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ORIGINAL ARICLE

Impact of teachers' professional learning on students' learning of multiplicative thinking

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Abstract
The shift from additive to multiplicative thinking is challenging for students. A professional learning program was developed that focused on an identified area of need by teachers, namely multiplicative thinking. Program content focused on concepts underpinning multiplicative thinking, pedagogical approaches, challenging tasks, and application to classroom practice. It was delivered via six 90-min modules in 13 participating schools across terms 2–4, as part of regular after school professional learning. Whilst all staff participated, the research focus was year 3–4 teachers. Students' historical data were collected across four years (2016–2019) to determine mean growth over time, in participating and non-participating schools. National Assessment Program Literacy and Numeracy (NAPLAN) and Mathematics Assessment Interview (MAI) data were used to determine the impact of the learning, as both assessments are administered annually. Analysis of year 4 students' longitudinal data showed greater mean growth in student learning over a 2-year period in schools involved in the learning and additional in-class coaching support, than students in non-participating schools. Our findings showed that targeted school-based professional learning, with in-class support from a knowledgeable other, leads to teachers' improved understanding of multiplicative thinking and subsequent pedagogical content knowledge to support student learning.

Keywords Multiplicative thinking · Practicing primary school teachers · Professional learning · Student achievement

Introduction

Learning to think multiplicatively is a key goal of mathematics teaching in the middle to upper primary years as it provides the foundation for understanding fractions, ratio, rate, percentage, and proportional reasoning in the middle years (Harel &

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Confrey, 1994; Siemon et al., 2005). Yet, recent research by Siemon et al. (2018) found that at least 25% and up to 55% of year 8 students in Australia have not developed facility to think multiplicatively. International studies report similar findings of a reliance on additive thinking (AT) by students in the middle years (Kosko, 2019; Kosko & Singh, 2018).

Potential explanations for why students have difficulty progressing to multiplicative thinking (MT) have been reported in the literature. These include equipping teachers with a greater understanding of the pathway to MT, and the nature of experiences that support younger students to develop it (Clark & Kamii, 1996). Others argue that the curriculum is placing too much emphasis on the traditional practice of MT beginning in the context of repeated addition, rather than reflecting a more holistic perspective that includes functional relations (Askew, 2018; Confrey, 1994). Another consideration is that teaching approaches used may be perpetuating a limited view of MT and limiting students' exposure to different multiplicative structures (Askew, 2018; Greer, 1988). Overall, the development of MT is critical to students' mathematical performance and teachers need to be equipped with the skills and expertise to support students' development of MT (Zwanch & Wilkins, 2021).

This paper reports the impact of teachers' professional learning (PL) on students' development of MT as assessed by the Mathematics Assessment Interview (MAI—a refinement of the Early Numeracy Interview, Clarke et al., 2002) and the National Assessment Program-Literacy and Numeracy (NAPLAN) (Australian Curriculum, Assessment and Reporting Authority, 2011). The aim of the PL was to support teachers' knowledge of MT and their pedagogical content knowledge (PCK) related to the multiplicative semantic structures. The research question underpinning this study was as follows: What is the impact of an on-site, PL program on students' development of MT?

Background

In this section, a review of the literature that informed this study is presented. Literature relating to MT, the pedagogical approaches that support students' development of MT and different multiplicative structures, are examined. As the study involved teacher PL, a critique of the literature was needed to identify the most appropriate model.

Multiplicative thinking

Much has been written about students' development of MT (e.g. Clark & Kamii, 1996; Steffe, 1994; Sullivan et al., 2001); the strategies students use to solve multiplicative problems (e.g. Downton & Sullivan, 2017; Mulligan & Watson, 1998); and reasons why some students are reliant on AT into their middle years of schooling—years 5 to 8 (e.g. Larsson et al., 2017; Siemon et al., 2005).

A key distinction between additive and MT is the level of abstraction required. Abstracting, in this context, is characterised by students moving beyond the need to create physical models, to forming mental images of a collection of objects as a composite unit (Sullivan et al., 2001). Clark and Kamii (1996) stated that the key distinction between additive and MT is the number of levels of abstractions and the number of inclusive relations required simultaneously in MT. In the example 4×3, two kinds of relationships are evident. The many-to-one correspondence between the four units of one and the one unit of three, and inclusive relations such as seeing that in each unit of three are three ones, and that in the product of 12 are four units of three. In essence, to think multiplicatively, one needs to "coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit" (Steffe, 1994, p. 19). The result of such an action is the product of a third composite unit (Tzur et al., 2010). This process is recognised as a major conceptual shift (Clark & Kamii, 1996; Clarke et al., 2002; Singh, 2000; Steffe, 1994; Tzur et al., 2010). Making such a shift includes recognising the relationship between the composite units and the role they play in a situation and that a third composite unit (the product) is produced as a result of this action. Several scholars argued that MT is more complex than AT and may take years to achieve (Clark & Kamii, 1996; Clarke et al., 2002; Lamon, 2005; Steffe, 1994; Vergnaud, 1983). Furthermore, teachers must understand the complexity involved to support students' development of MT (Zwanch & Wilkins, 2021).

The complexity of this transition is reflected in the findings of the Early Numeracy Research Project (Clarke et al., 2002). Reported findings indicated that 51% of year 2 children in trial schools (i.e. schools in which teachers received PL over a 3-year period) and 63% of children in reference schools (control group) were unable to abstract (simultaneously coordinate two composite units mentally, without the use of perceptual models), when solving multiplication tasks. Furthermore, Kosko and Singh (2018) found that 54% of year 3 students were pre-multiplicative (could not coordinate composite units and relied on count by ones strategy).

Pathway to multiplicative thinking

Whilst there is general agreement that the acquisition of the composite unit structure is the basis of students' development of MT (e.g. Anghileri & Johnson, 1992; Clark & Kamii, 1996; Steffe, 1994; Sullivan et al., 2001), there is less agreement as to how children develop MT. Some scholars (e.g. Anghileri, 1989; Mulligan & Mitchelmore, 1997) found that students either used modelling (physical objects, fingers, or drawings) or calculations, such as unitary counting, skip counting, and repeated addition, to known and derived multiplicative facts, whereas others (e.g. Askew, 2018; Confrey, 1994; Davydov, 1992; Steffe, 1994; Vergnaud, 1983) suggested that this is only one pathway to MT.

Davydov (1992) proposed that an alternative way to conceptualise a multiplicative situation was to think of something "taken so many times". In doing so, students explore the meaning of the numbers as part of the process. Others have strongly argued that students in primary schools should experience functional relationships between two variables (Askew, 2018; Nunes & Bryant, 2009; Vergnaud, 1983). Moreover, Askew found that young children (aged 7–8 years) can successfully engage with the functional aspect of MT, and consideration should be given to this in primary schools.

Confrey (1994) argued that the majority of the literature places too much emphasis on a repeated addition model of multiplication that has counting as its base. She claimed that counting or repeated addition–based models do not sufficiently explain “many of the actions of young children that can be seen as multiplicative” (p. 292). Confrey proposed the operation of splitting, defined as “an action of creating simultaneously multiple versions of an original” (p. 292). She argued that splitting is independent of counting and repeated addition and arises naturally in young children’s thinking when sharing, halving, and doubling, and is a precursor to ratio.

