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ORIGINAL PAPER

Self-derivations on the noncommutative Schwartz space

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Abstract We characterize derivations acting on the noncommutative Schwartz space.

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1 Introduction

The aim of the paper is twofold. First, we give a full proof of the characterization of boundedly approximately contractible Fréchet algebras. This result was announced (without proof) in [24]. Second, we apply this characterization to show that if $\delta\colon \mathcal{S} \to \mathcal{S}$ is a derivation acting on the noncommutative Schwartz space $\mathcal{S}$ then there is an element in the multiplier algebra of $\mathcal{S}$, say x, such that $\delta(a) = ax - xa$ for all $a \in \mathcal{S}$. Versions of for the so-called *smooth* **-algebras*—see [10, Section 4.8, Proposition]. To prove our derivation result we will need to rely at some point on the nuclearity (in the sense of Grothendieck) of the multiplier algebra of $\mathcal{S}$. Therefore we may say that, in a sense, all self-derivations on the noncommutative Schwartz space are inner. This result may be deduced from [25, Prop. 6.3.2]. The method of proof in [25] uses the theory of the so-called $\mathcal{O}^*$-algebras. Our approach will necessarily be different.

The noncommutative Schwartz space (the definition of which is given in the next section) is a specific m-convex Fréchet *-algebra. It is isomorphic (as a Fréchet *-algebra) to the algebra $L(\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R}))$ of linear and continuous operators from the space of tempered distributions into the Schwartz space of smooth rapidly decreasing functions (for more representations see [8, Th. 1.1]). It is also isomorphic (as a Fréchet space) to $\mathcal{S}(\mathbb{R})$.

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The noncommutative Schwartz space and the space $\mathcal{S}(\mathbb{R})$ itself play a role in a number of fields, e.g.: structure theory of Fréchet spaces and splitting of short exact sequences—[18, Part IV]; K-theory—[1,6,15,19]; C^*-dynamical systems—[13]; cyclic cohomology for crossed products—[26]; locally convex analogues of operator spaces—[11,12]; quantum mechanics, where it is called the *space of physical states* and its dual is the so-called *space of observables*—[9]. Recently, some progress in the investigation of the noncommutative Schwartz space has been made. This contains: functional calculus—[4]; description of closed, commutative, *-subalgebras—[3]; automatic continuity—[22]; amenability properties—[20,22]; Grothendieck inequality—[17]. Specifically, the multiplier algebra of the noncommutative Schwartz space has been identified and investigated in [5].

The paper is divided into four parts. In the next section we collect all the necessary definitions and basic properties of the objects involved. Section 3 contains a characterization of boundedly approximately contractible Fréchet algebras. In the last section we use this characterization (more precisely, the proof of it) to show the main result of the paper, i.e. we prove that if $\delta\colon \mathcal{S} \to \mathcal{S}$ is a derivation on the noncommutative Schwartz space then there exists an element x in the multiplier algebra of $\mathcal{S}$ such that $\delta(a) = ax - xa$ for all operators $a \in \mathcal{S}$.

General references are: for functional analysis—[18], for Banach algebra theory—[7] and for non-Banach algebra theory—[14].

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2 Notation and preliminaries

Let

$$s = \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}\colon\ |\xi|_n^2 := \sum_{j=1}^{+\infty} |\xi_j|^2 j^{2n} < +\infty \text{ for all } n \in \mathbb{N} \right\}$$

denote the so-called space of rapidly decreasing sequences. This space becomes Fréchet when endowed with the above defined sequence $(|\cdot|_n)_{n\in\mathbb{N}}$ of norms. It is isomorphic to the Schwartz space of test functions for tempered distributions or to the space of smooth functions on a compact, smooth manifold. The basis $(U_n)_{n\in\mathbb{N}}$ of zero neighbourhoods of s is defined by $U_n := \{\xi \in s\colon\ |\xi|_n \leqslant 1\}$. The topological dual of s is the so-called space of slowly increasing sequences

$$s' = \left\{ \eta = (\eta_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}\colon\ |\eta|_{-n}^2 := \sum_{j=1}^{+\infty} |\eta_j|^2 j^{-2n} < +\infty \text{ for some } n \in \mathbb{N} \right\}.$$

It is isomorphic to the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions on $\mathbb{R}^n$ or to the space $\mathcal{E}'(K)$ of distributions with support contained in a fixed, compact set $K \subset \mathbb{R}^n$.

