

# Complete invariance and compactness of the filled Julia sets of polynomial maps in $\mathbb{C}^N$

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**Abstract** We gather known results on escape radii and add some facts and examples concerning the subject of complete invariance and compactness of filled Julia sets.

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## 1 Introduction

For a self-map  $h : \mathbb{C}^N \rightarrow \mathbb{C}^N$  consider the set

$$K[h] := \left\{ z \in \mathbb{C}^N : (h^n(z))_{n=1}^\infty \text{ is bounded} \right\}$$

where  $\mathbb{C}^N$  is equipped with the Euclidean metric. The sequence  $(h^n(z))_{n=1}^\infty$  is called the *forward orbit* of  $z$ . For polynomials the set defined above coincides with the polynomially convex hull of the classical Julia set. The operation of taking the polynomially convex hull of a set in the one dimensional case means filling in all the holes and this is the explanation of the name *the filled Julia set* (or *the filled-in Julia set*) of  $h$ .

The Julia sets are actively studied for many self-maps of  $\mathbb{C}$  (polynomials, transcendental entire functions; see e.g. [1, 4, 7]),  $\mathbb{C}^2$  (generalized Hénon maps; see e.g. [8]) and  $\mathbb{C}^N$  (polynomial mappings with the Łojasiewicz exponents greater than 1; see [9]). They are completely invariant and one can ask whether this is always the case or if there exist mappings which

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fail to satisfy this condition. In our article we show examples of quite simple maps with the filled Julia sets not completely invariant.

Let us also note that Julia sets of rational mappings, which are self-maps of the Riemann sphere, are also actively studied (see e.g. [1]). However, the Riemann sphere equipped with the spherical metric is compact and therefore bounded. Thus an analogue of our definition of the filled Julia set in this case leads always to the entire sphere. That is why we will not consider this case.

Complete invariance can be stated in the language of the theory of fixed points. Namely, we consider two mappings (related to the self-map  $h$ ) defined by taking images and inverse images under  $h : \mathbb{C}^N \rightarrow \mathbb{C}^N$ :

$$\Phi_h : 2^{\mathbb{C}^N} \ni A \mapsto h(A) \in 2^{\mathbb{C}^N} \quad \text{and} \quad \Psi_h : 2^{\mathbb{C}^N} \ni A \mapsto h^{-1}(A) \in 2^{\mathbb{C}^N}.$$

Of these two mappings, the second one appears to be of more interest. Note that a set is completely invariant under  $h$  if and only if it is a fixed point of both mappings  $\Phi_h$  and  $\Psi_h$ . The family of all subsets, i.e.  $2^{\mathbb{C}^N}$ , seems to be too big, since  $\Phi_h$  and  $\Psi_h$  always have at least one fixed point there, namely the empty set.  $\Psi_h$  has also another fixed point, namely  $\mathbb{C}^N$ , which is also a fixed point for  $\Phi_h$  if  $h$  is surjective. Therefore, it seems to be natural to take restrictions of  $\Phi_h$  and  $\Psi_h$  to a smaller family. Define

$$\text{Comp}(\mathbb{C}^N) = \{K \subset \mathbb{C}^N : K \text{ is compact and non-empty}\}.$$

This family is a good candidate for  $\Phi_h$  in the case of a continuous mapping  $h$  and for  $\Psi_h$  in the case of a proper mapping  $h$ . Moreover,  $\text{Comp}(\mathbb{C}^N)$  is a complete metric space, if equipped with the Hausdorff metric, which is another advantage if we want to speak about fixed points. Therefore, and not only for this reason, we will try to find out when the filled Julia sets are compact.

Let us mention that compact Julia type sets are important examples e.g. in the (pluri)potential theory (see e.g. [9, 12, 18]) and also in the theory of analytic multifunctions (see [13] and the references given there).

It is known that the Julia sets of polynomials (or more generally of regular polynomial mappings) of degree at least 2 are compact and these of transcendental entire maps or of the Hénon maps are unbounded. In the investigations of the compact Julia sets so called escape radii (see [10, 20]) or escape criteria (see [5]) play an important role (see also [2], there is no mention of escape radii there but the geometric significance of one of them, namely  $r_c$ , is investigated).

