Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2022) 116:87
https://doi.org/10.1007/s13398-022-01228-2

ORIGINAL PAPER

Some new semi-exponential operators

Ulrich Abel[1] · **Vijay Gupta**[2] · **Meer Sisodia**[2]

Received: 13 November 2021 / Accepted: 27 February 2022 / Published online: 24 March 2022

Abstract
In the theory of approximation, linear operators play an important role. The exponential-type operators were introduced four decades ago, since then no new exponential-type operator was introduced by researchers, although several generalizations of existing exponential-type operators were proposed and studied. Very recently, the concept of semi-exponential operators was introduced and few semi-exponential operators were captured from the exponential-type operators. It is more difficult to obtain semi-exponential operators than the corresponding exponential-type operators. In this paper, we extend the studies and define semi-exponential Bernstein, semi-exponential Baskakov operators, semi-exponential Ismail–May operators related to $2x^{3/2}$ or x^3. Furthermore, we present a new derivation for the semi-exponential Post–Widder operators. In some examples, open problems are indicated.

Keywords Semi-exponential Bernstein polynomials · Semi-exponential Baskakov operators · Semi-exponential Ismail–May operators · Semi-exponential Post–Widder operators · Approximation by operators

Mathematics Subject Classification 41A35

1 Introduction

The exponential-type operators are important in the field of approximation theory. They were firstly considered by Ismail and May [4] in 1978. The exponential-type operators preserve

Meer Sisodia: Research Intern at NSUT.

✉ Ulrich Abel
ulrich.abel@mnd.thm.de

Vijay Gupta
vijaygupta2001@hotmail.com

Meer Sisodia
meer.sisodia@gmail.com

1 Technische Hochschule Mittelhessen, Fachbereich MND, Wilhelm-Leuschner-Straße 13, 61169 Friedberg, Germany

2 Department of Mathematics, Netaji Subhas University of Technology, Sector 3 Dwarka, New Delhi 110078, India

the linear functions. Many generalizations of exponential-type operators are available in the literature. Tyliba and Wachnicki [7] extended the definition of Ismail and May [4] by proposing a more general family of operators. For a non-negative real number β, they introduced the operators L_λ^β. For $\beta > 0$, they are not of exponential type but similar to exponential-type operators. Recently, Herzog [3] further extended the studies and termed such operators as semi-exponential type operators. Actually, an operator of the form

$$(L_\lambda f)(x) = \int_I W_\beta^L(\lambda, x, t)\, f(t)\, dt$$

is called a semi-exponential operator if its kernel $W_\beta^L(\lambda, x, t)$ satisfies the differential equation

$$\frac{\partial}{\partial x} W_\beta^L(\lambda, x, t) = \left(\frac{\lambda(t-x)}{p(x)} - \beta \right) W_\beta^L(\lambda, x, t). \tag{1}$$

In particular, for $\beta > 0$, one has $L_\lambda^\beta e_1 \neq e_1$, where $e_r(t) = t^r$ $(r = 0, 1, 2, \ldots)$. In the case $\beta = 0$, the operator $L_\lambda^{\beta=0}$ is simply the exponential-type operator studied by Ismail and May [4]. A collection of such operators may be found in the recent book [2, Ch. 1].

Choosing different functions $p(x)$ several exponential-type operators were captured in Ismail and May [4]. It is difficult to construct new exponential-type operators or the corresponding semi-exponential operators by just taking different functions $p(x)$. The essential obstacle is to fulfill the normalization condition

$$\int_I W_\beta^L(\lambda, x, t)\, dt = 1,$$

which means that L_λ^β preserves constant functions. Tyliba and Wachnicki [7] captured the semi-exponential operators of Weierstrass and Szász–Mirakyan operators, Herzog [3] got success to define the semi-exponential Post–Widder operators. We represent below the tabular form of known semi-exponential type operators available till date:

