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Maximal classes for families of lower and upper semicontinuous functions with a closed graph

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Abstract In this paper we characterize the following maximal classes for families of lower and upper semicontinuous functions with a closed graph: the maximal additive class, the maximal multiplicative class and the maximal classes with respect to maximum and minimum.

Keywords Functions with a closed graph · Lower semicontinuous functions · Upper semicontinuous functions · Sum of functions

Mathematics Subject Classification Primary 26A15; Secondary 54C08

1 Introduction

The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the real line, the set of rationals and the set of positive integers, respectively. The family of all functions from a set X into Y is denoted by Y^X. For each set $A \subset X$ its characteristic function is denoted by χ_A. In particular, $\chi_\emptyset$ stands for the zero constant function.

Let X be a topological space. The symbol X^d denotes the set of all accumulation points of X. For each set $A \subset X$ the symbols $\operatorname{int} A$ and $\operatorname{cl} A$ denote the interior and the closure of A, respectively. The spaces $\mathbb{R}$ and $X \times \mathbb{R}$ are considered with their standard topologies. We say that a function $f\colon X \to \mathbb{R}$ has a *closed graph*, if the graph of f, i.e., the set $\{(x, f(x)) : x \in X\}$ is a closed subset of the product $X \times \mathbb{R}$. We say that a function $f : X \to \mathbb{R}$ is lower (upper) semicontinuous at a point $x \in X$, if for each $\varepsilon > 0$ there is an open neighborhood U of x such that $f(z) > f(x) - \varepsilon$ ($f(z) < f(x) + \varepsilon$, respectively) for each $z \in U$. If $f : X \to \mathbb{R}$ is lower (upper) semicontinuous at each point $x \in X$, then we say that the function f is lower (upper, respectively) semicontinuous. Let $\mathscr{C}onst(X)$, $\mathcal{C}(X)$,

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$\mathcal{U}(X)$, $lsc(X)$, $usc(X)$ denote the class of all real-valued functions on X that are constant, continuous, have a closed graph, are lower and upper semicontinuous, respectively. Obviously $\mathcal{C}(X) \subset \mathcal{U}(X)$ (see also e.g. [5]) and $C(X) = lsc(X) \cap usc(X)$. For $\mathcal{F}(X)$ and $\mathcal{G}(X)$ nonempty subsets of $\mathbb{R}^X$ the symbol $\mathcal{FG}(X)$ denotes the class $\mathcal{F}(X) \cap \mathcal{G}(X)$. Further denote by $\mathcal{F}^+(X)$ the family of all nonnegative functions from $\mathcal{F}(X)$. Let $f \in \mathbb{R}^X$. The symbol $G(f)$ denotes the graph of f and the symbols $C(f)$ and $D(f)$ denote the sets of points of continuity and discontinuity of f, respectively. For each $y \in \mathbb{R}$ let $[f = y] = \{x \in X \colon f(x) = y\}$. Similarly we define the symbols $[f > y]$, $[f < y]$.

If $\mathcal{F} \subset \mathbb{R}^X$ is a family of functions, denote by

$$\mathcal{F} + \mathcal{F} \overset{\text{df}}{=} \{f \in \mathbb{R}^X : f = g + h \text{ for some } g, h \in \mathcal{F}\},$$

$$\mathcal{M}_a(\mathcal{F}) \overset{\text{df}}{=} \{f \in \mathbb{R}^X : (\forall_{g \in \mathcal{F}})\ f + g \in \mathcal{F}\},$$

$$\mathcal{M}_m(\mathcal{F}) \overset{\text{df}}{=} \{f \in \mathbb{R}^X : (\forall_{g \in \mathcal{F}})\ f \cdot g \in \mathcal{F}\},$$

$$\mathcal{M}_{\max}(\mathcal{F}) \overset{\text{df}}{=} \{f \in \mathbb{R}^X : (\forall_{g \in \mathcal{F}})\ \max(f, g) \in \mathcal{F}\},$$

$$\mathcal{M}_{\min}(\mathcal{F}) \overset{\text{df}}{=} \{f \in \mathbb{R}^X : (\forall_{g \in \mathcal{F}})\ \min(f, g) \in \mathcal{F}\}.$$

The above classes $\mathcal{M}_a(\mathcal{F})$, $\mathcal{M}_m(\mathcal{F})$, $\mathcal{M}_{\max}(\mathcal{F})$ and $\mathcal{M}_{\min}(\mathcal{F})$ are called the maximal additive class for $\mathcal{F}$, the maximal multiplicative class for $\mathcal{F}$, the maximal class with respect to maximum and minimum for $\mathcal{F}$, respectively.