More recent studies (Hackenberg & Tillema, 2009; Kosko, 2018, 2019; Tzur et al., 2013; Zwanch & Wilkins, 2021) examined the specific aspects of Steffe’s (1988, 1992, 1994) longitudinal study and his number sequence framework (Steffe, 2010) to gain a deeper sense of the stages involved in constructing MT in older students.

For example, Hackenberg and Tillema (2009) investigated how whole number multiplicative concepts are involved in year 6 students’ construction of fraction composition schemes. Informed by Steffe’s (1994) research, these authors identified the following conceptual framework for whole number multiplicative concepts.

> First multiplicative concept (MC1): coordination of two levels of units, involves enacting the coordination of the situation.
> Second multiplicative concept (MC2): the interiorization of two levels of units, involves anticipating two levels of units and coordinate three levels of units in activity (enacting with materials).
> Third multiplicative concept (MC3): interiorization of three levels of units, involves coordinating all three levels of units as mental operations.

These authors found that MC2 is the basis for students’ construction of a unit fraction composite scheme, and MC3 is necessary for students’ construction of a general fraction composite scheme.

Their conceptual framework builds on Steffe’s (1994) work and shows the subtle development of the composite unit coordination from enactment of one composite unit through to mental disembedding a unit into units of units and mentally coordinating three levels of units. Such insights can assist teachers to distinguish the level of MT evident in students’ responses, as well as being aware of students’ development of the composite unit coordination, and its importance.

In order to develop a sense of multiplicative relationships (i.e. how the numbers relate and interact with each other), students need many experiences with different multiplicative situations involving numbers for which counting alone will be inefficient (Callingham, 2003). Others not only concurred with this view, but also argued that part of a reliance on AT is attributed to students’ use of intuitive strategies, and the nature of numbers involved in a problem rather than on the underlying situation (e.g. Greer, 1988; Harel & Confrey, 1994; Thompson & Saldanha, 2003).

Routinely incorporating more difficult numbers that cannot be intuitively grasped in word problems and providing multi-step word problems have been recommended to encourage students to think more deeply about which operations to use and move beyond superficial strategies (Greer, 1988). Size of numbers and choice of tasks were a consideration in selecting tasks the teachers would use with their students for the study reported here.

Multiplicative semantic structures

Different classification schemes for multiplication and division word problems evident in the research literature are commonly referred to as semantic structures. These include equal groups, allocation/rate, rectangular array/area, multiplicative comparison (times-as-many/scale/multiplying factor), and Cartesian product (Anghileri & Johnson, 1992; Greer, 1992). Apart from Cartesian product, the other structures can be generalised to situations involving fractions and decimals. Greer argued that while the distinctions between models of situations are important pedagogically, the way in which a situation is interpreted depends on a student's perception of it. Key differences between these structures are the role and interpretation of the multiplier and multiplicand, the representations, and whether the structure reflects the commutative aspect of multiplication. Such differences are important considerations for teaching.

In the *equal groups* conceptualisation, the two numbers have different roles, one being the multiplier (number of groups), the other the multiplicand (number in each group). A critical step in students' development of MT is a focus on the number of groups rather than the number in each group. Such a shift leads to the recognition of the number of groups as a factor, and supports more efficient strategies and generalising (Siemon & Breed, 2006).

Rectangular array (area model) provides a visual representation of the mapping of two spaces into a third (Vergnaud, 1983) the product, which is a new quantity in its own right, rather than relationship between the two quantities (Schwartz, 1988; Vergnaud). Others contend that the array model encourages students to think about multiplication as a binary operation and makes the mathematical property of commutativity, intuitively acceptable (Barmby et al., 2009; Greer, 1992).

*Allocation/*rate or isomorphism of measures (Vergnaud, 1983) refers to situations involving a direct proportion between two measure spaces M1 and M2. In the example, "Jess buys four lollies that cost 15 cents each. How much does Jess have to pay?" the word "each" represents a hidden number (1) thus evoking a subtle rate situation: One lolly costs 15 cents; four lollies cost how much? Using a ratio table (Fig. 1) to represent the four items makes the relationships visible.

Viewing the situation vertically indicates a scalar relationship (multiplicative comparison) or a "within quantities" relation, as each is being scaled up by the same factor (×4): thinking four lollies cost four times as much as one lolly. In contrast, viewing it horizontally involves a functional relationship that is "between quantities"

Fig. 1 Representation of a simple rate problem

M_1 (lollies)	M_2 (cost)
1	15
4	?

relation (lollies and cost), and the mapping of the number of lollies onto the cost of the lollies (×15). Whilst the scalar relation can be thought of additively, the functional relation can only be thought of multiplicatively (Askew, 2018; Vergnaud, 1983).

Times-as-many/multiplicative comparison is the only ternary multiplicative relationship as there is only one variable or type of quantity (Vergnaud, 1997). In the example, "Jed has 6 times as many miniature cars as Sally. If Sally has 4 cars, how many cars does Jed have?" the only quantity (or variable) is miniature cars. The multiplicative factor may be considered the multiplier. Multiplicative comparison is a preliminary stage to ratio and relates directly to the nature of multiplication (Greer, 1992; Vergnaud, 1983).

Cartesian product, or product of measures (Vergnaud, 1983), is a different conceptualisation, as it involves the construction of a set of ordered pairs from two sets. In the example, "Jess has 3 skirts, and 4 shirts. How many different outfits can she wear?" the two sets contain three and four elements respectively, and the set of ordered pairs (Cartesian product) has 12 elements. The product is a new quantity in its own right, a characteristic shared with the rectangular array structure, rather than a relationship between the two quantities.

The description of each structure indicates that their representations (i.e. physical models) are different, but are also interconnected, which highlights the complexity associated with developing MT. Greer (1988) argued that experiences across these different problem types are necessary for students to develop MT and later proportional reasoning.

In summary, this critique of the literature highlights the complexities associated with the development of MT, and why access to such is a persistent barrier to many students' mathematical progress in the middle years. Several studies emphasised the need for teachers to understand the complexity associated with developing MT. In order to teach multiplication and division, teachers must first understand the nature of MT (Clark & Kamii, 1996; Zwanch & Wilkins, 2021). Other studies reported the need to reconsider the approaches to teaching multiplication (e.g. Askew, 2018; Siemon et al., 2018; Tzur et al., 2018). In particular, Askew (2018) argued that primary students' lack of development of MT may have more to do with the approaches to teaching multiplication than the students being "developmentally" ready. Furthermore, there is a need to break down the perception that the pathway to MT is through repeated addition, to broaden teachers' pedagogical and content knowledge to other multiplicative structures and to consider the functional nature of MT (Askew, 2018). In addition, a greater emphasis is needed on teaching multiplicative reasoning in the early years of primary school (Askew et al., 2019).

Teacher professional learning models

Tzur et al. (2013) highlighted the need for PL to focus on deepening teachers' understanding of concepts underpinning MT, carefully constructed tasks and a pedagogical approach that is aligned with a constructivist view of learning such as student-adaptive pedagogy. This study informed the content our study, whereas others we drew on focused on structural aspects and processes of the PL. These include the need to situate PL for teachers in realistic contexts, as part of the ongoing work in schools (Bruce et al., 2010; Clarke & Hollingsworth, 2002); iterative cycles of planning, practice, and reflecting (Clarke & Hollingsworth) spaced over a period of time (e.g. Desimone, 2009). Furthermore, Timperley et al. (2007) suggested that, "professional development that led to sustained better practice, had a focus on developing teachers' pedagogical content knowledge in sufficient depth to form the basis of principled decisions about practice" (p. xivi).