The *noncommutative Schwartz space* $\mathcal{S}$ is the space $L(s', s)$ of all linear and continuous operators from the dual of s into s itself, endowed with the topology of uniform convergence on bounded sets. This is a Fréchet space topology described by the sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of norms given by

$$\|x\|_n := \sup\{|x\xi|_n\colon\ \xi \in U_n^\circ\}, \tag{1}$$

where $U_n^\circ = \{\xi \in s'\colon\ |\xi|_n' \leqslant 1\}$ is the polar of the zero neighbourhood $U_n \subset s$. The identity map $\iota\colon s \hookrightarrow s'$ is a continuous embedding and defines multiplication in $\mathcal{S}$ by

$$xy := x \circ \iota \circ y, \quad x, y \in \mathcal{S}.$$

This multiplication is separately continuous therefore jointly continuous by [27, Th. 1.5]. Duality between s and s' is given by

$$\langle \xi, \eta \rangle := \sum_{j\in\mathbb{N}} \xi_j \overline{\eta_j}, \quad \xi \in s, \eta \in s' \tag{2}$$

and it leads to an involution map on $\mathcal{S}$ defined by

$$\langle x^*\xi, \eta \rangle := \langle \xi, x\eta \rangle, \quad x \in \mathcal{S},\ \xi, \eta \in s'.$$

With these operations the noncommutative Schwartz space $\mathcal{S}$ becomes an m-convex Fréchet *-algebra. It may be identified, via the *-algebra isomorphism $x \mapsto (\langle xe_j, e_i \rangle)_{i,j\in\mathbb{N}}$, $(e_j)_{j\in\mathbb{N}}$—standard unit vector basis, with the so-called algebra $\mathcal{K}^\infty$ of *rapidly decreasing matrices* defined as

$$\mathcal{K}^\infty := \Big\{x = (x_{i,j})_{i,j\in\mathbb{N}}\colon\ \|x\|_{n,\infty} := \sup_{i,j\in\mathbb{N}} |x_{ij}|(ij)^n < +\infty \quad \text{for all } n \in \mathbb{N}\Big\}. \tag{3}$$

The topological dual $\mathcal{S}'$ may be therefore identified with the space of the so-called *slowly increasing matrices*, i.e.

$$\mathcal{S}' = \Big\{\phi = (\phi_{ij})_{i,j\in\mathbb{N}} \mid \ \sup\{|\phi_{ij}|(ij)^{-k}\colon\ i, j \in \mathbb{N}\} < +\infty \quad \text{for some } k \in \mathbb{N}\Big\}.$$

The duality in the matrix language is given by the trace, i.e. if $x \in \mathcal{S}$, $\phi \in \mathcal{S}'$ then

$$\phi(x) := \langle\langle x, \phi \rangle\rangle = \sum_{i,j=1}^{+\infty} x_{ij}\overline{\phi_{ij}}. \tag{4}$$

More on the dual space $\mathcal{S}'$ may be found in [23]. The multiplier algebra $\mathcal{MS}$ of the noncommutative Schwartz space may be described in several ways, depending on the context—see [5, Corollary 4.4]. For our purposes the most convenient one will be to view the multiplier algebra as a *-algebra of all infinite scalar matrices $(x_{ij})_{i,j\in\mathbb{N}}$ such that for every $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ satisfying

$$\|x\|_{N,n} := \sup_{i,j\in\mathbb{N}} \left\{ |x_{ij}| \cdot \max\left\{\frac{i^N}{j^n}, \frac{j^N}{i^n}\right\}\right\} < \infty. \tag{5}$$

