv_5 = v_1v_4t + v_2v_3t + t^{10}$, $Dv_1 = Dv_2 = Dv_3 = Dv_4 = 0$
(ii) $Dv_5 = v_1v_4t + v_2v_3t + t^{10}$, $Dv_3 = v_1v_2t,\ Dv_4 = 0$
(iii) $Dv_5 = v_1v_4t + v_2v_3t + t^{10}$, $Dv_3 = 0,\ Dv_4 = v_1v_3t$
(iv) $Dv_5 = v_1v_4t + v_2v_3t + t^{10}$, $Dv_3 = v_1v_2t,\ Dv_4 = v_1v_3t.$

Although the cohomology algebra structures of them are very different, they are all c-symplectic with formal dimension 54. For example, the cohomology algebras of (i), (ii) and (iv) are given as

(i) $\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3, v_4)/(v_1v_4t + v_2v_3t + t^{10})$
(ii) $\mathbb{Q}[t, u_1, u_2] \otimes \Lambda(v_1, v_2, v_4)/(v_1v_4t + u_2t + t^{10}, v_2u_1 + v_1u_2, v_1v_2t, v_1u_1, v_2u_2)$
(iv) $\mathbb{Q}[t, u_1, u_2, u_3] \otimes \Lambda(v_1, v_2)/(u_2t + u_3t + t^{10}, v_2u_1 + v_1u_2, v_1v_2t, v_1u_1, v_2u_2, v_1u_3, u_1u_2, u_1u_3, u_1t)$, where $u_1 = [v_1v_3]$, $u_2 = [v_2v_3]$ and $u_3 = [v_1v_4]$.

Let $r_0(X)$ be the *rational toral rank* of X, which is the largest integer r such that an r-torus $T^r = S^1 \times \cdots \times S^1$($r$-factors) can act continuously on a space X' in the rational homotopy type of X with all its isotropy subgroups finite (almost free action) [9,10]. For example, $r_0(S^{k_1} \times \cdots \times S^{k_n}) = n$ when k_i are all odd and $r_0(\mathbb{C}P^n) = 0$. Pre-c-symplectic spaces are related to almost free toral actions. Indeed, for (1), there is a free S^1-action on a finite complex X' with $X'_{\mathbb{Q}} \simeq X_{\mathbb{Q}}$, from Halperin's Proposition 3.1 of Sect. 3. Here $X_{\mathbb{Q}}$ means the rationalization of X [11]. Thus we have the Borel fibration

$$X' \to ES^1 \times_{S^1} X' \to BS^1 \tag{3}$$

with $\dim H^*(ES^1 \times_{S^1} X'; \mathbb{Q}) < \infty$. It is rationally equivalent to (1). Namely,

Theorem 1.5 *A simply connected space X is pre-c-symplectic if and only if there is rationally an almost free circle action on X such that the orbit space is c-symplectic.*

In particular, we see that $r_0(X) > 0$ for a pre-c-symplectic space X. The being c-symplectic is surely a cohomological property. But the being pre-c-symplec depends on the dga and not simply on its cohomology. For example, when two spaces X_1 and X_2 are given by $X_1 = (S^3 \times S^8)\sharp(S^3 \times S^8)$ and $M(X_2) = (\Lambda(v_1, v_2, v_3), d)$ with $|v_1| = |v_2| = 3$, $|v_3| = 5$, $dv_1 = dv_2 = 0$ and $dv_3 = v_1v_2$, we have a graded algebra isomorphism

$$H^*(X_i; \mathbb{Q}) \cong \Lambda(x, y) \otimes \mathbb{Q}[w, u]/(xy, xu, xw + yu, yw, w^2, wu, u^2)$$

with $|x| = |y| = 3$ and $|w| = |u| = 8$ for $i = 1, 2$. When $i = 2$, $u = [v_1v_3]$ and $w = [v_2v_3]$. Recall that $r_0(X_1) = 0$ [17, Theorem 1.1(2)], so X_1 can not be pre-c-symplectic from Theorem 1.5, but X_2 is pre-c-symplectic (see Remark 1.3). The following proposition seems a special case of [21, Corollary 3.7, Theorem 5.2].

Proposition 1.6 *For a simply connected c-symplectic space Y, $r_0(Y) = 0$.*

If $ET^a \times_{T^a}^{\mu} X$ is c-symplectic for some T^a-action μ, then ($ET^{a-1} \times_{T^{a-1}}^{\tau} X$ is pre-c-symplectic for any restriction τ on T^{a-1} of μ and) $ET^b \times_{T^b}^{\tau} X$ ($a \neq b$) can not be c-symplectic for any restriction or extension τ on T^b of μ from Proposition 1.6. But notice that when X or $ET^a \times_{T^a}^{\mu} X$ is pre-c-symplectic, $ET^b \times_{T^b}^{\tau} X$ ($a < b$) may be pre-c-symplectic for an extension τ. It may complicate the being pre-c-symplectic than the being c-symplectic. For example, when $X \simeq_{\mathbb{Q}} S^3 \times S^3 \times S^7$ with $M(X) = (\Lambda(v_1, v_2, v_3), 0)$, X is pre-c-symplectic since the model $(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3), D)$ of (3) is given by $Dv_1 = Dv_2 = 0$ and $Dv_3 = v_1v_2t + t^4$. Indeed, then $fd(ES^1 \times_{S^1} X) = 12$ and $[t^6] \neq 0$ (see Example 3.6). On the other hand, for any almost free T^2-action on X, the Borel space $ET^2 \times_{T^2} X$ is also pre-c-symplectic since the model of (3) is given by Proposition 3.1 as

$$(\mathbb{Q}[t_3], 0) \to (\mathbb{Q}[t_1, t_2, t_3] \otimes \Lambda(v_1, v_2, v_3), D) \to (\mathbb{Q}[t_1, t_2] \otimes \Lambda(v_1, v_2, v_3), \overline{D})$$

where $(\mathbb{Q}[t_1, t_2] \otimes \Lambda(v_1, v_2, v_3), \overline{D}) = M(ET^2 \times_{T^2} X)$ and $Dv_1 = f_1, Dv_2 = f_2, Dv_3 = f_3$ with f_1, f_2, f_3 a regular sequence in $\mathbb{Q}[t_1, t_2, t_3]$ (see Corollary 3.3). Indeed, then $fd(ET^3 \times_{T^3} X) = fd(\mathbb{Q}[t_1, t_2, t_3] \otimes \Lambda(v_1, v_2, v_3), D) = 10$ and $\omega^5 \neq 0$ for $\omega = [\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3]$ for some $\lambda_i \in \mathbb{Q}$. Especially, Proposition 1.6 does not always deduce $r_0(X) = 1$ when X is pre-c-symplectic (cf. Theorem 1.2).

Recall the *Hasse diagram $\mathcal{H}(X)$ of rational toral ranks* for a simply connected space X [31], which is the Hasse diagram of a poset induced by ordering of the Borel fibrations of rationally almost free toral actions on X. When there exists a free t-toral action on a finite complex X' of same rational homotopy with X (Proposition 3.1), we can describe a point $P = [ET^t \times_{T^t} X']$ rationally presented by the Borel space $Y = ET^t \times_{T^t} X'$ in the lattice points of the quadrant I. The coordinate is

$$P := (s, t)\ ;\ \ 0 \leq s, t, \ \ s + t \leq r_0(X)$$

when $r_0(ET^t \times_{T^t} X') = r_0(X) - s - t$. In particular, the root $(0, 0)$ is presented by X itself. There is an order $P_i < P_j$ given by the existence of a rational fibration

$$Y_1 \to Y_2 \to BT^{t_2 - t_1}$$

for $P_i = [Y_1] = (s_1, t_1)$ and $P_j = [Y_2] = (s_2, t_2)$ with $s_1 \leq s_2$ and $t_1 < t_2$. It is also realized by a $T^{t_2-t_1}$-Borel fibration (Proposition 3.1). Then $\{P_i, <\}$ makes a poset and we denote its Hasse diagram as $\mathcal{H}(X)$. It may be useful to organize knowledge about almost free toral actions (often looks like the framework of a broken Japanese fan). Now, from Proposition 1.6, we immediately obtain a necessary condition for X to be pre-c-symplectic as