The Clarke and Hollingsworth (2002) model and Timperley et al. (2007) Inquiry and Knowledge Building Cycle (IKBC) informed the present study. Both emphasise that teacher knowledge is multifaceted and involves teacher theory of practice, beliefs, values, and pedagogy. Timperley and colleagues proposed that relevant PCK could be developed over an extended period of time through teacher inquiry. Unlike the Clarke and Hollingsworth model, Timperley et al. presented a model of PL that emphasised teacher professional inquiry as the first step in teacher professional development. A distinguishing factor of the IKBC is that it focuses first on identifying student needs and, consequently, teacher professional learning needs are identified from these needs. The development of teachers' PCK is, therefore, linked and contextualised to student learning needs.

Schools within the sector currently use the IKBC, so it was particularly relevant to our project. The leadership team and teachers in each school identified student learning needs from the analysis of student data (Timperley et al., 2009). A PL program to address identified needs was developed by the first author in conjunction with Teaching Educators (TE) employed by the school system. In using this term, we refer to an educator "with content-specific expertise" whose main role is to support teachers to improve the quality of teaching and learning of mathematics (Cobb et al., 2018, p. 113).

Supporting professional learning

PL offered to teachers needs to create a bridge between research and classroom practice (Kretlow et al., 2012). Both the Clarke and Hollingsworth (2002) and Timperley et al. (2009) models included an expert to support the intended learning in the classroom. Furthermore, a meta-analysis of PL (Yoon et al., 2007) found that an effective method of changing teacher practice and improving student learning incorporated a combination of PL followed by coaching of a knowledgeable other. Much has been written about the role of coaches to support teacher growth and pedagogical practice (e.g. Cobb, et al., 2018; De Paor, 2015;

Kretlow et al., 2012; Polly, 2012; Sharratt & Fullan, 2012; Teemant et al., 2011). A common finding from these studies—improvement in teachers' pedagogy—was a result of the support of an expert "knowledgeable other" (instructional coach), who provided in-classroom support to teachers as they enacted new pedagogical practices. Instructional coaches work in partnership with teachers to support the incorporation of research-based instructional practices into their teaching (Knight, 2009; Teemant et al., 2011). They support teachers to choose appropriate instructional approaches, model these practices in the classroom, observe the practice of teachers, and engage in feedback (Knight, 2009). The work of instructional coaches is to improve teacher efficacy and, consequently, to improve student outcomes (Kraft et al., 2018). TEs undertake such a role in this context of the study reported in this paper.

Instructional coaching is a way of providing PL alongside the daily work of teachers in the classroom. Critical to the effectiveness of the coaching activities is the need to "engage teachers in fundamental dialogue about mathematical content, mathematical learning and student understanding" (Campbell & Griffin, 2017, p. 163).

Knight (2017, as cited in Walsh et al., 2020) provided an approach to instructional coaching that included three stages:

1. identification, in which the instructional coach determines the goal for the work with the teacher to support the move from current practice to new practice;
2. learning within a partnership between coach and teacher through co-teaching, in which teaching strategies and developed and enhanced; and,
3. teacher improvement through implementation of new strategies and monitoring of the practice with the help of the coach.

Co-teaching is powerful in supporting teachers to develop sophisticated instructional practices by affording opportunities for "in-the-moment" reflection (Mason, 2002), and to enhance the depth of post-lesson reflective thinking (Eden, 2020). Increased responsiveness to student thinking within lessons was also noted as supporting teachers with developing adaptive practice (Eden). Sharratt and Fullan (2012) extended the notion of co-teaching to include instructional coaches employed by the school or, as in the current study, employed by an education system (TEs). The co-teaching cycle involves coaches and teachers co-planning, co-teaching, co-reflecting, and co-debriefing. It is recognised as having the potential to be "the most powerful way to improve teaching practice and to implement changes in assessment and instruction" (Sharratt & Fullan, p. 118–119). An important first step in the cycle is the process of co-planning, which is considered critical as it helps support teachers anticipate student responses to tasks (Stein et al., 2008). The instructional coach and class teacher co-teach lessons incorporating several opportunities during the lesson to ascertain student understanding. Post lesson, the instructional coach and classroom teacher co-debrief to discuss aspects of the lesson that could be modified, and the next steps for student learning (Sharratt & Fullan).

In sum, the research literature highlights the importance of a "knowledgeable other" to support teachers to implement research informed pedagogical practices. Co-teaching is a critical component of this support, as is establishing trust with teachers and working with them in their classrooms. In the current study the role of "knowledgeable other" is the TE who works in schools over a sustained period of time to support teachers to improve pedagogical practices and student learning outcomes.

Methodology

This quantitative study, comparing the results of student assessment data, was conducted in a metropolitan school system in Australia during 2016 and 2017. Two cohorts of primary schools were involved in the MT research project. These included 13 schools in 2016 and 22 schools in 2017. Principals of these schools chose to be involved, as they had identified MT as a focus for PL, based on students' MAI results. All students in years 1 to 6 were assessed by their class teacher at the start of each school year. MAI results in 2015 indicated that over 50% of year 3 and 38% of year 4 students in the school system had not progressed beyond using a physical model to solve basic multiplication and division problems (Growth Point 3, see Fig. 2). That is, students were reliant on physical objects and counting-based additive strategies.

Teaching educators

As stated earlier, TEs are employed by the school system to support mathematics instruction in schools. Five TEs work in partnership with school leadership teams on the school-identified mathematics goal using the IKBC (Timperley et al., 2009) to:

- deliver the big ideas from research related to mathematics content and pedagogical practices, for example, the use of challenging multiplication tasks;
- implement the co-teaching cycle through co-planning, co-teaching, and co-debriefing with teachers (Sharratt & Fullan, 2012); and

0. Not apparent.
1. Counting group items as ones.
2. Modelling multiplication and division (all objects perceived).
3. Partial modelling multiplication and division (some objects perceived).
4. Abstracting multiplication and division (no objects perceived).
5. Basic derived and intuitive strategies for multiplication.
6. Basic, derived and intuitive strategies for division.
7. Extending and applying multiplication and division.
8. Extending and applying multiplication and division to fractions and decimals.

Fig. 2 Strategies for multiplication and division growth points

- support teachers to experiment with changes of practice in the classroom by modelling, co-teaching, and observing.

In 2016, these TEs facilitated the PL in participating schools across three terms (terms 2–4) as part of each school's regular after-school meetings. They provided periodic in situ, that, is in-classroom coaching support for selected schools. In contrast, in 2017, TEs followed up the PL with in situ support 4–5 times per term for each project school. The decision to increase the support was informed by the 2016 MAI data, which showed greater growth in participating schools that had additional TE coaching support than those that received no TE support.

Professional learning modules

The research team, led by a university mathematic educator (first author), developed six PL modules with the TEs. Each PL module included the following structure.

1. Reading a professional article about the multiplicative structure in focus.
2. Analysing student data—work samples of tasks completed in mathematics lesson.
3. Reflecting on student work samples—the observations of multiplicative thinking in student responses were recorded in teacher reflective journals.
4. Solving a series of learning tasks focused on each multiplicative structure and discussing the underpinning mathematics and possible student responses.
5. Teaching the tasks from each module as a between module activity.