No.	Exponential operator	$p(x)$
(1)	Gauss–Weierstrass operators $(W_n f)(x)$	1
–	$(W_n f)(x) = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-n(t-x)^2}{2}\right) f(t)\, dt$	Exponential
–	$(W_n^\beta f)(x) = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-n(t-x-\beta/n)^2}{2}\right) f(t)\, dt$	Semi-exponential
(2)	Post–Widder operators $(P_n f)(x)$	x^2
–	$(P_n f)(x) = \frac{n^n}{\Gamma(n) x^n} \int_0^\infty e^{-nt/x} t^{n-1} f(t)\, dt$	Exponential
–	$(P_n^\beta f)(x) = \frac{n^n}{x^n e^{\beta x}} \sum_{k=0}^\infty \frac{(n\beta)^k}{k!\Gamma(n+k)} \int_0^\infty e^{-nt/x} t^{n+k-1} f(t)\, dt$	Semi-exponential
(3)	Szász–Mirakyan operators $(S_n f)(x)$	x
–	$(S_n f)(x) = \sum_{k=0}^\infty e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$	Exponential
–	$(S_n^\beta f)(x) = \sum_{k=0}^\infty e^{-(n+\beta)x} \frac{((n+\beta)x)^k}{k!} f\left(\frac{k}{n}\right)$	Semi-exponential

As pointed out earlier, one can obtain the exponential-type operator as the special $\beta = 0$ from semi-exponential operators, but the converse is not analogous. Here we capture some more semi-exponential operators viz. semi-exponential Bernstein polynomials, semi-exponential Baskakov operators, etc.

2 New semi-exponential operators

In this section, we establish some new exponential-type operators. In all listed cases it is possible to solve the differential equation (1) in the form $W_\beta^L(n,x,t) = A_L(n,t,\beta)\,y$, but it is difficult to find the normalization, i.e., the factor $A_L(n,t,\beta)$ of the solution y such that

$$\int_I W_\beta^L(n,x,t)\,dt = 1$$

or, in the discrete case,

$$\sum_{k=0}^{\infty} W_\beta^L(n,x,k/n) = 1,$$

respectively. Below we list some instances of $p(x)$, which were considered for well-known exponential-type operators.

2.1 Semi-exponential Bernstein operators

If we take $p(x) = x(1-x)$, then for a kernel $W_\beta^B(n,x,k/n) = A_B(n,k,\beta)\,y$, we have

$$y' = \frac{k-nx}{x(1-x)}y - \beta y,$$

where the derivative of y is with respect to the variable x. We conclude that

$$\frac{y'}{y} = k\left(\frac{1}{1-x} + \frac{1}{x}\right) - \frac{n}{1-x} - \beta,$$
$$\log y = \log(1-x)^{n-k} + \log x^k - \beta x,$$

implying

$$y = x^k(1-x)^{n-k}e^{-\beta x}.$$

In order to have normalization

$$\sum_{k=0}^{\infty} W_\beta^B(n,x,k/n) = \sum_{k=0}^{\infty} A_B(n,k,\beta)\,x^k(1-x)^{n-k}e^{-\beta x} = 1.$$

we evaluate $A_B(n,k,\beta)$ from the equation

$$\sum_{k=0}^{\infty} A_B(n,k,\beta)\left(\frac{x}{1-x}\right)^k = e^{\beta x}(1-x)^{-n}.$$

For $0 \le x < 1$, put $z = x/(1-x)$. Then $x = z/(1+z)$, and for any positive integer n, the generating function of the sequence $(A_B(n,k,\beta))_{k=0}^{\infty}$

$$\sum_{k=0}^{\infty} A_B(n,k,\beta)\,z^k = e^{\beta z/(1+z)}(1+z)^n$$

is analytic, for $|z| < 1$, with an essential singularity at $z = -1$. Hence, it can be developed as a power series in the disk $|z| < 1$. The series

$$e^{\beta z/(1+z)}(1+z)^n = \sum_{j=0}^{\infty} \frac{(\beta z)^j}{j!}(1+z)^{n-j}$$

