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Gröbner–Shirshov bases and their calculation

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Abstract In this survey we give an exposition of the theory of Gröbner–Shirshov bases for associative algebras, Lie algebras, groups, semigroups, Ω-algebras, operads, etc. We mention some new Composition-Diamond lemmas and applications.

Keywords Gröbner basis · Gröbner–Shirshov basis · Composition-Diamond lemma · Congruence · Normal form · Braid group · Free semigroup · Chinese monoid · Plactic monoid · Associative algebra · Lie algebra · Lyndon–Shirshov basis · Lyndon–Shirshov word · PBW theorem · Ω-algebra · Dialgebra · Semiring · Pre-Lie algebra · Rota–Baxter algebra · Category · Module

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Abbreviations

CD-lemma	Composition-Diamond lemma
GS basis	Gröbner–Shirshov basis
LS word (basis)	Lyndon–Shirshov word (basis)
ALSW(X)	The set of all associative Lyndon–Shirshov words in X
NLSW(X)	The set of all non-associative Lyndon–Shirshov words in X
PBW theorem	The Poincare–Birkhoff–Witt theorem
X^*	The free monoid generated by X
$[X]$	The free commutative monoid generated by X
X^{**}	The set of all non-associative words (u) in X
$gp\langle X\|S\rangle$	The group generated by X with defining relations S
sgp$\langle X\|S\rangle$	The semigroup generated by X with defining relations S
k	A field
K	A commutative algebra over k with unity
$k\langle X\rangle$	The free associative algebra over k generated by X
$k\langle X\|S\rangle$	The associative algebra over k with generators X and defining relations S
S^c	A Gröbner–Shirshov completion of S
$Id(S)$	The ideal generated by a set S
$\bar{s}$	The maximal word of a polynomial s with respect to some ordering $<$
$Irr(S)$	The set of all monomials avoiding the subword $\bar{s}$ for all $s \in S$
$k[X]$	The polynomial algebra over k generated by X
$Lie(X)$	The free Lie algebra over k generated by X
$Lie_K(X)$	The free Lie algebra generated by X over a commutative algebra K

1 Introduction

In this survey we review the method of Gröbner–Shirshov[1] (GS for short) bases for different classes of linear universal algebras, together with an overview of calculation of these bases in a variety of specific cases.

Shirshov (also spelled Širšov) in his pioneering work ([207], 1962) posed the following fundamental question:

How to find a linear basis of a Lie algebra defined by generators and relations?

He gave an infinite algorithm to solve this problem using a new notion of the composition (later the 's-polynomial' in Buchberger's terminology [65,66]) of two Lie

[1] Though Shirshov [207] 1962 was the first to come up with the idea of a 'Gröbner–Shirshov basis' for Lie and non-commutative polynomial algebras, his paper became practically unknown outside Russia. In the meantime, Buchberger's 'Gröbner basis' (Thesis 1965 [65], paper 1970 [66]) for (commutative) polynomials became very popular in science. As a result, the first author suggested the name 'Gröbner–Shirshov basis' for non-commutative and non-associative polynomials. For (commutative) differential polynomials an analogous, or better to say, closely related 'basis' is called a Ritt–Kolchin characteristic set, due to Ritt [193] 1950 and Kolchin [140] 1973, and rediscovered by Wu [219] 1978.

polynomials and a new notion of completion of a set of Lie polynomials (adding nontrivial compositions; the critical pair/completion (cpc-) algorithm in the later terminology of Knuth and Bendix [138] and Buchberger [67,68]).

Shirshov's algorithm goes as follows. Consider a set $S \subset Lie(X)$ of Lie polynomials in the free algebra $k\langle X\rangle$ on X over a field k (the algebra of non-commutative polynomials on X over k). Denote by S' the superset of S obtained by adding all non-trivial Lie compositions ('Lie s-polynomials') of the elements of S. The problem of triviality of a Lie polynomial modulo a finite (or recursive) set S can be solved algorithmically using Shirshov's Lie reduction algorithm from his previous paper [203], 1958. In general, an infinite sequence

$$S \subseteq S' \subseteq S'' \subseteq \cdots \subseteq S^{(n)} \subseteq \ldots$$

of Lie multi-compositions arises. The union S^c of this sequence has the property that every Lie composition of elements of S^c is trivial modulo S^c. This is what is now called a Lie GS basis.

Then a new 'Composition-Diamond lemma[2] for Lie algebras' (Lemma 3 in [207]) implies that the set $Irr(S^c)$ of all S^c-irreducible (or S^c-reduced) basic Lie monomials $[u]$ in X is a linear basis of the Lie algebra $Lie(X|S)$ generated by X with defining relations S. Here a basic Lie monomial means a Lie monomial in a special linear basis of the free Lie algebra $Lie(X) \subset k\langle X\rangle$, known as the Lyndon–Shirshov (LS for short) basis (Shirshov [207] and Chen–Fox–Lyndon [72], see below). An LS monomial $[u]$ is called S^c-irreducible (or S^c-reduced) whenever u, the associative support of $[u]$, avoids the word $\bar{s}$ for all $s \in S$, where $\bar{s}$ is the maximal word of s as an associative polynomial (in the deg-lex ordering). To be more precise, Shirshov used his reduction algorithm at each step $S, S', S'', \ldots$. Then we have a direct system $S \to S' \to S'' \to \ldots$ and $S^c = \varinjlim S^{(n)}$ is what is now called a *minimal GS basis* (a minimal GS basis is not unique, but a reduced GS basis is, see below). As a result, Shirshov's algorithm gives a solution to the above problem for Lie algebras.

Shirshov's algorithm, dealing with the word problem, is an infinite algorithm like the Knuth–Bendix algorithm [138], 1970 dealing with the identity problem for every variety of universal algebras.[3] The initial data for the Knuth–Bendix algorithm is the defining identities of a variety. The output of the algorithm, if any, is a 'Knuth–Bendix basis' of identities of the variety in the class of all universal algebras of a given signature (not a GS basis of defining relations, say, of a Lie algebra).

Shirshov's algorithm gives linear bases and algorithmic decidability of the word problem for one-relation Lie algebras [207], (recursive) linear bases for Lie algebras with (finite) homogeneous defining relations [207], and linear bases for free products of Lie algebras with known linear bases [208]. He also proved the Freiheitssatz (freeness theorem) for Lie algebras [207] (for every one-relation Lie algebra $Lie(X|f)$,

[2] The name 'Composition-Diamond lemma' combines the Neuman Diamond Lemma [172], the Shirshov Composition Lemma [207] and the Bergman Diamond Lemma [11].

[3] We use the standard algebraic terminology 'the word problem', 'the identity problem', see Kharlampovich, Sapir [136] for instance.

the subalgebra $\langle X\backslash\{x_{i_0}\}\rangle$, where x_{i_0} appears in f, is a free Lie algebra). The Shirshov problem [207] of the decidability of the word problem for Lie algebras was solved negatively in [21]. More generally, it was proved [21] that some recursively presented Lie algebras with undecidable word problem can be embedded into finitely presented Lie algebras (with undecidable word problem). It is a weak analogue of the Higman embedding theorem for groups [115]. The problem [21] whether an analogue of the Higman embedding theorem is valid for Lie algebras is still open. For associative algebras a similar problem [21] was solved positively by Belyaev [10]. A simple example of a Lie algebra with undecidable word problem was given by Kukin [142].

Actually, a similar algorithm for associative algebras is implicit in Shirshov's paper [207]. The reason is that he treats $Lie(X)$ as the subspace of Lie polynomials in the free associative algebra $k\langle X\rangle$. Then to define a Lie composition $\langle f, g\rangle_w$ of two Lie polynomials relative to an associative word $w = \mathrm{lcm}(\bar{f}, \bar{g})$, he defines firstly the associative composition (non-commutative 's-polynomial') $(f, g)_w = fb - ag$, with $a, b \in X^*$. Then he inserts some brackets $\langle f, g\rangle_w = [fb]_{\bar{f}} - [ag]_{\bar{g}}$ by applying his special bracketing lemma of [203]. We can obtain S^c for every $S \subset k\langle X\rangle$ in the same way as for Lie polynomials and in the same way as for Lie algebras ('CD-lemma for associative algebras') to infer that $Irr(S^c)$ is a linear basis of the associative algebra $k\langle X|S\rangle$ generated by X with defining relations S. All proofs are similar to those in [207] but much easier.

Moreover, the cases of semigroups and groups presented by generators and defining relations are just special cases of associative algebras via semigroup and group algebras. To summarize, Shirshov's algorithm gives linear bases and normal forms of elements of every Lie algebra, associative algebra, semigroup or group presented by generators and defining relations! The algorithm works in many cases (see below).

The theory of Gröbner bases and Buchberger's algorithm were initiated by Buchberger (Thesis [65] 1965, paper [66] 1970) for commutative associative algebras. Buchberger's algorithm is a finite algorithm for finitely generated commutative algebras. It is one of the most useful and famous algorithms in modern computer science.

Shirshov's paper [207] was in the spirit of the program of Kurosh (1908–1972) to study non-associative (relatively) free algebras and free products of algebras, initiated in Kurosh's paper [143], 1947. In that paper he proved non-associative analogs of the Nielsen–Schreier and Kurosh theorems for groups. It took quite a few years to clarify the situation for Lie algebras in Shirshov's papers [200], 1953 and [207], 1962 closely related to his theory of GS bases. It is important to note that Kurosh's program quite unexpectedly led to Shirshov's theory of GS bases for Lie and associative algebras [207].