The natural topology on $\mathcal{MS}$ is that of a PLS-space. More precisely, there is a topological *-algebra isomorphism

$$\mathcal{MS} = \mathrm{proj}_{N\in\mathbb{N}} \mathrm{ind}_{n\in\mathbb{N}} \ell_\infty\big((a_{ij;N,n})_{i,j\in\mathbb{N}}\big), \tag{6}$$

where

$$a_{ij;N,n} := \max\left\{\frac{i^N}{j^n}, \frac{j^N}{i^n}\right\}$$

and

$$\ell_\infty\big((a_{ij;N,n})_{i,j\in\mathbb{N}}\big) := \Big\{(x_{ij})_{i,j\in\mathbb{N}}\colon\ \sup_{i,j\in\mathbb{N}} |x_{ij}|a_{ij;N,n} < \infty\Big\}$$

is a weighted ℓ_∞-space of bounded, doubly indexed sequences. It ia also useful to view $\mathcal{MS}$ as an algebra of all linear and continuous operators $x\colon s \to s$ which extend (necessarily uniquely) to linear and continuous operators $\tilde{x}\colon s' \to s'$. In other words, $\mathcal{MS} = L(s) \cap L(s')$ and the base of zero neighbourhoods in $\mathcal{MS}$ is given by the family $\{U \cap V\}$, where U, respectively V runs through the base of zero neighbourhoods in $L(s)$, respectively in $L(s')$.

3 Boundedly approximately contractible Fréchet algebras

The notion of an amenable/contractible algebra admits a number of weakened versions see e.g. [2,16]. We will deal with the following one. A Fréchet algebra A is *boundedly approximately contractible* if for every A-bimodule X and every continuous derivation $\delta\colon A \to X$ there is a net $(x_\alpha)_\alpha \subset X$ such that

$$\delta(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \qquad \forall\, a \in A$$

and the set of inner derivations $(\mathrm{ad}_{x_\alpha})_\alpha \subset L(A, X)$ is equicontinuous. By ad_x we denote the so-called *inner derivation* acting by $a \mapsto a \cdot x - x \cdot a$, $a \in A$. If A is a unital algebra then by $\mathbf{1}$ we denote the unit element in A and if A is a non-unital algebra then by $A^\#$ we denote its unitization.

Proposition 1 *Let A be a non-unital Fréchet algebra. Then A is boundedly approximately contractible if and only if $A^\#$ is boundedly approximately contractible.*

Proof Sufficiency. Let $\delta\colon A \to X$ be a continuous derivation into an A-bimodule X. By defining $\delta(\mathbf{1}) := 0$ and the module operations $\mathbf{1} \cdot x = x \cdot \mathbf{1} := x$, $x \in X$, we get a continuous derivation $\delta\colon A^\# \to X$ into an $A^\#$-bimodule X. By assumption $\delta = \lim_\alpha \mathrm{ad}_{x_\alpha}$ and $(\mathrm{ad}_{x_\alpha})_\alpha \subset L(A^\#, X)$ is equicontinuous. Since A is a complemented ideal in $A^\#$, the net $(\mathrm{ad}_{x_\alpha})_\alpha$ is also equicontinuous in $L(A, X)$.

Necessity. Let $\delta\colon A^\# \to X$ be a continuous derivation. By [16, Lemma 2.2] we may assume X is unital, i.e. $\mathbf{1} \cdot x = x \cdot \mathbf{1} = x$ for all $x \in X$. Consequently, $\delta(\mathbf{1}) = 0$. Since $\delta\,|_A$ is a continuous derivation, we get by assumption $\delta\,|_A = \lim_\alpha \mathrm{ad}_{x_\alpha}$. Therefore

$$\begin{aligned}\delta(a + \lambda\mathbf{1}) = (\delta\,|_A)(a) &= \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a + \lambda\mathbf{1} \cdot x_\alpha - x_\alpha \cdot \lambda\mathbf{1}) \\ &= \lim_\alpha ((a + \lambda\mathbf{1}) \cdot x_\alpha - x_\alpha \cdot (a + \lambda\mathbf{1}))\end{aligned}$$

and the net $(\mathrm{ad}_{x_\alpha})_\alpha \subset L(A^\#, X)$ is equicontinuous. □

We are now going to give a characterization of boundedly approximately contractible Fréchet algebras. It was first stated (without proof) in [24]. Necessity in the case of Banach algebras was proved in [2, Th. 2.5]. The method of proof (sufficiency part) will be of particular importance for us. Before proceeding to this let us recall few basic definitions. If X, Y are Fréchet spaces with the respective non-decreasing sequences of seminorms giving their topologies and $T\colon X \to Y$ is a linear and continuous operator then the *characteristic of continuity* of T is a function $\sigma_T : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$, $\sigma_T(k)$ is defined to be the smallest number n satisfying

$$\exists\, C > 0\ \forall\, x \in X\colon\ \|Tx\|_k \leqslant C\|x\|_n. \tag{7}$$

