A model categorical approach to group completion of E_n-algebras

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Abstract A group completion functor Q is constructed in the category of algebras in simplicial sets over a cofibrant E_n-operad $\mathcal{M}$. It is shown that Q defines a Bousfield–Friedlander simplicial model category on $\mathcal{M}$-algebras.

0 Introduction

A space X is of the homotopy type of an n-fold loop space if and only if it carries an action of an E_n-operad which induces a group-like H-structure on X. Since algebras over a suitable E_n-operad form a model category [27] it is natural to consider the localization of these model structures with respect to some variant of group completion functor Q. The aim of this paper is to construct such a Bousfield–Friedlander Q-structure for algebras over a cofibrant E_n-operad $\mathcal{M}$ in simplicial sets. The fibrant objects turn out to be essentially the group complete objects whose underlying simplicial sets are fibrant. Hence, in the light of the delooping results in [6,19], the Q-local homotopy category may be viewed as the homotopy category of n-fold loop spaces. The results of this paper play a role in our current joint work [14] with Zig Fiedorowicz and Rainer Vogt on n-fold monoidal categories [2].

The paper is organised as follows. In Sect. 1, we construct a group completion functor $\bar{Q}$ together with a coaugmentation $\bar{q} : 1 \to \bar{Q}$ in topological algebras over a cofibrant E_n-operad. For technical reasons, we use a mixture of classical and model

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categorical homotopy theory. Since an E_n-space is in general not a monoid we have to change operads. For this step, we rely on a recent theorem of Morton Brun, Zig Fiedorowicz and Rainer Vogt on tensor products of operads [10]. This result is used to generalize and adapt an argument of May, from $n = \infty$ to all n. In the next section, we recall Bousfield's improved axioms for the existence of a localized model structure defined by a coaugmented functor Q. It is shown that a Q-local model structure exists for algebras over a cofibrant E_n-operad $\mathcal{M}$ in simplicial sets. The functor Q is induced from the topological version $\bar{Q}$ defined in Sect. 1. In a nutshell, the strategy for the proof of the main result Theorem 2.7. may be described as follows. In order to verify Bousfield's axioms we have to use properties of the classifying space functor B which figures in the classical group completion ΩBM of a topological monoid M. On the other hand, to construct a coaugmented functor in E_n-algebras we have to rely on May's machine and on the cofibrancy of the operad $\mathcal{M}$. The point is, that we need a natural transformation $\bar{q}$ which commutes with the operad action on the nose and not only up to coherent homotopy. So we have to merge the best of two worlds in our construction.

In [1] the authors establish a related result. They consider the model category of algebras over the theory which encodes all natural maps between products of n-fold loop spaces and prove a recognition theorem for n-fold loop spaces with respect to this theory. Also, in unpublished work, Bousfield generalizes Segal's approach [25] to infinite loop space theory to n-fold loop spaces [8]. However, for the applications which we have in mind, the close connection to the operad of little n-cubes is essential. We will use freely the language of model categories. Besides the original source [23] there are now some more recent books on this subject [15–17]. For general background information on operads the reader may consult [18].

In this paper, we will work in the ground categories Top, Top_*, SS, SS_* of k-spaces, based k-spaces [32], simplicial sets and based simplicial sets. For an operad M in Top or SS we write Top_*^M and SS_*^M for the categories of M-algebras.

1 Group completions of E_n-algebras

The group completion of a topological monoid M is the loop space of the classifying space ΩBM. We use Milgram's version of the classifying space construction in this paper [22]. There is a natural map

$$M \overset{\kappa}{\to} \Omega BM$$

which is well known to be an A_∞-map [6]. This means κ commutes with the action of an A_∞-operad up to coherent homotopy. If X is an A_∞-space there is a functor [6] which replaces X by an equivalent monoid MX and the group completion of X can be defined as ΩBMX. For H-spaces whose multiplication satisfies a weak form of homotopy commutativity there is a homological version of group completion which we recall in the form given in [20].