is convergent for all complex z different from -1. It follows that, for $|z| < 1$,

$$e^{\beta z/(1+z)} (1+z)^n = \sum_{j=0}^{\infty} \frac{(\beta z)^j}{j!} \sum_{\ell=0}^{\infty} \binom{n-j}{\ell} z^\ell = \sum_{k=0}^{\infty} z^k \sum_{j+\ell=k} \binom{n-j}{\ell} \frac{\beta^j}{j!},$$

where the binomial coefficient is to be read as $\binom{n-j}{0} = 1$ and $\binom{n-j}{\ell} = (\ell!)^{-1} \prod_{\nu=0}^{\ell-1} (n-j-\nu)$, for $\ell \in \mathbb{N}$. We have

$$\left(B_n^\beta f\right)(x) = e^{-\beta x} \sum_{k=0}^{\infty} A_B(n,k,\beta)\, x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \qquad \left(0 \le x < \frac{1}{2}\right),$$

where

$$A_B(n,k,\beta) = \sum_{j=0}^{k} \binom{n-j}{k-j} \frac{\beta^j}{j!}.$$

Thus the semi-exponential Bernstein polynomials B_n^β map a function f on $[0,+\infty)$ to a function $B_n^\beta f$ defined on $[0,1/2)$, whenever the sum is convergent. It can be shown that the operators B_n^β apply to all polynomials. In the special case $\beta = 0$ we have $j = 0$ and $\ell = k$ such that $A_B(n,k,\beta) = \binom{n}{k}$. Hence, the sum defining $B_n^{\beta=0} f$ is finite, and we get the Bernstein polynomials.

The operators B_n^β can be rewritten in the alternative form

$$\left(B_n^\beta f\right)(x) = e^{-\beta x} \sum_{j=0}^{\infty} \frac{\beta^j}{j!} x^j (1-x)^{n-j} \sum_{k=0}^{\infty} \binom{n-j}{k} \left(\frac{x}{1-x}\right)^k f\left(\frac{j+k}{n}\right)$$

$\left(0 \le x < \frac{1}{2}\right)$. The latter representation immediately reveals the special case $\beta = 0$.

2.2 Semi-exponential Baskakov operators

If we take $p(x) = x(1+x)$, then for a kernel $W_\beta^V(n,x,k/n) = A_V(n,k,\beta)\, y$, we have

$$y' = \frac{k-nx}{x(1+x)} y - \beta y,$$

where the derivative of y is with respect to the variable x. We conclude that

$$\frac{y'}{y} = k\left(\frac{1}{x} - \frac{1}{1+x}\right) - \frac{n}{1+x} - \beta,$$
$$\log y = -k \log(1+x) + k \log x - n \log(1+x) - \beta x,$$

implying

$$y = \frac{x^k}{(1+x)^{n+k}} e^{-\beta x}.$$

In order to have normalization

$$\sum_{k=0}^{\infty} W_\beta^V(n,x,k/n) = \sum_{k=0}^{\infty} A_V(n,k,\beta) \frac{x^k}{(1+x)^{n+k}} e^{-\beta x} = 1.$$

Put, for $x \geq 0$, $z = x/(1+x)$. Then $x = z/(1-z)$. We obtain

$$\begin{aligned}\sum_{k=0}^{\infty} A_V(n,k,\beta)\, z^k &= e^{\beta z/(1-z)}(1-z)^{-n} \\ &= \sum_{j=0}^{\infty} \frac{(\beta z)^j}{j!}(1-z)^{-n-j} \\ &= \sum_{j=0}^{\infty} \frac{(\beta z)^j}{j!} \sum_{\ell=0}^{\infty} \binom{n+j-1+\ell}{\ell} z^{\ell} \\ &= \sum_{k=0}^{\infty} z^k \sum_{j+\ell=k} \binom{n+k-1}{\ell} \frac{\beta^j}{j!}.\end{aligned}$$