The topology of the projective tensor product $X\widehat{\otimes}Y$ is given by the sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of seminorms defined by

$$\|u\|_n := \inf\left\{\sum_{j=1}^k \|x_j\|_n\|y_j\|_n :\ u=\sum_{j=1}^k x_j\otimes y_j,\ x_j\in X, y_j\in Y, k\in\mathbb{N}\right\}.$$

If X, Y are algebras then

$$\Big(\sum_{i=1}^m x_i\otimes y_i\Big)\cdot\Big(\sum_{j=1}^n a_j\otimes b_j\Big) := \sum_{i,j} x_ia_j\otimes y_ib_j$$

and if they both are involutive then

$$\Big(\sum_{i=1}^m x_i\otimes y_i\Big)^* := \sum_{i=1}^m x_i^*\otimes y_i^*.$$

If A is a Fréchet algebra then the product map $\pi: A\widehat{\otimes}A\to A$ is defined by

$$\pi\Big(\sum_{j=1}^k a_j\otimes b_j\Big) := \sum_{j=1}^k a_jb_j$$

and extended by density. With respect to the projective tensor product topology, the product map is continuous.

Theorem 1 *Let A be a Fréchet algebra with the topology given by a non-decreasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms. TFAE:*

(1) *A is boundedly approximately contractible,*
(2) *there are nets $(M_\alpha)_\alpha\subset A\widehat{\otimes}A$, $(F_\alpha)_\alpha, (G_\alpha)_\alpha\subset A$, a function $\sigma:\mathbb{N}\to\mathbb{N}$ and a sequence $(C_k)_k$ of positive constants satisfying:*
   (i) *$\pi(M_\alpha)=F_\alpha+G_\alpha$ for all α,*
   (ii) *$\lim_\alpha aF_\alpha=\lim_\alpha G_\alpha a=a$ for all $a\in A$,*
   (iii) *$\|aF_\alpha\|_k\leqslant C_k\|a\|_{\sigma(k)},\ \|G_\alpha a\|_k\leqslant C_k\|a\|_{\sigma(k)}$ for all α, all $a\in A$ and all $k\in\mathbb{N}$,*
   (iv) *$\lim_\alpha(a\cdot M_\alpha-M_\alpha\cdot a-a\otimes G_\alpha+F_\alpha\otimes a)=0$ for all $a\in A$,*
   (v) *$\|a\cdot M_\alpha-M_\alpha\cdot a-a\otimes G_\alpha+F_\alpha\otimes a\|_k\leqslant C_k\|a\|_{\sigma(k)}$ for all α, all $a\in A$ and all $k\in\mathbb{N}$.*

Proof (1) $\Rightarrow$ (2): this is similar to the proof of [2, Th. 2.5].
(2) $\Rightarrow$ (1): by Proposition 1 we may assume A is unital. Define

$$d_\alpha := M_\alpha - F_\alpha\otimes\mathbf{1}-\mathbf{1}\otimes G_\alpha+\mathbf{1}\otimes\mathbf{1}\in A\widehat{\otimes}A$$

and observe that

$$\begin{aligned} a\cdot d_\alpha-d_\alpha\cdot a &= (a\cdot M_\alpha-M_\alpha\cdot a-a\otimes G_\alpha+F_\alpha\otimes a)\\ &\quad+(a-aF_\alpha)\otimes\mathbf{1}+\mathbf{1}\otimes(G_\alpha a-a)\to 0. \end{aligned}\tag{8}$$

Moreover,

$$\|a\cdot d_\alpha-d_\alpha\cdot a\|_k\leqslant 3C_k\|a\|_{\sigma(k)}\qquad\forall\,\alpha.\tag{9}$$

Using assumption (i) we obtain

$$\pi(d_\alpha)=\pi(M_\alpha)-F_\alpha-G_\alpha+\mathbf{1}=\mathbf{1}\qquad\forall\,\alpha.\tag{10}$$