A step in Kurosh's program was made by his student Zhukov in his Ph.D. Thesis [226], 1950. He algorithmically solved the word problem for non-associative algebras. In a sense, it was the beginning of the theory of GS bases for non-associative algebras. The main difference with the future approach of Shirshov is that Zhukov did not use a linear ordering of non-associative monomials. Instead he chose an arbitrary monomial of maximal degree as the 'leading' monomial of a polynomial. Also, for non-associative algebras there is no 'composition of intersection' ('s-polynomial'). In this sense it cannot be a model for Lie and associative algebras.[4]

Shirshov, also a student of Kurosh's, defended his Candidate of Sciences Thesis [199] at Moscow State University in 1953. His thesis together with the paper that followed [203], 1958 may be viewed as a background for his later method of GS bases. In the thesis, he proved the free subalgebra theorem for free Lie algebras (now known as Shirshov–Witt theorem, see also Witt [218], 1956) using the elimination process rediscovered by Lazard [149], 1960. He used the elimination process later [203], 1958 as a general method to prove the properties of regular (LS) words, including an algorithm of (special) bracketing of an LS word (with a fixed LS subword). The latter algorithm is of some importance in his theory of GS bases for Lie algebras (particularly in the definition of the composition of two Lie polynomials). Shirshov also proved the free subalgebra theorem for (anti-) commutative non-associative algebras [202], 1954. He used that later in [206], 1962 for the theory of GS bases of (commutative, anti-commutative) non-associative algebras. Shirshov (Thesis [199], 1953) found the ('Hall–Shirshov') series of bases of a free Lie algebra (see also [205] 1962, the first issue of Malcev's Algebra and Logic).[5]

The LS basis is a particular case of the Shirshov or Hall–Shirshov series of bases (cf. Reutenauer [190], where this series is called the 'Hall series'). In the definition of his series, Shirshov used Hall's inductive procedure (see Ph. Hall [114], 1933, Hall [113], 1950): a non-associative monomial $w = ((u)(v))$ is a basic monomial whenever

(1) (u), (v) are basic;
(2) $(u) > (v)$;
(3) if $(u) = ((u_1)(u_2))$ then $(u_2) \leq (v)$.

However, instead of ordering by the degree function (Hall words), he used an arbitrary linear ordering of non-associative monomials satisfying

$$((u)(v)) > (v).$$

[4] After his Ph.D. Thesis of 1950, Zhukov moved to the present Keldysh Institute of Applied Mathematics (Moscow) to do computational mathematics. Godunov in 'Reminiscence about numerical schemes', arxiv.org/pdf/0810.0649, 2008, mentioned his name in relation to the creation of the famous Godunov numerical method. So, Zhukov was a forerunner of two important computational methods!

[5] It must be pointed out that Malcev (1909–1967) inspired Shirshov's works very much. Malcev was an official opponent (referee) of his (second) Doctor of Sciences Dissertation at MSU in 1958. The first author, Bokut, remembers this event at the Science Council Meeting, chaired by Kolmogorov, and Malcev's words "Shirshov's dissertation is a brilliant one!". Malcev and Shirshov worked together at the present Sobolev Institute of Mathematics in Novosibirsk since 1959 until Malcev's sudden death at 1967, and have been friends despite the age difference. Malcev headed the Algebra and Logic Department (by the way, the first author is a member of the department since 1960) and Shirshov was the first deputy director of the institute (whose director was Sobolev). In those years, Malcev was interested in the theory of algorithms of mathematical logic and algorithmic problems of model theory. Thus, Shirshov had an additional motivation to work on algorithmic problems for Lie algebras. Both Maltsev and Kurosh were delighted with Shirshov's results of [207]. Malcev successfully nominated the paper for an award of the Presidium of the Siberian Branch of the Academy of Sciences (Sobolev and Malcev were the only Presidium members from the Institute of Mathematics at the time).

For example, in his Thesis [199], 1953 he used the ordering by the content of monomials (the *content* of, say, the monomial $(u) = ((x_2x_1)((x_2x_1)x_1))$ is the vector $(x_2, x_2, x_1, x_1, x_1)$). Actually, the content $\widehat{u}$ of (u) may be viewed as a commutative associative word that equals u in the free commutative semigroup. Two contents are compared lexicographically (a proper prefix of a content is greater than the content).

If we use the lexicographic ordering, $(u) \succ (v)$ if $u \succ v$ lexicographically (with the condition $u \succ uv,\ v \neq 1$), then we obtain the LS basis.[6] For example, for the alphabet x_1, x_2 with $x_2 \succ x_1$ we obtain basic Lyndon–Shirshov monomials by induction:

$$x_2,\ x_1,\ [x_2x_1],\ [x_2[x_2x_1]] = [x_2x_2x_1],\ [[x_2x_1]x_1] = [x_2x_1x_1],$$
$$[x_2[x_2x_2x_1]] = [x_2x_2x_2x_1],\ [x_2[x_2x_1x_1]] = [x_2x_2x_1x_1],$$
$$[[x_2x_1x_1]x_1] = [x_2x_1x_1x_1],\ [[x_2x_1][x_2x_1x_1]] = [x_2x_1x_2x_1x_1],$$

and so on. They are exactly all Shirshov regular (LS) Lie monomials and their associative supports are exactly all Shirshov regular words with a one-to-one correspondence between two sets given by the Shirshov elimination (bracketing) algorithm for (associative) words.

Let us recall that an elementary step of Shirshov's elimination algorithm is to join the minimal letter of a word to previous ones by bracketing and to continue this process with the lexicographic ordering of the new alphabet. For example, suppose that $x_2 \succ x_1$. Then we have the succession of bracketings

$$\begin{aligned}
&x_2x_1x_2x_1x_1x_2x_1x_1x_1x_1x_2x_1x_1,\\
&\quad [x_2x_1][x_2x_1x_1][x_2x_1x_1x_1][x_2x_1x_1],\\
&\quad [x_2x_1][[x_2x_1x_1][x_2x_1x_1x_1]][x_2x_1x_1],\\
&\quad [[x_2x_1][[x_2x_1x_1][x_2x_1x_1x_1]]][x_2x_1x_1],\\
&\quad [[[x_2x_1][[x_2x_1x_1][x_2x_1x_1x_1]]][x_2x_1x_1]];\\
&\quad x_2x_1x_1x_1x_2x_1x_1x_2x_1x_2x_2x_1,\\
&\quad [x_2x_1x_1x_1][x_2x_1x_1][x_2x_1]x_2[x_2x_1],\\
&\quad [x_2x_1x_1x_1][x_2x_1x_1][x_2x_1][x_2[x_2x_1]];\\
&\quad x_2x_1x_1x_1 \prec x_2x_1x_1 \prec x_2x_1 \prec x_2x_2x_1.
\end{aligned}$$

By the way, the second series of partial bracketings illustrates Shirshov's factorization theorem [203] of 1958 that every word is a non-decreasing product of LS words (it is often mistakenly called Lyndon's theorem, see [12]).

The Shirshov special bracketing [203] goes as follows. Let us give as an example the special bracketing of the LS word $w = x_2x_2x_1x_1x_2x_1x_1x_1$ with the LS subword $u = x_2x_2x_1$. The Shirshov standard bracketing is

$$[w] = [x_2[[[x_2x_1]x_1][x_2x_1x_1x_1]]].$$

[6] The Lyndon–Shirshov basis for the alphabet x_1, x_2 is different from the above Shirshov content basis starting with monomials of degree 7.

The Shirshov special bracketing is

$$[w]_u = [[[u]x_1][x_2x_1x_1x_1]].$$

In general, if $w = aub$ then the Shirshov standard bracketing gives $[w] = [a[uc]d]$, where $b = cd$. Now, $c = c_1 \cdots c_t$, each c_i is an LS-word, and $c_1 \preceq \cdots \preceq c_t$ in the lex ordering (Shirshov's factorization theorem). Then we must change the bracketing of $[uc]$:

$$[w]_u = [a[\ldots[[u][c_1]]\ldots[c_t]]d]$$

The main property of $[w]_u$ is that $[w]_u$ is a monic associative polynomial with the maximal monomial w; hence, $\overline{[w]_u} = w$.

Actually, Shirshov [207], 1962 needed a 'double' relative bracketing of a regular word with two disjoint LS subwords. Then he implicitly used the following property: every LS subword of $c = c_1 \cdots c_t$ as above is a subword of some c_i for $1 \le i \le t$.

Shirshov defined regular (LS) monomials [203], 1958, as follows: $(w) = ((u)(v))$ is a regular monomial iff:

(1) w is a regular word;
(2) (u) and (v) are regular monomials (then automatically $u \succ v$ in the lex ordering);
(3) if $(u) = ((u_1)(u_2))$ then $u_2 \preceq v$.

Once again, if we formally omit all Lie brackets in Shirshov's paper [207] then essentially the same algorithm and essentially the same CD-lemma (with the same but much simpler proof) yield a linear basis for associative algebra presented by generators and defining relations. The differences are the following:

– no need to use LS monomials and LS words, since the set X^* is a linear basis of the free associative algebra $k\langle X\rangle$;
– the definition of associative composition for monic polynomials f and g,

$$(f, g)_w = fb - ag, \quad w = \bar{f}b = a\bar{g}, \quad deg(w) < deg(\bar{f}) + deg(\bar{g}),$$

or

$$(f, g)_w = f - agb, \quad w = \bar{f} = a\bar{g}b, \quad w, a, b \in X^*,$$

are much simpler than the definition of Lie composition for monic Lie polynomials f and g,

$$\langle f, g\rangle_w = [fb]_{\bar{f}} - [ag]_{\bar{g}}, \quad w = \bar{f}b = a\bar{g}, \quad deg(w) < deg(\bar{f}) + deg(\bar{g}),$$

or

$$\langle f, g\rangle_w = f - [agb]_{\bar{g}}, \quad w = \bar{f} = a\bar{g}b, \quad w, a, b, \bar{f}, \bar{g} \in X^*,$$

where $[fb]_{\bar{f}}$, $[ag]_{\bar{g}}$, and $[agb]_{\bar{g}}$ are the Shirshov special bracketings of the LS words w with fixed LS subwords $\bar{f}$ and $\bar{g}$ respectively.