Thus, the semi-exponential Baskakov operators can be defined by

$$\begin{aligned}\left(V_n^{\beta} f\right)(x) &= \sum_{k=0}^{\infty} b_{n,k}^{\beta}(x)\, f\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^{\infty} A_V(n,k,\beta) \frac{x^k}{(1+x)^{n+k}} e^{-\beta x} f\left(\frac{k}{n}\right),\end{aligned}$$

where

$$A_V(n,k,\beta) = \sum_{j+\ell=k} \binom{n+k-1}{\ell} \frac{\beta^j}{j!} = \sum_{j+\ell=k} \frac{(n+j)_{\ell}}{k!} \binom{k}{j} \beta^j.$$

In special case $\beta = 0$ we have $j = 0$ and $\ell = k$ such that we get the Baskakov operators.

2.3 Semi-exponential Ismail–May operators related to $2x^{3/2}$

If we take $p(x) = 2x^{3/2}$, then for a kernel $W_{\beta}^{U}(n,x,t) = A_U(n,t,\beta)\, y$, we have

$$y' = \frac{n(t-x)}{2x^{3/2}} y - \beta y,$$

where the derivative of y is with respect to the variable x. We conclude that

$$\begin{aligned}\frac{y'}{y} &= \frac{nt}{2x^{3/2}} - \frac{n}{2\sqrt{x}} - \beta, \\ \log y &= \frac{-nt}{\sqrt{x}} - n\sqrt{x} - \beta x,\end{aligned}$$

implying

$$y = \exp\left(\frac{-nt}{\sqrt{x}} - n\sqrt{x} - \beta x\right).$$

Our target is to obtain $A_U(n,t,\beta)$ in order to have normalization

$$\int_0^{\infty} A_U(n,t,\beta)\, y\, dt = 1.$$

If we put, for abbreviation, $s = n/\sqrt{x}$, the normalization condition takes the form

$$\int_0^\infty A_U(n,t,\beta)\, e^{-st} dt = \exp\left(\frac{n^2}{s} + \beta\frac{n^2}{s^2}\right) \qquad (s > 0).$$

Since

$$\exp\left(\frac{n^2}{s} + \beta\frac{n^2}{s^2}\right) = \sum_{k=0}^\infty \left(\frac{n}{s}\right)^k \sum_{\substack{i,j\ge 0,\\ i+2j=k}} \frac{n^i \beta^j}{i!j!} \qquad (s \ne 0)$$

we obtain

$$A_U(n,t,\beta) = \delta(t) + \sum_{k=1}^\infty \frac{n^k t^{k-1}}{\Gamma(k)} \sum_{\substack{i,j\ge 0,\\ i+2j=k}} \frac{n^i \beta^j}{i!j!} \qquad (s > 0),$$

where $\delta(t)$ denotes Dirac's delta function. Hence, the operators are defined by

$$\left(U_n^\beta f\right)(x) = e^{-n\sqrt{x}-\beta x} f(0) + e^{-n\sqrt{x}-\beta x} \int_0^\infty \hat{A}_U(n,t,\beta) \exp\left(-\frac{nt}{\sqrt{x}}\right) f(t)\, dt$$

with

$$\hat{A}_U(n,t,\beta) = \sum_{k=0}^\infty \frac{(nt)^k}{k!} \sum_{\substack{i,j\ge 0,\\ i+2j=k+1}} \frac{n^{i+1}\beta^j}{i!j!} \qquad (s > 0).$$

Thus, the semi-exponential operator, related to $2x^{3/2}$, takes the form

$$\left(U_n^\beta f\right)(x) = e^{-n\sqrt{x}-\beta x} f(0) + e^{-n\sqrt{x}-\beta x} \sum_{k=0}^\infty \frac{n^k}{k!} \left(\sum_{\substack{i,j\ge 0,\\ i+2j=k+1}} \frac{n^{i+1}\beta^j}{i!j!} \right) \times \int_0^\infty t^k \exp\left(-\frac{nt}{\sqrt{x}}\right) f(t)\, dt.$$