Content was informed by the literature that highlighted the importance of students understanding multiplication beyond the "equal groups" structure (Askew, 2018; Barmby et al., 2009; Confrey, 1994; Greer, 1992; Vergnaud, 1983); the early development of MT, in particular the importance of the construction and coordination of composite units (Clark & Kamii, 1996; Hackenberg & Tillema, 2009; Steffe, 1994; Sullivan et al., 2001); and pathway to MT. Through the process of co-constructing the modules with a mathematics education academic, the TEs built their content knowledge and confidence to deliver the modules to all staff in each project school. To ensure consistency, each module included facilitator notes, a PowerPoint presentation including specific tasks and pedagogical practices teachers would explore during each module, and a resource pack for teachers.

Each module included challenging tasks (Sullivan et al., 2011) related to the content and ways to adapt and extend tasks (Sullivan et al., 2006). Throughout each module, important ideas about learning mathematics with understanding (exploring, reasoning, questioning, justifying, and reflecting) were discussed and modelled. Other components included teacher reflection, and pedagogies to explore in classrooms (e.g. teacher noticing, holding back from telling, giving students time to engage in productive struggle, use of probing questions, student choice in how they represented their thinking, and sharing solution methods). The elements of the launch, explore, summarise lesson structure (Sullivan et al., 2016) were modelled in the modules and in classrooms.

Table 1 presents an overview of each module's content and an example of a task that teachers explored in the module, and then with their students in the classroom. These tasks were designed for years 3–4 students and modified versions of these tasks were provided for Foundation (first year of school)–year 2, and years 5–6 students.

Participants

Table 2 provides information about the schools participating in the project. Both cohorts included a diverse range of schools based on socio-economic status (SES), English as an Additional Language/Dialect (EAL/D), and school size. The 2017 cohort had a slightly wider range of SES than 2016 including one school with the lowest possible SES ranking. There was also a range of students with EAL/D, in both cohorts. Only information for each years 3 and 4 cohort is included, as their historical data was used for the research.

Whilst the PL and research focused on years 3 and 4 teachers and students, all teachers from Foundation to year 6 participated in the PL with the expectation that the learning would be adapted for their students. Teachers' experience ranged from early career teachers to very experienced teachers.

Table 1 Overview of the content of the PL modules and an example of a task

Module	Focus	Task example
1	Overview of course and multiplicative interview	A Blue whale is about 34 m long. A crocodile is about 4.3 m long. A Blue whale is about how many times as long as a crocodile?
2	Overview of multiplicative structures	106 people were going to the netball gala day. They travelled in vans that could take six people. What is the least number of vans needed? Solve it in two different ways
3	Rectangular arrays	This carrot patch is being harvested. The farmer asked his workers to find the best way to work out how many carrots are still to be picked. How might you work out how many carrots are in the patch? (Students are shown a picture of a partially picked patch.)
4	Times-as-many	Jess collected twice as many shells as Angie. They collected 30 shells altogether. How many shells did Angie collect?
5	Allocation and rate	It costs $18 to buy 12 cans of soft drink. How many cans would you get for $30?
6	Analysis of interview data and reflection of student learning	A shop sells Ghost Drop lollies at ten for 35 cents. How many Ghost Drops could I buy with $1.40?

Table 2 Information pertaining to schools participating in the project

Year	*Number of schools*	*Size range (enrolments)*	*Total number of years 3 & 4 student participants*	*SES ranking range (2016 census)*	*SES ranking mean (2016 census)*	*English as an Additional Language (EALD range*
2016	13	168–764	1480	94–115	100.9	0.5–87.6%
2017	22	121–815	2695	78–120	101.2	0–93%

Data collection

Data were collected from three different sources: (1) teacher survey data, the findings of which were reported earlier (see Downton et al., 2018); (2) historical growth point (GP) data from the MAI; and (3) NAPLAN data. The MAI is a clinical interview designed to assess primary students' learning in the number domains of counting, place value, addition and subtraction strategies, and multiplication and division strategies. It is linked to a research-based framework of growth points (Gervasoni et al., 2017) that describes a pathway through which students' mathematical thinking can be viewed and analysed (Clarke et al., 2005). MAI data are used to compare growth over time and to identify trends.

MAI data are collected each year by the whole system and allow historical data over time to be analysed. NAPLAN is a national assessment conducted in May each year to assess all students across Australia in years 3, 5, 7, and 9 on fundamental disciplines of literacy and numeracy. Both assessments are administered annually by all schools in the system and, therefore, provided longitudinal data for analysis of MT.

MAI data for both cohorts represent data for year 4 students in February 2016 and 2017 (administered before the PL commenced) through to year 6 in February 2018 and 2019, that is MAI data for years 3, 4, 5, and 6. NAPLAN data was analysed for each cohort in year 3 (the year before the PL) and in year 5 (the year after the PL). NAPLAN and MAI data sets, therefore, represent the achievements of the same students both prior to and post the PL.

Data analysis

MAI data reported in this paper were for students whose data could be matched from year 4 (2016) to year 6 (2018) and year 4 (2017) to year 6 (2019). These data were analysed using the Growth Point Framework (Clarke et al., 2002) for the multiplication and division domain (Fig. 2). Growth Point 4 is the point at which students can abstract—a critical stage in the development of MT (Sullivan et al., 2001). For this study, students who achieved GP 4 and above on the MAI were categorised as multiplicative thinkers, and those who achieved GP 3 or below were categorised as additive thinkers in multiplicative situations. Direct comparisons of the percentage of students at lower GPs (0, 1, 2, and 3) were made between school involved in the MT project (MT) and non-participating schools (notMT). Comparisons between MT and notMT schools were also made pertaining to the percentage of students at higher GPs (4, 5, 6, 7, and 8).

In addition, the change in each student's growth point for this domain was identified and the cohort was disaggregated by the level of increase in the growth points. This measure was used to gauge the overall level of growth for each student.

NAPLAN numeracy results are reported using a national achievement scale. Student raw scores on tests are converted to a NAPLAN "scale score" so the scores can be located on the national scale. NAPLAN reporting scales are constructed so that any given scale score represents the same level of achievement over time within a domain. For example, a score of 700 in numeracy in 1 year represents the same level of numeracy achievement in other years. Prior to 2018, the NAPLAN data for numeracy was disaggregated and reported against two scales, Number, Patterns and Algebra (NPA), and Data, Space and Measurement (DSM). From 2018 onwards, the disaggregation into the two sub-scales of numeracy was not reported.

NAPLAN scores for the 2016 year 4 cohort were matched from their 2015 (year 3) to their 2017 (year 5) scores on the NPA scales. These data were then classified into three groups to compare growth: schools participating with TE support, schools participating with no TE support, and schools not participating (notMT). Two levels of data analysis were conducted. The first level involved calculating mean growth for each cohort, the second to identify any levels of significance between the means using analysis of variance (ANOVA). A similar analysis was repeated using the DSM scale.

NAPLAN scores were not reported separately on these scales from 2018, as similar analysis was not possible for the 2017 year 4 cohort. This was a limitation of our study.

Results

In this section, results of the analysis of MAI data are presented, followed by those of NAPLAN data. Whilst the analysis of student data is the focus of this paper, it is also necessary to draw on the analysis of the teacher survey data (2016) and TE observations to contextualise the findings relating to the student data.