– The definition of elimination of the leading word $\bar{s}$ of an associative monic polynomial s is straightforward: $a\bar{s}b \to a(r_s)b$ whenever $s = \bar{s} - r_s$ and $a, b \in X^*$. However, for Lie polynomials, it is much more involved and uses the Shirshov special bracketing: $f \to f - [agb]_{\bar{g}}$ whenever $\bar{f} = a\bar{g}b$.

We can formulate the main idea of Shirshov's proof as follows. Consider a complete set S of monic Lie polynomials (all compositions are trivial). If $w = a_1\bar{s_1}b_1 = a_2\bar{s_2}b_2$, where $w, a_i, b_i \in X^*$ and w is an LS word, while $s_1, s_2 \in S$, then the Lie monomials $[a_1s_1b_1]_{\overline{s_1}}$ and $[a_2s_2b_2]_{\overline{s_2}}$ are equal modulo the smaller Lie monomials in $Id(S)$:

$$[a_1s_1b_1]_{\overline{s_1}} = [a_2s_2b_2]_{\overline{s_2}} + \sum_{i>2} \alpha_i [a_is_ib_i]_{\overline{s_i}},$$

where $\alpha_i \in k,\ s_i \in S$ and $\overline{[a_is_ib_i]_{\overline{s_i}}} = a_i\bar{s_i}b_i < w$. Actually, Shirshov proved a more general result: if $\overline{(a_1s_1b_1)} = a_1\overline{s_1}b_1$ and $\overline{(a_2s_2b_2)} = a_2\overline{s_2}b_2$ with $w = a_1\overline{s_1}b_1 = a_2\overline{s_2}b_2$ then

$$(a_1s_1b_1) = (a_2s_2b_2) + \sum_{i>2} \alpha_i (a_is_ib_i),$$

where $\alpha_i \in k,\ s_i \in S$ and $\overline{(a_is_ib_i)} = a_i\bar{s_i}b_i < w$. Below we call a Lie polynomial (asb) a Lie normal S-word provided that $\overline{(asb)} = a\bar{s}b$.

This is precisely where he used the notion of composition and other notions and properties mentioned above.

It is much easier to prove an analogue of this property for associative algebras (as well as commutative associative algebras): given a complete monic set S in $k\langle X\rangle$ ($k[X]$), for $w = a_1\bar{s_1}b_1 = a_2\bar{s_2}b_2$ with $a_i, b_i \in X^*$ and $s_1, s_2 \in S$ we have

$$a_1s_1b_1 = a_2s_2b_2 + \sum_{i>2} \alpha_i a_is_ib_i,$$

where $\alpha_i \in k,\ s_i \in S$ and $a_i\bar{s_i}b_i < w$.

Summarizing, we can say with confidence that the work (Shirshov [207]) implicitly contains the CD-lemma for associative algebras as a simple exercise that requires no new ideas. The first author, Bokut, can confirm that Shirshov clearly understood this and told him that "the case of associative algebras is the same". The lemma was formulated explicitly in Bokut [22], 1976 (with a reference to Shirshov's paper [207]), Bergman [11], 1978, and Mora [171], 1986.

Let us emphasize once again that the CD-Lemma for associative algebras applies to every semigroup $P = \mathrm{sgp}\langle X|S\rangle$, and in particular to every group, by way of the semigroup algebra kP over a field k. The latter algebra has the same generators and defining relations as P, or $kP = k\langle X|S\rangle$. Every composition of the binomials $u_1 - v_1$ and $u_2 - v_2$ is a binomial $u - v$. As a result, applying Shirshov's algorithm to a set of semigroup relations S gives rise to a complete set of semigroup relations S^c. The S^c-irreducible words in X constitute the set of normal forms of the elements of P.

Before we go any further, let us give some well-known examples of algebra, group, and semigroup presentations by generators and defining relations together with linear bases, normal forms, and GS bases for them (if known). Consider a field k and a commutative ring or commutative k-algebra K.

– The Grassman algebra over K is

$$K\langle X|x_i^2=0, x_ix_j+x_jx_i=0,\ i>j\rangle.$$

The set of defining relations is a GS basis with respect to the deg-lex ordering. A K-basis is

$$\{x_{i_1}\cdots x_{i_n}|x_{i_j}\in X,\ j=1,\dots,n,\ i_1<\cdots<i_n,\ n\geq 0\}.$$

– The Clifford algebra over K is

$$K\langle X|x_ix_j+x_jx_i=a_{ij},\ 1\leq i,j\leq n\rangle,$$

where (a_{ij}) is an $n\times n$ symmetric matrix over K. The set of defining relations is a GS basis with respect to the deg-lex ordering. A K-basis is

$$\{x_{i_1}\cdots x_{i_n}|x_{i_j}\in X,\ j=1,\dots,n,\ n\geq 0,\ i_1<\cdots<i_n\}.$$

– The universal enveloping algebra of a Lie algebra L is

$$U_K(L)=K\left\langle X|x_ix_j-x_jx_i=\sum\alpha_{ij}^k x_k,\ i>j\right\rangle.$$

If L is a free K-module with a well-ordered K-basis

$$X=\{x_i|i\in I\},\ [x_ix_j]=\sum\alpha_{ij}^k x_k,\ i>j,\ i,j\in I,$$

then the set of defining relations is a GS basis of $U_K(L)$. The PBW theorem follows: $U_K(L)$ is a free K-module with a K-basis,

$$\{x_{i_1}\cdots x_{i_n}\,|\,i_1\leq\cdots\leq i_n,\ i_t\in I,\ t=1,\dots,n,\ n\geq 0\}.$$

– Kandri-Rody and Weispfenning [122] invented an important class of (noncommutative polynomial) 'algebras of solvable type', which includes universal enveloping algebras. An algebra of solvable type is

$$R=k\langle X|s_{ij}=x_ix_j-x_jx_i-p_{ij},\ i>j,\ p_{ij}<x_ix_j\rangle,$$

and the compositions $(s_{ij},s_{jk})_w=0$ modulo (S,w), where $w=x_ix_jx_k$ with $i>j>k$. Here p_{ij} is a noncommutative polynomial with all terms less than x_ix_j. They created a theory of GS bases for every algebra of this class; thus, they found a linear basis of every quotient of R.

- A general presentation $U_k(L) = k\langle X|S^{(-)}\rangle$ of a universal enveloping algebra over a field k, where $L = Lie(X|S)$ with $S \subset Lie(X) \subset k\langle X\rangle$ and $S^{(-)}$ is S as a set of associative polynomials. PBW theorem in a Shirshov's form. The following conditions are equivalent:
  - (i) the set S is a Lie GS basis;
  - (ii) the set $S^{(-)}$ is a GS basis for $k\langle X\rangle$;
  - (iii) a linear basis for $U_k(L)$ consists of words $u_1u_2\cdots u_n$, where u_i are S-irreducible LS words with $u_1 \preceq u_2 \preceq \cdots \preceq u_n$ (in the lex-ordering), see [56,57];
  - (iv) a linear basis for L consists of the S-irreducible LS Lie monomials $[u]$ in X;
  - (v) a linear basis for $U_k(L)$ consists of the polynomials $u = [u_1]\cdots[u_n]$, where $u_1 \preceq \cdots \preceq u_n$ in the lex ordering, $n \geq 0$, and each $[u_i]$ is an S-irreducible non-associative LS word in X.
- Free Lie algebras $Lie_K(X)$ over K. Hall, Shirshov, and Lyndon provided different linear K-bases for a free Lie algebra (the Hall–Shirshov series of bases, in particular, the Hall basis, the Lyndon–Shirshov basis, the basis compatible with the free solvable (polynilpotent) Lie algebra) [194], see also [15]. Two anticommutative GS bases of $Lie_K(X)$ were found in [34,37], which yields the Hall and Lyndon–Shirshov linear bases respectively.
- The Lie k-algebras presented by Chevalley generators and defining relations of types $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7$, and E_8. Serre's theorem provides linear bases and multiplication tables for these algebras (they are finite dimensional simple Lie algebras over k). Lie GS bases for these algebras are found in [49–51].
- The Coxeter group

$$W = \mathrm{sgp}\langle S|s_i^2 = 1,\ m_{ij}(s_i, s_j) = m_{ji}(s_j, s_i)\rangle$$

  for a given Coxeter matrix $M = (m_{ij})$. Tits [210] (see also [14]) algorithmically solved the word problem for Coxeter groups. Finite Coxeter groups are presented by 'finite' Coxeter matrices $A_n, B_n, D_n, G_2, F_4, E_6, E_7, E_8, H_3$, and H_4. Coxeter's theorem provides normal forms and Cayley tables (these are finite groups generated by reflections). GS bases for finite Coxeter groups are found in [58].
- The Iwahory–Hecke (Hecke) algebras H over K differ from the group algebras $K(W)$ of Coxeter groups in that instead of $s_i^2 = 1$ there are relations $(s_i - q_i^{1/2})(s_i + q_i^{1/2}) = 0$ or $(s_i - q_i)(s_i + 1) = 0$, where q_i are units of K. Two K-bases for H are known; one is natural, and the other is the Kazhdan–Lusztig canonical basis [155]. The GS bases for the Iwahory–Hecke algebras are known for the finite Coxeter matrices. A deep connection of the Iwahory–Hecke algebras of type A_n and braid groups (as well as link invariants) was found by Jones [116].
- Affine Kac–Moody algebras [117]. The Kac–Gabber theorem provides linear bases for these algebras under the symmetrizability condition on the Cartan matrix. Using this result, Poroshenko found the GS bases of these algebras [178–180].
- Borcherds–Kac–Moody algebras [61–63,117]. GS bases are not known.
- Quantum enveloping algebras (Drinfeld, Jimbo). Lusztig's theorem [154] provides linear canonical bases of these algebras. Different approaches were developed by