Definition 1.1 An H-space X is called admissible if X is homotopy associative and if left translation by any given element of X is homotopic to right translation with the same element. An H-map between admissible H-spaces

$$g : X \to Y$$

is called a homological group completion of X if Y is group-like and the unique morphism of k-algebras

$$\bar{g}_* : H_*(X; k)\left[\pi_0^{-1} X\right] \to H_*(Y; k)$$

which extends g_* is an isomorphism for all commutative coefficient rings k.

The group completion theorem [21,24] (see also [20]) asserts that κ is a homological group completion for a well pointed topological monoid M in case the multiplications of M and ΩBM are admissible.

Our goal in this section is to construct a coaugmented functor $\bar{Q}$ in topological E_n-algebras which is closely related to a classical group completion. It will turn out that this functor is a homological group completion for $n > 1$. Many of our arguments are adaptions of the ones given by May for the case $n = \infty$ in [20], complemented by model categorical considerations, which are needed to ensure that the natural maps

$$\bar{q} : X \to \bar{Q}X$$

are E_n-homomorphisms.

Next we recall some notions and results from the literature.

Definition 1.2 A topological operad M is called:

(i)   well-pointed if the inclusion

$$\{id\} \to M_1$$

is a closed cofibration,

(ii) reduced if $M_0 = *$,

(iii) Σ-free if M_k is Σ_k-free and $M_k \to M_k/\Sigma_k$ is a numerable principal Σ_k-bundle [6],

(iv) E_n-operad if there exists a chain of maps of operads

$$M = B^0 \xrightarrow{f^0} B^1 \xleftarrow{f^1} \cdots \to B^r \xleftarrow{f^r} \mathcal{C}_n$$

such that

$$f_i^s : B_i^s \to B_i^{s+1}$$

are Σ_i-equivariant homotopy equivalences, and where $\mathcal{C}_n$ is the operad of little n-cubes [6].

A topological or simplicial operad M is called:

(v) Σ-cofibrant if the underlying collection is cofibrant in the model category of collections [4].

(vi) cofibrant if it is cofibrant in the model category of operads [4].

Note that $\mathcal{C}_n$ is Σ-free but not known to be Σ-cofibrant. In general, notions of cofibrancy related to classical homotopy theory, whose model categorical incarnation in Top is the Strøm structure [30], are weaker than the ones related to the Quillen model structure [23]. There are natural ways to replace a given operad by a well-pointed [33] or a cofibrant one. For example, the Boardman–Vogt W-construction [6] serves as a cofibrant replacement functor. This is shown in [33] in the setting of the cofibration category of topological operads with underlying Strøm structure. The model categorical case is treated in [5]. There is also a reduced version of W [6,5] which is cofibrant in the model category of reduced operads. In the following, we will restrict attention to reduced operads. Algebras over reduced operads have only one 0-ary operation. In the based context this operation is assumed to coincide with the basepoint. In this situation, the relevant monad associated with a reduced operad is defined by certain identifications related to the base points [19].

Let C_n be the monad defined by $\mathcal{C}_n$ on pointed spaces. There is a morphism of monads [19, 5.2.]

$$\alpha_n : C_n \to \Omega^n \Sigma^n.$$

such that the induced map of $\mathcal{C}_n$-algebras

$$C_n X \to \Omega^n \Sigma^n X$$

is a homological group completion for $n > 1$ as has been shown by Cohen and Segal [11,26].

In the proof of the proposition below will make use of May's two-sided bar construction [19] and of the results in [19, Appendix], [20, Appendix].

Proposition 1.3 *Let M be a cofibrant, reduced topological E_n-operad. Then there is a functor*

$$G : Top_*^M \to Top_*$$

such that GX is a topological monoid, and a natural transformation g, with values in H-maps, from the forgetfull functor

$$U : Top_*^M \to Top_*$$

to G such that

$$g : X \to GX$$

is a homological group completion for well pointed X in case n > 1.