Let now $\delta\colon A \to X$ be a continuous derivation into a unital A-bimodule X (this restriction is justified by [16, Lemma 2.2]). Define

$$D\colon A\widehat{\otimes}A \to X, \qquad D(a\otimes b) := a\cdot\delta(b)$$

and extend linearly. Clearly, D is continuous. Put

$$x_\alpha := D(d_\alpha) \tag{11}$$

and observe that if we denote

$$d_\alpha = \sum_{n=1}^{\infty} a_n^\alpha \otimes b_n^\alpha$$

then

$$x_\alpha = \sum_{n=1}^{\infty} a_n^\alpha \cdot \delta(b_n^\alpha).$$

Now we choose an arbitrary element $a \in A$ and make the following computation:

$$\begin{aligned}
&a\cdot x_\alpha - x_\alpha\cdot a - \delta(a)\\
&\qquad = \sum_n aa_n^\alpha\cdot\delta(b_n^\alpha) - \sum_n a_n^\alpha\cdot\delta(b_n^\alpha)\cdot a - \sum_n a_n^\alpha b_n^\alpha\cdot\delta(a)\\
&\qquad = \sum_n aa_n^\alpha\cdot\delta(b_n^\alpha) - \sum_n a_n^\alpha\cdot\delta(b_n^\alpha a)\\
&\qquad = D(a\cdot d_\alpha - d_\alpha\cdot a).
\end{aligned}$$

The first equality follows by (10) and the fact that X is unital while the second equality follows by the derivation rule. Continuity of D and (8) imply

$$\delta(a) = \lim_\alpha (a\cdot x_\alpha - x_\alpha\cdot a) \qquad \forall\, a\in A.$$

Equicontinuity of $(\mathrm{ad}_{x_\alpha})_\alpha \subset L(A, X)$ follows by (9). Indeed, for every $k\in\mathbb{N}$ we have

$$\begin{aligned}
\|a\cdot x_\alpha - x_\alpha\cdot a\|_k &\leqslant \|\delta(a)\|_k + \|D(a\cdot d_\alpha - d_\alpha\cdot a)\|_k\\
&\leqslant A_k\|a\|_{\sigma_\delta(k)} + 3B_kC_k\|a\|_{\sigma(\sigma_D(k))}
\end{aligned}$$

and the constants A_k, B_k satisfy (7) for $\delta\colon A\to X$ and $D\colon A\widehat{\otimes}A \to X$, respectively.

4 Self-derivations on $\mathcal{S}$

We start by making several observations. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ be a sequence of operators in the noncommutative Schwartz space defined by

$$u_n\colon s' \to s, \qquad u_n((\xi_j)_j) := (\xi_1, \ldots \xi_n, 0, 0, \ldots).$$

We can also view each u_n as an infinite matrix

$$u_n = \begin{pmatrix} I_n & 0\\ 0 & 0\end{pmatrix},$$

where I_n is the $n\times n$ identity matrix. From [22, Prop. 2] it follows that $(u_n)_n$ is a sequential approximate identity.

If δ is a derivation on a 'matrix algebra' then it has to have a specific form on matrix units $(e_{ij})_{i,j\in\mathbb{N}}$. Indeed, since

$$\delta(e_{ij}) = \delta(e_{ij})e_{jj} + e_{ij}\delta(e_{jj}) = e_{ii}\delta(e_{ij}) + \delta(e_{ii})e_{ij},$$

the matricial form of $\delta(e_{ij}) = (t^{ij}_{pq})_{p,q\in\mathbb{N}}$ is

$$\begin{pmatrix} & & & t^{ij}_{1j} & & \\ & \mathbf{0} & & \vdots & \mathbf{0} & \\ & & & t^{ij}_{i-1,j} & & \\ t^{jj}_{j1} & \dots & t^{jj}_{j,j-1} & t^{ij}_{ij} & t^{jj}_{j,j+1} & \dots \\ & & & t^{ij}_{i+1,j} & & \\ & \mathbf{0} & & \vdots & \mathbf{0} & \end{pmatrix} = \begin{pmatrix} & & & t^{ii}_{1i} & & \\ & \mathbf{0} & & \vdots & \mathbf{0} & \\ & & & t^{ii}_{i-1,i} & & \\ t^{ij}_{i1} & \dots & t^{ij}_{i,j-1} & t^{ij}_{ij} & t^{ij}_{i,j+1} & \dots \\ & & & t^{ii}_{i+1,i} & & \\ & \mathbf{0} & & \vdots & \mathbf{0} & \end{pmatrix}, \tag{12}$$