## 2 Complete invariance

In this section and the next one we investigate a more general setting. Consider a metric space  $(X, d)$  and a self-map  $h : X \rightarrow X$ . We say that a set  $A \subset X$  is *bounded* if there exist a point  $a \in X$  and a positive number  $R$  such that  $A \subset \overline{B}(a, R) := \{x \in X : d(a, x) \leq R\}$ . Set

$$K[h] := \{x \in X : (h^n(x))_{n=1}^\infty \text{ is bounded}\}.$$

As we mentioned in the introduction, this notion is not of interest if  $X$  is bounded, because then  $K[h] = X$ .

We say that a set  $A \subset X$  is *completely invariant* under  $h$  if

$$h(A) = A = h^{-1}(A).$$

Is  $K[h]$  completely invariant (under  $h$ )? Three inclusions are always easily satisfied.

**Proposition 2.1** *Let  $h : X \rightarrow X$  be a mapping. Then  $K[h] = h^{-1}(K[h])$  and  $h(K[h]) \subset K[h]$ .*

*Proof*  $h(x) \in K[h] \iff (h^{n+1}(x))_{n=1}^\infty$  bounded  $\iff x \in K[h]$ .  
 Consequently  $h(K[h]) = h(h^{-1}(K[h])) \subset K[h]$ . □

The third inclusion does not follow suit.

*Remark 2.2* In general  $K[h] \subset h(K[h])$  does not hold.

To see this consider the following situation

*Example 2.3* Let  $B = X \setminus A$  where  $A, B \neq \emptyset$ . Let  $f : A \rightarrow B$  be arbitrary and  $g : B \rightarrow B$  be such that  $K[g] \neq \emptyset$ . Define  $h : X \rightarrow X$  by

$$h|_A = g \circ f \quad \text{and} \quad h|_B = g.$$

If  $f(A) \subset K[g]$ , then  $K[h] = A \cup K[g]$  and  $h(K[h]) \subset K[g]$ .

One of the simplest examples of this type of function is

$$h : \mathbb{R} \ni x \mapsto |x| \in \mathbb{R}.$$

Namely, put  $A := (-\infty, 0)$ ,  $f(x) = -x$  and  $g(x) = x$ . Another example is

$$Q : \mathbb{C}^2 \ni (z, w) \mapsto (z, zw - 1) \in \mathbb{C}^2,$$

where  $A := \{0\} \times (\mathbb{C} \setminus \{-1\})$ ,  $f(0, w) = (0, -1)$  and  $g(z, w) = (z, zw - 1)$ .

We state now a necessary and sufficient condition for the complete invariance.

**Proposition 2.4** *Let  $h : X \rightarrow X$  be a mapping. Then  $K[h]$  is completely invariant under  $h$  if and only if  $K[h] \subset h(X)$ .*

*Proof* In view of Proposition 2.1 it suffices to show that  $K[h] \subset h(K[h])$  if and only if  $K[h] \subset h(X)$ . To get the nontrivial implication it is enough to look at the forward orbit of the point  $y \in h^{-1}(x)$  for any  $x \in K[h]$ . □

We have an immediate consequence.

**Corollary 2.5** *If  $h : X \rightarrow X$  is surjective, then  $K[h]$  is completely invariant under  $h$ .* □

This result is obvious even without Proposition 2.4. Namely, in the case where  $h$  is surjective, if a set  $A$  is backward invariant, i.e. the equality  $h^{-1}(A) = A$  holds, then the complete invariance of  $A$  follows from general properties of images and inverse images (see [1]). It is natural to ask whether surjectivity is a necessary condition. This is not the case. The easiest example can be given by a nonsurjective mapping  $h$  with empty  $K[h]$ , e.g.

$$h : \mathbb{C}^2 \ni (z, w) \mapsto (z + 1, zw - 1) \in \mathbb{C}^2.$$

But let us give another one.

*Example 2.6* The logistic function

$$h : \mathbb{R} \ni x \mapsto 4x(1 - x) \in \mathbb{R}$$

is not surjective and  $K[h] = [0, 1]$  is completely invariant under  $h$ .

Since surjectivity is a sufficient condition for complete invariance, it is worth mentioning the following well known fact which is helpful, e.g. in the case of polynomial mappings (see [19, Proposition 15.1.5]).

**Theorem 2.7** *If  $h : \mathbb{C}^N \rightarrow \mathbb{C}^N$  is holomorphic and proper, then it is surjective.* □

The proof of this result can be found in [19] (just before Proposition 15.1.5). But let us say that the assertion can be derived from two facts: a simple one stating that each continuous proper mapping is closed and a less simple one saying that a proper holomorphic map between open subsets of  $\mathbb{C}^n$  is open (cf. [16, page 76 and Proposition V.7.1]).