Theorem 1.7 *If X is pre-c-symplectic, then there exists the point $P = (r_0(X) - 1, 1)$ in $\mathcal{H}(X)$.*

It schematically gives a necessary condition for the existence of a c-symplectic space $Y = ES^1 \times_{S^1} X'$ with $X'_{\mathbb{Q}} \simeq X_{\mathbb{Q}}$, in all classes (associated with rational toral ranks) of orbit spaces of rational almost free toral actions on X. When X is pre-c-symplectic, the points $(r_0(X)-i, i)$ of $\mathcal{H}(X)$, i.e., the leaves of the Hasse diagram, may be presented by c-symplectic models. For example, the point $(0, 3)$ is surely presented by them when $X \simeq_{\mathbb{Q}} S^3 \times S^3 \times S^7$ as we see in above. Also see Examples 3.7 and 3.8. When a pre-c-symplectic space X is a product of n odd-spheres, we can easily check that there are at least the points $(2, 1), (2, 2), \dots, (2, n-2)$ in $\mathcal{H}(X)$. When a c-symplectic space is a homogeneous space as in [20], it presents the point $(0, r_0(X))$ of $\mathcal{H}(X)$ for some pure space X with $\pi_2(X) \otimes \mathbb{Q} = 0$ (see Remark 3.9). On the other hand, any c-symplectic space Y presents $(r_0(X)-1, 1)$ of $\mathcal{H}(X)$ for some pre-c-symplectic space X with $\dim \pi_2(X) \otimes \mathbb{Q} = \dim \pi_2(Y) \otimes \mathbb{Q} - 1$.

Remark 1.8 The converse of Theorem 1.7 is not true. For example, put $X = S^3 \times S^3 \times S^9 \times S^{11} \times S^{13} \times S^{15} \times S^{19}$, which is not pre-c-symplectic from Theorem 1.2 since $k_3 + k_4 = 9 + 11 > 19 = k_7$ $(n = 7)$. But there is a point $P = (r_0(X)-1, 1) = (6, 1)$ in $\mathcal{H}(X)$ presented by a model $(\mathbb{Q}[t] \otimes \Lambda(v_1, \dots, v_7), D)$ with the differential $Dv_1 = \cdots = Dv_4 = 0$, $Dv_5 = v_2v_3t$, $Dv_6 = v_1v_4t$, $Dv_7 = v_1v_6t + v_2v_5t^2 + t^{10}$ in (4) for $H^*(X;\mathbb{Q}) = \Lambda(v_1, \dots, v_7)$ with $|v_1| = |v_2| = 3$, $|v_3| = 9$, $|v_4| = 11$, $|v_5| = 13$, $|v_6| = 15$ and $|v_7| = 19$. We can directly check $r_0(\mathbb{Q}[t] \otimes \Lambda(v_1, \dots, v_7), D) = 0$ from Proposition 3.1.

This paper is purely a Sullivan model approach to the opening question restricted on c-symplectic structures in the simply connected case. Then we see that the ratio of degrees in elliptic model structure (homotopy rank type [25]) play an important role to be pre-c-symplectic. It consists of three sections. In Sect. 2, we give the proof of Theorem 1.2 and see some related topics. In particular, we see in Theorem 2.6 that a space is pre-c-symplectic imposes a restrict on the degrees when its rational homotopy group is finite oddly generated. In Sect. 3, we prove Proposition 1.6 under a Halperin's criterion (Proposition 3.1) and see some examples of $\mathcal{H}(X)$ when X is pre-c-symplectic in the cases of $r_0(X) \leq 5$.

2 Proof and related topics

In the following Lemmas 2.1 and 2.2, we assume that $M(X) = (\Lambda(v_1, v_2, \dots, v_n), d)$ where $|v_i| = k_i$ are odd for all i and $1 < k_1 \leq \cdots \leq k_n$ for an odd integer n. The symbol $(f_1, \dots, f_k)$ means the ideal of $\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \dots, v_n)$ generated by elements $f_1, \dots, f_k$ and '$f \sim g$' means the D-cocycles f and g are cohomologuous in $(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \dots, v_n), D)$ of (2); i.e., $[f] = [g]$ in $H^*(Y;\mathbb{Q})$.

Lemma 2.1 *If* $(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \dots, v_n), D)$ *is c-symplectic, then we can put* D *up to dga-isomorphisms so that*

(i) $Dv_i \in (v_1, \dots, v_{i-1})$ *for all* $i < n$,
(ii) $Dv_n = f - \lambda t^{(k_n+1)/2}$ *for some* $f \in (v_1, v_2, \dots, v_{n-1})$ *and* $\lambda \neq 0 \in \mathbb{Q}$,
(iii) $v_1v_2 \dots v_{n-1} \cdot t^{(k_n-1)/2} \sim \lambda t^{(fd(X)-1)/2}$ *for some* $\lambda \neq 0 \in \mathbb{Q}$.

Proof (i) Suppose that there is an element v_i with $i < n$ such that $Dv_i = g - \lambda t^{(k_i+1)/2}$ for some $g \in (v_1, \dots, v_{i-1})$ and $\lambda \neq 0 \in \mathbb{Q}$. Then $\dim H^*(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \dots, v_i), D) < \infty$ and $fd(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \dots, v_i), D) = k_1 +$

$\cdots + k_i - 1$ [8]. Therefore we deduce $t^{a/2+1} \sim 0$; i.e., $[t^{a/2+1}] = 0$ for some $a < fd(X) - 1 = k_1 + \cdots + k_n - 1$. It contradicts the definition of a c-symplectic space.

(ii) It is required from (i) and $\dim H^*(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \ldots, v_n), D) < \infty$.

(iii) The element $v_1 v_2 \ldots v_{n-1}$ is a D-cocycle from $Dv_1 = Dv_2 = 0$ and (i). It is not D-exact from (ii). Then we have $[v_1 v_2 \ldots v_{n-1}] \cdot [t^a] = \lambda[t^{(fd(X)-1)/2}]$ in $H^*(\mathbb{Q}[t] \otimes \Lambda(v_1, \ldots, v_n), D)$ for $a = (fd(X) - 1 - k_1 - \cdots - k_{n-1})/2 = (k_n - 1)/2$ from the Poincaré duality property. □

Lemma 2.2 *Suppose that* $(\mathbb{Q}[t]\otimes\Lambda(v_1, v_2, \ldots, v_n), D)$ *satisfies* $Dv_n = f - t^{(|v_n|+1)/2}$ *for some* $f = g_1 t^{a_1} + \cdots + g_k t^{a_k}$ *with monomials* $g_i \in \Lambda(v_1, \ldots, v_{n-1})$ *and* $a_i \geq 0$. *If it is c-symplectic, then* $g_{i_1} \ldots g_{i_m} \neq 0 \in (v_1 v_2 \ldots v_{n-1})$ *for some* $g_{i_1}, \ldots, g_{i_m}$ $(m \leq k)$.