In the special case $\beta = 0$, the definition reduces to the Ismail–May operator of exponential type

$$\begin{aligned}\left(U_n^{\beta=0} f\right)(x) &= e^{-n\sqrt{x}} f(0) + e^{-n\sqrt{x}} \sum_{k=0}^\infty \frac{n^k}{k!} \frac{n^{k+2}}{(k+1)!} \int_0^\infty t^k \exp\left(-\frac{nt}{\sqrt{x}}\right) f(t)\, dt \\ &= e^{-n\sqrt{x}} \left\{ f(0) + n \int_0^\infty e^{-nt/\sqrt{x}} t^{-1/2} I_1\left(2n\sqrt{t}\right) f(t)\, dt \right\},\end{aligned}$$

where $I_1(x)$ is modified Bessel function of the first kind. Further results on the operators $U_n^{\beta=0}$ can be found in [1].

2.4 Semi-exponential Post–Widder operators

Although the semi-exponential Post–Widder operators were captured in [3, Eq. (10)], using Laplace transform, we provide an alternative approach that is shorter. We proceed as follows.

If we take $p(x) = x^2$, then for a kernel $W_\beta^P\,(n,x,t) = A_P\,(n,t,\beta)\,y$, we have

$$\begin{aligned} y' &= \frac{n(t-x)}{x^2}y - \beta y, \\ \frac{y'}{y} &= ntx^{-2} - nx^{-1} - \beta, \\ \log y &= \frac{-nt}{x} - n\log x - \beta x, \\ y &= e^{-nt/x}x^{-n}e^{-\beta x}. \end{aligned}$$

For normalization, we look for a function $A_P\,(n,t,\beta)$ such that

$$\int_0^\infty W_\beta^P\,(n,x,t)\,dt = \int_0^\infty A_P\,(n,t,\beta)\,e^{-nt/x}x^{-n}e^{-\beta x}dt = 1.$$

Putting

$$A_P\,(n,t,\beta) = \sum_{k=0}^\infty a_k t^{k+\alpha}$$

we have to choose coefficients a_k such that

$$\sum_{k=0}^\infty a_k \int_0^\infty t^{k+\alpha}e^{-nt/x}dt = x^n e^{\beta x}.$$

This is equivalent to

$$\sum_{k=0}^\infty a_k \Gamma\,(k+\alpha+1)\left(\frac{x}{n}\right)^{k+\alpha+1} = \sum_{k=0}^\infty \frac{\beta^k}{k!}x^{k+n}.$$

It follows that $\alpha = n-1$ and

$$a_k = n^{k+n}\frac{\beta^k}{k!\Gamma\,(k+n)}.$$

Hence, $A_P\,(n,t,\beta)$ is given by

$$A_P\,(n,t,\beta) = n^n\sum_{k=0}^\infty \frac{(n\beta)^k}{k!\Gamma\,(k+n)}t^{k+n-1}.$$

Thus, semi-exponential Post–Widder operators take the form

$$\left(P_n^\beta f\right)(x) = \frac{n^n}{e^{\beta x}x^n}\sum_{k=0}^\infty \frac{(n\beta)^k}{k!}\frac{1}{\Gamma(n+k)}\int_0^\infty t^{n+k-1}e^{-nt/x}f(t)dt.$$

Observing that $A_P\,(n,t,\beta) = n\,(nt/\beta)^{(n-1)/2}\,I_{n-1}\left(2\sqrt{n\beta t}\right)$, where I_n denotes the modified Bessel function of the first kind, we obtain the alternative representation

$$\left(P_n^\beta f\right)(x) = \frac{n}{x^n e^{\beta x}}\int_0^\infty \left(\frac{nt}{\beta}\right)^{(n-1)/2} e^{-nt/x}I_{n-1}\left(2\sqrt{n\beta t}\right)f\,(t)\,dt.$$