Ringel [191,192], Green [110], and Kharchenko [131–135]. GS bases of quantum enveloping algebras are unknown except for the case A_n, see [55,86,195,220].
- Koszul algebras. The quadratic algebras with a basis of standard monomials, called PBW-algebras, are always Koszul (Priddy [184]), but not conversely. In different terminology, PBW-algebras are algebras with quadratic GS bases. See [177].
- Elliptic algebras (Feigin, Odesskii) These are associative algebras presented by n generators and $n(n-1)/2$ homogeneous quadratic relations for which the dimensions of the graded components are the same as for the polynomial algebra in n variables. The first example of this type was Sklyanin algebra (1982) generated by x_1, x_2, and x_3 with the defining relations $[x_3, x_2] = x_1^2$, $[x_2, x_1] = x_3^2$, and $[x_1, x_3] = x_2^2$. See [175]. GS bases are not known.
- Leavitt path algebras. GS bases for these algebras are found in Alahmedi et al. [2] and applied by the same authors to determine the structure of the Leavitt path algebras of polynomial growth in [3].
- Artin braid group Br_n. The Markov–Artin theorem provides the normal form and semi-direct structure of the group in the Burau generators. Other normal forms of Br_n were obtained by Garside, Birman–Ko–Lee, and Adjan–Thurston. GS bases for Br_n in the Artin–Burau, Artin–Garside, Birman–Ko–Lee, and Adjan–Thurston generators were found in [23–25,89] respectively.
- Artin–Tits groups. They differ from Coxeter groups in the absence of the relations $s_i^2 = 1$. Normal forms are known in the spherical case, see Brieskorn, Saito [64]. GS bases are not known except for braid groups (the Artin–Tits groups of type A_n).
- The groups of Novikov–Boon type (Novikov [173], Boon [60], Collins [97], Kalorkoti [118–121]) with unsolvable word or conjugacy problem. They are groups with standard bases (standard normal forms or standard GS bases), see [16–18,77].
- Adjan's [1] and Rabin's [187] constructions of groups with unsolvable isomorphism problem and Markov properties. A GS basis is known for Adjan's construction [26].
- Markov's [161] and Post's [183] semigroups with unsolvable word problem. The GS basis of Post's semigroup is found in [223].
- Markov's construction of semigroups with unsolvable isomorphism problem and Markov properties. The GS basis for the construction is not known.
- Plactic monoids. A theorem due to Richardson, Schensted, and Knuth provides a normal form of the elements of these monoids (see Lothaire [151]). New approaches to plactic monoids via GS bases in the alphabets of row and column generators are found in [29].
- The groups of quotients of the multiplicative semigroups of power series rings with topological quadratic relations of the type $k\langle\langle x, y, z, t | xy = zt\rangle\rangle$ embeddable (without the zero element) into groups but in general not embeddable into division algebras (settling a problem of Malcev). The relative standard normal forms of these groups found in [19,20] are the reduced words for what was later called a *relative* GS basis [59].

To date, the method of GS bases has been adapted, in particular, to the following classes of linear universal algebras, as well as for operads, categories, and semirings. Unless stated otherwise, we consider all linear algebras over a field k. Following the terminology of Higgins and Kurosh, we mean by a ((differential) associative) Ω-

algebra a linear space ((differential) associative algebra) with a set of multi-linear operations Ω:

- Associative algebras, Shirshov [207], Bokut [22], Bergman [11];
- Associative algebras over a commutative algebra, Mikhalev and Zolotykh [170];
- Associative Γ-algebras, where Γ is a group, Bokut and Shum [59];
- Lie algebras, Shirshov [207];
- Lie algebras over a commutative algebra, Bokut et al. [31];
- Lie p-algebras over k with char $k = p$, Mikhalev [166];
- Lie superalgebras, Mikhalev [165, 167];
- Metabelian Lie algebras, Chen and Chen [75];
- Quiver (path) algebras, Farkas et al. [101];
- Tensor products of associative algebras, Bokut et al. [30];
- Associative differential algebras, Chen et al. [76];
- Associative $(n-)$conformal algebras over k with char $k = 0$, Bokut et al. [45], Bokut et al. [43];
- Dialgebras, Bokut et al. [38];
- Pre-Lie (Vinberg–Koszul–Gerstenhaber, right (left) symmetric) algebras, Bokut et al. [35],
- Associative Rota–Baxter algebras over k with char $k = 0$, Bokut et al. [32];
- L-algebras, Bokut et al. [33];
- Associative Ω-algebras, Bokut et al. [41];
- Associative differential Ω-algebras, Qiu and Chen [185];
- Ω-algebras, Bokut et al. [33];
- Differential Rota–Baxter commutative associative algebras, Guo et al. [111];
- Semirings, Bokut et al. [40];
- Modules over an associative algebra, Golod [108], Green [109], Kang and Lee [123, 124], Chibrikov [90];
- Small categories, Bokut et al. [36];
- Non-associative algebras, Shirshov [206];
- Non-associative algebras over a commutative algebra, Chen et al. [81];
- Commutative non-associative algebras, Shirshov [206];
- Anti-commutative non-associative algebras, Shirshov [206];
- Symmetric operads, Dotsenko and Khoroshkin [98].

At the heart of the GS method for a class of linear algebras lies a CD-lemma for a free object of the class. For the cases above, the free objects are the free associative algebra $k\langle X\rangle$, the doubly free associative $k[Y]$-algebra $k[Y]\langle X\rangle$, the free Lie algebra $Lie(X)$, and the doubly free Lie $k[Y]$-algebra $Lie_{k[Y]}(X)$. For the tensor product of two associative algebras we need to use the tensor product of two free algebras, $k\langle X\rangle \otimes k\langle Y\rangle$. We can view every semiring as a double semigroup with two associative products $\cdot$ and $\circ$. So, the CD-lemma for semirings is the CD-lemma for the semiring algebra of the free semiring $Rig(X)$. The CD-lemma for modules is the CD-lemma for the doubly free module $\mathrm{Mod}_{k\langle Y\rangle}(X)$, a free module over a free associative algebra. The CD-lemma for small categories is the CD-lemma for the 'free partial k-algebra' $kC\langle X\rangle$ generated by an oriented graph X (a sequence $z_1 z_2 \cdots z_n$, where $z_i \in X$, is a partial word in X iff it is a path; a partial polynomial is a linear combination of partial words with the same source and target).

All CD-lemmas have essentially the same statement. Consider a class $\mathbf{V}$ of linear universal algebras, a free algebra $\mathbf{V}(X)$ in $\mathbf{V}$, and a well-ordered k-basis of terms $N(X)$ of $\mathbf{V}(X)$. A subset $S \subset \mathbf{V}(X)$ is called a GS basis if every *composition* of the elements of S is *trivial* (vanishes upon the elimination of the leading terms $\bar{s}$ for $s \in S$). Then the following conditions are equivalent:

(i) S is a GS basis.
(ii) If $f \in Id(S)$ then the leading term $\bar{f}$ contains the subterm $\bar{s}$ for some $s \in S$.
(iii) The set of S-irreducible terms is a linear basis for the $\mathbf{V}$-algebra $\mathbf{V}\langle X|S\rangle$ generated by X with defining relations S.

In some cases ($(n-)$ conformal algebras, dialgebras), conditions (i) and (ii) are not equivalent. To be more precise, in those cases we have $(i) \Rightarrow (ii) \Leftrightarrow (iii)$.

Typical compositions are compositions of intersection and inclusion. Shirshov [206,207] avoided inclusion composition. He suggested instead that a GS basis must be minimal (the leading words do not contain each other as subwords). In some cases, new compositions must be defined, for example, the composition of left (right) multiplication. Also, sometimes we need to combine all these compositions. We present here a new approach to the definition of a composition, based on the concept of the least common multiple $\mathrm{lcm}(u, v)$ of two terms u and v.

In some cases (Lie algebras, (n-) conformal algebras) the 'leading' term $\bar{f}$ of a polynomial $f \in \mathbf{V}(X)$ lies outside $\mathbf{V}(X)$. For Lie algebras, we have $\bar{f} \in k\langle X\rangle$, for ($n$-) conformal algebras $\bar{f}$ belongs to an 'Ω-semigroup'.

Almost all CD-lemmas require the new notion of a 'normal S-term'. A term (asb) in $\{X, \Omega\}$, where $s \in S$, with only one occurrence of s is called a normal S-term whenever $\overline{(asb)} = (a(\bar{s})b)$. Given $S \subset k\langle X\rangle$, every S-word (that is, an S-term) is a normal S-word. Given $S \subset Lie(X)$, every Lie S-monomial (Lie S-term) is a linear combination of normal Lie S-terms (Shirshov [207]).

One of the two key lemmas asserts that if S is complete under compositions of multiplication then every element of the ideal generated by S is a linear combination of normal S-terms. Another key lemma says that if S is a GS basis and the leading words of two normal S-terms are the same then these terms are the same modulo lower normal S-terms. As we mentioned above, Shirshov proved these results [207] for $Lie(X)$ (there are no compositions of multiplication for Lie and associative algebras).

This survey continues our surveys with Kolesnikov, Fong, Ke, and Shum [27,28, 42,46,52,53], Ufnarovski's survey [213], and the book of the first named author and Kukin [54].