MAI results—year 4, 2016 cohort

Increase in multiplicative thinkers

Table 3 shows the distribution of GPs for students as they progressed from year 4, 2016, to year 6, 2018, for the 13 schools involved in the MT project in 2016. These schools are coded as MT; schools not involved are coded as notMT. Only students with assessment data in each of the 3 years were included in the analysis. Consequently, the data represent a sample of the entire population (approximately 2,800

Table 3 Distribution of growth points by numbers of students' 2016-2018 MAI data

	Multiplicative Thinking Project (MT)						Not Multiplicative Thinking Project (notMT)					
	Year 4 2016		Year 5 2017		Year 6 2018		Year 4 2016		Year 5 2017		Year 6 2018	
GP	N	%	N	%	N	%	N	%	N	%	N	%
8	0	0.0%	7	1.2%	26	4.6%	3	0.4%	8	1.0%	19	2.4%
7	4	0.7%	13	2.3%	30	5.3%	1	0.1%	5	0.6%	24	3.0%
6	7	1.2%	34	6.0%	58	10.3%	14	1.8%	49	6.1%	94	11.8%
5	45	8.0%	87	15.4%	119	21.1%	50	6.3%	112	14.0%	164	20.5%
4	118	20.9%	209	37.0%	201	35.6%	211	26.4%	301	37.7%	285	35.7%
3	135	23.9%	103	18.2%	69	12.2%	175	21.9%	135	16.9%	117	14.6%
2	218	38.6%	98	17.3%	59	10.4%	288	36.0%	169	21.2%	86	10.8%
1	31	5.5%	10	1.8%	3	0.5%	43	5.4%	16	2.0%	5	0.6%
0	7	1.2%	4	0.7%	0	0.0%	14	1.8%	4	0.5%	5	0.6%
Total	565	100%	565	100%	565	100%	799	100%	799	100%	799	100%

students). Reading the table horizontally shows the number and percentage of students who achieved that GP in each year. GP4 is a critical stage in students' development of MT; below this GP, students are relying on counting based strategies or AT. Rows representing GP4 to GP8 are shaded to indicate students at these growth points are considered multiplicative thinkers.

As seen by the GP distribution (columns 3 and 9), there was little difference between the cohorts in 2016, with 69% of students (MT) and 65% (notMT) still on GP3 or less. By 2018, this had reduced to 23% (MT) and 27% (notMT).

Table 4 presents the aggregated data from Table 3 pertaining to multiplicative thinkers' for cohorts in years 4 and 6 MT and notMT groups, presented as percentages only.

Table 4 Percentage of students using multiplicative strategies (≥ GP4) by cohort with 95% confidence intervals

Cohort	*Group*	*Percentage multiplicative thinkers*	*Lower 95% CI*	*Upper 95% CI*
Year 4 2016	*MT*	*30.8%*	*27.9%*	*34.6%*
Year 4 2016	*notMT*	*34.9%*	*31.6%*	*38.2%*
Year 6 2018	*MT*	*76.8%*	*73.3%*	*80.3%*
Year 6 2018	*notMT*	*73.3%*	*70.3%*	*76.4%*

There is a 46% increase in the proportion of multiplicative thinkers for the MT group, which was then compared with a 38% increase in multiplicative thinkers for the notMT group (see Fig. 3). A Z-statistic was calculated to compare the two population proportions to test for significance. The Z-statistic was calculated to be 2.8 ($p = 0.0025$), which is significant at the alpha level of 0.05. Therefore, we reject the null hypothesis that there was no difference in the proportions between the two groups. Thus, we have evidence that the 8% increase in multiplicative thinkers between the two groups is an actual difference.

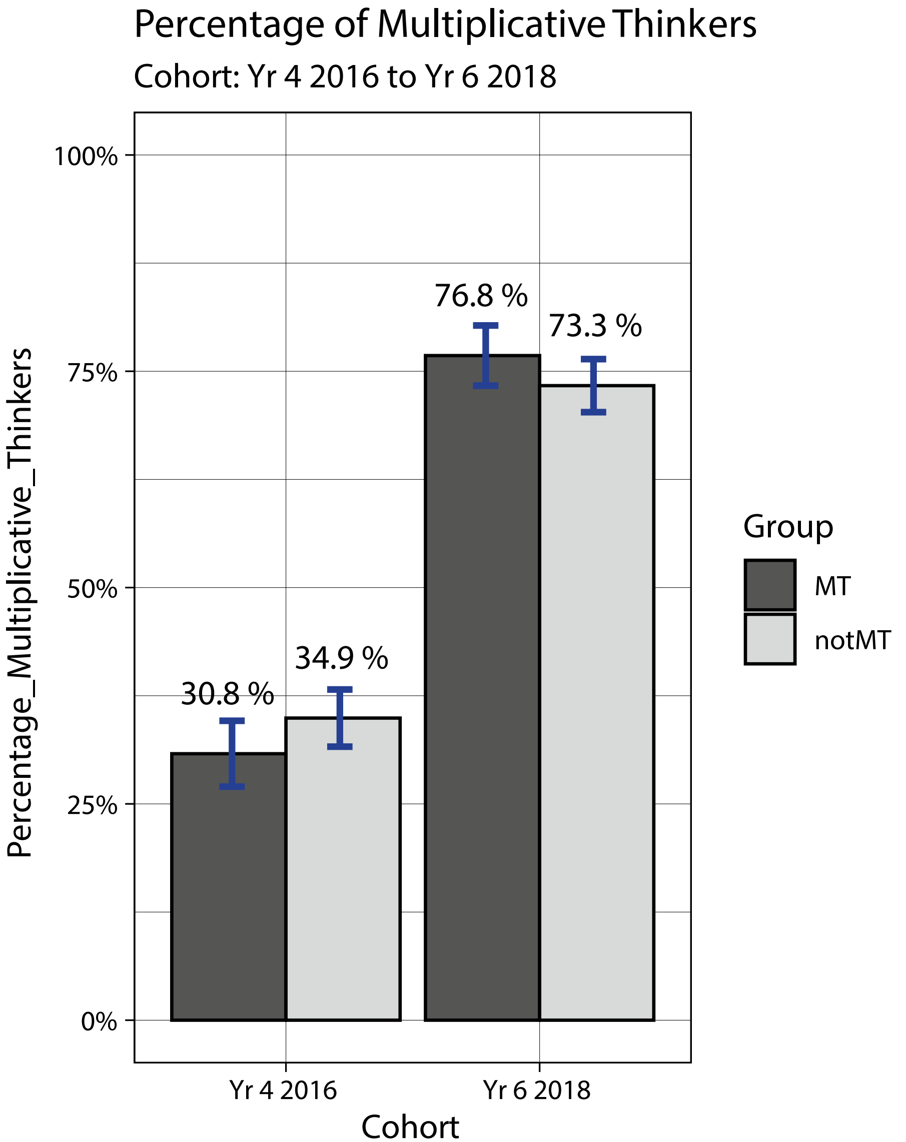

Fig. 3 Percentage of multiplicative thinkers in each cohort with 95% confidence intervals shown

Student growth in multiplicative thinking

The MAI data was disaggregated by the increase in the number of growth points for each student between year 4, 2016, and year 6, 2018. The "on the way growth point" (OTWGP) for the start of year 4 is 3 and the OTWGP for the start of year 6 is 5. Hence, an increase of 2 growth points would be considered a measure of the typical growth of a student over the 2-year period.