2.5 Semi-exponential Ismail–May operators related to $x\,(1+x)^2$

If we take $p(x) = x\,(1+x)^2$, then for a kernel $W_\beta^R(n, x, k/n) = A_R(n, k, \beta)\, y$, we have

$$\begin{aligned} y' &= \frac{k - nx}{x\,(1+x)^2}\, y - \beta y, \\ \frac{y'}{y} &= k\left(\frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2}\right) - \frac{n}{(1+x)^2} - \beta, \\ \log y &= k \log x - k \log(1+x) + \frac{n+k}{1+x} - \beta x, \end{aligned}$$

implying

$$y = \left(\frac{x}{1+x}\right)^k \exp\left(\frac{n+k}{1+x}\right) e^{-\beta x}.$$

If we put $y = x/(1+x)$ the normalization condition reads

$$\sum_{k=0}^{\infty} A_R(n, k, \beta) \left(y e^{1-y}\right)^k = \exp\left(\beta \frac{y}{1-y} - n(1-y)\right).$$

Now we put $z = y e^{1-y}$, so we have the inverse $y = -W(-z/e)$, where W denotes the Lambert W function, i.e., the inverse of $z \mapsto z e^z$. Hence, $A_R(n, k, \beta)$ are the coefficients of the power series

$$\sum_{k=0}^{\infty} A_R(n, k, \beta)\, z^k = \exp\left(-\beta \frac{W(-z/e)}{1 + W(-z/e)} - n\,(1 + W(-z/e))\right),$$

which is convergent in a neighborhood of $z = 0$. Following Ismail and May [4, Eq. (3.13)] we take advantage of the identity [6, p. 348]

$$e^{nw} = \sum_{k=0}^{\infty} \frac{n\,(n+k)^{k-1}}{k!} \left(w e^{-w}\right)^k \qquad (n \neq 0),$$

which is an easy consequence of the Lagrange expansion theorem. With $w = -W(-z/e)$ we obtain

$$e^{-nW(-z/e)} = \sum_{k=0}^{\infty} \frac{n\,(n+k)^{k-1}}{k!} \left(-W(-z/e)\, e^{W(-z/e)}\right)^k = \sum_{k=0}^{\infty} \frac{n\,(n+k)^{k-1}}{k!} \left(\frac{z}{e}\right)^k.$$

It follows

$$\sum_{k=0}^{\infty} A_R(n, k, \beta)\, z^k = \exp\left(-n - \beta \frac{W(-z/e)}{1 + W(-z/e)}\right) \sum_{k=0}^{\infty} \frac{n\,(n+k)^{k-1}}{k!} \left(\frac{z}{e}\right)^k,$$

i.e., $A_R(n, k, \beta)$ is the coefficient of z^k in the latter power series expansion. The semi-exponential operators related to $x(1+x)^2$, take the form

$$\left(R_n^\beta f\right)(x) = e^{-\beta x} \sum_{k=0}^{\infty} A_R(n, k, \beta) \left(\frac{x}{1+x}\right)^k \exp\left(\frac{n+k}{1+x}\right) f\left(\frac{k}{n}\right).$$

In the special case $\beta = 0$ we have

$$A_R(n, k, \beta = 0) = \frac{n(n+k)^{k-1}}{k!} e^{-(n+k)}$$

and the operators reduce to

$$(R_n f)(x) = \exp\left(\frac{-nx}{1+x}\right) \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k \exp\left(\frac{-kx}{1+x}\right) f\left(\frac{k}{n}\right)$$

[4, Eq. (3.14)]. As Ismail and May remarked, the substitution $y = x/(1+x)$ leads to the operators

$$(R_n^* f)(y) = e^{-ny} \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}}{k!} \left(ye^{-y}\right)^k f\left(\frac{k}{n+k}\right) \qquad (y \in (0,1))$$

It may be considered as an open problem to find a closed form of the coefficients $A_R(n, k, \beta)$.