Note that the polynomial mapping  $Q : \mathbb{C}^2 \ni (z, w) \mapsto (z, zw - 1) \in \mathbb{C}^2$  (cf. Example 2.3) is not proper. To see this, it is enough to consider the inverse image of  $\{(0, -1)\}$  under this mapping.

### 3 Escape radii

If a set is compact, it is bounded and closed. Therefore, we will speak about boundedness and closedness now. We will start with a formula for the set  $K[h]$  associated with a self-map  $h : X \rightarrow X$  in a special case.

**Proposition 3.1** *Let  $h : X \rightarrow X$  be a mapping. If  $K[h]$  is bounded, then for every bounded set  $E$  containing  $K[h]$  we have  $K[h] = \bigcap_{n=1}^{\infty} h^{-n}(E)$ .*

( $h^{-n}(E)$  denotes the inverse image of  $E$  under the  $n$ -th iterate  $h^n$  of  $h$ ).

*Proof* Proposition 2.1 yields  $K[h] = h^{-n}(K[h]) \subset h^{-n}(E)$  and in consequence  $K[h] \subset \bigcap h^{-n}(E)$ . The inverse inclusion is obvious, since  $E$  is bounded. □

In contrast to this, let us now look at an example where  $K[h]$  is not closed. One can take  $h : \mathbb{R} \rightarrow \mathbb{R}$  defined as in Example 2.3 with use of

$$f : (-\infty, 0) \ni x \mapsto \frac{2x - 1}{x - 1} \in [0, \infty) \quad \text{and} \quad g : [0, \infty) \ni x \mapsto (x - 2)^2 \in [0, \infty).$$

Then  $K[h] = (-\infty, 0) \cup [1, 3]$ .

However, if  $h$  is continuous and  $K[h]$  is bounded, this set has to be closed. We have namely the following consequence of Proposition 3.1.

**Corollary 3.2** *If  $h : X \rightarrow X$  is continuous and  $K[h]$  is bounded, then it is closed.*

*Proof* Proposition 3.1 yields  $K[h] = \bigcap_{n=1}^{\infty} h^{-1}(\overline{K[h]})$ . □

There is another consequence of Proposition 3.1.

**Corollary 3.3** *Let  $h : X \rightarrow X$  be a mapping and  $K[h]$  be bounded. Let  $A \subset X$  be bounded such that  $h^{-1}(A) = A$ . Then  $A \subset K[h]$ .*

*Proof*  $K[h] = \bigcap_{n=1}^{\infty} (h^{-n}(A) \cup h^{-n}(K[h])) = A \cup K[h]$ . □

If there exists  $a \in X$  and  $R > 0$  such that

$$\forall x \in X : d(a, x) > R \implies \lim_{n \rightarrow \infty} d(a, h^n(x)) = \infty, \tag{3.1}$$

then  $K[h] \subset \overline{B}(a, R) = \{x \in X : d(a, x) \leq R\}$  and Proposition 3.1 yields  $K[h] = \bigcap_{n=1}^{\infty} h^{-n}(\overline{B}(a, R))$ . A number  $R$  satisfying (3.1) is an *escape radius* for the mapping  $h$ . It is obvious that, if we know one escape radius for a mapping, then any number greater than it is also an escape radius for the map.

Note finally that if  $h : X \rightarrow X$  is a continuous mapping having an escape radius, then  $K[h]$  is closed and bounded. In particular, if a continuous mapping  $h : \mathbb{C}^N \rightarrow \mathbb{C}^N$  has an escape radius, then its filled Julia set is compact.

### 4 Escape radii for polynomial self-maps

In this section  $\mathbb{C}^N$  is equipped with the norm  $|z| = \max(|z_1|, \dots, |z_N|)$  and we consider only polynomial (hence continuous) maps. With one exception, the point  $a$  from formula (3.1) will here be 0.