Students engaged through their school's involvement in the MT project had 29% of the cohort show an increase of 2 growth points compared with 24% of students in non-engaged schools (Fig. 4). A positive difference also exists between MT and notMT schools for increases of 3, 4, and 5 growth points of 2%, 1%, and 1% respectively. Conversely, a greater proportion of students from notMT schools exhibited some degree of stagnation in their learning, being defined as less than a 2-growth point increase when compared with students from MT schools.

Increase in multiplicative thinkers

Growth point data in Table 5 reflects the MAI analysis for the 22 schools involved in the PL in 2017. Data tracks the progression of student achievement in the multiplication and division strategies domain from year 4, 2017, to year 6, 2019. Their data is compared in Tables 5 and 6 against schools that had not participated in the PL. Only students with assessment data in each of the 3 years are included in the analysis. Prior to the project, more students were on GP3 or less (64%) in MT schools than notMT schools (60%). However, by 2019, this had reduced to 22% (MT) and 24% (notMT).

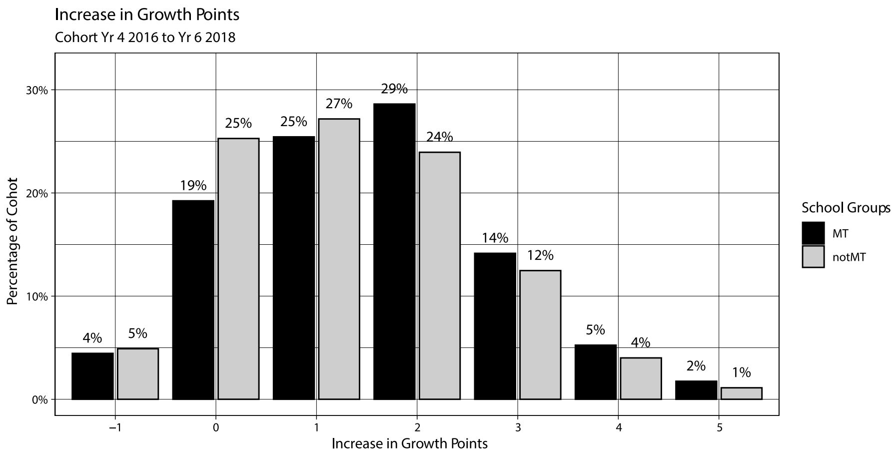

Fig. 4 Percentage of cohort by increase in growth points from year 4 to year 6

Table 5 Distribution of growth points by numbers of students' 2017–2019 MAI data

	Multiplicative Thinking Project (MT)						*Not Multiplicative Thinking Project (notMT)*					
	Year 4 ***2017***		**Year 5** ***2018***		**Year 6** ***2019***		**Year 4** ***2017***		**Year 5** ***2018***		**Year 6** ***2019***	
GP	***N***	*%*	***N***	*%*	***N***	*%*	***N***	*%*	***N***	*%*	***N***	*%*
8	*4*	*0.4%*	*18*	*1.6%*	*82*	*7.3%*	*2*	*0.2%*	*12*	*1.1%*	*55*	*5.3%*
7	*8*	*0.7%*	*42*	*3.7%*	*65*	*5.8%*	*6*	*0.6%*	*16*	*1.5%*	*49*	*4.7%*
6	*34*	*3.0%*	*64*	*5.7%*	*111*	*9.8%*	*19*	*1.8%*	*79*	*7.5%*	*136*	*13.0%*
5	*88*	*7.8%*	*191*	*16.9%*	*230*	*20.4%*	*88*	*8.4%*	*139*	*13.3%*	*206*	*19.7%*
4	*275*	*24.4%*	*374*	*33.2%*	*390*	*34.6%*	*298*	*28.5%*	*386*	*36.9%*	*346*	*33.0%*
3	*250*	*22.2%*	*193*	*17.1%*	*142*	*12.6%*	*233*	*22.3%*	*201*	*19.2%*	*129*	*12.3%*
2	*419*	*37.1%*	*226*	*20.0%*	*97*	*8.6%*	*346*	*33.0%*	*194*	*18.5%*	*113*	*10.8%*
1	*35*	*3.1%*	*17*	*1.5%*	*7*	*0.6%*	*43*	*4.1%*	*11*	*1.1%*	*11*	*1.1%*
0	*15*	*1.3%*	*3*	*0.3%*	*4*	*0.4%*	*12*	*1.1%*	*9*	*0.9%*	*2*	*0.2%*
Total	*1128*	*100%*	*1128*	*100%*	*1128*	*100%*	*1047*	*100%*	*1047*	*100%*	*1047*	*100%*

Table 6 Percentage of students using multiplicative strategies (≥ GP4) by cohort with 95% confidence intervals

Cohort	Group	Percentage multiplicative thinkers	Lower 95% CI	Upper 95% CI
Year 4 2017	*MT*	*36.3%*	*33.5%*	*39.1%*
Year 4 2017	*notMT*	*39.4%*	*36.5%*	*42.4%*
Year 6 2019	*MT*	*77.8%*	*75.4%*	*80.2%*
Year 6 2019	*notMT*	*75.6%*	*73.0%*	*78.2%*

Table 6 presents the aggregated data from Table 5 pertaining to multiplicative thinkers' for cohorts in years 4 and 6 MT and notMT groups, presented as percentages only.