2.6 Semi-exponential Ismail–May operators related to x^3

If we take $p(x) = x^3$, then for a kernel $W_\beta^Q(n, x, t) = A_Q(n, t, \beta)\, y$, we have

$$\begin{aligned} y' &= \frac{n(t-x)}{x^3} y - \beta y, \\ \frac{y'}{y} &= n\left(\frac{t}{x^3} - \frac{1}{x^2}\right) - \beta, \\ \log y &= n\left(-\frac{t}{2x^2} + \frac{1}{x}\right) - \beta x. \end{aligned}$$

Thus

$$y = \exp\left(\frac{n}{x} - \frac{nt}{2x^2} - \beta x\right).$$

If we put $s = n/\left(2x^2\right)$ the normalization condition reads

$$\int_0^\infty A_Q(n, t, \beta)\, e^{-st} dt = \exp\left(\beta\sqrt{\frac{n}{2s}} - \sqrt{2ns}\right),$$

such that $A_Q(n, t, \beta)$ is the inverse Laplace transform $\mathcal{L}^{-1}$ of $\exp\left(\beta\sqrt{n/(2s)} - \sqrt{2ns}\right)$. We have

$$\mathcal{L}^{-1}\left\{\exp\left(1/\sqrt{s}\right) - 1\right\} = \sum_{k=1}^{\infty} \frac{1}{k!} \mathcal{L}^{-1}\left\{s^{-k/2}\right\} = \sum_{k=1}^{\infty} \frac{1}{k!\,\Gamma(k/2)} t^{k/2-1},$$

which implies

$$\mathcal{L}^{-1}\left\{\exp\left(\beta\sqrt{\frac{n}{2s}}\right)\right\} = \delta(t) + \sum_{k=1}^{\infty} \frac{1}{k!\,\Gamma(k/2)} \left(\frac{n\beta^2}{2}\right)^{k/2} t^{k/2-1}, \tag{2}$$

where $\delta(t)$ denotes Dirac's delta function. It is well known that

$$\mathcal{L}^{-1}\left\{\exp\left(-\sqrt{2ns}\right)\right\} = \sqrt{\frac{n}{2\pi}}\, t^{-3/2} e^{-n/(2t)}. \tag{3}$$

We will take advantage of the convolution formula

$$\mathcal{L}^{-1}\left\{\mathcal{L}\left\{g\right\}\mathcal{L}\left\{h\right\}\right\} = g * h,$$

where

$$\left(g * h\right)(t) = \int_0^t g\left(t-u\right)h\left(u\right)du.$$

Combining Eqs. (2) and (3)

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\exp\left(\beta\sqrt{n/\left(2s\right)} - \sqrt{2ns}\right)\right\} \\
&= \sqrt{\frac{n}{2\pi}}\int_0^t u^{-3/2}e^{-n/(2u)}\delta\left(t-u\right)du \\
&\quad + \sqrt{\frac{n}{2\pi}}\sum_{k=1}^{\infty}\frac{1}{k!\Gamma\left(k/2\right)}\left(\frac{n\beta^2}{2}\right)^{k/2}\int_0^t u^{-3/2}e^{-n/(2u)}\left(t-u\right)^{k/2-1}du.
\end{aligned}$$

Thus, semi-exponential operators related to $p\left(x\right) = x^3$ take the form

$$\left(Q_n^{\beta} f\right)(x) = e^{n/x-\beta x}\int_0^{\infty} A_Q\left(n,t,\beta\right)e^{-nt/\left(2x^2\right)}f\left(t\right)dt,$$

where

$$\begin{aligned}
A_Q\left(n,t,\beta\right) = \sqrt{\frac{n}{2\pi}}\Bigg(& t^{-3/2}e^{-n/(2t)} + \sum_{k=1}^{\infty}\frac{1}{k!\Gamma\left(k/2\right)}\left(\frac{n\beta^2}{2}\right)^{k/2} \\
&\times \int_0^t u^{-3/2}e^{-n/(2u)}\left(t-u\right)^{k/2-1}du\Bigg).
\end{aligned}$$

In the special case $\beta = 0$ we have

$$A_Q\left(n,t,\beta=0\right) = \sqrt{\frac{n}{2\pi}}t^{-3/2}e^{-n/(2t)}$$

and the operators reduce to

$$\left(Q_n^{\beta=0} f\right)(x) = \sqrt{\frac{n}{2\pi}}e^{n/x}\int_0^{\infty} t^{-3/2}\exp\left(-\frac{n}{2t} - \frac{nt}{2x^2}\right)f\left(t\right)dt$$

[4, Eq. (3.11)].