#### 4.1 Polynomials of one complex variable

Let us start with quadratic polynomials. It is enough to consider

$$P_c : \mathbb{C} \ni z \mapsto z^2 + c \in \mathbb{C}, \quad c \in \mathbb{C}.$$

In [5] it is shown that  $\max(|c|, 2)$  is an escape radius for  $P_c$ . It is however far from being the smallest. In [1, Exercise 1.6.4] the escape radius

$$r_c := (1 + \sqrt{1 + 4|c|})/2$$

is derived. For the proof we refer the readers to [2]. Moreover,  $r_c$  is the smallest of the radii  $R$  such that  $|z| > R$  implies  $|P_c(z)| > R$  (see [2, Section 2]).

Let us mention in passing that in [5] an escape radius  $r_\lambda := 1 + (1/|\lambda|)$  for the logistic functions  $L_\lambda : \mathbb{C} \ni z \mapsto \lambda z(1 - z) \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus \{0\}$ , is given.

In the case of cubic polynomials it is enough to consider mappings

$$C_{a,b} : \mathbb{C} \ni z \mapsto z^3 + az + b \in \mathbb{C}, \quad a, b \in \mathbb{C}.$$

The author of [5] proposes an escape radius  $r_{a,b} := \max(|b|, \sqrt{|a| + 2})$ . From the proof it easily follows that, if  $b = 0$ , then one can take  $r_a := \sqrt{|a| + 1}$ . If  $a = 0$  another escape radius  $r_b := 1 + \sqrt[3]{|b|}$  was proposed in [14, Proof of Proposition 5.2]. It is smaller than  $r_{0,b}$  for  $b < 5\sqrt{2} - 7$  or  $b > 2\sqrt{2}$ .

In general,

$$R = \frac{1 + |a_d| + \dots + |a_0|}{|a_d|} \tag{4.1}$$

is an escape radius for a polynomial  $P(z) = a_d z^d + \dots + a_0$  of degree  $d \geq 2$  (see e.g. [20, page 387] or [10, Lemma 1]). Note that this escape radius depends continuously on the polynomial (to be precise, on its coefficients). It is by no means optimal. If we consider  $p_n(z) = z^2 + \frac{1}{n}z^n$ , then using (4.1) we obtain the escape radius  $R = 2n + 1$ . However, if  $n \geq 5$ , then  $r = 2$  is an escape radius for  $p_n$  since  $|p_n(z)| \geq 2|z|$  if  $|z| \geq 2$ .

Let us also mention another way of obtaining an escape radius in the case of connected filled Julia set. Namely, each polynomial of degree  $d \geq 2$  can be conjugated to a monic and centered polynomial, i.e. a polynomial of the form  $P(z) = z^d + a_{d-2}z^{d-2} + \dots + a_0$ . If  $K[P]$  is connected, then its diameter is not greater than

$$D = \sqrt{8 \left(1 + \frac{|a_{d-2}|}{d}\right)}$$

(see [20, Theorem 3]). Thus one can take  $a \in K[P]$  and for it (this time not necessarily for 0) the number  $D$  will be an escape radius (because for polynomials the forward orbits of points are either bounded or tending to infinity).

### 4.2 Polynomial mappings of several complex variables

Now let us turn to higher dimensions. Note that, if the degree of a polynomial of one complex variable is at least 2, then its filled Julia set is compact. For a polynomial self-map of  $\mathbb{C}^N$  with  $N > 1$  it does not suffice to have the degree greater than 1. The easiest specimen can be obtained by a modification of the last mapping in Example 2.3. For any  $k \geq 2$  we can e.g. consider the mapping

$$Q_k : \mathbb{C}^2 \ni (z, w) \mapsto (z^k, zw^{k-1} - 1) \in \mathbb{C}^2.$$

Then  $\text{deg}(Q_k) = k$  and  $\{(0, w) : w \in \mathbb{C}\} \subset K[Q_k]$ .

However,  $Q_k$  is not surjective (hence not proper by Theorem 2.7). The question is whether there is a surjective (proper) polynomial map of degree  $k$  with unbounded filled Julia set. The answer to this question is yes and in fact there are quite many such examples. It suffices to take an elementary polynomial automorphism of  $\mathbb{C}^2$  of the form

$$\mathbb{C}^2 \ni (z, w) \mapsto (z, w + z^k) \in \mathbb{C}^2$$

or a generalized Hénon mapping of the form

$$\mathbb{C}^2 \ni (z, w) \mapsto (w, z + w^k) \in \mathbb{C}^2$$

(we take here just a special examples of those maps, see [6]). These maps are surjective, proper, of degree  $k$  and their filled Julia sets are unbounded (for the first type once again the complex line  $z = 0$  is contained in the filled Julia sets, the situation for the second type is more complicated, see [8]). Note that they are polynomial automorphisms, i.e. their inverses are also polynomial mappings.