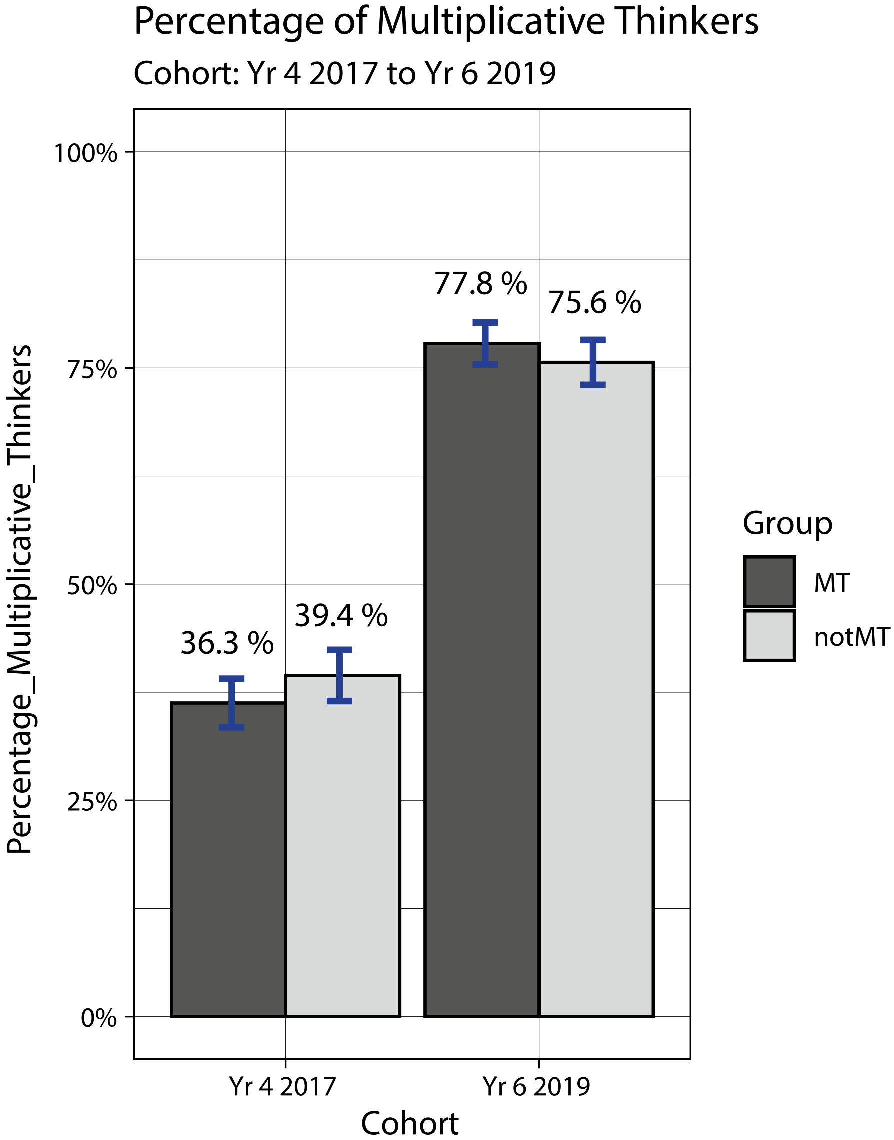

Fig. 5 Percentage of multiplicative thinkers in each cohort with 95% confidence intervals shown

Figure 5 shows a 42% increase in multiplicative thinkers in the MT group compared with the 36% increase in the notMT group. A *Z*-statistic was calculated to be 2.57 ($p=0.005$), which is significant at an alpha of 0.05. Therefore, we can reject the null hypothesis that there was no difference in the increase of multiplicative thinkers between the two groups. Thus, we have evidence that the 6% difference between the two groups is statistically significant.

Student growth in multiplicative thinking

The patterns of growth for the year 4, 2017, to year 6, 2019, cohort (see Fig. 6) are similar to the previous cohort. A greater proportion of students in MT schools exhibited growth of 2 or more growth points when compared with students in notMT schools. Greater stagnation, defined as less than a 2-growth point increase, was observed for notMT schools as compared with MT schools.

NAPLAN data analysis

NAPLAN data were collected and grouped according to three groups of schools: TE support, no TE support, and notMT. The analysis was completed for the NPA and the DSM scales. For each of the analyses, a table of the difference in means is provided, followed by a one-way analysis of variance (ANOVA). Finally, a post hoc analysis using the Tukey HSD test is provided to examine differences between the groups, if any.

As seen in Table 7, the cohort with TE support had the largest mean growth of 94.07 across 2 years from 2015 to 2017; those with no TE in-classroom support had a mean growth of 88.25, whereas notMT schools had a mean growth of 84.28.

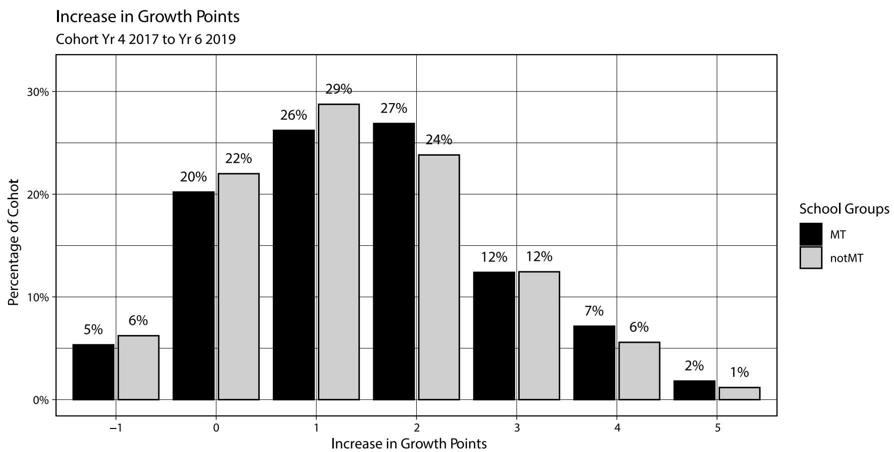

Fig. 6 Percentage of cohort by increase in growth points from year 4, 2017, to year 6, 2019

Table 7 Differences in mean growth between groups 2015 to 2017 NAPLAN (NPA)

Groups of schools	Number of students	Mean of growth between scores	Standard deviation
TE support	278	94.07	65.7
No TE support	361	88.25	56.5
notMT	2159	84.28	66.5

At an alpha of 0.05, the analysis of variance (Table 8) showed a significant difference among the groups, F (2, 2795) = 3.0725, p = 0.046. Post hoc analysis using the Tukey HSD test was conducted to examine differences between the groups (see Table 9).

Statistically significant difference "Between Groups" at p = 0.049 was between the TE support ($\bar{x}$ = 94.07) and notMT group ($\bar{x}$ = 84.28) for the NPA scale. The effect size for the TE support group was calculated at d = 0.70 compared to d = 0.64 for the notMT group.

The cohort with TE support had the largest mean growth across the 2 years for DSM scale (see Table 10). Unlike NPA (Table 7), the mean growth for notMT was larger than no TE support schools.

At an alpha of 0.05, the analysis of variance (Table 11) showed a significant difference among the groups, F (2, 2795) = 4.108, p = 0.016. Post hoc analysis using the Tukey HSD test was conducted to examine differences between the groups (see Table 12).

Statistically significant difference "Between Groups" at p = 0.013 was between the TE support ($\bar{x}$ = 105.68) and no TE Support group ($\bar{x}$ = 92.33) for the DSM scale.

Note that the PL focus did not cover any of the content outcomes for the DSM strand explicitly, and there was no statistically significant difference between the TE-supported schools and non-project schools.

In sum, the results of the analysis of MAI and NAPLAN student data indicate greater growth in student learning of MT in schools that participated in PL combined with additional TE in-class coaching support. The results also show greater student growth in MT for schools that participated with no additional support. When student data of participating schools is disaggregated into schools that had in-class TE coaching support, compared with those that only participated in PL sessions, the results indicate that there was a statistically significant difference. More specifically, student growth

Table 8 Analysis of variance

Source of variation	SS	df	MS	F	P-value	F crit
Between groups	26,147	2	13,073	3.0725	0.04645	2.998
Within groups	11,892,615	2795	4254			
Total	11,918,762	2797				

Table 9 Tukey HSD post hoc analysis

Group comparison	Diff	Lower	Upper	*p* adj
notMT/no TE support	−3.9654	12.6630	4.7323	0.5333
TE support/no TE support	5.8226	−6.3829	18.028	0.5025
TE support/notMT	9.7880	0.0412	19.5348	0.0487*

*Significant at alpha 0.05

Table 10 Differences in mean growth between groups 2015 to 2017 NAPLAN (DSM)

Groups of schools	Number of students	Mean of growth between scores	Standard deviation
TE support	278	105.68	60.2
No TE support	361	92.33	58.0
notMT	2159	99.16	58.8

Table 11 Analysis of variance

Source of variation	SS	df	MS	F	P-value	F crit
Between groups	28,481	2	14,240	4.108	0.016	2.998
Within groups	9,687,998	2795	3466			
Total	9,716,480	2797				

Table 12 Tukey HSD post hoc analysis

Group comparison	Diff	Lower	Upper	*p* adj
notMT/no TE support	6.829	−1.021	14.679	0.103
TE support/no TE support	13.344	2.328	24.360	0.013*
TE support/notMT	6.515	−2.282	15.312	0.192

*Significant at alpha 0.05

was greater in schools where there was a combination of PL and in-class TE coaching support. In contrast, there was no statistical difference in growth in these schools for the DSM strand.