2.7 Semi-exponential Ismail–May operators related to $1 + x^2$

If we take $p\left(x\right) = 1 + x^2$, then for a kernel $W_{\beta}^{T}\left(n,x,t\right) = A_T\left(n,t,\beta\right)y$, we have

$$\begin{aligned}
y' &= \frac{n(t-x)}{1+x^2}y - \beta y, \\
\frac{y'}{y} &= \frac{nt}{1+x^2} - \frac{nx}{1+x^2} - \beta, \\
\log y &= nt\arctan x - \frac{n}{2}\log\left(1+x^2\right) - \beta x,
\end{aligned}$$

implying

$$y = e^{nt \arctan x - \beta x} \left(1 + x^2\right)^{-n/2}.$$

The operators related to $1 + x^2$ take the form

$$\left(T_n^\beta f\right)(x) = \frac{e^{-\beta x}}{\left(1 + x^2\right)^{n/2}} \int_{-\infty}^{\infty} A_T(n, t, \beta)\, e^{nt \arctan x} f(t) dt$$

To have the normalization, we need

$$\int_{-\infty}^{\infty} A_T(n, t, \beta)\, e^{nt \arctan x} dt = e^{\beta x} \left(1 + x^2\right)^{n/2}.$$

If we put $s = n \arctan x$, this is equivalent to

$$\int_{-\infty}^{\infty} A_T(n, t, \beta)\, e^{st} dt = \frac{e^{\beta \tan(s/n)}}{\cos^n(s/n)}.$$

Using the identity [5, Section 9, p. 46] (see [4, Lemma 3.3])

$$\int_{-\infty}^{\infty} \left| \Gamma\left(\frac{\lambda + it}{2}\right) \right|^2 e^{st} dt = \frac{\pi \Gamma(\lambda)}{2^{\lambda-2} \cos^\lambda s} \qquad (\lambda > 0, -\pi/2 < s < \pi/2)$$

Ismail and May [4, Eq. (3.10)] obtained in the special case $\beta = 0$,

$$A_T(n, t, \beta = 0) = \frac{2^{n-2} n}{\pi \Gamma(n)} \left| \Gamma\left(n \frac{1 + it}{2}\right) \right|^2.$$

The main target to find a closed expression for $A_T(n, t, \beta)$, for general $\beta > 0$, may be considered as an open problem.

Acknowledgements The authors thank both anonymous reviewers for valuable suggestions that led to a better exposition of the paper.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Abel, U., Gupta, V.: Rate of convergence of exponential type operators related to $p(x) = 2x^{3/2}$ for functions of bounded variation. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **114**, Art. 188 (2020). https://doi.org/10.1007/s13398-020-00919-y
2. Gupta, V., Rassias, M.T.: Computation and Approximation. Ser. Springer Briefs in Mathematics. Springer Nature Switzerland AG, New York (2021)
3. Herzog, M.: Semi-exponential operators. Symmetry **13**, 637 (2021). https://doi.org/10.3390/sym13040637
4. Ismail, M., May, C.P.: On a family of approximation operators. J. Math. Anal. Appl. **63**, 446–462 (1978)
5. Oberhettinger, F.: Tabellen zur Fourier Transformation. Springer, Berlin (1957)

6. Polya, G., Szegö, G.: Problems and Theorems in Analysis, vol. 1 (English translation). Springer, New York (1972)
7. Tyliba, A., Wachnicki, E.: On some class of exponential type operators. Ann. Soc. Math. Pol. Ser. I Comment. Math. **45**, 59–73 (2005)