Proof From the assumption, for $M := (|v_n| + 1)/2$, we have

$$g_1 t^{a_1} + \cdots + g_k t^{a_k} \sim t^M.$$

Suppose $g_{i_1} \ldots g_{i_m} \neq 0$. By the multiplication of $t^{M-a_{i_1}}$ on the both sides, we have

$$g_{i_1} g_{i_2} t^{a_{i_2}} + \cdots = g_{i_1}(g_1 t^{a_1} + \cdots + g_k t^{a_k}) + \cdots \sim g_{i_1} t^M + \cdots \sim t^{2M-a_{i_1}}.$$

Again by the multiplication of $t^{M-a_{i_2}}$ on the both sides, we have

$$g_{i_1} g_{i_2} g_{i_3} t^{a_{i_3}} + \cdots \sim t^{3M-a_{i_1}-a_{i_2}}.$$

Iterate the multiplication of $t^{M-a_{i_j}}$ to $j = m - 1$. Then we have

$$g_{i_1} g_{i_2} \ldots g_{i_m} t^{a_{i_m}} + \cdots \sim t^{mM-a_{i_1}-\cdots-a_{i_{m-1}}}.$$

Finally we have

$$g_{i_1} g_{i_2} \ldots g_{i_m} t^{M-1} + \cdots \sim t^{(m+1)M-a_{i_1}-\cdots-a_{i_m}-1} = t^{(|g_{i_1}|+\cdots+|g_{i_m}|+|v_n|-1)/2}.$$

If $g_{i_1} \ldots g_{i_m} = \lambda v_1 v_2 \ldots v_{n-1}$ for some $\lambda \neq 0 \in \mathbb{Q}$, then

$$(\lambda + \cdots) v_1 v_2 \ldots v_{n-1} t^{M-1} \sim t^{(k_1+k_2+\cdots+k_n-1)/2} = t^{(fd(X)-1)/2}$$

and it makes a non-zero class of $H^{fd(X)-1}(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \ldots, v_n), D)$ when $\lambda + \cdots \neq 0$. If there are no such elements $g_{i_1}, g_{i_2}, \ldots, g_{i_m}$, then $(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \ldots, v_n), D)$ is not c-symplectic from Lemma 2.1(iii). □

Proof of Theorem 1.2. The "*if*" part: We can define the model $(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \ldots, v_n), D)$ of (2) by putting $Dv_1 = \cdots = Dv_{n-1} = 0$ and

$$Dv_n = v_1 v_{n-1} t^{a_1} + v_2 v_{n-2} t^{a_2} + \cdots + v_{(n-1)/2} v_{(n+1)/2} t^{a_{n-1}} - t^{a_n}$$

for suitable a_i. Then $v_1 v_{n-1} t^{a_1} + v_2 v_{n-2} t^{a_2} + \cdots + v_{(n-1)/2} v_{(n+1)/2} t^{a_{n-1}} \sim t^{a_n}$ deduces, by iterated multiplications of t,

$$v_1 \cdots v_{n-1} t^{(k_n - 1)/2} \sim t^{(\dim X - 1)/2},$$

where the left side is not D-exact. Thus $(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \ldots, v_n), D)$ is c-symplectic

The "*only if*" part: From Lemma 2.1(ii), we can put

$$Dv_n = \sum_{i=1}^{r} g_i t^{n_i} - t^{(k_n - 1)/2}$$

with $g_1, \ldots, g_r$ some monomials in $\Lambda(v_1, \ldots, v_{n-1})$ and $n_i = (|v_n| - |g_i| + 1)/2$. From Lemma 2.2, there is the set

$$S := \{ v_{i_1}, v_{j_1}, \ldots, v_{i_{(n-1)/2}}, v_{j_{(n-1)/2}} \}$$

such that $S = \{v_1, \ldots, v_{n-1}\}$ and that there are indexes l_k for $k = 1, \ldots, (n-1)/2$ such that g_{l_k} contains the term $v_{i_k} v_{j_k}$; i.e., $g_{l_k} \in (v_{i_k} v_{j_k})$. Then

$$|v_{i_k}| + |v_{j_k}| = |v_{i_k} v_{j_k}| \le |g_{l_k}| < |v_n|$$

for $k = 1, \ldots, (n-1)/2$. From Proposition 2.4 below, we have $|v_1| + |v_{n-1}| < |v_n|$, $|v_2| + |v_{n-2}| < |v_n|, \cdots$ and $|v_{(n-1)/2}| + |v_{(n+1)/2}| < |v_n|$. □

Lemma 2.3 *Let $S = \{a_1, a_2, \ldots, a_{2n}\}$ be a set of real numbers with $a_1 \le a_2 \le \cdots \le a_{2n}$. For any partition*

$$\mathcal{T} = \{\{a_{i_1}, a_{j_1}\}, \{a_{i_2}, a_{j_2}\}, \ldots, \{a_{i_n}, a_{j_n}\}\}$$

of S into 2-subsets, where $i_k, j_k \in \{1, 2, \ldots, 2n\}$ and $i_k \neq j_k$ for $k = 1, 2, \ldots, n$, there exists an element $\{a_{i_k}, a_{j_k}\}$ of $\mathcal{T}$ such that

$$\begin{cases} a_1 + a_{2n} \le a_{i_k} + a_{j_k} \\ a_2 + a_{2n-1} \le a_{i_k} + a_{j_k} \\ \quad \cdots \\ a_n + a_{n+1} \le a_{i_k} + a_{j_k}. \end{cases}$$

Proof We show the result by induction on the positive integer n. For $n = 1$, the statement is true since $a_1 + a_2 \le a_1 + a_2$. Assume the statement is true for $n - 1$. We must prove the assertion is also true for n. Let

$$\mathcal{T} = \{\{a_{i_1}, a_{j_1}\}, \{a_{i_2}, a_{j_2}\}, \ldots, \{a_{i_n}, a_{j_n}\}\}$$

be any partition of S into 2-subsets and let $\{a_i, a_{2n}\}(1 \le i \le 2n-1)$ be an element of $\mathcal{T}$ containing a_{2n}.

Case of $a_n \le a_i$. Then we have

$$\begin{cases} a_1 + a_{2n} \le a_n + a_{2n} \le a_i + a_{2n} \\ a_2 + a_{2n-1} \le a_n + a_{2n} \le a_i + a_{2n} \\ \quad \cdots \\ a_n + a_{n+1} \le a_n + a_{2n} \le a_i + a_{2n}, \end{cases}$$

hence we may take $\{a_{i_k}, a_{j_k}\}$ as $\{a_i, a_{2n}\}$.

Case of $a_i \le a_{n-1}$. Then we have

$$\begin{cases} a_1 + a_{2n} \le a_i + a_{2n} \\ a_2 + a_{2n-1} \le a_i + a_{2n} \\ \quad \cdots \\ a_i + a_{2n+1-i} \le a_i + a_{2n}. \end{cases} \tag{$*$}$$

We consider $\mathcal{T}' = \mathcal{T} \backslash \{a_i, a_{2n}\}$. Since $\sharp\mathcal{T}' = n-1$ ($\sharp$ denotes the cardinality of a set), we can apply the induction hypothesis to $\mathcal{T}'$. Since $a_1 \le a_2 \le \cdots \le a_{i-1} \le a_{i+1} \le \cdots \le a_{2n-1}$, there exsits an element $\{a_{i_k}, a_{j_k}\}$ of $\mathcal{T}'$ such that

$$\begin{cases} a_1 + a_{2n} \le a_{i_k} + a_{j_k} \\ a_2 + a_{2n-1} \le a_{i_k} + a_{j_k} \\ \quad \cdots \\ a_{i-1} + a_{2n-i+1} \le a_{i_k} + a_{j_k} \\ a_{i+1} + a_{2n-i} \le a_{i_k} + a_{j_k} \\ \quad \cdots \\ a_n + a_{n+1} \le a_{i_k} + a_{j_k}. \end{cases} \tag{$**$}$$

From ($*$) and ($**$), we conclude that

$$\begin{cases} a_1 + a_{2n} \le a_i + a_{2n} \\ a_2 + a_{2n-1} \le a_i + a_{2n} \\ \quad \cdots \\ a_{i-1} + a_{2n-i+1} \le a_i + a_{2n} \\ a_{i+1} + a_{2n-i} \le a_{i_k} + a_{j_k} \\ \quad \cdots \\ a_n + a_{n+1} \le a_{i_k} + a_{j_k}. \end{cases}$$

If we put $Max\{a_i, +a_{2n}, a_{i_k} + a_{j_k}\} = a_s + a_t$, then $\{a_s, a_t\}$ satisfies the desired inequality. □

From this lemma, we have immediately

Proposition 2.4 (cf. [26, Proposition 1.1]) *Let $S = \{a_1, a_2, \ldots, a_{2n}\}$ be a set of positive integers with $a_1 \le a_2 \le \cdots \le a_{2n}$. Assume that there exsits a positive integer N such that*

$$\begin{cases} a_{i_1} + a_{j_1} \le N \\ a_{i_2} + a_{j_2} \le N \\ \quad \cdots \\ a_{i_n} + a_{j_n} \le N \end{cases}$$

for a partition

$$\mathcal{T} = \{\{a_{i_1}, a_{j_1}\}, \{a_{i_2}, a_{j_2}\}, \ldots, \{a_{i_n}, a_{j_n}\}\}$$

of S into 2-subsets, where $i_k, j_k \in \{1, 2, \ldots, 2n\}$ and $i_k \neq j_k$ for $k = 1, 2, \ldots, n$. Then we have the following inequality:

$$\begin{cases} a_1 + a_{2n} \le N \\ a_2 + a_{2n-1} \le N \\ \quad \cdots \\ a_n + a_{n+1} \le N. \end{cases}$$

In [26], we can see various versions of Proposition 2.4.