Insights from teacher survey data and TE reflections

Responses from the 2016 teacher survey, teacher reflective diary, and TE reflections of in situ support are included to situate the findings relating to student data.

There are definitely learnings from this experience that I would pursue in my future teaching these include: **asking challenging questions** that use specific language i.e. **times as many**, word problems presented in different ways (not just the typical form), use of **arrays** to show multiplication **and** division, highlight **proportional reasoning** in the solution, the concept of composites. *Year 4 Teacher*

Involvement in this project has challenged the staff to think differently about the questions they are asking of the students and the task they are designing. This has resulted in a great improvement in teacher questioning and student learning as evidenced by our 2016 Naplan results, which saw a great improvement in the targeted areas of Multiplication and Division. *Lead teacher*

The language of times-as-many was challenging for students initially but once they had more experience with tasks like this, I saw a shift in the strategies they used and they were using multiplicative language and making connections between multiplication and division. *Year 3 teacher's diary*

Watching our TE model, listening to the language she uses and applying that into my own practice and modelling to colleagues. Knowing how and when to prompt and use multiplicative strategies to weave it into the concept has been extremely helpful. *Lead teacher*

TE reflections

Before we began the professional learning, teachers reported that it was difficult to hold back from telling students what to do.

Teachers we worked with in classrooms were able to confidently participate in conversations around student problem solving as their MCK improved.

Lead teachers commented that conversations in the staffroom around the tasks, had teachers talking confidently about students' Multiplicative thinking.

Teachers used the mantra "Use what you know, to work out what you don't know", shared in the PL. This was stated by a student I interviewed in the post interview.

This small sample of responses provides insights into the impact of the PL, role of TE in-class support, and a teacher's own growth in understanding of how students develop MT.

Discussion

A key goal in teaching MT is to shift students from counting-based strategies to multiplicative strategies. In order to achieve this goal, teachers require an understanding of the key conceptual underpinnings of multiplicative thinking (Clark & Kamii, 1996; Zwanch & Wilkins, 2021), the stages involved, the different multiplicative semantics structures, and PCK (Askew, 2018; Siemon et al., 2018). These components were illustrated and exemplified in each module of the PL in the MT project (Downton et al., 2018).

To consider the impact of the PL on student learning, we drew on existing longitudinal student data, rather than relying solely on teacher self-reported data, their diaries, and TE reflections. Three main findings relating to students' development of MT were evident.

1. Students in schools where teachers engaged in PL with ongoing TE support had the greatest mean growth over the 2 years as evident in both the MAI data and NAPLAN results. The increase was statistically significant.
2. More students achieved the higher GPs in schools that received the PL than those in not-project schools.
3. Students in project schools also demonstrated less reliance on additive strategies evidenced by the reduced number of students in the lower GPs (GP 3 or below) compared to students in not-project schools.

Four factors relating to the PL contributed to this growth in student learning. First, the staff in each MT school made a commitment to participate. Doing so provided an opportunity for a shared understanding of effective pedagogies and a theoretical trajectory of learning pertaining to MT, to be developed across the school.

Second, the PL was targeted and carefully designed to incorporate pedagogies, professional readings, challenging tasks, and content related to multiplicative structures that were less familiar, or unfamiliar, to the teachers, namely times-as-many, allocation/rate, and Cartesian product (Greer, 1988; Thompson & Saldanha, 2003). A key feature of the PL was its delivery as modules over an extended period of time by TEs in participating schools. The effectiveness of spacing the PL over a period of time (Desimone, 2009; Timperley et al., 2007) provided sufficient time for teachers to explore the tasks and pedagogies in their classrooms, then reflect on the learning when analysing student work samples. Reflecting in this way assisted teachers to realise the capabilities of students when challenged. Furthermore, the cyclical nature of the PL enabled teachers to reflect on the impact of their new knowledge and practice on student learning.

Third, and central to the improvement in student growth, was the additional in-class TE support, as each teacher enacted the different tasks and pedagogies. A critical aspect of this support was developing teachers' expertise to notice particular student thinking and respond in the moment using enabling or extending prompts. As indicated in the literature, this between-module classroom enactment of the new practice (Clarke & Hollingsworth, 2002) is more effective when teachers are able to work with knowledgeable others (Timperley et al., 2009). TE support was enacted through the implementation of the co-teaching cycle of co-planning, co-teaching, and co-debriefing (Sharratt & Fullan, 2012), which Sharratt (2019) maintained is critical to improving student learning outcomes.

Fourth, as a consequence of the teacher PL, students engaged with more challenging tasks related to different multiplicative semantic structures in their mathematics lessons, and used more efficient strategies. As reported in the literature, students need to experience the different multiplicative problem types, multi-step problems, and engage with more complex number combinations that cannot be intuitively grasped, in order to shift from a reliance on counting based strategies (Askew, 2018; Callingham, 2003; Downton & Sullivan, 2017; Greer, 1988).

Concluding comments

This study investigated the impact of a structured school-based PL program on students' development of MT. The findings indicate that greater mean growth in student learning over a 2-year period was evident in participating schools with TE support. Critical to this development is teachers' PCK (Downton et al., 2018), in particular their awareness of the need to gradually remove physical prompts and encourage students to form mental images of multiplicative situations (Sullivan et al., 2001).

In addition to the key features and models of PL identified in the literature, this study highlights the value of providing a carefully designed ongoing school-based PL conducted by knowledgeable leaders who also provide regular support to teachers. Such support is characterised by a cycle of planning, practice, and reflection (Bruce et al., 2010), and the acknowledgment that sustained growth in teacher practice and subsequent student learning takes time and requires a whole school commitment (Clarke & Hollingsworth, 2002). Providing opportunities for such support and collaboration reflects the third recommendation of the Gonski et al. (2018) report which detailed the importance of creating, supporting, and valuing a profession of expert educators. Specifically, that Australian education should "Create the conditions and culture to enable and encourage more professional collaboration, observation, feedback and mentoring amongst teachers" (p. 3).

Several avenues for further research arise from this study relating to students' transition from additive to MT, in particular the importance of exploring different multiplicative structures from an early age to support this transition. Opportunities for further research relating to PL include the following: how to sustain the impact of PL programs in schools; the impact of ongoing in situ PL on teacher knowledge and practice, and student learning; and the optimum frequency and level of TE (external coaching) support offered to schools that will have the greatest impact on student learning.

We acknowledge two limitations of this study: the lack of opportunity to analyse subsequent NAPLAN data, and lack of resources and time to conduct teacher interviews and classroom observations.

Overall, the findings highlight the importance of providing a PL program targeted to an identified need, in this instance, teachers' knowledge of MT and PCK, to improve students' learning. A unique component of this PL model was the ongoing in-classroom support from a "knowledgeable other", who facilitated the PL within the school.

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Declarations

Ethical statement This research had ethical approval from Monash University in 2016–2021 with approval number CF16/1677–2016000873 and Diocese of Parramatta approval 270416. Informed consent was granted from principals and teachers. Student and parent consent was not required as all the student data was historical data collected annually by the Catholic Education Diocese of Parramatta, and the data was de-identified.

Competing interests The authors declare no competing interests.

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