Proof In a first step we change operads since we need a monoid structure to go along the algebra structure. Let $\mathcal{A}$ be the operad whose algebras are topological monoids and $\mathcal{A} \otimes \mathcal{C}_{n-1}$ the tensor product of operads [6]. There is an operad map

$$\gamma : M \to \mathcal{A} \otimes \mathcal{C}_{n-1}$$

whose underlying maps are homotopy equivalences. This is the case since M is cofibrant, $\mathcal{A} \otimes \mathcal{C}_{n-1}$ is an E_n-operad [10, Theorem C] and because $\mathcal{C}_n$ is of the homotopy type of a CW-complex [2]. The operad $\mathcal{A} \otimes \mathcal{C}_{n-1}$ is well-pointed and Σ-free [10]. Define G by

$$GX = \Omega_M BB(\mathcal{A} \otimes \mathcal{C}_{n-1}, M, X)$$

where $B(\mathcal{A} \otimes \mathcal{C}_{n-1}, M, X)$ is the two sided bar construction, Ω_M the Moore loop functor, and define g as the composition

$$X \overset{\tau}{\to} B(M, M, X) \overset{B(\gamma,1,1)}{\longrightarrow} B(\mathcal{A} \otimes \mathcal{C}_{n-1}, M, X)$$

$$B(\mathcal{A} \otimes \mathcal{C}_{n-1}, M, X) \overset{\bar{\kappa}}{\to} \Omega_M BB(\mathcal{A} \otimes \mathcal{C}_{n-1}, M, X).$$

Here $\bar{\kappa}$ is the natural inclusion and τ is the right inverse of the augmentation

$$X \overset{\epsilon}{\leftarrow} B(M, M, X)$$

which is a M-homomorphism and strong deformation retraction. So τ is a homotopy equivalence and a M-map in the sense of [6]. In particular it is an H-map. Note that $B(\mathcal{A} \otimes \mathcal{C}_{n-1}, M, X)$ is an $\mathcal{A} \otimes \mathcal{C}_{n-1}$-space and hence it is a monoid in $\mathcal{C}_{n-1}$-spaces.

The M-morphism $B(\gamma, 1, 1)$ is a homotopy equivalence for well pointed X. In order to see this we consider the map induced by γ between the simplicial spaces whose realizations are the bar constructions in question

$$B_*(M, M, X) \overset{B(\gamma,1,1)_*}{\longrightarrow} B_*(\mathcal{A} \otimes \mathcal{C}_{n-1}, M, X).$$

These simplicial spaces are proper [19]. To see this we apply [19, A.10.]. The assumptions made there hold since by A.7. loc.cit. the monad defined by $\mathcal{A} \otimes \mathcal{C}_{n-1}$ is an admissible M-functor. We apply [20, A.4.] which states that a map between proper simplicial spaces which is a homotopy equivalence in any given simplicial degree induces a homotopy equivalence after realization. Since M and $\mathcal{A} \otimes \mathcal{C}_{n-1}$ are both Σ-free the maps $B(\gamma, 1, 1)_m$ are indeed homotopy equivalences [20, A.2.], [6, A.3.4.].

So the composition of the first two maps whose composition make up g is a homotopy equivalence and an H-map between admissible H-spaces. The last assertion follows from the group completion theorem applied to the third map since, using well known facts, one can replace Ω by Ω_M. $\qquad\square$

Proposition 1.4 *Let M be a cofibrant, reduced topological E_n-operad and X a well pointed M-algebra. Then*

(i) the composition

$$X \xrightarrow{\tau} B(M, M, X) \xrightarrow{B(\alpha_n \pi_n, 1, 1)} B(\Omega^n \Sigma^n, M, X)$$

is a homological group completion for $n > 1$.

(ii) the spaces GX and $B(\Omega^n \Sigma^n, M, X)$ are naturally weakly equivalent by H-maps for all $n > 0$.