In 1987 Menkyna [7] characterized the maximal additive and multiplicative classes for the family of functions with a closed graph. He proved that $\mathcal{M}_a(\mathcal{U}(X)) = \mathcal{C}(X)$ for a topological space X [7, Theorem 1] and $\mathcal{M}_m(\mathcal{U}(X)) = \{f \in \mathcal{C}(X) : [f = 0]\ \textit{is an open set}\}$ for a locally compact normal topological space X [7, Theorem 2]. Let $\mathcal{Q}(X)$ denote the family of all quasi-continuous functions from a topological space X to $\mathbb{R}$. Recall that $f \in \mathcal{Q}(X)$ if and only if for each $x \in X$, $\varepsilon > 0$ and for each neighbourhood U of x there is a nonempty open set $V \subset U$ such that $|f(x) - f(y)| < \varepsilon$ for each $y \in V$. In 2008 Sieg [8] considered real functions defined on $\mathbb{R}$ and showed that $\mathcal{M}_a(\mathcal{QU}(\mathbb{R})) = \mathcal{C}(\mathbb{R})$, $\mathcal{M}_m(\mathcal{QU}(\mathbb{R})) = \{f \in \mathcal{C}(\mathbb{R}) : f = \chi_\emptyset$ or $f(x) \neq 0$ for all $x \in \mathbb{R}\}$ and $\mathcal{M}_{\max}(\mathcal{QU}(\mathbb{R})) = \mathcal{M}_{\min}(\mathcal{QU}(\mathbb{R})) = \emptyset$. In 2014 Szczuka (see [9,10]) characterized the following maximal classes for lower and upper semicontinuous strong Świątkowski functions and lower and upper semicontinuous extra strong Świątkowski functions: the maximal additive class, the maximal multiplicative class and the maximal classes with respect to maximum. She proved, among others, that if $\mathcal{F}$ denotes the family of lower semicontinuous strong Świątkowski real functions defined on $\mathbb{R}$, then $\mathcal{M}_a(\mathcal{F}) = \mathscr{C}onst$ [9, Theorems 3.1], $\mathcal{M}_m(\mathcal{F}) = \mathscr{C}onst^+$ [9, Theorem 3.2] and $\mathcal{M}_{\max}(\mathcal{F}) = \mathscr{C}onst$ [9, Theorem 3.3].

In this paper we deal with the families of lower and upper semicontinuous functions with a closed graph. We obtain the following results:

- $\mathcal{M}_a(\mathcal{U}lsc(X)) = \mathcal{U}lsc(X)$, where X is a topological space (Theorem 2.5),
- $\mathcal{M}_a(\mathcal{U}usc(X)) = \mathcal{U}usc(X)$, where X is a topological space (Theorem 3.3),
- $\mathcal{M}_m(\mathcal{U}lsc(X)) = \{f \in \mathcal{C}(X) : [f = 0]$ is an open set and $f(x) \geq 0$ for all $x \in X\} = \mathcal{M}_m(\mathcal{U}usc(X))$, where X is a perfectly normal topological space such that $X = X^d$ (Theorems 2.7, 3.4),
- $\mathcal{M}_{\max}(\mathcal{U}lsc(X)) = \mathcal{U}lsc(X)$, where X is a topological space (Theorem 2.10),
- $\mathcal{M}_{\min}(\mathcal{U}usc(X)) = \mathcal{U}usc(X)$, where X is a topological space (Theorem 3.5),
- $\mathcal{M}_{\min}(\mathcal{U}lsc(X)) = \mathcal{M}_{\max}(\mathcal{U}usc(X)) = \emptyset$, where X is a perfectly normal topological space such that $X^d \neq \emptyset$ (Corollary 2.15, Theorems 3.6).