where the only non-zero entries lie in the i-th row and j-th column. Observe that in the (i, j)-th entry the 'missing' summand t^{jj}_{jj} (on the left) and t^{ii}_{ii} (on the right) is zero. Indeed, e_{kk} is a projection for every $k \in \mathbb{N}$ therefore by [7, Prop. 1.8.2]

$$t^{kk}_{kk} = \langle\langle e_{kk}, \delta(e_{kk})\rangle\rangle = \langle\langle e_{kk}, e_{kk}\delta(e_{kk})e_{kk}\rangle\rangle = 0.$$

From this we obtain that $\delta(u_n)$ is of the form

$$\begin{pmatrix} & & & t^{11}_{1,n+1} & t^{11}_{1,n+2} & \dots \\ & \mathbf{0} & & \vdots & \vdots & \dots \\ & & & t^{nn}_{n,n+1} & t^{nn}_{n,n+2} & \dots \\ t^{11}_{n+1,1} & \dots & t^{nn}_{n+1,n} & & & \\ t^{11}_{n+2,1} & \dots & t^{nn}_{n+2,n} & & & \\ \vdots & \vdots & \vdots & & \mathbf{0} & \end{pmatrix}. \tag{13}$$

Observe that by [21, Prop. 2] we have an $n \times n$ zero matrix in the upper left corner. The last matrix we are interested in is $\frac{1}{n}\sum_{i,j=1}^{n} e_{ji}\delta(e_{ij})$. It is of the form

$$\begin{pmatrix} 0 & t^{11}_{12} & t^{11}_{13} & \dots & t^{11}_{1n} & t^{11}_{1,n+1} & \dots \\ -t^{11}_{21} & 0 & t^{22}_{23} & \dots & t^{22}_{2n} & t^{22}_{2,n+1} & \dots \\ -t^{11}_{31} & -t^{22}_{32} & 0 & \dots & t^{33}_{3n} & t^{33}_{3,n+1} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ -t^{11}_{n1} & -t^{22}_{n2} & -t^{33}_{n3} & \dots & 0 & t^{nn}_{n,n+1} & \dots \\ & & & & & & \\ & & \mathbf{0} & & & \mathbf{0} & \end{pmatrix}. \tag{14}$$

Observe that the upper triangular entries follow by adding rows appearing on the left hand side of (12) while the lower ones follow by adding columns appearing on the right hand side of (12). This is indeed the case, since it follows from (13) that for any $p, q = 1, \dots, n$ we have

$$\left\langle\!\left\langle e_{pq}, \frac{1}{n}\sum_{i,j=1}^{n}\Big(e_{ji}\delta(e_{ij}) + \delta(e_{ji})e_{ij}\Big)\right\rangle\!\right\rangle = \langle\langle e_{pq}, \delta(u_n)\rangle\rangle = 0.$$

Before proceeding to the main result of the paper we will need the following fact.

Proposition 2 *The multiplication map* $M\colon \mathcal{S}\times\mathcal{MS}\to\mathcal{S}$ *is separately continuous.*

Proof Fix $y\in\mathcal{MS}$ and define the map $M_y\colon\mathcal{S}\to\mathcal{S}$ by $M_y(x):=xy$. To show its continuity, let $N\in\mathbb{N}$. By (5) there is $n\in\mathbb{N}$ such that $\|y\|_{N,n}<\infty$ (recall that we can always take $n\geqslant N$). This leads to the following estimations:

$$\begin{aligned}\|xy\|_{N,\infty}&\leqslant\sup_{i,j\in\mathbb{N}}\sum_{k=1}^{\infty}|x_{ik}||y_{kj}|(ij)^N\\&\leqslant\sup_{i,j\in\mathbb{N}}\sum_{k=1}^{\infty}k^{-2}|x_{ik}|(ik)^{n+2}|y_{kj}|\frac{j^N}{k^n}\\&\leqslant C\|y\|_{N,n}\|x\|_{n+2,\infty},\end{aligned}$$