Consider now a polynomial mapping  $P : \mathbb{C}^N \rightarrow \mathbb{C}^N$  of degree  $d \geq 2$ . It can be written in the form  $P = H_d + H_{d-1} + \dots + H_0$ , where  $H_j$  is the homogeneous part of  $P$  of degree  $j$ ,  $j \in \{0, 1, \dots, d\}$ . The mapping  $P$  is called *regular*, if  $H_d^{-1}(0) = \{0\}$ . (Note that every polynomial of one complex variable of degree  $d \geq 2$  is regular.) If  $P : \mathbb{C}^N \rightarrow \mathbb{C}^N$  is regular, then

$$R = \frac{1 + \inf_{|z|=1} |H_d(z)| + \|H_{d-1}\|_C + \dots + \|H_0\|_C}{\inf_{|z|=1} |H_d(z)|}$$

(where  $\|f\|_C := \sup_{|z|=1} |f(z)|$ ) is an escape radius for  $P$  (see [10, Lemma 1]). This is a straightforward generalization of (4.1). Let us also underline that this escape radius plays a crucial role in the definitions of some generalizations of the filled Julia sets (see [11]).

However, there are also polynomial mappings which are not regular but whose filled Julia sets are compact. Let  $P : \mathbb{C}^N \rightarrow \mathbb{C}^N$  be a polynomial mapping. The so-called *Lojasiewicz exponent of  $P$  (at infinity)* defined by the formula

$$\mathcal{L}_\infty(P) := \sup \left\{ \delta \in \mathbb{R} : \liminf_{|z| \rightarrow \infty} \frac{|P(z)|}{|z|^\delta} > 0 \right\}$$

describes the growth of the mapping at infinity. For more information on this notion we refer the reader to a recent survey [15] and the references given there. Note that the Łojasiewicz exponent can take the value  $-\infty$ , if there exists no  $\delta$  satisfying the inequality (for a characterization of mappings with the Łojasiewicz exponent greater than  $-\infty$  see [15, Proposition 1]). It can be also negative, e.g.  $\mathcal{L}_\infty(Q) = -1$  if  $Q$  is the mapping known from Example 2.3, namely  $Q : \mathbb{C}^2 \ni (z, w) \mapsto (z, zw - 1) \in \mathbb{C}^2$  (see [3, Remark 11.5] or [15, Example 4]). Of course in the one dimensional case the Łojasiewicz exponent coincides with the degree of the polynomial. Therefore, this problem is interesting only if  $N > 1$ . Furthermore, if  $P$  is regular, then  $\mathcal{L}_\infty(P) = \text{deg}(P)$  ([17, Example (1.7)]). The polynomial mapping  $P$  is proper if and only if  $\mathcal{L}_\infty(P) > 0$  ([15, Proposition 8]). If  $P$  is a polynomial automorphism of  $\mathbb{C}^N$ , then  $\mathcal{L}_\infty(P) = 1/\text{deg}(P^{-1}) \leq 1$  ([17, Example (1.8)]). For example, the Łojasiewicz exponent of a Hénon map is thus not greater than 1. However, polynomial automorphisms are not the only mappings whose Łojasiewicz exponents are positive but not greater than one. For instance, if  $P : \mathbb{C}^2 \ni (z, w) \mapsto (z^2, z^4 + w^2) \in \mathbb{C}^2$ , then  $P(1, 1) = P(-1, -1)$  and  $\mathcal{L}_\infty(P) = 1$  (see [17, Example (1.9)]).

Consider now a polynomial mapping  $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$  satisfying  $\mathcal{L}_\infty(P) > 1$ . Klimek introduced in [9] a subfamily  $\mathcal{R}$  of  $\text{Comp}(\mathbb{C}^N)$  and a metric  $\Gamma$  on this subfamily such that  $(\mathcal{R}, \Gamma)$  is a complete metric space (see [9, Theorem 1]). Furthermore, the mapping

$$\Psi_P|_{\mathcal{R}} : \mathcal{R} \ni A \mapsto P^{-1}(A) \in \mathcal{R}$$

turns out to be well defined and moreover a contraction with ratio  $1/\mathcal{L}_\infty(P)$  (see [9, Proof of Theorem 2]). Therefore, in view of the Banach contraction principle it has a unique fixed point which is the filled Julia set  $K[P]$  (see [9, Theorem 2 and Corollary 6]). This situation is therefore one of the best possible. In consequence, we obtain not only compactness of  $K[P]$  but also another nice pluripotential property of the set.