Sketch of Proof The assertions in (i) and (ii) in the case $n > 1$ follow by the same argument as in Theorem 2.3.(ii) in [20], once one replaces Theorem 2.2 in loc.cit. by the result of Cohen and Segal mentioned above. To settle the case $n = 1$, we appeal to Thomason's result on the uniqueness of delooping machines [31], which implies that the May delooping $B(\Sigma, M, X)$ and $BB(\mathcal{A} \otimes \mathcal{C}_{n-1}, M, X)$ are homotopy equivalent for well pointed X. Since there is a weak equivalence

$$\rho : B(\Omega\Sigma, M, X) \to \Omega B(\Sigma, M, X)$$

[19, 13.1(iii)] the assertion for $n = 1$ is proved.   □

Definition 1.5 A model category $\mathcal{D}$ is called right proper if every pullback of a weak equivalence along a fibration is a weak equivalence, left proper if every pushout of a weak equivalence along a cofibration is a weak equivalence, and proper if it is right and left proper.

As shown in [4,27,28] the model structures on Top_*^M and SS_*^M, for a Σ-cofibrant operad M, are transferred [12] along the free M-algebra functor from Top_* and SS_* and are cofibrantly generated. Moreover, it was shown by Spitzweck [28, Theorem 4 in section 4] that, for a cofibrant operad M, the transferred structure is right proper and the pushout in Top_*^M of a weak equivalence along a cofibration is a weak equivalence, provided the source is cofibrant in the underlying model category of Top_*. In SS_*^M this holds unconditionally. Hence, the model category of algebras over a cofibrant operad in simplicial sets is proper.

Now we turn to the construction of the functor $\bar{Q}$:

Theorem 1.6 *Let $\mathcal{M}$ be a reduced cofibrant simplicial operad such that the topological realization $M = |\mathcal{M}|$ is an E_n-operad. Then there is a functor*

$$\bar{Q} : Top_*^M \to Top_*^M$$

and a natural transformation

$$\bar{q} : 1 \to \bar{Q}$$

in Top_*^M which is a homological group completion for algebras which are cofibrant in Top_* if $n > 1$. If $X \in Top_*^M$ is cofibrant in Top_*, then $\bar{Q}X$ is so as well and is naturally weakly equivalent to GX by H-maps for all n.

Proof Let X be an M-algebra which is cofibrant as a pointed space. Note that X and M are both well pointed the later because geometric realization respects cofibrations. The operad M is cofibrant since $\mathcal{M}$ is and it is Σ-cofibrant by [4, 4.3.] and consequently also Σ-free [6, Appendix 3.]. Hence, $B(\alpha_n\pi_n, 1, 1)$ is a homological group completion if $n > 1$ by 1.4. Put

$$HX = B(M, M, X)$$

and

$$KX = B(\Omega^n\Sigma^n, M, X)$$

for short. These functors come with natural transformations of M-algebras

$$\epsilon : H \to 1$$

and

$$\eta = B(\alpha_n\pi_n, 1, 1) : H \to K$$

where the underlying map of ϵ is a homotopy equivalence. Apply the natural CW-approximation $T = |S|$ to the simplicial spaces whose realization is the diagram

$$X \xleftarrow{\epsilon} HX \xrightarrow{\eta} KX$$

and realize the diagram to

$$TX = X_T \xleftarrow{\epsilon_T} H_T X \xrightarrow{\eta_T} K_T X.$$

By [13, 4.9.] TZ is a M-algebra with cellular action and the natural map $\phi : TZ \to Z$ is a weak equivalence of M-algebras for every M-algebra Z. It follows from [19, 11.4.] that

$$X \xleftarrow{\phi} TX \xleftarrow{\epsilon_T} H_T X \xrightarrow{\eta_T} K_T X$$

is a diagram of M-algebras with all underlying spaces cofibrant.
   Let

$$H_T X \xrightarrow{i} U \xrightarrow{j} K_T X$$

be a natural factorization of η_T into cofibration i and trivial fibration j in the model category of M-algebras.