where we have denoted $C:=\sum_k k^{-2}$. On the other hand, for a fixed $x\in\mathcal{S}$ we define the map $M_x\colon\mathcal{MS}\to\mathcal{S}$ by $M_x(y):=xy$. Since by (6) we have

$$\mathcal{MS}=\mathrm{proj}_{N\in\mathbb{N}}\mathrm{ind}_{n\in\mathbb{N}}\ell_\infty\big((a_{ij;N,n})_{i,j\in\mathbb{N}}\big),$$

continuity of M_x means that (see [18, Propositions 22.6, 24.7]) for every $N\in\mathbb{N}$ there is $M\in\mathbb{N}$ such that for any $m\in\mathbb{N}$ there exists $C>0$ satisfying

$$\|xy\|_{N,\infty}\leqslant C\|y\|_{M,m}\qquad(y\in\mathcal{MS}).$$

Following the computations for the map M_y above we get

$$\|xy\|_{N,\infty}\leqslant C\|x\|_{m+2,\infty}\|y\|_{M,m}\qquad(y\in\mathcal{MS})$$

with the same constant $C=\sum_k k^{-2}$. □

Now we are ready to characterize self-derivations on the noncommutative Schwartz space. Recall from [22, Theorem 13] that every such derivation is continuous.

Theorem 2 *If* $\delta\colon\mathcal{S}\to\mathcal{S}$ *is a derivation on the noncommutative Schwartz space then there exists an element* x *in the multiplier algebra* $\mathcal{MS}$ *of the noncommutative Schwartz space such that*

$$\delta(a)=ax-xa\qquad\forall\,a\in\mathcal{S}.$$

Proof From [22, Th. 16] it follows that the noncommutative Schwartz space satisfies condition (2) of Theorem 1 with the following sequences:

$$M_n:=u_n\otimes u_n+\frac{1}{n}\sum_{i,j=1}^{n}e_{ij}\otimes e_{ji},\ \ F_n=G_n=u_n,\ \ \sigma(k)=3k+1,\ \ C_k=4.$$

Let now $\delta\colon\mathcal{S}\to\mathcal{S}$ be a derivation. By (11) and the proof of Theorem 1 we can observe that

$$\delta(a)=\lim_{n\to\infty}(ax_n-x_na)\qquad\forall\,a\in\mathcal{S},\tag{15}$$

where

$$x_n=u_n\delta(u_n)-\delta(u_n)+\frac{1}{n}\sum_{i,j=1}^{n}e_{ji}\delta(e_{ij})\in\mathcal{S}.$$

If we now denote $\delta(e_{ij}) = (t_{pq}^{ij})_{p,q\in\mathbb{N}}$ then by (13) and (14) we get that

$$x_n = \begin{pmatrix} 0 & t_{12}^{11} & t_{13}^{11} & \dots & t_{1n}^{11} & t_{1,n+1}^{11} & \dots \\ -t_{21}^{11} & 0 & t_{23}^{22} & \dots & t_{2n}^{22} & t_{2,n+1}^{22} & \dots \\ -t_{31}^{11} & -t_{32}^{22} & 0 & \dots & t_{3n}^{33} & t_{3,n+1}^{33} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ -t_{n1}^{11} & -t_{n2}^{22} & -t_{n3}^{33} & \dots & 0 & t_{n,n+1}^{nn} & \dots \\ -t_{n+1,1}^{11} & -t_{n+1,2}^{22} & -t_{n+1,3}^{33} & \dots & -t_{n+1,n}^{nn} & & \\ -t_{n+2,1}^{11} & -t_{n+2,2}^{22} & -t_{n+2,3}^{33} & \dots & -t_{n+2,n}^{nn} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \mathbf{0} & \end{pmatrix}. \tag{16}$$

In general, the sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ need not be bounded in $\mathcal{S}$. We will show that—however—it is bounded in (the weaker topology of) the multiplier algebra $\mathcal{MS}$. To this end, observe that continuity of δ implies that

$$\forall\, K \in \mathbb{N}\ \exists\, k \in \mathbb{N}, C_K > 0\ \forall\, x \in \mathcal{S}: \qquad \|\delta(x)\|_{K,\infty} \leqslant C_K \|x\|_{k,\infty}$$