2 Lower semicontinuous functions with a closed graph

We start with a following proposition.

Proposition 2.1 *Let X be a topological space. A function $f : X \to \mathbb{R}$ has the closed graph if and only if for each $x \in X$ and for each $m \in \mathbb{N}$ there is a neighbourhood V of x such that $f(z) \in (-\infty, -m) \cup (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty)$ for each $z \in V$.*

Proof The implication ($\Leftarrow$) we can find in [2] (see p. 118, lines 11–14). The implication ($\Rightarrow$) immediately follows from [6] or [1, Proposition 1]: if $f \in \mathcal{U}(X)$, then for each $x \in X$ and each neighborhood U of $f(x)$ such that $Y \setminus U$ is compact there is an neighborhood V of x such that $f(V) \subset U$. Now, it is sufficient to take $U = (-\infty, -m) \cup (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty)$. Observe that, the equivalence of this proposition also immediately follows from [1, Proposition 2]. □

From above and the definitions of the class lsc we obtain:

Lemma 2.2 *Let X be a topological space. A function $f : X \to \mathbb{R}$ is lower semicontinuous function with a closed graph if and only if for each $x \in X$ and for each $m \in \mathbb{N}$ there is a neighbourhood V of x such that $f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty)$ for each $z \in V$.*

Proof First, assume that for each $x \in X$ and for each $m \in \mathbb{N}$ there is a neighbourhood V of x such that $f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty)$ for each $z \in V$. Then, by Proposition 2.1, $f \in \mathcal{U}(X)$. Now, we will show that $f \in lsc(X)$. Let $x \in X$ and $\varepsilon > 0$. We choose $m \in \mathbb{N}$ such that $m \geq \max\{\frac{1}{\varepsilon}, f(x) - \varepsilon\}$. There is a neighbourhood V of x such that $f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty) \subset (f(x) - \varepsilon, \infty)$ for each $z \in V$ and consequently $f \in lsc(X)$.

Now, let $f \in \mathcal{U}lsc(X)$. Fix $x \in X$ and $m \in \mathbb{N}$. Since $f \in lsc(X)$, there is a neighbourhood V_1 of x such that $f(z) \in (f(x) - 1/m, \infty)$ for each $z \in V_1$. We consider two cases.

First, assume that $f(x) \geq 0$. Since $f \in \mathcal{U}(X)$, there is a neighbourhood V_2 of x such that $f(z) \in (-\infty, -m) \cup (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty)$ for each $z \in V_2$ (see Proposition 2.1). Let $V = V_1 \cap V_2$ and let $z \in V$. Then $f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty)$.

Now, assume that $f(x) < 0$. We choose $k \in \mathbb{N}$ such that $k \geq \max\{m, -f(x) + \frac{1}{m}\}$. Since $f \in \mathcal{U}(X)$, there is a neighbourhood V_2 of x such that $f(z) \in (-\infty, -k) \cup (f(x) - 1/k, f(x) + 1/k) \cup (k, \infty)$ for each $z \in V_2$. Let $V = V_1 \cap V_2$ and let $z \in V$. Since $k \geq m$, $-k \leq f(x) - \frac{1}{m}$ and $\frac{1}{k} \leq \frac{1}{m}$, we have $f(z) \in \big((f(x) - 1/k, f(x) + 1/k) \cup (k, \infty)\big) \subset \big((f(x) - 1/m, f(x) + 1/m) \cup (m, \infty)\big)$. This completes the proof. □

The next lemma follows from Proposition 2.1 and Lemma 2.2.

Lemma 2.3 *Let X be a topological space. Then $\mathcal{U}^{+}(X) \subset \mathcal{U}lsc(X)$.*

Now, we will characterize the class of the sums of lower semicontinuous functions with a closed graph.

Lemma 2.4 *Let X be a topological space. Then $\mathcal{U}lsc(X) + \mathcal{U}lsc(X) = \mathcal{U}lsc(X)$.*

Proof Let $f, g \in \mathcal{U}lsc(X)$. Fix $x \in X$ and $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $1/k < 1/(2m + |f(x)| + |g(x)|)$. By Lemma 2.2, there exists a neighbourhood V of x such that $f(z) \in$

$(f(x)-1/k, f(x)+1/k)\cup(k,\infty)$ and $g(z)\in(g(x)-1/k, g(x)+1/k)\cup(k,\infty)$ for each $z\in V$. Let $z\in V$. We consider four cases.