Define $\bar{Q}X$ to be the pushout in M-algebras of $i, \phi\epsilon_T$

$$
\begin{array}{ccc}
H_T X & \xrightarrow{\phi\epsilon_T} & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{q}} \\
U & \xrightarrow{h} & \bar{Q}X
\end{array}
$$

By [28, Theorem 4 in section 4], h is a weak equivalence hence $\bar{Q}X$ with the induced natural transformation of M-algebras $\bar{q} : 1 \to \bar{Q}$ is a group completion of X in the category of M-algebras. The space $\bar{Q}X$ is cofibrant because the induced map $\bar{q} : X \to \bar{Q}X$ is a cofibration of algebras since i is one. But a cofibration of algebras is a cofibration of spaces by [28, Theorem 4 in section 4]. Now X was assumed to be a cofibrant space hence $\bar{Q}X$ is one as well. □

Remark 1.7 The assumption that M is the realization of a simplicial operad is not severe because there is a Quillen equivalence between simplicial and topological operads [4].

2 The Q-structure on $\mathcal{M}$-algebras

Let $\mathcal{D}$ be a proper model category and $Q : \mathcal{D} \to \mathcal{D}$ a coaugmented functor.

Following Bousfield and Friedlander, we say that a morphism $f : X \to Y$ in $\mathcal{D}$ is a Q-equivalence if Qf is a weak equivalence, a Q-cofibration if f is a cofibration, and a Q-fibration if f satisfies the right lifting property with respect to Q-trivial cofibrations.

Consider the following axioms:

(A1) for each weak equivalence $f : X \to Y$ in $\mathcal{D}$ the map

$$Qf : QX \to QY$$

is a weak equivalence;

(A2) for each object in $X \in \mathcal{D}$ the maps

$$q_{QX}, Qq_X : QX \to QQX$$

are weak equivalences;

(A3) for each pullback square

$$
\begin{array}{ccc}
V & \xrightarrow{k} & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
W & \xrightarrow{h} & Y
\end{array}
$$

in $\mathcal{D}$, with f a fibration of fibrant objects such that $q : X \to QX, q : Y \to QY$ and $Qh : QW \to QY$ are weak equivalences, the map

$$QV \overset{Qk}{\to} QX$$

is a weak equivalence.

It is a theorem of Bousfield [7, 9.3.] that in case (A1)–(A3) hold, then the three classes of maps given above define a proper model category on $\mathcal{D}$. We will apply this theorem, or more precisely its proof, to the category $SS_*^{\mathcal{M}}$ where $\mathcal{M}$ a cofibrant operad whose realization is an E_n-operad. As for the stable model structure of Γ-spaces constructed in [9] axiom (A3) does not hold in full generality. So we have to adapt the arguments in [9] to the situation at hand. There is a Q-structure even for non proper $\mathcal{D}$. This follows from [7, 9.5] and [29]. However, we had to rely on the weak form of left properness of Top_*^{M} for the proof of 1.6.

Lemma 2.1 *Let $M = |\mathcal{M}|$ be as in 1.6. and $\bar{Q}, \bar{q}$ the coaugmented functor constructed in section 1. Then the pair $\bar{Q}, \bar{q}$ satisfies (A1) and (A2) if X and Y are cofibrant spaces.*

Proof By 1.6., we may replace $\bar{Q}X$ by $GX = \Omega_M B B(\mathcal{A} \otimes \mathcal{C}_{n-1}, M, X)$ in the argument. Then the assertion follows from some well known properties of Ω_M and B (see [6, Chapter VI.]). Property (A1) is satisfied since the functors Ω_M and B preserve weak equivalences. For a connected space X of the homotopy type of a CW-complex the natural evaluation map

$$e : B\Omega_M X \to X$$

$$e(t_1, x_1, \ldots, t_k, x_k) = \omega\left(\sum_{i=1}^{k}(1 - t_1 * t_2 * \cdots * t_i)a_i\right)$$

is a homotopy equivalence [6, 6.15] where $x_i = (\omega_i, a_i) \in \Omega_M X$, $t_1 * t_2 = t_1 + t_2 - t_1 t_2$, and $(\omega, \sum_{i=1} a_i) = x_1 \cdot x_2 \cdot \ldots \cdot x_k$. The functor $\Omega_M B$ is a monad up to homotopy with structure morphisms $\Omega_M e$ and $\bar{\kappa}$. In particular $(\Omega_M e)(\bar{\kappa}_{\Omega_M BX})$ and $(\Omega_M e)(\Omega_M \bar{\kappa}_X)$ are homotopic to the identity. Property (A2) follows. $\square$