From the proof of Lemma 2.2, we have

Proposition 2.5 *Suppose that $M(X) = (\Lambda(v_1, v_2, \ldots, v_n), d)$ with all $|v_i|$ odd and that $(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, \ldots, v_n), D)$ satisfies $Dv_n = f - t^{(|v_n|+1)/2}$ for some $f = g_1 t^{a_1} + \cdots + g_k t^{a_k}$ with monomials $g_j = \lambda_j v_{j_1} \cdots v_{j_{m_j}} \in \Lambda(v_1, \ldots, v_{n-1})$, $\lambda_j \neq 0 \in \mathbb{Q}$ and $a_j \geq 0$. If $\prod_{j=1}^{k} v_{j_1} \cdots v_{j_{m_j}} \neq 0 \in (v_1 v_2 \cdots v_{n-1})$, then it is c-symplectic.*

From the proof of the "*only if*" part of Theorem 1.2, we have

Theorem 2.6 *Suppose that $M(X) = (\Lambda(v_1, v_2, \ldots, v_n), d)$ with all $|v_i|$ odd and $1 < |v_1| \le |v_2| \le \cdots \le |v_n|$. If X is pre-c-symplectic, then n is odd and $|v_1|+|v_{n-1}| \le |v_n| + 1$, $|v_2| + |v_{n-2}| \le |v_n| + 1, \ldots, |v_{(n-1)/2}| + |v_{(n+1)/2}| \le |v_n| + 1$.*

Question 2.7 *What is the necessary and sufficient condition for a model $(\Lambda(v_1, v_2, \ldots, v_n), d)$ with all $|v_i|$ odd to be pre-c-symplectic?*

Proof of Corollary 1.4 The rational types of compact connected simple Lie groups are given as

$$\begin{array}{ll} A_n & (3, 5, \ldots, 2n + 1), \\ B_n & (3, 7, \ldots, 4n - 1), \\ C_n & (3, 7, \ldots, 4n - 1), \\ D_n & (3, 7, \ldots, 4n - 5, 2n - 1), \\ G_2 & (3, 11), \\ F_4 & (3, 11, 15, 23), \\ E_6 & (3, 9, 11, 15, 17, 23), \\ E_7 & (3, 11, 15, 19, 23, 27, 35), \\ E_8 & (3, 15, 23, 27, 35, 39, 47, 59) \end{array}$$

(see [23]). For A_n, even if n is odd, we have $3 + (2n-1) = 2n+1$, which does not satisfy the condition of Theorem 1.2. It is obvious that B_n (C_n) and E_7 satisfy the condition of Theorem 1.2 as

$$3 + 4(n-1) - 1 < 4n-1,\ 7 + 4(n-2) - 1 < 4n-1, \dots, (2n-3) \\ +(2n+1) < 4n-1 \quad \text{and} \quad 3+27 < 35,\ 11+23 < 35,\ 15+19 < 35,$$

respectively. Since the ranks of G_2, F_4, E_6 and E_8 are even, they are not pre-c-symplectic. Finally we check D_n. Put an odd integer $n = 2k+1 (k \geq 1)$. Assume there is an integer N as in Proposition 2.4 for the set $S = \{3, 7, \dots, 8k-5, 4k+1\}$. Then $N = 4n-5 = 4(2k+1)-5 = 8k-1$. Sorting elements of S into increasing order, we have

$$a_1 = 3 \leq a_2 = 7 \leq \cdots \leq a_k = 4k-1 \leq a_{k+1} = 4k+1 \leq a_{k+2} = 4k+3 \\ \leq \cdots \leq a_{2k-1} = 8k-9 \leq a_{2k} = 8k-5.$$

Then $a_k + a_{k+1} = (4k-1) + (4k+1) = 8k > N$. It contradicts Proposition 2.4. Therefore, Theorem 1.2 does not hold for D_n. □

Example 2.8 Even when a space X is a product of odd-spheres, the c-symplectic spaces whose pre-c-symplectic space is X are various. For example, when $X = S^3 \times S^5 \times S^9 \times S^{15} \times S^{33}$, there are at least the following twenty rational homotopy types of c-symplectic models with the differential $Dv_1 = Dv_2 = 0$ and

(1) $Dv_5 = v_1v_4t^8 + v_2v_3t^{10} + t^{17},\ Dv_3 = Dv_4 = 0$
(2) $Dv_5 = v_1v_4t^8 + v_2v_3t^{10} + t^{17},\ Dv_3 = 0,\ Dv_4 = v_1v_2t^4$
(3) $Dv_5 = v_1v_4t^8 + v_2v_3t^{10} + t^{17},\ Dv_3 = 0,\ Dv_4 = v_1v_3t^2$
(4) $Dv_5 = v_1v_4t^8 + v_2v_3t^{10} + t^{17},\ Dv_3 = v_1v_2t,\ Dv_4 = 0$
(5) $Dv_5 = v_1v_4t^8 + v_2v_3t^{10} + t^{17},\ Dv_3 = v_1v_2t,\ Dv_4 = v_1v_3t$
(6) $Dv_5 = v_1v_2t^{13} + v_3v_4t^5 + t^{17},\ Dv_3 = Dv_4 = 0$
(7) $Dv_5 = v_1v_2t^{13} + v_3v_4t^5 + t^{17},\ Dv_3 = 0,\ Dv_4 = v_1v_3t^2$
(8) $Dv_5 = v_1v_2t^{13} + v_3v_4t^5 + t^{17},\ Dv_3 = 0,\ Dv_4 = v_2v_3t$
(9) $Dv_5 = v_1v_3t^{11} + v_2v_4t^7 + t^{17},\ Dv_3 = Dv_4 = 0$
(10) $Dv_5 = v_1v_3t^{11} + v_2v_4t^7 + t^{17},\ Dv_3 = 0,\ Dv_4 = v_1v_2t^4$
(11) $Dv_5 = v_1v_3t^{11} + v_2v_4t^7 + t^{17},\ Dv_3 = 0,\ Dv_4 = v_2v_3t$
(12) $Dv_5 = v_1v_3t^{11} + v_2v_4t^7 + t^{17},\ Dv_3 = v_1v_2t,\ Dv_4 = 0$
(13) $Dv_5 = v_1v_3t^{11} + v_2v_4t^7 + t^{17},\ Dv_3 = v_1v_2t,\ Dv_4 = v_2v_3t$
(14) $Dv_5 = v_1v_2v_3v_4t + t^{17},\ Dv_3 = Dv_4 = 0$
(15) $Dv_5 = v_1v_2v_3v_4t + t^{17},\ Dv_3 = 0,\ Dv_4 = v_1v_2t^4$
(16) $Dv_5 = v_1v_2v_3v_4t + t^{17},\ Dv_3 = 0,\ Dv_4 = v_1v_3t^2$
(17) $Dv_5 = v_1v_2v_3v_4t + t^{17},\ Dv_3 = 0,\ Dv_4 = v_2v_3t$
(18) $Dv_5 = v_1v_2v_3v_4t + t^{17},\ Dv_3 = v_1v_2t,\ Dv_4 = 0$
(19) $Dv_5 = v_1v_2v_3v_4t + t^{17},\ Dv_3 = v_1v_2t,\ Dv_4 = v_1v_3t^2$
(20) $Dv_5 = v_1v_2v_3v_4t + t^{17},\ Dv_3 = v_1v_2t,\ Dv_4 = v_2v_3t$