We know already that the set is compact but we nevertheless want to speak about an escape radius for such a mapping. It was shown in [17, Proposition (1.6)] that the supremum in the definition of the Łojasiewicz exponent is actually a maximum for a proper polynomial mapping, i.e.

$$\exists C, r > 0 : |z| \geq r \implies |P(z)| \geq C|z|^{\mathcal{L}_\infty(P)}. \tag{4.2}$$

It is clear that, if we take  $R > r$ , then the estimate on the right hand side of (4.2) above will still hold for  $|z| \geq R$ . We are ready to prove

**Proposition 4.1** *Let  $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$  be a polynomial mapping such that  $\mathcal{L}_\infty(P) > 1$ . Set  $\varrho := \mathcal{L}_\infty(P) - 1$  and take  $C, r$  from (4.2). Then*

$$R := \max \left( r, \left( \frac{2}{C} \right)^{1/\varrho} \right),$$

*is an escape radius for  $P$ .*

*Proof* From (4.2) it follows that  $|P(z)| \geq CR^\varrho|z| \geq 2|z|$ , and by induction  $|P^n(z)| \geq 2^n|z|$ , for  $|z| \geq R, n \in \mathbb{N}$ . □

Note that in general we cannot do the same for those proper polynomial mappings with Łojasiewicz exponent not greater than 1. The filled Julia sets of the elementary polynomial automorphisms or of the generalized Hénon maps are unbounded and the Łojasiewicz exponent of these mappings is smaller than 1. However, there exist proper polynomial mappings with the Łojasiewicz exponents not greater than 1 and with compact Julia sets. Some of them appear in our last result.

**Proposition 4.2** Let  $p : \mathbb{C} \rightarrow \mathbb{C}$  be a polynomial of one complex variable,  $a \in \mathbb{C} \setminus \{0\}$  and  $k, m \in \{2, 3, 4, \dots\}$ . Set  $M = \max\{|p(w)| : |w| \leq 1\}$ . Then

$$R = \max\left(1, \frac{1 + \sqrt{1 + 4|a|M}}{2|a|}\right)$$

is an escape radius for a polynomial mapping of the form

$$P : \mathbb{C}^2 \ni (z, w) \mapsto (az^k + p(w), w^m) \in \mathbb{C}^2.$$

*Proof* Fix an  $\varepsilon > 0$  and put

$$R_\varepsilon = \max\left(1, \frac{1 + \varepsilon + \sqrt{(1 + \varepsilon)^2 + 4|a|M}}{2|a|}\right).$$

Assume that  $\max(|z|, |w|) > R_\varepsilon$ . We consider two cases:

1. If  $|w| > 1$ , then  $|P^n(z, w)| \geq |w|^{mn} \rightarrow \infty$  if  $n \rightarrow \infty$ .
2. Assume now that  $|w| \leq 1$ . Then  $|z| > R_\varepsilon$  and

$$|P(z, w)| \geq |a||z|^k - M \geq |a||z|^2 - M > (1 + \varepsilon)|z| > R_\varepsilon$$

by the choice of  $R_\varepsilon$ . Since  $|z| = |(z, w)|$ , we see by induction that

$$|P^n(z, w)| \geq (1 + \varepsilon)^n |(z, w)| > (1 + \varepsilon)^n R_\varepsilon \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Since  $\varepsilon$  was arbitrarily chosen,  $|P^n(z, w)| \rightarrow \infty$  as  $n \rightarrow \infty$  for any  $(z, w)$  such that  $|(z, w)| > R$ . □

*Remark 4.3* If we took  $k = m = 1$  and  $p$  to be of degree at least 2, we would obtain an elementary polynomial automorphism (see [6]). In this case the filled Julia set is not bounded, hence there cannot be any escape radii. On the other hand with  $k$  and  $m$  greater than 1, if  $p(w) = q(w^m)$  for some polynomial  $q$ , then the mapping  $P$  above is a composition of  $(z, w) \mapsto (z^k, w^m)$  with an elementary polynomial automorphism. Note also that the mapping  $P$  is regular if and only if  $k = m \geq \deg(p)$  (recall that  $k, m \geq 2$  by the assumptions). Moreover, the mappings of the form (represented in Proposition 4.2)

$$(z, w) \mapsto (z^k + w^{sm}, w^m)$$

have the Łojasiewicz exponents equal to  $\min(m, k/s)$  (see [17, Example 1.9]), hence if  $s \geq k$ , then the Łojasiewicz exponents of the mappings is not greater than 1 and they are not covered by Proposition 4.1.