Recall that the adjoint pair $|-|, S$ geometric realization and the singular functor induces an adjoint pair [13] which will be denoted by the same symbols $|-|, S$

$$|-| : SS_*^{\mathcal{M}} \rightleftarrows Top_*^{M} : S .$$

Define a functor

$$Q : SS_*^{\mathcal{M}} \to SS_*^{\mathcal{M}}$$

by $QX = S\bar{Q}|X|$ and a natural transformation $q : 1 \to Q$ by the composition

$$X \overset{\eta_X}{\to} S|X| \overset{S\bar{q}_{|X|}}{\to} S\bar{Q}|X|$$

where η is the unit of the adjunction. The proof of the following lemma is left as an exercise. It proceeds by reduction to 2.1. using well known properties of the pair $|-|, S$.

Lemma 2.2 *The pair Q, q satisfies (A1) and (A2).*

Denote the subcategories of Top_*^M whose objects have underlying spaces which are cofibrant by Top_{*c}^M and write $SS_{*f}^{\mathcal{M}}$ for the subcategory of SS_*^M whose objects have fibrant underlying simplicial sets. Let Ab be the category of abelian groups. We may consider $A \in Ab$ as a topological M-algebra in the obvious way. This defines an inclusion functor $i : Ab \to Top_{*c}^M$. The assignment $M \to SiM$ defines a functor $S_{ab} : Ab \to SS_{*f}^{\mathcal{M}}$ from abelian groups to $SS_*^{\mathcal{M}}$.

Lemma 2.3 *The functor S_{ab} is right adjoint to $\pi_0 Q$.*

Proof First note that by the adjunction between simplicial and topological algebras there is a bijection:

$$Hom_{SS_*^{\mathcal{M}}}(X, S_{ab}A) \to Hom_{Top_*^M}(|X|, A).$$

For any topological M-algebra Y the map $Y \to \pi_0 Y$ where $\pi_0 Y$ carries the quotient topology is a morphism of M-algebras. In case the space underlying Y is of the homotopy type of a CW-complex the topology on $\pi_0 Y$ is the discrete one. This holds since the path components of a CW-complex are open and closed. In particular this applies to an algebra which is cofibrant as a space because a generalized CW-complex is homotopy equivalent to a genuine CW-complex by cellular approximation. Now $|X|$ and $\bar{Q}|X|$ are cofibrant spaces. It follows that any morphism $|X| \to A$ factorizes uniquely over $|X| \to \pi_0|X| \to \pi_0 \bar{Q}|X|$ and this is the claim. □

We need the following fact whose statement and proof are parallel to the ones of [9, 5.4.].

Lemma 2.4 *Every morphism $f : X \to Y$ in $SS_*^{\mathcal{M}}$ can be factored as*

$$X \xrightarrow{u} Z \xrightarrow{v} Y$$

where $\pi_0 Qu : \pi_0 QX \to \pi_0 QZ$ is onto and v is a Q-fibration.

Proof We define inductively a descending filtration of $\mathcal{M}$-algebras

$$Y = C^0 \supset C^1 \supset \ldots C^\alpha \supset \ldots$$

indexed by the ordinals such that $f(X) \subset C^\alpha$ and $C^\alpha \subset Y$ is a Q-fibration as follows. Suppose C^α is found define $C^{\alpha+1}$ as the pullback

$$
\begin{array}{ccc}
C^{\alpha+1} & \longrightarrow & S_{ab}G^\alpha \\
\downarrow & & \downarrow \\
C^\alpha & \longrightarrow & S_{ab}\pi_0\bar{Q}|C^\alpha|
\end{array}
$$

where G^α is the image of $\pi_0 Qf : \pi_0 QX \to \pi_0 QC^\alpha$ and the map at the bottom is the composite

$$C^\alpha \to S|C^\alpha| \to S\bar{Q}|C^\alpha| \to S_{ab}\pi_0\bar{Q}|C^\alpha|.$$

We claim that $S_{ab}G^\alpha \subset S_{ab}\pi_0 QC^\alpha$ is a Q-fibration and hence $C^{\alpha+1} \subset C^\alpha$ is one as well. To see this let

$$
\begin{array}{ccc}
A & \longrightarrow & S_{ab}G^\alpha \\
\downarrow & & \downarrow \\
B & \longrightarrow & S_{ab}\pi_0 QC^\alpha
\end{array}
$$

be a commuting diagram with the vertical map on the left a Q-trivial cofibration.