If $f(z)>k$ and $g(z)>k$, then evidently $(f+g)(z)>m$.

If $f(z)>k$ and $g(z)\in(g(x)-1/k, g(x)+1/k)$, then

$$(f+g)(z)>k+g(x)-1/k>2m+|f(x)|+|g(x)|+g(x)-1/k>m.$$

Similarly, $g(z)>k$ and $f(z)\in(f(x)-1/k, f(x)+1/k)$, implies $(f+g)(z)>m$.

Now, let $f(z)\in(f(x)-1/k, f(x)+1/k)$ and $g(z)\in(g(x)-1/k, g(x)+1/k)$. Then, we have

$$|(f+g)(z)-(f+g)(x)|\le|f(z)-f(x)|+|g(z)-g(x)|<2/k<1/m.$$

It follows that $f+g\in\mathcal{U}lsc(X)$. □

Theorem 2.5 *Let X be a topological space. Then* $\mathcal{M}_a(\mathcal{U}lsc(X))=\mathcal{U}lsc(X)$.

Proof Since $\chi_\emptyset\in\mathcal{U}lsc(X)$, we conclude that $\mathcal{M}_a(\mathcal{U}lsc(X))\subset\mathcal{U}lsc(X)$. The inclusion $\mathcal{U}lsc(X)\subset\mathcal{M}_a(\mathcal{U}lsc(X))$ follows from Lemma 2.4. □

Now, recall the following lemma [7, Lemma 2], which will be applied in this paper.

Proposition 2.6 *Let X be a topological space and let* $f\in\mathcal{C}(X)$. *Then the function* $g: X\to\mathbb{R}$ *defined by the formula*

$$g(x)=\begin{cases}\frac{1}{f(x)}, & \text{if } x\in[f\neq 0],\\ 0, & \text{if } x\in[f=0].\end{cases}$$

has the closed graph.

Theorem 2.7 *Let X be a normal topological space such that each singleton is* G_δ*-set. Then*

$$\mathcal{M}_m(\mathcal{U}lsc(X))=\{f\in\mathcal{C}(X):[f=0]\ \text{is an open set and}\ [f<0]^d=\emptyset\}.$$

Proof We will prove this theorem in four parts. First, we will show that $\mathcal{M}_m(\mathcal{U}lsc(X))\subset\mathcal{C}(X)$. Let $f\in\mathcal{M}_m(\mathcal{U}lsc(X))$. Since $\chi_\mathbb{R}, -\chi_\mathbb{R}\in\mathcal{U}lsc(X)$, we have $f\in lsc(X)$ and $-f\in lsc(X)$. Consequently $f\in lsc(X)\cap usc(X)=\mathcal{C}(X)$.

Now, we assume that the function $f\in\mathcal{C}(X)$ and the set $[f=0]$ is not open. We will show that $f\notin\mathcal{M}_m(\mathcal{U}lsc(X))$ (The proof of this part is similar to the second part of the proof of [7, Theorem 2]). Define the function $g: X\to\mathbb{R}$ by the formula

$$g(x)=\begin{cases}\frac{1}{|f(x)|}, & \text{if } x\in[f\neq 0],\\ 0, & \text{if } x\in[f=0].\end{cases}$$

By Proposition 2.6 the function g has the closed graph. Moreover g is non-negative function and consequently, by Lemma 2.3, $g\in\mathcal{U}lsc(X)$. Now, we will show that $f\cdot g\notin\mathcal{U}lsc(X)$.

Since the set $[f=0]$ is not open, there is $x_0\in[f=0]$ such that for each open neighbourhood V of x_0 there is $x_V\in V\cap[f\neq 0]$. Notice that $(f\cdot g)(x_V)\in\{-1,1\}$ for each neighbourhood V of x_0 and $(f\cdot g)(x_0)=0$. By Proposition 2.1, $f\cdot g\notin\mathcal{U}(X)$.