The paper is organized as follows. Section 2 is for associative algebras, Sect. 3 is for semigroups and groups, Sect. 4 is for Lie algebras, and the short Sect. 5 is for Ω-algebras and operads.[7]

To conclude this introduction, we give some information about the work of Shirshov; for more on this, see the book [209]. Shirshov (1921–1981) was a famous Russian

[7] The first definitions of the symmetric operad were given by Kurosh's student Artamonov under the name 'clone of multilinear operations' in 1969, see Kurosh [144] and Artamonov [4], cf. Lambek (1969) [146] and May (1972) [162].

mathematician. His name is associated with notions and results on the Gröbner–Shirshov bases, the Composition-Diamond lemma, the Shirshov–Witt theorem, the Lazard–Shirshov elimination, the Shirshov height theorem, Lyndon–Shirshov words, Lyndon–Shirshov basis (in a free Lie algebra), the Hall–Shirshov series of bases, the Cohn–Shirshov theorem for Jordan algebras, Shirshov's theorem on the Kurosh problem, and the Shirshov factorization theorem. Shirshov's ideas were used by his students Efim Zelmanov to solve the restricted Burnside problem and Aleksander Kemer to solve the Specht problem.

1.1 Digression on the history of Lyndon–Shirshov bases and Lyndon–Shirshov words

Lyndon [156], 1954, defined standard words, which are the same as Shirshov's regular words [203], 1958. Unfortunately, the papers (Lyndon [156]) and (Chen et al. [72], 1958) were practically unknown before 1983. As a result, at that time almost all authors (except four who used the names Shirshov and Chen–Fox–Lyndon, see below) refer to the basis and words as Shirshov regular basis and words, cf. for instance [8,9,96,188,212,224]. To the best of our knowledge, none of the authors mentioned Lyndon's paper [156] as a source of 'Lyndon words' before 1983(!).

In the following papers the authors mentioned both (Chen et al. [72]) and (Shirshov [203]) as a source of 'Lyndon–Shirshov basis' and 'Lyndon–Shirshov words':

– Schützenberger and Sherman [196], 1963;
– Schützenberger [197], 1965;
– Viennot [217], 1978;
– Michel [163], 1975; [164], 1976.

The authors of [196] thank Cohn for pointing out Shirshov's paper [203]. They also formulate Shirshov's factorization theorem [203]. They mention [72,203] as a source of 'LS words'. Schützenberger also mentions [197] Shirshov's factorization theorem, but in this case he attributes it to both Chen et al. [72] and Shirshov [203]. Actually, he cites [72] by mistake, as that result is absent from the paper, see Berstel and Perrin [12].[8]

Starting with the book of Lothaire, *Combinatorics on words* ([151], 1983), some authors called the words and basis 'Lyndon words' and 'Lyndon basis'; for instance, see Reutenauer, Free Lie algebras ([190], 1993).

2 Gröbner–Shirshov bases for associative algebras

In this section we give a proof of Shirshov's CD-lemma for associative algebras and Buchberger's theorem for commutative algebras. Also, we give the Eisenbud–Peeva–

[8] From [12]: "A famous theorem concerning Lyndon words asserts that any word w can be factorized in a unique way as a non-increasing product of Lyndon words, i.e. written $w = x_1x_2 \dots x_n$ with $x_1 \geq x_2 \geq \cdots \geq x_n$. This theorem has imprecise origin. It is usually credited to Chen et al., following the paper of Schützenberger [197] in which it appears as an example of factorization of free monoids. Actually, as pointed out to one of us by Knuth in 2004, the reference [72] does not contain explicitly this statement."

Sturmfels lifting theorem, the CD-lemmas for modules (following Kang and Lee [124] and Chibrikov [90]), the PBW theorem and the PBW theorem in Shirshov's form, the CD-lemma for categories, the CD-lemma for associative algebras over commutative algebras and the Rosso–Yamane theorem for $U_q(A_n)$.

2.1 Composition-Diamond lemma for associative algebras

Let k be a field, $k\langle X\rangle$ be the free associative algebra over k generated by X and X^* be the free monoid generated by X, where the empty word is the identity, denoted by 1. Suppose that X^* is a well-ordered set. Take $f \in k\langle X\rangle$ with the leading word $\bar{f}$ and $f = \alpha \bar{f} - r_f$, where $0 \neq \alpha \in k$ and $\overline{r_f} < \bar{f}$. We call f *monic* if $\alpha = 1$.

A well-ordering $>$ on X^* is called a *monomial ordering* whenever it is compatible with the multiplication of words, that is, for all $u, v \in X^*$ we have

$$u > v \Rightarrow w_1 u w_2 > w_1 v w_2, \text{ for all } w_1, \ w_2 \in X^*.$$

A standard example of monomial ordering on X^* is the deg-lex ordering, in which two words are compared first by the degree and then lexicographically, where X is a well-ordered set.

Fix a monomial ordering $<$ on X^* and take two monic polynomials f and g in $k\langle X\rangle$. There are two kinds of compositions:

(i) If w is a word such that $w = \bar{f} b = a \bar{g}$ for some $a, b \in X^*$ with $|\bar{f}| + |\bar{g}| > |w|$ then the polynomial $(f, g)_w = fb - ag$ is called the *intersection composition* of f and g with respect to w.

(ii) If $w = \bar{f} = a \bar{g} b$ for some $a, b \in X^*$ then the polynomial $(f, g)_w = f - agb$ is called the *inclusion composition* of f and g with respect to w.

Then $\overline{(f, g)_w} < w$ and $(f, g)_w$ lies in the ideal $Id\{f, g\}$ of $k\langle X\rangle$ generated by f and g.

In the composition $(f, g)_w$, we call w an ambiguity (or the least common multiple $\mathrm{lcm}(\bar{f}, \bar{g})$, see below).

Consider $S \subset k\langle X\rangle$ such that very $s \in S$ is monic. Take $h \in k\langle X\rangle$ and $w \in X^*$. Then h is called *trivial* modulo (S, w), denoted by

$$h \equiv 0 \mod (S, w),$$

if $h = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in k$, $a_i, b_i \in X^*$, and $s_i \in S$ with $a_i \overline{s_i} b_i < w$.

The elements asb, $a, b \in X^*$, and $s \in S$ are called S-words.

A monic set $S \subset k\langle X\rangle$ is called a GS basis in $k\langle X\rangle$ with respect to the monomial ordering $<$ if every composition of polynomials in S is trivial modulo S and the corresponding w.

A set S is called a minimal GS basis in $k\langle X\rangle$ if S is a GS basis in $k\langle X\rangle$ avoiding inclusion compositions; that is, given $f, g \in S$ with $f \neq g$, we have $\bar{f} \neq a \bar{g} b$ for all $a, b \in X^*$.

Put

$$Irr(S) = \{u \in X^* | u \neq a\bar{s}b, s \in S, a, b \in X^*\}.$$

The elements of $Irr(S)$ are called S-irreducible or S-reduced.

A GS basis S in $k\langle X\rangle$ is reduced provided that $\mathrm{supp}(s) \subseteq Irr(S\backslash\{s\})$ for every $s \in S$, where $\mathrm{supp}(s) = \{u_1, u_2, \ldots, u_n\}$ whenever $s = \sum_{i=1}^{n} \alpha_i u_i$ with $0 \neq \alpha_i \in k$ and $u_i \in X^*$. In other words, each u_i is an $S\backslash\{s\}$-irreducible word.

The following lemma is key for proving the CD-lemma for associative algebras.

Lemma 1 *If S is a GS basis in $k\langle X\rangle$ and $w = a_1\overline{s_1}b_1 = a_2\overline{s_2}b_2$, where $a_1, b_1, a_2, b_2 \in X^*$ and $s_1, s_2 \in S$, then $a_1s_1b_1 \equiv a_2s_2b_2 \mod (S, w)$.*

Proof There are three cases to consider.

Case 1 Assume that the subwords $\bar{s}_1$ and $\bar{s}_2$ of w are disjoint, say, $|a_2| \geq |a_1| + |\bar{s}_1|$. Then, $a_2 = a_1\bar{s}_1c$ and $b_1 = c\bar{s}_2b_2$ for some $c \in X^*$, and so $w_1 = a_1\bar{s}_1c\bar{s}_2b_2$. Now,

$$\begin{aligned} a_1s_1b_1 - a_2s_2b_2 &= a_1s_1c\bar{s}_2b_2 - a_1\bar{s}_1cs_2b_2 \\ &= a_1s_1c(\bar{s}_2 - s_2)b_2 + a_1(s_1 - \bar{s}_1)cs_2b_2. \end{aligned}$$

Since $\overline{\overline{s_2} - s_2} < \bar{s}_2$ and $\overline{s_1 - \overline{s_1}} < \bar{s}_1$, we conclude that

$$a_1s_1b_1 - a_2s_2b_2 = \sum_i \alpha_i u_i s_1 v_i + \sum_j \beta_j u_j s_2 v_j$$

with $\alpha_i, \beta_j \in k$ and S-words $u_is_1v_i$ and $u_js_2v_j$ satisfying $u_i\bar{s}_1v_i, u_j\bar{s}_2v_j < w$.

Case 2 Assume that the subword $\bar{s}_1$ of w contains $\bar{s}_2$ as a subword. Then $\bar{s}_1 = a\bar{s}_2b$ with $a_2 = a_1a$ and $b_2 = bb_1$, that is, $w = a_1a\bar{s}_2bb_1$ for some S-word as_2b. We have

$$a_1s_1b_1 - a_2s_2b_2 = a_1s_1b_1 - a_1as_2bb_1 = a_1(s_1 - as_2b)b_1 = a_1(s_1, s_2)_{\overline{s_1}}b_1.$$

The triviality of compositions implies that $a_1s_1b_1 \equiv a_2s_2b_2 \mod (S, w)$.