It is enough to show existence of a filler for the adjoint diagram

$$
\begin{array}{ccc}
\pi_0 QA & \longrightarrow & G^\alpha \\
\downarrow & & \downarrow \\
\pi_0 QB & \longrightarrow & \pi_0 QC^\alpha
\end{array}
$$

The morphism on the left is an isomorphism between discrete abelian groups. Hence the searched for filler exists. Note that $f(X) \subset C^{\alpha+1}$ and that $C^{\alpha+1} \to Y$ is a fibration since fibrations are closed under composition. For a limit ordinal λ such that $f(X) \subset C^\alpha$ for all $\alpha < \lambda$ define $C^\lambda = \lim_{\alpha < \lambda} C^\alpha$. For sufficiently large α one has $C^\alpha = C^{\alpha+1}$ and then $\pi_0 Qf : \pi_0 QX \to \pi_0 QC^\alpha$ is onto. Now put $Z = C^\alpha$. □

The next proposition establishes a weak form of axiom (A3) for the pair Q, q.

Proposition 2.5 *Let*

$$
\begin{array}{ccc}
V & \xrightarrow{k} & X \\
g\downarrow & & \downarrow f \\
W & \xrightarrow{h} & Y
\end{array}
$$

be a pullback square of $\mathcal{M}$-algebras with f a fibration of fibrant objects such that the maps $q : X \to QX$, $q : Y \to QY$ and $Qh : QW \to QY$ are weak equivalences and with $\pi_0 Qf : \pi_0 QX \to \pi_0 QY$ onto. Then $Qk : QV \to QX$ is a weak equivalence.

Proof Since $SG|Z|$ and QZ are naturally weakly equivalent simplicial sets we may replace $\bar{Q}$ by G in the argument. Consider the square of bisimplicial sets which results by application of the functor

$$Z \to SB_*G|Z|$$

to the square above and note that $SB_* = B_*S$ since S commutes with products. Here B_*X is the simplicial bar construction for topological or simplicial monoids X which has the powers X^n in simplicial degree n. Now we wish to apply [9, B.4.]. This theorem gives conditions on a square of bisimplicial sets which imply that the realization of the square is a homotopy pullback square in simplicial sets.

We have to check that these conditions hold in our situation.

First, we need to verify that the squares defined by $SG|X|^m$, $SG|Y|^m$, $SG|V|^m$, $SG|W|^m$ are homotopy fibre squares. This is the case since these spaces are naturally weakly equivalent to X^m, Y^m, V^m, W^m and these form a fibre square since X, Y, V, W form one by assumption.

Second, we need that the so called π_*-Kan condition [9, B.3.] holds for $B_*SG|X|$, $B_*SG|Y|$ and that the map induced by f

$$\alpha : \pi_0^v B_* SG|X| \to \pi_0^v B_* SG|Y|$$

is a fibration. Here $\pi_*^v(Z)$ denotes the vertical homotopy of a bisimplicial set Z. For simplicial spaces Z which are simple in each degree the π_*-Kan condition is equivalent to the following condition [9, B.3.1]: The obvious map

$$\beta : \pi_t^v Z_{free} \to \pi_o^v Z_{free}$$

is a fibration for each $t \geq 1$. Where for a simplicial set U, the symbol $\pi_t U_{free}$ stands for the set of unpointed homotopy classes from S^t to $|U|$. By assumption, $|X|$, $|Y|$ are group-like H-spaces. It follows that B_*SX, B_*SY are indeed simple in each degree. Moreover, it follows that all the maps in question are surjective homomorphism of simplicial groups and hence are fibrations.   □

A cofibrant fibrant approximation for a map $f : X \to Y$ in a model category is a commuting diagram

$$
\begin{array}{ccc}
X & \xrightarrow{u} & \widehat{X} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\
Y & \xrightarrow{v} & \widehat{Y}
\end{array}
$$

with trivial cofibrations u, v and $\widehat{X}$, $\widehat{Y}$ fibrant. It is true that cofibrant fibrant approximation always exist and one can choose $\hat{f}$ as a fibration [16, 8.1.23.]