for $|v_1| = 3, |v_2| = 5, |v_3| = 9, |v_4| = 15, |v_5| = 33$. Note that only (1), (6), (9) and (14) are two stage models and formal; i.e., the minimal model is formally constructed from its cohomology [8,20]. Note that (1)–(20) make a poset structure as in [32]. For example, we have "(5) < (3) < (1) < (14) < (0)" where the maximal element (0) is given by $Dv_1 = \cdots = Dv_5 = 0$ (the model of X). For a product $S^{k_1} \times S^{k_2} \times S^{k_3} \times S^{k_4} \times S^{k_5}$ of odd spheres with $k_1 \leq \cdots \leq k_5$, the inequations that

$$k_1 + k_2 < k_3, \quad k_2 + k_3 < k_4, \quad k_1 + k_2 + k_3 + k_4 < k_5$$

make the most c-symplectic models. Conversely, when

$$k_1 + k_2 > k_4, \quad k_2 + k_4 > k_5$$

the c-symplectic model is uniquely determined up to dga-isomorphism. For example, when $(k_1, \ldots, k_5) = (3, 5, 5, 7, 11)$,

$$Dv_1 = \cdots = Dv_4 = 0, \quad Dv_5 = v_1v_4t + v_2v_3t + t^6.$$

Remark 2.9 Put the set C-Symp(X) := {rational homotopy types of c-symplectic spaces in (1) with the fibre X}. Then C-Symp$(X) = \phi$ if X is not pre-c-symplectic. For example, $\sharp$C-Symp$(S^{k_1} \times S^{k_2} \times S^{k_3}) \leq 1$ when k_i are odd, $\sharp$C-Symp$(Sp(5)) \geq 4$ (see §1) and $\sharp$C-Symp$(S^3 \times S^5 \times S^9 \times S^{15} \times S^{33}) \geq 20$ (see Example 2.8). When Y is c-symplectic and X is pre-c-symplectic, $Y \times X$ is pre-c-symplectic and there is an inclusion C-Symp$(X) \subset$ C-Symp$(Y \times X)$ as sets. For example, C-Symp$(S^3) = \{S^2_{\mathbb{Q}}\}$ (one point) and C-Symp$(S^2 \times S^3)$ is

$$\{(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3), D_a)\ ;\ D_a v_1 = 0, D_a v_2 = tv_1, D_a v_3 = v_1^2 + at^2,\ a \in \mathbb{Q}^*\}/\simeq$$

$\cong \mathbb{Q}^*/\mathbb{Q}^{*2}$ for $\mathbb{Q}^* := \mathbb{Q} - 0$, $|v_1| = 2$ and $|v_2| = |v_3| = 3$ as a set [24], which is infinite. Also we can give an equivalence relation in the rational homotopy types of simply connected c-symplectic spaces, that is, put $Y \sim Y'$ for two c-symplectic spaces Y and Y' when there are certain finite maps

$$Y \leftarrow X_1 \rightarrow Y_1 \leftarrow X_2 \rightarrow \cdots \rightarrow Y_{n-1} \leftarrow X_n \rightarrow Y'$$

which are fibre inclusions of (1) (Y_i are c-symplectic). It satisfies the laws of reflectance, symmetry and transitivity. For example, the models (1),. . .,(20) in Example 2.8 are all equivalent.

Remark 2.10 Recall the rational LS category $\mathrm{cat}_0(Y)$ of a simply connected space Y [8,27]. It is equal to the Toomer's invariant of Y (the biggest s for which there is a non trivial class in $H^*(Y;\mathbb{Q}) = H^*(\Lambda W)$ represented by a cycle in $\Lambda^{\geq s} W$) when Y is a rationally Poincaré duality space(r.P.d.s.) [7]. For a simply connected space X with $\dim H^*(X;\mathbb{Q}) < \infty$, put

$$c(X) = \sup\left\{ \frac{2\mathrm{cat}_0(Y)}{fd(X) - 1} \mid \text{fibrations } X \to Y \to K(\mathbb{Z}, 2) \text{ where } Y \text{ are r.P.d.s.} \right\},$$

where $c(X) := 0$ if no such space Y exists for X. Then $c(X)$ is a rational number with $0 \leq c(X) \leq 1$. In particular, (i) $c(X) = 0$ if X is c-symplectic, (ii) $c(X) = 1$ if and only if X is pre-c-symplectic and (iii) $c(X) \leq c(X \times Y)$ for any c-symplectic space Y. For example, when $X_n = S^7 \times S^7 \times S^{2n+1}$, $c(X_n)$ is given as

n	1	2	3	4	5	6	7	8	9	$\cdots$
$c(X_n)$	$\frac{5}{8}$	$\frac{5}{9}$	$\frac{1}{2}$	$\frac{6}{11}$	$\frac{7}{12}$	$\frac{8}{13}$	1	1	1	$\cdots$

When $X_n = S^3 \times S^{2n}$, $c(X_n) = 2/(n+1)$ and $\lim_n c(X_n) = 0$. When $X_n = S^3 \times S^{2n+1}$, $c(X_n) = (2n+2)/(2n+3)$. Though X_n is not pre-c-symplectic for any n, we have $\lim_n c(X_n) = 1$.

Example 2.11 For any product of odd-spheres $X = S^{k_1} \times \cdots \times S^{k_n}$ with n odd and $k_1 \leq \cdots \leq k_n$, the product $X \times \mathbb{C}P^N$ is pre-c-symplectic if $k_1 + k_{n-1} \leq 2N$, $k_2 + k_{n-2} \leq 2N, \cdots, k_{(n-1)/2} + k_{(n+1)/2} \leq 2N$ and $k_n \leq 2N+1$. Indeed, we can put $Dx = Dv_1 = \cdots = Dv_{n-1} = 0$, $Dv_n = x^{(k_n-1)/2}t$ and

$$Dy = x^{N+1} + v_1 v_{n-1} t^* + \cdots + v_{(n-1)/2} v_{(n+1)/2} t^* + t^{N+1}$$

for $M(\mathbb{C}P^N) = (\Lambda(x, y), d)$ with $|x| = 2$, $dx = 0$ and $dy = x^{N+1}$. Then $[t^a] \neq 0$ for $a = (k_1 + \cdots + k_n - 1)/2 + N$.

Remark 2.12 What additional properties of a c-symplectic space Y (or model $M(Y)$) can be deduced from the pre-c-symplectic space X in (1)? A c-symplectic space Y of $fd(Y) = 2m$ is said that it satisfies the hard Lefschetz condition with respect to the c-symplectic class t when the maps

$$\cup t^k : H^{m-k}(Y; \mathbb{Q}) \to H^{m+k}(Y; \mathbb{Q}) \quad 1 \leq k \leq m$$

are isomorphisms [29]. For example, a compact Kähler manifold satisfies the hard Lefschetz condition [29] [9, Theorem 4.35]. As well as when $(\mathbb{Q}[t] \otimes \Lambda V, D)$ of (2) is c-symplectic, whether or not it satisfies the hard Lefschetz condition depends on D. For example, when $H^*(X; \mathbb{Q}) = \Lambda(v_1, v_2, v_3, v_4, v_5)$ with $|v_1| = |v_2| = 3$, $|v_3| = |v_4| = 5$ and $|v_5| = 11$, put $Dv_1 = \cdots = Dv_4 = 0$ and

(a) $Dv_5 = v_1 v_2 t^3 + v_3 v_4 t + t^6$
(b) $Dv_5 = v_1 v_4 t^2 + v_2 v_3 t^2 + t^6$,

which are both c-symplectic with $m = 13$. Then (a) satisfies the hard Lefschetz condition but (b) does not. Indeed,

Case of (a) When $k = 10$, $Ker(\cup t^{10} : H^3(Y; \mathbb{Q}) \to H^{23}(Y; \mathbb{Q})) = 0$ since $[v_1 t^{10}] = -[v_1(v_1 v_2 t^3 + v_3 v_4 t)t^4] = -[v_1 v_3 v_4 t^5] \neq 0$. When $k = 8$, $Ker(\cup t^8 : H^5(Y; \mathbb{Q}) \to H^{21}(Y; \mathbb{Q})) = 0$ since $[v_3 t^8] = -[v_3(v_1 v_2 t^3 + v_3 v_4 t)t^2] = -[v_1 v_2 v_3 t^5] \neq 0$. When $k \neq 8,\ 10$, we can easily check $Ker(\cup t^k) = 0$.