In the third part of the proof, suppose that $f\in\mathcal{C}(X)$, the set $[f=0]$ is open and $[f<0]^d\neq\emptyset$. We will prove that $f\notin\mathcal{M}_m(\mathcal{U}lsc(X))$. Let $x_0\in[f<0]^d$. Then there is a net $(x_\gamma)_{\gamma\in\Gamma}$ of elements of X such that $x_\gamma\to x_0$, $x_\gamma\neq x_0$ and $f(x_\gamma)<0$ for every $\gamma\in\Gamma$.

Notice that, since $f \in \mathcal{C}(X)$ and the set $[f = 0]$ is open, we have $f(x_0) < 0$. By Urysohn Lemma there is a continuous function $h : X \to [0, 1]$ such that $[h = 0] = \{x_0\}$.

Define the function $g : X \to \mathbb{R}$ by the formula

$$g(x) = \begin{cases} \frac{1}{h(x)}, & \text{if x} \neq \text{x}_0, \\ 0, & \text{if x} = \text{x}_0. \end{cases}$$

Observe that, by Proposition 2.6 and Lemma 2.3, $g \in \mathcal{U}lsc(X)$. Moreover $f \cdot g \notin lsc(X)$, because the net $((f \cdot g)(x_\gamma))_{\gamma \in \Gamma}$ diverges to $-\infty$ (recall that $f(x_\gamma) \to f(x_0) < 0$).

In the last part suppose that $f \in \mathcal{C}(X)$, the set $[f = 0]$ is open, $[f < 0]^d = \emptyset$ and $g \in \mathcal{U}lsc(X)$. Then, by [7, Theorem 2], $(f \cdot g) \in \mathcal{U}(X)$ (see also the third part of the proof of [7, Theorem 2]). It is enough to show that $(f \cdot g) \in lsc(X)$. Let $x_0 \in X$. If $f(x_0) \leq 0$, then the function $f \cdot g$ is continuous at x_0 and consequently $f \cdot g$ is a lower semicontinuous at this point. Indeed, if $f(x_0) = 0$, then by the assumption $[f = 0] = \text{int}[f = 0]$, we have $x_0 \in \text{int}[f = 0] \subset \text{int}[f \cdot g = 0]$ and if $f(x_0) < 0$, then x_0 is a isolated point of X. Finally, assume that $f(x_0) > 0$. Since $f \in \mathcal{C}(X)$, there is an open neighborhood U of x_0 such that $U \subset [f > 0]$. Since $g \in lsc(X)$, f is continuous and positive function on U, the function $f \cdot g$ is a lower semicontinuous at x_0. The proof is complete. □

It is easy to see that from above for $X = \mathbb{R}$ we have the following corollary.

Corollary 2.8 $\mathcal{M}_m(\mathcal{U}lsc(\mathbb{R})) = \{f \in \mathcal{C}(\mathbb{R}) : f = \chi_\emptyset \textit{ or } f(x) > 0 \textit{ for all } x \in \mathbb{R}\}$.

Lemma 2.9 *Let X be a topological space and let $f, g \in \mathcal{U}lsc(X)$. Then the real function $h = \max\{f, g\}$ defined on X is a lower semicontinuous function with a closed graph.*

Proof Let $f, g \in \mathcal{U}lsc(X)$. We will use Lemma 2.2. Fix $x \in X$ and $m \in \mathbb{N}$. Then there exists a neighbourhood V of x such that $f(z) \in (f(x) - 1/m, f(x) + 1/m) \cup (m, \infty)$ and $g(z) \in (g(x) - 1/m, g(x) + 1/m) \cup (m, \infty)$ for each $z \in V$. We assume that $f(x) \geq g(x)$ (The case $f(x) < g(x)$ is analogous). Then $h(x) = f(x)$ and it is easy to see that $h(z) \in (h(x) - 1/m, h(x) + 1/m) \cup (m, \infty)$ for each $z \in V$. So, $h \in \mathcal{U}lsc(X)$. □

Theorem 2.10 *Let X be a topological space. Then $\mathcal{M}_{\max}(\mathcal{U}lsc(X)) = \mathcal{U}lsc(X)$.*