Case 3 Assume that the subwords $\bar{s}_1$ and $\bar{s}_2$ of w have a nonempty intersection. We may assume that $a_2 = a_1a$ and $b_1 = bb_2$ with $w = \bar{s}_1b = a\bar{s}_2$ and $|w| < |\bar{s}_1| + |\bar{s}_2|$. Then, as in Case 2, we have $a_1s_1b_1 \equiv a_2s_2b_2 \mod (S, w)$. □

Lemma 2 *Consider a set $S \subset k\langle X\rangle$ of monic polynomials. For every $f \in k\langle X\rangle$ we have*

$$f = \sum_{u_i \leq \bar{f}} \alpha_i u_i + \sum_{a_j\overline{s_j}b_j \leq \bar{f}} \beta_j a_j s_j b_j$$

where $\alpha_i, \beta_j \in k$, $u_i \in Irr(S)$, and $a_js_jb_j$ are S-words. So, $Irr(S)$ is a set of linear generators of the algebra $k\langle X|S\rangle$.

Proof Induct on $\bar{f}$. □

Theorem 1 (The CD-lemma for associative algebras) *Choose a monomial ordering $<$ on X^*. Consider a monic set $S \subset k\langle X\rangle$ and the ideal $Id(S)$ of $k\langle X\rangle$ generated by S. The following statements are equivalent:*

(i) *S is a Gröbner–Shirshov basis in $k\langle X\rangle$.*
(ii) *$f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$.*
(iii) *$Irr(S) = \{u \in X^* | u \neq a\bar{s}b, s \in S, a, b \in X^*\}$ is a linear basis of the algebra $k\langle X|S\rangle$.*

Proof (i)⇒(ii). Assume that S is a GS basis and take $0 \neq f \in Id(S)$. Then, we have $f = \sum_{i=1}^{n} \alpha_i a_i s_i b_i$ where $\alpha_i \in k$, $a_i, b_i \in X^*$, and $s_i \in S$. Suppose that $w_i = a_i \overline{s_i} b_i$ satisfy

$$w_1 = w_2 = \cdots = w_l > w_{l+1} \geq \cdots .$$

Induct on w_1 and l to show that $\overline{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$. To be more precise, induct on (w_1, l) with the lex ordering of the pairs.

If $l = 1$ then $\overline{f} = \overline{a_1 s_1 b_1} = a_1 \overline{s_1} b_1$ and hence the claim holds. Assume that $l \geq 2$. Then $w_1 = a_1 \overline{s_1} b_1 = a_2 \overline{s_2} b_2$. Lemma 1 implies that $a_1 s_1 b_1 \equiv a_2 s_2 b_2 \mod (S, w_1)$. If $\alpha_1 + \alpha_2 \neq 0$ or $l > 2$ then the claim follows by induction on l. For the case $\alpha_1 + \alpha_2 = 0$ and $l = 2$, induct on w_1. Thus, (ii) holds.

(ii)⇒(iii). By Lemma 2, $Irr(S)$ generates $k\langle X|S\rangle$ as a linear space. Suppose that $\sum_i \alpha_i u_i = 0$ in $k\langle X|S\rangle$, where $0 \neq \alpha_i \in k$ and $u_i \in Irr(S)$. It means that $\sum_i \alpha_i u_i \in Id(S)$ in $k\langle X\rangle$. Then $\overline{\sum_i \alpha_i u_i} = u_j \in Irr(S)$ for some j, which contradicts (ii).

(iii)⇒(i). Given $f, g \in S$, Lemma 2 and (iii) yield $(f, g)_w \equiv 0 \mod (S, w)$. Therefore, S is a GS basis. □

A new exposition of the proof of Theorem 1 (CD-lemma for associative algebras).

Let us start with the concepts of non-unique *common multiple* and *least common multiple* of two words $u, v \in X^*$. A common multiple $\mathrm{cm}(u, v)$ means that $\mathrm{cm}(u, v) = a_1 u b_1 = a_2 v b_2$ for some $a_i, b_i \in X^*$. Then $\mathrm{lcm}(u, v)$ means that some $\mathrm{cm}(u, v)$ contains some $\mathrm{lcm}(u, v)$ as a subword: $\mathrm{cm}(u, v) = c \cdot \mathrm{lcm}(u, v) \cdot d$ with $c, d \in X^*$, where u and v are the same subwords in both sides. To be precise,

$$\begin{aligned} \mathrm{lcm}(u, v) \in \{&ucv, c \in X^* \text{ (a trivial } \mathrm{lcm}(u, v));\\ &u = avb,\ a, b \in X^* \text{ (an inclusion } \mathrm{lcm}(u, v));\\ &ub = av,\ a, b \in X^*,\ |ub| < |u| + |v| \text{ (an intersection } \mathrm{lcm}(u, v))\}. \end{aligned}$$

Define the general composition $(f, g)_{\mathrm{lcm}(\bar{f}, \bar{g})}$ of monic polynomials $f, g \in k\langle X\rangle$ as

$$(f, g)_{\mathrm{lcm}(\bar{f}, \bar{g})} = \mathrm{lcm}(\bar{f}, \bar{g})|_{\bar{f} \mapsto f} - \mathrm{lcm}(\bar{f}, \bar{g})|_{\bar{g} \mapsto g}.$$

The only difference with the previous definition of composition is that we include the case of trivial $\mathrm{lcm}(\bar{f}, \bar{g})$. However, in this case the composition is trivial,

$$(f, g)_{\bar{f} c \bar{g}} \equiv 0 \mod (\{f, g\}, \bar{f} c \bar{g}).$$

It is clear that if $a_1\bar{f}b_1 = a_2\bar{g}b_2$ then, up to the ordering of f and g,

$$a_1 f b_1 - a_2 g b_2 = c \cdot (f, g)_{\mathrm{lcm}(\bar{f},\bar{g})} \cdot d.$$

This implies Lemma 1. The main claim (i)⇒(ii) of Theorem 1 follows from Lemma 1.

Shirshov algorithm. If a monic subset $S \subset k\langle X\rangle$ is not a GS basis then we can add to S all nontrivial compositions, making them monic. Iterating this process, we eventually obtain a GS basis S^c that contains S and generates the same ideal, $Id(S^c) = Id(S)$. This S^c is called the GS completion of S. Using the reduction algorithm (elimination of the leading words of polynomials), we may obtain a minimal GS basis S^c or a reduced GS basis.

The following theorem gives a linear basis for the ideal $Id(S)$ provided that $S \subset k\langle X\rangle$ is a GS basis.

Theorem 2 *If $S \subset k\langle X\rangle$ is a Gröbner–Shirshov basis then, given $u \in X^*\backslash Irr(S)$, by Lemma 2 there exists $\widehat{u} \in kIrr(S)$ with $\overline{\widehat{u}} < u$ (if $\widehat{u} \neq 0$) such that $u - \widehat{u} \in Id(S)$ and the set $\{u - \widehat{u} | u \in X^*\backslash Irr(S)\}$ is a linear basis for the ideal $Id(S)$ of $k\langle X\rangle$.*

Proof Take $0 \neq f \in Id(S)$. Then by the CD-lemma for associative algebras, $\bar{f} = a_1\overline{s_1}b_1 = u_1$ for some $s_1 \in S$ and $a_1, b_1 \in X^*$, which implies that $\bar{f} = u_1 \in X^*\backslash Irr(S)$. Put $f_1 = f - \alpha_1(u_1 - \widehat{u_1})$, where α_1 is the coefficient of the leading term of f and $\overline{\widehat{u_1}} < u_1$ or $\widehat{u_1} = 0$. Then $f_1 \in Id(S)$ and $\overline{f_1} < \bar{f}$. By induction on $\bar{f}$, the set $\{u - \widehat{u} | u \in X^*\backslash Irr(S)\}$ generates $Id(S)$ as a linear space. It is clear that $\{u - \widehat{u} | u \in X^*\backslash Irr(S)\}$ is a linearly independent set. □

Theorem 3 *Choose a monomial ordering $>$ on X^*. For every ideal I of $k\langle X\rangle$ there exists a unique reduced Gröbner–Shirshov basis S for I.*

Proof Clearly, a Gröbner–Shirshov basis $S \subset k\langle X\rangle$ for the ideal $I = Id(S)$ exists; for example, we may take $S = I$. By Theorem 1, we may assume that the leading terms of the elements of S are distinct. Given $g \in S$, put

$$\Delta_g = \{f \in S | f \neq g \quad \text{and} \quad \overline{f} = a\bar{g}b \text{ for some } a, b \in X^*\}$$

and $S_1 = S\backslash \cup_{g\in S} \Delta_g$.

For every $f \in Id(S)$ we show that there exists an $s_1 \in S_1$ such that $\overline{f} = a\overline{s_1}b$ for some $a, b \in X^*$.

In fact, Theorem 1 implies that $\overline{f} = a'\bar{h}b'$ for some $a', b' \in X^*$ and $h \in S$. Suppose that $h \in S\backslash S_1$. Then we have $h \in \cup_{g\in S}\Delta_g$, say, $h \in \Delta_g$. Therefore, $h \neq g$ and $\bar{h} = a\bar{g}b$ for some $a, b \in X^*$. We claim that $\bar{h} > \bar{g}$. Otherwise, $\bar{h} < \bar{g}$. It follows that $\bar{h} = a\bar{g}b > a\bar{h}b$ and so we have the infinite descending chain

$$\bar{h} > a\bar{h}b > a^2\bar{h}b^2 > a^3\bar{h}b^3 > \dots,$$

which contradicts the assumption that $>$ is a well ordering.