Case of (b) When $k = 10$, $Ker(\cup t^{10} : H^3(Y; \mathbb{Q}) \to H^{23}(Y; \mathbb{Q})) \neq 0$. Indeed, $[v_1] \in Ker(\cup t^{10})$ since

$$
\begin{aligned}
[v_1 t^{10}] &= -[v_1(v_1 v_4 t^2 + v_2 v_3 t^2) t^4] = -[v_1 v_2 v_3 t^6] \\
&= [v_1 v_2 v_3 (v_1 v_4 t^2 + v_2 v_3 t^2)] = 0.
\end{aligned}
$$

Remark 2.13 When a map $g : (Y_1, w_1) \to (Y_2, w_2)$ between simply connected c-symplectic spaces induces $H^*(g)(w_2) = w_1$; i.e., a *c-symplectic map*, there is a map between fibrations:

$$
\begin{array}{ccccc}
X_1 & \longrightarrow & Y_1 & \longrightarrow & K(\mathbb{Z}, 2) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \| \\
X_2 & \longrightarrow & Y_2 & \longrightarrow & K(\mathbb{Z}, 2)
\end{array}
$$

where $f : X_1 \to X_2$ is the induced map between pre-c-symplectic spaces. Conversely, when is a map $f : X_1 \to X_2$ between pre-c-symplectic spaces extended to a c-symplectic map; i.e., a *pre-c-symplectic map*? Refer [27] in the case of self homotopy equivalences.

3 Rational toral ranks

If an r-torus T^r acts on a simply connected space X by $\mu : T^r \times X \to X$, there is the Borel fibration

$$X \to ET^r \times_{T^r} X \to BT^r,$$

where $ET^r \times_{T^r} X$ is the orbit space of the action $g(e, x) = (e \cdot g^{-1}, g \cdot x)$ on the product $ET^r \times X$ for $g \in T^r$. Note that $ET^r \times_{T^r} X$ is rational homotopy equivalent to the T^r-orbit space of X when μ is an almost free toral action [9]. The above Borel fibration is rationally given by the KS model

$$(\mathbb{Q}[t_1, \ldots, t_r], 0) \to (\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V, D) \to (\Lambda V, d) \tag{4}$$

where with $|t_i| = 2$ for $i = 1, \ldots, r$, $Dt_i = 0$ and $Dv \equiv dv$ modulo the ideal $(t_1, \ldots, t_r)$ for $v \in V$. It is a generalization of (2). Recall Halperin's

Proposition 3.1 [10, Proposition 4.2] *Suppose that X is a simply connected CW-complex with* $\dim H^*(X; \mathbb{Q}) < \infty$. *Put $M(X) = (\Lambda V, d)$. Then $r_0(X) \geq r$ if and only if there is a KS model* (4) *satisfying* $\dim H^*(\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V, D) < \infty$. *Moreover, if $r_0(X) \geq r$, then T^r acts freely on a finite complex X' that has the same rational homotopy type as X and $M(ET^r \times_{T^r} X') \cong (\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V, D)$.*

Proof of Proposition 1.6 Put the formal dimension of Y as $2n$. Then there is an element $[\omega] \in H^2(Y; \mathbb{Q})$ with $[\omega]^n \neq 0$. Suppose $r_0(Y) > 0$. From Proposition 3.1, there is a finite complex Y' with $Y'_{\mathbb{Q}} \simeq Y_{\mathbb{Q}}$ and there is a free S^1-action on Y'. Thus we have the Borel fibration $Y' \overset{i}{\to} ES^1 \times_{S^1} Y' \to BS^1$, where $[\omega]$ is a restriction of an element $[u]$ of $H^2(ES^1 \times_{S^1} Y'; \mathbb{Q})$; i.e., $i^*([u]) = [w]$. Since the formal dimension of $ES^1 \times_{S^1} Y'$ is $2n - 1$, we have $[u]^n = 0$. This is a contradiction. □

Recall the following proposition induced by [13, Lemma 2.12].

Proposition 3.2 [33, Lemma 2.1] *When X is the product of n odd-spheres, the second row of $\mathcal{H}(X)$ is empty, that is, there is no point $P = (1, *)$ in $\mathcal{H}(X)$ for $* = 1, 2, \ldots, n-1$.*

Corollary 3.3 *For a fibration $S^{k_1} \times \cdots \times S^{k_n} \to X \to \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$ ($n-1$-factors) with $k_1, \ldots, k_n$ odd, X is pre-c-symplectic if* $\dim H^*(X; \mathbb{Q}) < \infty$.

Proof Put $M(S^{k_1} \times \cdots \times S^{k_n}) = (\Lambda(v_1, \ldots, v_n), 0)$. We show that the model $M(X) = (\mathbb{Q}[t_1, \ldots, t_{n-1}] \otimes \Lambda(v_1, \ldots, v_n), D)$ is pre-c-symplectic. From Proposition 3.2 [13, Lemma 2.12], there is a KS model (2)

$$(\mathbb{Q}[t_n], 0) \to (\mathbb{Q}[t_1, \ldots, t_n] \otimes \Lambda(v_1, \ldots, v_n), D') \to (\mathbb{Q}[t_1, \ldots, t_{n-1}] \otimes \Lambda(v_1, \ldots, v_n), D)$$

such that the formal dimension of $B := (\mathbb{Q}[t_1, \ldots, t_n] \otimes \Lambda(v_1, \ldots, v_n), D')$ is $N := |v_1| + \cdots + |v_n| - n$. It is formal and the cohomology algebra is

$$\mathbb{Q}[t_1, \ldots, t_n]/(D'v_1, \ldots, D'v_n)$$

where $D'v_1, \ldots, D'v_n$ is a regular sequence in $\mathbb{Q}[t_1, \ldots, t_n]$. Then $(\lambda_1 t_1 + \cdots + \lambda_n t_n)^{N/2}$ is the fundamental class of $H^*(B)$ for an element $\lambda_1 t_1 + \cdots + \lambda_n t_n \in H^2(B)$ with $\lambda_i \in \mathbb{Q}$. □

Thus, when X is a product of n odd-spheres, the point $(0, n-1)$ in $\mathcal{H}(X)$ is surely presented by pre-c-symplectic models and the point $(0, n)$ is by c-symplectic models. In the following examples, $P_0 = (0, 0) = [X]$.

Example 3.4 For a pre-c-symplectic space X with $r_0(X) = 1$, the Hasse diagram $\mathcal{H}(X)$ is (uniquely) given as

where the point P_1 is presented by a c-symplectic model. For example, when $X = S^{2n+1}$, $P_1 = (0, 1) = [\mathbb{C}P^n]$.

When $M(X) = (\Lambda(v_1, \ldots, v_{2n+1}), d)$ with

$$dv_i = 0 \ (i < 2n+1), \quad dv_{2n+1} = v_1 \ldots v_{2j_1} + \cdots + v_{2j_{k-1}+1} \ldots v_{2j_k} \ \ (2j_k = 2n),$$

we can put $Dv_i = 0$ for $i \neq 2n+1$ and

$$Dv_{2n+1} = v_1 \ldots v_{2j_1} + \cdots + v_{2j_{k-1}+1} \ldots v_{2j_k} + t^{|v_{2n+1}|+1/2}.$$

Then it is formal and c-symplectic from Proposition 2.5.