Proof The inclusion $\mathcal{U}lsc(X) \subset \mathcal{M}_{\max}(\mathcal{U}lsc(X))$ follows from Lemma 2.9. So, we will only prove that $\mathcal{M}_{\max}(\mathcal{U}lsc(X)) \subset \mathcal{U}lsc(X)$. Let $f : X \to \mathbb{R}$ be a function such that $f \notin \mathcal{U}lsc(X)$. We choose $x_0 \in X$ and $m \in \mathbb{N}$, such that $m \geq f(x_0) + \frac{1}{m}$ and for each open neighborhood V of x_0 there is $x \in V$ such that $f(x) \leq m$ and $f(x) \notin (f(x_0) - \frac{1}{m}, f(x_0) + \frac{1}{m})$. We will show that $f \notin \mathcal{M}_{\max}(\mathcal{U}lsc(X))$. Let $c = f(x_0) - \frac{1}{m}$. Define the function $g : X \to \mathbb{R}$ by $g \overset{\text{df}}{=} c$. Clearly $g \in \mathcal{U}lsc(X)$. Denote $h = \max\{f, g\}$. We will prove that $h \notin \mathcal{U}lsc(X)$. Notice that $h(x_0) = f(x_0)$. Observe that, for each open neighborhood V of x_0 there is $x_V \in V$ such that $f(x_V) \in (-\infty, c] \cup [f(x_0) + \frac{1}{m}, m]$ and consequently $h(x_V) \in \{h(x_0) - \frac{1}{m}\} \cup [h(x_0) + \frac{1}{m}, m]$. By Proposition 2.1, $h \notin \mathcal{U}(X)$. This completes the proof. □

Theorem 2.11 *Let X be a topological space such that $\mathcal{U}(X) \neq \mathcal{C}(X)$. Then $\mathcal{M}_{\min}(\mathcal{U}lsc(X)) = \emptyset$.*

Proof Let $f \in \mathbb{R}^X$. We will show that there is a function $g \in \mathcal{U}lsc(X)$ such that the function $h = \min\{f, g\} \notin \mathcal{U}lsc(X)$.

Let $g_1 : X \to \mathbb{R}$ be a function with a closed graph and let $x_0 \in D(g_1)$. Put $g_2 = |g_1|$. Then $g_2 \in \mathcal{U}lsc$ and there is a net $(x_\gamma)_{\gamma \in \Gamma}$ of elements of X which converges to the point x_0 and a net $(g_2(x_\gamma))_{\gamma \in \Gamma}$ diverges to ∞. We consider two cases.

If $x_0 \in C(f)$, we define the function $g : X \to \mathbb{R}$ by $g(x) \stackrel{\text{df}}{=} g_2(x) - g_2(x_0) + f(x_0) - 1$. It is easy to see that $g \in \mathcal{U}lsc(X)$. Let $h = \min\{f, g\}$. Then $h(x_0) = g(x_0) = f(x_0) - 1$ and there is $\gamma_0 \in \Gamma$ such that $h(x_\gamma) = f(x_\gamma)$ for each $\gamma > \gamma_0$. Consequently $(x_0, f(x_0)) \in \operatorname{cl} G(h) \setminus G(h)$ and $h \notin \mathcal{U}(X)$.

Now, let $x_0 \in D(f)$. There is $\varepsilon > 0$ such that for each neighbourhood V of x_0 there is $z \in V$ such that $f(z) \notin (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Define the function $g : X \to \mathbb{R}$ by $g(x) \stackrel{\text{df}}{=} f(x_0) + \varepsilon$. Let $h = \min\{f, g\}$. Then $h(x_0) = f(x_0)$ and for each neighbourhood V of x_0 there is $z \in V$ such that $h(z) \in (-\infty, h(x_0) - \varepsilon] \cup \{h(x_0) + \varepsilon\}$. By Proposition 2.2, $h \notin \mathcal{U}lsc(X)$ □

It is easy to see that

Remark 1 Let X be a topological space such that $\mathcal{U}(X) = \mathcal{C}(X)$. Then $\mathcal{M}_{\min}(\mathcal{U}lsc(X)) = \mathcal{C}$.

Now, we recall the definition of a P-space [4, pp. 62–63] and two propositions given by Wójtowicz and Sieg [11, Theorem 1 and Corollary 1].