(continuity with respect to the topology given by the norms defined in (3)). Applying this to matrix units $(e_{ij})_{i,j\in\mathbb{N}}$ we get

$$\forall\, K \in \mathbb{N}\ \exists\, k \in \mathbb{N}, C_K > 0\ \forall\, i, j, p, q \in \mathbb{N}: \qquad |t_{pq}^{ij}| \leqslant C_K \frac{(ij)^k}{(pq)^K}.$$

Consequently, for any $i, j \in \mathbb{N}$ this implies

$$|t_{ij}^{ii}| \leqslant C_K \frac{i^{2k}}{(ij)^K} \leqslant C_K \frac{i^{2k}}{j^K}$$

and

$$|t_{ij}^{jj}| \leqslant C_K \frac{j^{2k}}{(ij)^K} \leqslant C_K \frac{j^{2k}}{i^K}.$$

We can now estimate entries of the matrices x_n, $n \in \mathbb{N}$. From (16) we know that

$$\langle\langle e_{ij}, x_n\rangle\rangle = \begin{cases} t_{ij}^{ii}, & \text{if } i < j \text{ and i} \leqslant \text{n} \\ -t_{ij}^{jj}, & \text{if } i > j \text{ and j} \leqslant \text{n} \\ 0, & \text{if } i \geqslant n \text{ and j} \geqslant \text{n} \end{cases}$$

and for any $K, k \in \mathbb{N}$

$$\max\left\{\frac{i^K}{j^k}, \frac{j^K}{i^k}\right\} = \begin{cases} \frac{j^K}{i^k}, & \text{if } i < j \\ \frac{i^K}{j^k}, & \text{if } i > j \end{cases}.$$

Therefore for every $n \in \mathbb{N}$ and any $i, j \in \mathbb{N}$ we obtain that

$$\left|\langle\langle e_{ij}, x_n\rangle\rangle\right| \max\left\{\frac{i^K}{j^{2k}}, \frac{j^K}{i^{2k}}\right\} \leqslant C_K. \tag{17}$$

Using the notation from p. 3 we may write that for every $K \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\ell_\infty\big((a_{ij;K,2k})_{i,j\in\mathbb{N}}\big)$. By [18, Lemma 24.2] $(x_n)_n$ is bounded in $\mathcal{MS}$. Since the multiplier algebra is nuclear by [5, Cor. 4.2], it then follows (see [18, Corollary 28.5, Lemma 24.19]) that the sequence $(x_n)_n$ has a convergent subsequence, say $x := \lim_n x_{k_n}$. Now we wish to apply (15). The only point is that the limit in (15) is taken in $\mathcal{S}$ while $(x_{k_n})_n$ tends to x in the (weaker) topology of $\mathcal{MS}$. Observe, however, that by Proposition 2 the multiplication map $M\colon \mathcal{S} \times \mathcal{MS} \to \mathcal{S}$ is separately continuous. Consequently,

$$\delta(a) = ax - xa \qquad \forall\, a \in \mathcal{S}.$$

Alternatively (without involving nuclearity of the multiplier algebra) one might wish to find the explicit form of the limit operator $x \in \mathcal{MS}$. By (16) we can observe that $\lim_n x_n = x$, where

$$x = \begin{pmatrix} 0 & t_{12}^{11} & t_{13}^{11} & \dots \\ -t_{21}^{11} & 0 & t_{23}^{22} & \dots \\ -t_{31}^{11} & -t_{32}^{22} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Indeed,

$$\langle\langle e_{ij}, x - x_n\rangle\rangle = \begin{cases} t_{ij}^{ii}, & \text{if } n < i < j \\ -t_{ij}^{jj}, & \text{if } n < j < i \\ 0, & \text{if } i \leqslant n \text{ or } j \leqslant n \text{ or } i = j \end{cases}$$

and by (17) we obtain

$$\sup_{i,j\in\mathbb{N}} \left\{ \left|\langle\langle e_{ij}, x - x_n\rangle\rangle\right| \max\left\{ \frac{i^K}{j^{2k+1}}, \frac{j^K}{i^{2k+1}} \right\} \right\} \leqslant C_K \frac{1}{n} \to 0.$$

In other words, for every $K \in \mathbb{N}$ there is another $k \in \mathbb{N}$ so that $\lim_n \|x - x_n\|_{K,k} = 0$ which gives exactly convergence of a sequence in a PLS-space. □

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