When $M(X) = (\Lambda(v_1, \ldots, v_n), d)$ with $|v_1| = |v_2| = 3, |v_3| = 5, \ldots, |v_n| = 2n-1$ and

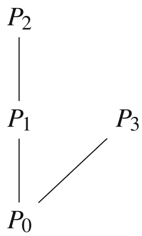
$$dv_1 = dv_2 = 0,\ dv_3 = v_1v_2,\ dv_4 = v_1v_3, \ldots, dv_n = v_1v_{n-1}$$

for an odd integer $n > 2$, we can put $Dv_i = dv_i$ for $i \neq n$ and

$$Dv_n = v_1v_{n-1} + v_2v_{n-2}t - v_3v_{n-3}t + \cdots + (-1)^a v_av_{a+1}t + t^n$$

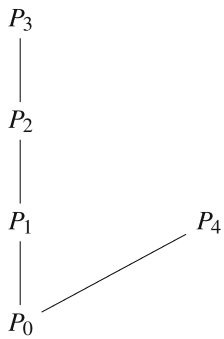
for $a = (n-1)/2$. Then $D \circ D = 0$ and it is c-symplectic from Proposition 2.5.

Example 3.5 For a pre-c-symplectic space X with $r_0(X) = 2$, the Hasse diagram $\mathcal{H}(X)$ is uniquely given as



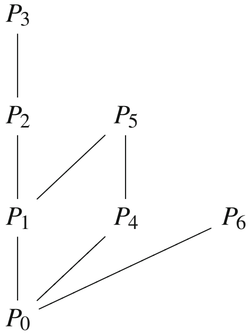
which has the point $P_3 = (1, 1)$ from Theorem 1.7. For example, it is given when $M(X) = (\Lambda(v_1, v_2, v_3, v_4, v_5), d)$ where $dv_1 = dv_2 = dv_3 = 0, dv_4 = v_1v_2$ and $dv_5 = v_1v_3$ with $|v_1| = |v_2| = 3, |v_3| = 7, |v_4| = 5, |v_5| = 9$. Then $P_2 = (0, 2) = [(\mathbb{Q}[t_1, t_2] \otimes \Lambda(v_1, v_2, v_3, v_4, v_5), D)]$ where $Dv_1 = Dv_2 = Dv_3 = 0, Dv_4 = v_1v_2 + t_1^3$ and $Dv_5 = v_1v_3 + t_2^5$. Also $P_3 = [(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3, v_4, v_5), D)]$ where $Dv_1 = Dv_2 = Dv_3 = 0, Dv_4 = v_1v_2$ and $Dv_5 = v_1v_3 + v_2v_4t + t^5$, which is c-symplectic from Proposition 2.5. Indeed, $[t^{13}] = [v_1v_2v_3v_4t^4] \neq 0$.

Example 3.6 (see [31, Examples 3.5, 3.6]) Suppose that X with $r_0(X) = 3$ is pre-c-symplectic. When $X = S^{k_1} \times S^{k_2} \times S^{k_3}$, from Theorem 1.7 and Proposition 3.2, the Hasse diagram $\mathcal{H}(X)$ is uniquely given as



which has the point $P_4 = (2, 1)$. For example, when $(k_1, k_2, k_3) = (3, 3, 7)$, $P_1 = [S^2 \times S^3 \times S^7]$, $P_2 = [S^2 \times S^2 \times S^7]$ and $P_3 = [S^2 \times S^2 \times \mathbb{C}P^3]$. Here $P_4 = (2, 1) = [Y]$ is given by the model $M(Y) = (\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3), D)$ with $Dv_1 = Dv_2 = 0$ and $Dv_3 = v_1v_2t + t^4$, which is c-symplectic.

Next put $M(X) = (\Lambda V, d) = (\Lambda(v_1, v_2, v_3, v_4, v_5), d)$ with $dv_1 = dv_2 = dv_4 = dv_5 = 0$ and $dv_3 = v_1v_2$. If $|v_1| = |v_2| = 3$, $|v_3| = 5$, $|v_4| = 9$ and $|v_5| = 13$, then $\mathcal{H}(X)$ is given as

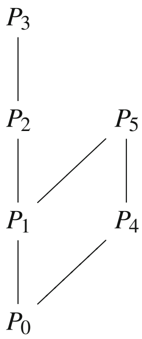


where $P_3 = [(\mathbb{Q}[t_1, t_2, t_3] \otimes \Lambda V, D)]$ with $Dv_3 = v_1v_2 + t_2^3$, $Dv_4 = t_1^5$, $Dv_5 = t_3^7$, $P_4 = [(\mathbb{Q}[t_1] \otimes \Lambda V, D)]$ with $Dv_3 = v_1v_2$, $Dv_4 = v_1v_3t_1 + t_1^5$, $Dv_5 = 0$, $P_5 = [(\mathbb{Q}[t_1, t_2] \otimes \Lambda V, D)]$ with $Dv_3 = v_1v_2$, $Dv_4 = v_1v_3t_1 + t_1^5$, $Dv_5 = t_2^7$ and $P_6 = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ with $Dv_4 = 0$, $Dv_3 = v_1v_2$, $Dv_5 = v_2v_4t + v_1v_3t^3 + t^7$. Here $Dv_1 = Dv_2 = 0$ for all. This model presenting $P_6 = (2, 1)$ makes X to be pre-c-symplectic from Proposition 2.5. Indeed, $[t^{16}] = [v_1v_2v_3v_4t^6] \neq 0$ for $fd(\mathbb{Q}[t] \otimes \Lambda V, D) = 32$.

If $|v_1| = |v_2| = 3$, $|v_3| = 5$, $|v_4| = 9$ and $|v_5| = 11$, it satisfies the necessary condition of Theorem 2.6 that $3 + 9 \leq 11 + 1$ and $3 + 5 \leq 11 + 1$. But we can easily check that there is no point $P_6 = (2, 1)$ since $Dv_5 \in (t, v_1, v_2, v_3)$ in any dga $(\mathbb{Q}[t] \otimes \Lambda V, D)$ from degree reason. Indeed, then $r_0(\mathbb{Q}[t] \otimes \Lambda V, D) > 0$ since we can put $D_2(v_4) = t_2^5$ and $D_2(v_i) = D(v_i)$ for $i \neq 4$ as a relative model of (4)

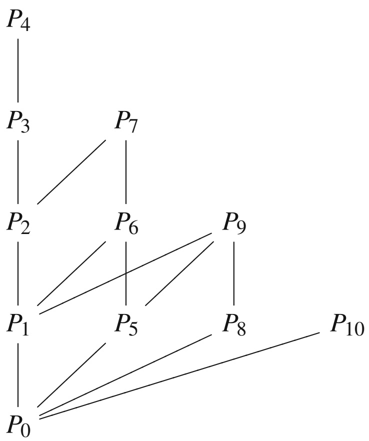
$$(\mathbb{Q}[t_2], 0) \to (\mathbb{Q}[t_2, t] \otimes \Lambda V, D_2) \to (\mathbb{Q}[t] \otimes \Lambda V, D)$$

with $\dim H^*(\mathbb{Q}[t_2, t] \otimes \Lambda V, D_2) < \infty$. Thus $\mathcal{H}(X)$ is given as



and X is not pre-c-symplectic from Theorem 1.7.