Definition 1 We say that a completely regular (Tychonoff) space X is a P-space if every G_δ-subset (F_σ-subset) of X is open (closed); equivalently, every co-zero subset of X is closed.

Proposition 2.12 *Let X be a completely regular space. Then $\mathcal{U}(X) = \mathcal{C}(X)$ if and only if X is a P-space.*

Proposition 2.13 *Let X be a perfectly normal or first countable space, or a locally compact space. Then $\mathcal{U}(X) \neq \mathcal{C}(X)$ if and only if X is non-discrete.*

From Proposition 2.12, Theorem 2.11 and Remark 1 we obtain the following Corollary.

Corollary 2.14 *Let X be a nonempty completely regular space. Then $\mathcal{M}_{\min}(\mathcal{U}lsc(X)) = \emptyset$ if and only if X is not a P-space.*

Moreover, using Proposition 2.13 and Theorem 2.11 we conclude that

Corollary 2.15 *Let X be a non-discrete perfectly normal or first countable space, or a locally compact space. Then $\mathcal{M}_{\min}(\mathcal{U}lsc(X)) = \emptyset$.*

Finally, observe that we can extend the lists (see e.g. [11, Theorem 1]) of equivalent conditions for X to be a P-space as follows:

Corollary 2.16 *Let X be a nonempty completely regular space. Then X is a P-space if and only if $\mathcal{M}_{\min}(\mathcal{U}lsc(X)) \neq \emptyset$.*

3 Upper semicontinuous functions with a closed graph

First, we recall some basic property of the functions with a closed graph [3, Proposition 2]

Proposition 3.1 *Let X be a topological space. Let α be a real number. If $f \in \mathcal{U}(X)$, then $\alpha \cdot f \in \mathcal{U}(X)$.*

From above and the definitions of the classes $lsc(X)$ and $usc(X)$ we obtain:

Proposition 3.2 *Let X be a topological space. For each function $f \in \mathbb{R}^X$ we have $f \in \mathcal{U}usc(X)$ if and only if $(-f) \in \mathcal{U}lsc(X)$.*

Now, we will characterize the following maximal classes for the family of upper semicontinuous functions with a closed graph: the maximal additive class, the maximal multiplicative class and the maximal classes with respect to maximum and minimum.

Theorem 3.3 *Let X be a topological space. Then $\mathcal{M}_a(\mathcal{U}usc(X)) = \mathcal{U}usc(X)$.*

Proof Observe that, by Proposition 3.2, $f \in \mathcal{M}_a(\mathcal{U}usc(X))$ if and only if $-f \in \mathcal{M}_a(\mathcal{U}lsc(X))$. Using Theorem 2.5 and again Proposition 3.2, we conclude that $\mathcal{M}_a(\mathcal{U}usc(X)) = \mathcal{U}usc(X)$. □

The next theorem follows from Proposition 3.2.

Theorem 3.4 *Let X be a topological space. Then $\mathcal{M}_m(\mathcal{U}usc(X)) = \mathcal{M}_m(\mathcal{U}lsc(X))$.*

Theorem 3.5 *Let X be a topological space. Then $\mathcal{M}_{\min}(\mathcal{U}usc(X)) = \mathcal{U}usc(X)$.*

Proof Since $-\min\{f, g\} = \max\{-f, -g\}$ for each functions $f, g \in \mathbb{R}^X$, by Proposition 3.2, we conclude that $f \in \mathcal{M}_{\min}(\mathcal{U}usc(X))$ if and only if $-f \in \mathcal{M}_{\max}(\mathcal{U}lsc(X))$. Now, using Theorem 2.10 and again Proposition 3.2, we obtain that $\mathcal{M}_{\min}(\mathcal{U}usc(X)) = \mathcal{U}usc(X)$. □

It is easy to see that using Theorem 2.11, Remark 1 and the equivalence $f \in \mathcal{M}_{\max}(\mathcal{U}usc(X))$ if and only if $-f \in \mathcal{M}_{\min}(\mathcal{U}lsc(X))$, we conclude that:

Theorem 3.6 *Let X be a topological space. Then $\mathcal{M}_{\max}(\mathcal{U}usc(X)) = \mathcal{M}_{\min}(\mathcal{U}lsc(X))$.*

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