Suppose that $g \notin S_1$. Then, by the argument above, there exists $g_1 \in S$ such that $g \in \Delta_{g_1}$ and $\overline{g} > \overline{g_1}$. Since $>$ is a well ordering, there must exist $s_1 \in S_1$ such that $\overline{f} = a_1\overline{s_1}b_1$ for some $a_1, b_1 \in X^*$.

Put $f_1 = f - \alpha_1 a_1 s_1 b_1$, where α_1 is the coefficient of the leading term of f. Then $f_1 \in Id(S)$ and $\overline{f} > \overline{f_1}$.

By induction on $\overline{f}$, we know that $f \in Id(S_1)$, and hence $I = Id(S_1)$. Moreover, Theorem 1 implies that S_1 is clearly a minimal GS basis for the ideal $Id(S)$.

Assume that S is a minimal GS basis for I.

For every $s \in S$ we have $s = s' + s''$, where $\mathrm{supp}(s') \subseteq Irr(S\backslash\{s\})$ and $s'' \in Id(S\backslash\{s\})$. Since S is a minimal GS basis, it follows that $\overline{s} = \overline{s'}$ for every $s \in S$.

We claim that $S_2 = \{s' | s \in S\}$ is a reduced GS basis for I. In fact, it is clear that $S_2 \subseteq Id(S) = I$. By Theorem 1, for every $f \in Id(S)$ we have $\overline{f} = a_1\overline{s_1}b_1 = a_1\overline{s_1'}b_1$ for some $a_1, b_1 \in X^*$.

Take two reduced GS bases S and R for the ideal I. By Theorem 1, for every $s \in S$,

$$\overline{s} = a\overline{r}b, \quad \overline{r} = c\overline{s_1}d$$

for some $a, b, c, d \in X^*$, $r \in R$, and $s_1 \in S$, and hence $\overline{s} = ac\overline{s_1}db$. Since $\overline{s} \in \mathrm{supp}(s) \subseteq Irr(S\backslash\{s\})$, we have $s = s_1$. It follows that $a = b = c = d = 1$, and so $\overline{s} = \overline{r}$.

If $s \neq r$ then $0 \neq s - r \in I = Id(S) = Id(R)$. By Theorem 1, $\overline{s-r} = a_1\overline{r_1}b_1 = c_1\overline{s_2}d_1$ for some $a_1, b_1, c_1, d_1 \in X^*$ with $\overline{r_1}, \overline{s_2} < \overline{s} = \overline{r}$. This means that $s_2 \in S\backslash\{s\}$ and $r_1 \in R\backslash\{r\}$. Noting that $\overline{s-r} \in \mathrm{supp}(s) \cup \mathrm{supp}(r)$, we have either $\overline{s-r} \in \mathrm{supp}(s)$ or $\overline{s-r} \in \mathrm{supp}(r)$. If $\overline{s-r} \in \mathrm{supp}(s)$ then $\overline{s-r} \in Irr(S\backslash\{s\})$, which contradicts $\overline{s-r} = c_1\overline{s_2}d_1$; if $\overline{s-r} \in \mathrm{supp}(r)$ then $\overline{s-r} \in Irr(R\backslash\{r\})$, which contradicts $\overline{s-r} = a_1\overline{r_1}b_1$. This shows that $s = r$, and then $S \subseteq R$. Similarly, $R \subseteq S$. □

Remark 1 In fact, a reduced GS basis is unique (up to the ordering) in all possible cases below.

Remark 2 Both associative and Lie CD-lemmas are valid when we replace the base field k by an arbitrary commutative ring K with identity because we assume that all GS bases consist of monic polynomials. For example, consider a Lie algebra L over K which is a free K-module with a well-ordered K-basis $\{a_i | i \in I\}$. With the deg-lex ordering on $\{a_i | i \in I\}^*$, the universal enveloping associative algebra $U_K(L)$ has a (monic) GS basis

$$\left\{a_i a_j - a_j a_i = \sum \alpha_{ij}^t a_t | i > j,\ i, j \in I\right\},$$

where $\alpha_{ij}^t \in K$ and $[a_i, a_j] = \sum \alpha_{ij}^t a_t$ in L, and the CD-lemma for associative algebras over K implies that $L \subset U_K(L)$ and

$$\{a_{i_1} \cdots a_{i_n} | i_1 \leq \cdots \leq i_n,\ n \geq 0,\ i_1, \ldots, i_n \in I\}$$

is a K-basis for $U_K(L)$.

In fact, for the same reason, all CD-lemmas in this survey are valid if we replace the base field k by an arbitrary commutative ring K with identity. If this is the case then claim (iii) in the CD-lemma should read: $K(X|S)$ is a free K-module with a K-basis $Irr(S)$. But in the general case, Shirshov's algorithm fails: if S is a monic set then S', the set obtained by adding to S all non-trivial compositions, is not a monic set in general, and the algorithm may stop with no result.

2.2 Gröbner bases for commutative algebras and their lifting to Gröbner–Shirshov bases

Consider the free commutative associative algebra $k[X]$. Given a well ordering $<$ on $X = \{x_i | i \in I\}$,

$$[X] = \{x_{i_1} \dots x_{i_t} | i_1 \leq \dots \leq i_t,\ i_1, \dots, i_t \in I,\ t \geq 0\}$$

is a linear basis for $k[X]$.

Choose a monomial ordering $<$ on $[X]$. Take two monic polynomials f and g in $k[X]$ such that $w = \mathrm{lcm}(\bar{f}, \bar{g}) = \bar{f}a = \bar{g}b$ for some $a, b \in [X]$ with $|\bar{f}| + |\bar{g}| > |w|$ (so, $\bar{f}$ and $\bar{g}$ are not coprime in $[X]$). Then $(f, g)_w = fa - gb$ is called the s-polynomial of f and g.

A monic subset $S \subseteq k[X]$ is called a Gröbner basis with respect to the monomial ordering $<$ whenever all s-polynomials of two arbitrary polynomials in S are trivial modulo S and corresponding w.

An argument similar to the proof of the CD-lemma for associative algebras justifies the following theorem due to Buchberger.

Theorem 4 (Buchberger Theorem) *Choose a monomial ordering $<$ on $[X]$. Consider a monic set $S \subset k[X]$ and the ideal $Id(S)$ of $k[X]$ generated by S. The following statements are equivalent:*

(i) *S is a Gröbner basis in $k[X]$.*
(ii) *$f \in Id(S) \Rightarrow \bar{f} = \bar{s}a$ for some $s \in S$ and $a \in [X]$.*
(iii) *$Irr(S) = \{u \in [X] | u \neq \bar{s}a, s \in S, a \in [X]\}$ is a linear basis for the algebra $k[X|S] = k[X]/Id(S)$.*

Proof Denote by $\mathrm{lcm}(u, v)$ be the usual (unique) least common multiple of two commutative words $u, v \in [X]$:

$$\begin{aligned} &\mathrm{lcm}(u, v) \in \{uv \text{ (the trivial lcm}(u, v));\\ &\quad au = bv,\ a, b \in [X],\ |au| < |u| + |v| \text{ (the nontrivial lcm}(u, v))\}. \end{aligned}$$

If $\mathrm{cm}(u, v) = a_1 u = a_2 v$ is a common multiple of u and v then $\mathrm{cm}(u, v) = b \cdot \mathrm{lcm}(u, v)$.

The s-polynomial of two monic polynomials f and g is

$$(f, g)_{\mathrm{lcm}(\bar{f}, \bar{g})} = \mathrm{lcm}(\bar{f}, \bar{g})|_{\bar{f} \mapsto f} - \mathrm{lcm}(\bar{f}, \bar{g})|_{\bar{g} \mapsto g}.$$

An analogue of Lemma 1 is valid for $k[X]$ because if $a_1\bar{s}_1 = a_2\bar{s}_2$ for two monic polynomials s_1 and s_2 then

$$a_1 s_1 - a_2 s_2 = b \cdot (s_1, s_2)_{\mathrm{lcm}(\bar{s}_1, \bar{s}_2)}.$$

Lemma 1 implies the main claim (i)⇒(ii) of Buchberger's theorem. □

Theorem 5 *Given an ideal I of $k[X]$ and a monomial ordering $<$ on $[X]$, there exists a unique reduced Gröbner basis S for I. Moreover, if X is finite then so is S.*

Eisenbud et al. [99] constructed a GS basis in $k\langle X\rangle$ by lifting a commutative Gröbner basis for $k[X]$ and adding all commutators. Write $X = \{x_1, x_2, \dots, x_n\}$ and put

$$S_1 = \{h_{ij} = x_i x_j - x_j x_i \,|\, i > j\} \subset k\langle X\rangle.$$

Consider the natural map $\gamma : k\langle X\rangle \to k[X]$ carrying x_i to x_i and the lexicographic splitting of γ, which is defined as the k-linear map

$$\delta : k[X] \to k\langle X\rangle, \quad x_{i_1}x_{i_2}\cdots x_{i_r} \mapsto x_{i_1}x_{i_2}\cdots x_{i_r} \quad \text{if} \quad i_1 \le i_2 \cdots \le i_r.$$

Given $u \in [X]$, we express it as $u = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$, where $l_i \ge 0$, using an arbitrary monomial ordering on $[X]$.

Following [99], define an ordering on X^* using the ordering $x_1 < x_2 < \cdots < x_n$ as follows: given $u, v \in X^*$, put

$$u > v \ \text{ if } \ \gamma(u) > \gamma(v) \ \text{ in } \ [X] \ \text{ or } \ (\gamma(u) = \gamma(v) \,\text{and}\, u >_{lex} v).$$

It is easy to check that this is a monomial ordering on X^* and $\overline{\delta(s)} = \delta(\bar{s})$ for every $s \in k[X]$. Moreover, $v \ge \delta(u)$ for every $v \in \gamma^{-1}(u)$.