Example 3.7 Put $M(X) = (\Lambda(v_1, v_2, v_3, v_4, v_5, v_6, v_7), d)$ with $dv_1 = dv_2 = dv_3 = dv_4 = dv_7 = 0, dv_5 = v_1v_2, dv_6 = v_1v_3$ and $|v_1| = |v_2| = |v_3| = 3, |v_4| = |v_5| = |v_6| = 5, |v_7| = 9$. Then $r_0(X) = 4$ and $\mathcal{H}(X)$ is given as



where the edge P_5P_9 $(P_5 < P_9)$ is given by $Dv_i = dv_i$ for $i \neq 4, 7$,

$$Dv_7 = v_1v_6t_1 + v_2v_5t_2 + t_1^5, \quad Dv_4 = t_2^3$$

and $P_{10} = (3, 1)$ is presented by $Dv_i = dv_i$ for $i \neq 7$,

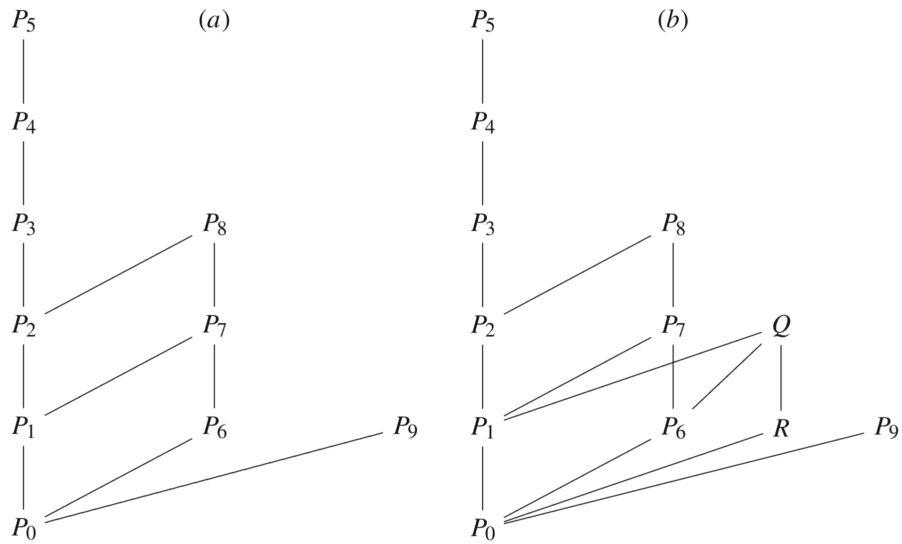
$$Dv_7 = v_1v_6t + v_2v_5t + v_3v_4t + t^5,$$

which is c-symplectic from Proposition 2.5. Also P_7 is presented by a c-symplectic model with $Dv_i = dv_i$ for $i = 1, 2, 3$,

$$Dv_7 = v_1v_6t_i + t_i^5, \; Dv_5 = v_1v_2 + t_j^3, \; Dv_4 = t_k^3,$$

which gives the sequence of orders $P_0 < P_5 < P_6 < P_7$ when $(i, j, k) = (1, 2, 3)$ or $(1, 3, 2)$. Also $P_0 < P_1 < P_6 < P_7$ when $(i, j, k) = (2, 1, 3)$ or $(3, 1, 2)$ and $P_0 < P_1 < P_2 < P_7$ when $(i, j, k) = (2, 3, 1)$ or $(3, 2, 1)$.

Example 3.8 When the product of five odd-spheres $X = S^{k_1} \times S^{k_2} \times S^{k_3} \times S^{k_4} \times S^{k_5}$ is pre-c-symplectic, there are (at least) the following two Hasse diagrams (a) and (b) that have the point $P_9 = (4, 1)$.



For example, (a) is given when $X = S^3 \times S^3 \times S^3 \times S^3 \times S^9$ and (b) is given when $X = S^3 \times S^3 \times S^7 \times S^{11} \times S^{15}$. They satisfy the condition of Theorem 1.2. The point R of (b) is presented by the model, for example, with $Dv_1 = Dv_2 = Dv_5 = 0$, $Dv_3 = v_1 v_2 t_1$ and $Dv_4 = v_1 v_3 t_1 + t_1^6$. The point Q of (b) is presented by the model, for example, with $Dv_1 = Dv_2 = 0$, $Dv_3 = v_1 v_2 t_1$, $Dv_4 = v_1 v_3 t_1 + t_1^6$ and $Dv_5 = t_2^8$. The points P_6 of (a), (b) are presented by the model, for example, with $Dv_1 = Dv_2 = Dv_3 = Dv_4 = 0$ and $Dv_5 = v_1 v_4 t^{(k_5-k_1-k_4+1)/2} + t^{(k_5-1)/2}$. Finally, the points P_9 of (a), (b) are presented by the model, for example, $Dv_1 = Dv_2 = Dv_3 = Dv_4 = 0$, $(a): \ Dv_5 = v_1 v_4 t^2 + v_2 v_3 t^2 + t^5$ and $(b): \ Dv_5 = v_1 v_4 t + v_2 v_3 t^3 + t^8$, which are c-symplectic models. In these examples of X, three points P_5, P_8 and P_9 are presented by c-symplectic models, in (a) and (b). In particular, for $M(S^3 \times S^3 \times S^3 \times S^3 \times S^9) = (\Lambda V, 0)$ giving (a), the c-symplectic model $(\mathbb{Q}[t_1, t_2, t_3] \otimes \Lambda V, D)$ with $(*)$:

$$Dv_1 = Dv_2 = 0, \ Dv_3 = t_i^2, \ Dv_4 = t_j^2, \ Dv_5 = v_1 v_2 t_k^2 + t_k^5,$$

where $\{i, j, k\} = \{1, 2, 3\}$, presents P_8 and its process of fibrations gives the sequence of orders $P_0 < P_1 < P_2 < P_8$, $P_0 < P_1 < P_7 < P_8$ or $P_0 < P_6 < P_7 < P_8$. On the other hands, the c-symplectic model $(\mathbb{Q}[t_1, t_2, t_3] \otimes \Lambda V, D)$ of Lupton–Oprea [20, Example 2.12] with $(**)$:

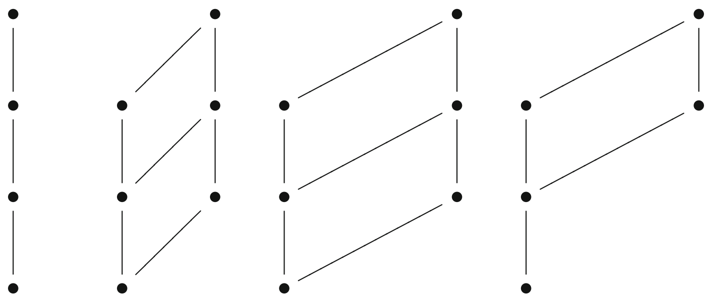
$$Dv_1 = t_i^2, Dv_2 = t_i t_j, Dv_3 = t_j^2, Dv_4 = t_j t_k, Dv_5 = t_k^5 + (v_1 t_j - t_i v_2)(v_3 t_k - t_j v_4)$$

presents P_8 but can not give $P_0 < P_6 < P_7 < P_8$, especially since $v_1 t_1^2 v_4 = \overline{D}(-v_1 v_3 v_4)$ in $(\mathbb{Q}[t_1] \otimes \Lambda V, \overline{D})$ when $j = 1$. Notice that the model of $(*)$ is formal but $(**)$ is not.

Remark 3.9 Simply connected c-symplectic spaces Y are schematically classified by the following diagrams $\mathcal{P}(Y)$ with respect to rational toral ranks. When $\dim \pi_2(Y) \otimes \mathbb{Q} = n$ with $M(Y) = (\Lambda U, d_U)$, there is the relative model

$$(\mathbb{Q}[t_1, \ldots, t_n], 0) \to (\Lambda U, d_U) \to (\Lambda V, d);\ V^2 = 0$$

with $|t_i| = 2$ and $U = V \oplus \mathbb{Q}(t_1, \ldots, t_n)$. Then Y presents a point (leaf) in $\mathcal{H}(\Lambda V, d)$ with certain sequences $[(\Lambda V, d)] < \cdots < [Y]$ of orders which are given by compositions of fibrations. Glue all such paths $[(\Lambda V, d)] - \cdots - [Y]$ from $[(\Lambda V, d)]$ to $[Y]$ in $\mathcal{H}(\Lambda V, d)$ and denote it as $\mathcal{P}(Y)$. For example, in the case of $n = 3$, we can concretely find the following four types of $\mathcal{P}(Y)$ in this paper:



which are in Examples 3.6, 3.7, 3.8$(a)(*)$ and 3.8$(a)(**)$, respectively. If a c-symplectic space is a homogeneous space, it is the first type from $r_0(X) \leq -\chi_\pi(X) := \dim \pi_{odd}(X) \otimes \mathbb{Q} - \dim \pi_{even}(X) \otimes \mathbb{Q}$ for an elliptic space X [2] and [20, Corollary 2.3].

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