Before we proceed we have to recall one more result of Bousfield. The proof of [7, 9.3.] gives us:

Proposition 2.6 *Let $\mathcal{D}$ be a proper model category and Q, q a coaugmented functor for which (A1)-(A2) holds. Moreover, let $f : X \to Y$ be a map in $\mathcal{D}$ such that (A3) holds for one (and hence any) fibration $\widehat{Qf}$ in a cofibrant fibrant approximation of Qf. Then f can be factored into ji with trivial Q-cofibration i and Q-fibration j.*

We are now ready to prove the main result of the paper:

Theorem 2.7 *Let $\mathcal{M}$ and Q be as in 1.6. The category $SS_*^{\mathcal{M}}$ with the Q-structure is a left proper simplicial model category. Moreover, a morphism $f : X \to Y$ in $SS_*^{\mathcal{M}}$ is a Q-fibration if it is a fibration and*

$$
\begin{array}{ccc}
X & \xrightarrow{q_X} & QX \\
f \downarrow & & \downarrow Qf \\
Y & \xrightarrow{q_Y} & QY
\end{array}
$$

is a homotopy fibre square. In case $\pi_0 Qf$ is onto this condition is also necessary.

Proof Since (A1) holds by 2.2. the axioms of a model category are satisfied by [9, A.8] except maybe the trivial cofibration fibration part of the factorization axiom.

Let $f : X \to Y$ be a map of $\mathcal{M}$-algebras. Factor $f = vu$ as in 2.4. in Q-fibration v and with $\pi_0 Qu$ onto. It is enough to factor u into trivial Q-cofibration and Q-fibration. By 2.5. and 2.6 this can be done.

To verify the left properness let

$$
\begin{array}{ccc}
V & \xrightarrow{k} & X \\
i \downarrow & & \downarrow j \\
W & \xrightarrow{h} & Y
\end{array}
$$

be a pushout diagram with Q-equivalence k and Q-cofibration i. Factor k into $k = fg$ with Q-cofibration g and trivial Q-fibration f. Then g is a trivial Q-cofibration and f is a trivial fibration in the underlying model category. The former by definition the later by [9, A.8.(ii)]. That the pushout of g along i is a Q-equivalence follows directly from the axioms of a model category. The pushout of f along the induced cofibration $\bar{i}$ is a weak equivalence and hence a Q-equivalence by the left properness of the underlying model category. The model structure on $SS_*^{\mathcal{M}}$ is simplicial by [27]. Hence, the Q-strucure is simplicial as well by [7, 9.7.] whose proof does not use (A3). The sufficiency of the stated condition follows from [9, A.9.]. For the last statement, note that (A3) holds for f by 2.5. Now the proof proceeds as in [9, A.10.]. □

Corollary 2.8 *An object $X \in SS_*^{\mathcal{M}}$ is fibrant in the Q-structure if and only if the underlying simplicial set is fibrant and q_X is a weak equivalence.*

Remark 2.9 The Q-structure on $SS_*^{\mathcal{M}}$ is not right proper. This follows from the fact that (A3) does not hold in general. An example which shows this can be found in [9] on page 109.

Remark 2.10 One can show that the $\bar{Q}$-structure induced on M-algebras satisfies axioms which are slightly weaker than those of a cofibration category but still strong enough to induce a well defined homotopy category. Most of the axioms hold only if the source (and sometimes the target) of the morphisms are cofibrant spaces. One has to use the modifications for some of the arguments in [9] which were already hinted on by Bousfield in [7, 9.5.].

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