We would also like to find an escape radius for a mapping of the form

$$\mathbb{C}^2 \ni (z, w) \mapsto (w^m, az^k + p(w)) \in \mathbb{C}^2$$

but we do not know how to do it yet. For these mappings we could repeat the statements of the last remark with ‘a generalized Hénon map’ (see [6]) put in all places where ‘an elementary polynomial automorphism’ was used.

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## References

1. Beardon, A.F.: Iteration of rational functions. Complex analytic dynamical systems. In: Graduate Texts in Mathematics, vol. 132. Springer, New York (1991)
2. Beardon, A.F., Rippon, P.J.: A remark on the shape of quadratic Julia sets. *Nonlinearity* **7**, 1277–1280 (1994)
3. Chądzyński, J., Krasieński, T.: On the Łojasiewicz exponent at infinity for polynomial mappings of  $\mathbb{C}^2$  into  $\mathbb{C}^2$  and components of polynomial automorphisms of  $\mathbb{C}^2$ . *Ann. Polon. Math.* **57**, 291–302 (1992)
4. Devaney, R.L.: An introduction to chaotic dynamical systems. In: Addison-Wesley Studies in Nonlinearity, 2nd ed. Addison-Wesley Publishing Company, Redwood City (1989)
5. Devaney, R.L.: A first course in chaotic dynamical systems. Theory and experiment. In: Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company, Reading (1992)
6. Friedland, S., Milnor, J.: Dynamical properties of plane polynomial automorphisms. *Ergodic Theory Dynam. Syst.* **9**, 67–99 (1989)
7. Hua, X., Yang, C.: Dynamics of transcendental functions. In: Asian Mathematics Series, vol. 1. Gordon and Breach Science Publishers, Amsterdam (1998)
8. Hubbard, J.H., Oberste-Vorth, R.W.: Hénon mappings in the complex domain. I. The global topology of dynamical space. *Inst. Hautes Études Sci. Publ. Math.* **79**, 5–46 (1994)
9. Klimek, M.: Metrics associated with extremal plurisubharmonic functions. *Proc. Am. Math. Soc.* **123**, 2763–2770 (1995)
10. Klimek, M.: Iteration of analytic multifunctions. *Nagoya Math. J.* **162**, 19–40 (2001)
11. Klimek, M., Kosek, M.: Composite Julia sets generated by infinite polynomial arrays. *Bull. Sci. Math.* **127**, 885–897 (2003)
12. Kosek, M.: Hölder continuity property of filled-in Julia sets in  $\mathbb{C}^n$ . *Proc. Am. Math. Soc.* **125**, 2029–2032 (1997)
13. Kosek, M.: Dynamical analytic multifunctions. *RACSAM Rev. R. Acad. A* **108**, 465–473 (2014)
14. Kosek, M.: Hölder exponents of the Green functions of planar polynomial Julia sets. *Ann. Math. Pura Appl.* **4**(193), 359–368 (2014)
15. Krasieński, T.: On the Łojasiewicz exponent at infinity of polynomial mappings. *Acta Math. Vietnam* **32**, 189–203 (2007)
16. Łojasiewicz, S.: Introduction to complex analytic geometry. In: Translated from the Polish by Maciej Klimek. Birkhäuser, Basel (1991)
17. Płoski, A.: On the growth of proper polynomial mappings. *Ann. Polon. Math.* **45**, 297–309 (1985)
18. Protin, F.: Condition LS pour les ensembles de Julia dans  $\mathbb{C}^n$  (2015). **(Preprint)**
19. Rudin, W.: Function theory in the unit ball of  $\mathbb{C}^n$ . Reprint of the 1980 edition. In: Classics in Mathematics. Springer, Berlin (2008)
20. Yang, G.: Some geometric properties of Julia sets and filled-in Julia sets of polynomials. *Complex Var. Theory Appl.* **47**, 383–391 (2002)