Consider an arbitrary ideal L of $k[X]$ generated by monomials. Given $m = x_{i_1}x_{i_2}\cdots x_{i_r} \in L$, $i_1 \le i_2 \cdots \le i_r$, denote by $U_L(m)$ the set of all monomials $u \in [x_{i_1+1}, \dots, x_{i_r-1}]$ such that neither $u x_{i_2}\cdots x_{i_r}$ nor $u x_{i_1}\cdots x_{i_{r-1}}$ lie in L.

Theorem 6 ([99]) *Consider the orderings on $[X]$ and X^* defined above. If S is a minimal Gröbner basis in $k[X]$ then $S' = \{\delta(us) | s \in S, u \in U_L(\bar{s})\} \cup S_1$ is a minimal Gröbner–Shirshov basis in $k\langle X\rangle$, where L is the monomial ideal of $k[X]$ generated by $\bar{S}$.*

Jointly with Yongshan Chen [30], we generalized this result to lifting a GS basis $S \subset k[Y] \otimes k\langle X\rangle$, see Mikhalev and Zolotykh [170], to a GS basis of $Id(S, [y_i, y_j]$ for all $(i, j))$ of $k\langle Y\rangle \otimes k\langle X\rangle$.

Recall that for a prime number p the Gauss ordering on the natural numbers is described as $s \le_p t$ whenever $\binom{t}{s} \not\equiv 0 \mod p$. Let $\le_0 = \le$ be the usual ordering on the natural numbers. A monomial ideal L of $k[X]$ is called p-Borel-fixed whenever it satisfies the following condition: for each monomial generator m of L, if m is divisible by x_j^t but no higher power of x_j then $(x_i/x_j)^s m \in L$ for all $i < j$ and $s \le_p t$.

Thus, we have the following Eisenbud–Peeva–Sturmfels lifting theorem.

Theorem 7 ([99]) *Given an ideal I of $k[X]$, take $L = Id(\bar{f}, f \in I)$ and $J = \gamma^{-1}(I) \subset k\langle X\rangle$.*

(i) *If L is 0-Borel-fixed then a minimal Gröbner–Shirshov basis of J is obtained by applying δ to a minimal Gröbner basis of I and adding commutators.*
(ii) *If L is p-Borel-fixed for some p then J has a finite Gröbner–Shirshov basis.*

Proof Assume that L is p-Borel-fixed for some p. Take a generator $m = x_{i_1}x_{i_2}\cdots x_{i_r}$ of L, where $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_r}$, and suppose that $x_{i_r}^t$ is the highest power of x_{i_r} dividing m. Since $t \leq_p t$, it follows that $x_l^t m/x_{i_r}^t \in L$ for $l < i_r$. This implies that $x_l^t m/x_{i_r} \in L$ for $l < i_r$, and hence, every monomial in $U_L(m)$ satisfies $deg_{x_l}(u) < t$ for $i_1 < l < i_r$. Thus, $U_L(m)$ is a finite set, and the result follows from Theorem 6. In particular, if $p = 0$ then $U_L(m) = 1$. □

In characteristic $p \geq 0$ observe that if the field k is infinite then after a generic change of variables L is p-Borel-fixed. Then Theorems 6 and 7 imply

Corollary 1 ([99]) *Consider an infinite field k and an ideal $I \subset k[X]$. After a general linear change of variables, the ideal $\gamma^{-1}(I)$ in $k\langle X\rangle$ has a finite Gröbner–Shirshov basis.*

2.3 Composition-Diamond lemma for modules

Consider $S, T \subset k\langle X\rangle$ and $f, g \in k\langle X\rangle$. Kang and Lee define [123] the composition of f and g as follows.

Definition 1 ([123,127])

(a) If there exist $a, b \in X^*$ such that $w = \overline{f}a = b\overline{g}$ with $|w| < |\overline{f}| + |\overline{g}|$ then the intersection composition is defined as $(f, g)_w = fa - bg$.
(b) If there exist $a, b \in X^*$ such that $w = a\overline{f}b = \overline{g}$ then the inclusion composition is defined as $(f, g)_w = afb - g$.
(c) The composition $(f, g)_w$ is called *right-justified* whenever $w = \overline{f} = a\overline{g}$ for some $a \in X^*$.

If $f - g = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j$, where $\alpha_i, \beta_j \in k$, $a_i, b_i, c_j \in X^*$, $s_i \in S$, and $t_j \in T$ with $a_i\bar{s}_i b_i < w$ and $c_j\bar{t}_j < w$ for all i and j, then we call $f - g$ *trivial* with respect to S and T and write $f \equiv g \mod (S, T; w)$.

Definition 2 ([123,124]) A pair (S, T) of monic subsets of $k\langle X\rangle$ is called a *GS pair* if S is closed under composition, T is closed under right-justified composition with respect to S, and given $f \in S$, $g \in T$, and $w \in X^*$ such that if $(f, g)_w$ is defined, we have $(f, g)_w \equiv 0 \mod (S, T; w)$. In this case, say that (S, T) is a *GS pair* for the A-module ${}_AM = {}_A\, k\langle X\rangle/(k\langle X\rangle T + Id(S))$, where $A = k\langle X|S\rangle$.

Theorem 8 (Kang and Lee [123,124], the CD-lemma for cyclic modules) *Consider a pair (S, T) of monic subsets of $k\langle X\rangle$, the associative algebra $A = k\langle X|S\rangle$ defined by S, and the left cyclic module ${}_AM = {}_A\, k\langle X\rangle/(k\langle X\rangle T + Id(S))$ defined by (S, T). Suppose that (S, T) is a Gröbner–Shirshov pair for the A-module ${}_AM$ and $p \in k\langle X\rangle T + Id(S)$. Then $\bar{p} = a\bar{s}b$ or $\bar{p} = c\bar{t}$, where $a, b, c \in X^*$, $s \in S$, and $t \in T$.*

Applications of Theorem 8 appeared in [125–127].

Take two sets X and Y and consider the free left $k\langle X\rangle$-module $\mathrm{Mod}_{k\langle X\rangle}\langle Y\rangle$ with $k\langle X\rangle$-basis Y. Then $\mathrm{Mod}_{k\langle X\rangle}\langle Y\rangle = \oplus_{y\in Y} k\langle X\rangle y$ is called a *double-free* module. We now define the GS basis in $\mathrm{Mod}_{k\langle X\rangle}\langle Y\rangle$. Choose a monomial ordering $<$ on X^*, and a well-ordering $<$ on Y. Put $X^*Y = \{uy | u \in X^*,\ y \in Y\}$ and define an ordering $<$ on X^*Y as follows: for any $w_1 = u_1y_1, w_2 = u_2y_2 \in X^*Y$,

$$w_1 < w_2 \Leftrightarrow u_1 < u_2 \quad \text{or} \quad u_1 = u_2,\ y_1 < y_2$$

Given $S \subset \mathrm{Mod}_{k\langle X\rangle}\langle Y\rangle$ with all $s \in S$ monic, define composition in S to be only inclusion composition, which means that $\bar{f} = a\bar{g}$ for some $a \in X^*$, where $f, g \in S$. If $(f, g)_{\bar{f}} = f - ag = \sum \alpha_i a_i s_i$, where $\alpha_i \in k$, $a_i \in X^*$, $s_i \in S$, and $a_i\overline{s}_i < \bar{f}$, then this composition is called trivial modulo $(S, \bar{f})$.

Theorem 9 (Chibrikov [90], see also [78], the CD-lemma for modules) *Consider a non-empty set $S \subset mod_{k\langle X\rangle}\langle Y\rangle$ with all $s \in S$ monic and choose an ordering $<$ on X^*Y as before. The following statements are equivalent:*

(i) *S is a Gröbner–Shirshov basis in* $\mathrm{Mod}_{k\langle X\rangle}\langle Y\rangle$.
(ii) *If $0 \neq f \in k\langle X\rangle S$ then $\overline{f} = a\overline{s}$ for some $a \in X^*$ and $s \in S$.*
(iii) *$Irr(S) = \{w \in X^*Y | w \neq a\overline{s},\ a \in X^*,\ s \in S\}$ is a linear basis for the quotient* $\mathrm{Mod}_{k\langle X\rangle}\langle Y|S\rangle = \mathrm{Mod}_{k\langle X\rangle}\langle Y\rangle/k\langle X\rangle S$.

Outline of the proof. Take $u \in X^*Y$ and express it as $u = u^X y_u$ with $u^X \in X^*$ and $y_u \in Y$. Put

$$\mathrm{cm}(u, v) = a^X u = b^X v, \quad \mathrm{lcm}(u, v) = u = d^X v,$$

where $y_u = y_v$. Up to the order of u and v, we have $cm(u, v) = c \cdot \mathrm{lcm}(u, v)$.

The composition of two monic elements $f, g \in \mathrm{Mod}_{k\langle X\rangle}(Y)$ is

$$(f, g)|_{\mathrm{lcm}(\bar{f},\bar{g})} = \mathrm{lcm}(\bar{f}, \bar{g})|_{\bar{f}\mapsto f} - \mathrm{lcm}(\bar{f}, \bar{g})|_{\bar{g}\mapsto g}.$$

If $a_1\bar{s}_1 = a_2\bar{s}_2$ for monic s_1 and s_2 then $a_1s_1 - a_2s_2 = c \cdot (s_1, s_2)_{\mathrm{lcm}(\bar{s}_1,\bar{s}_2)}$. This gives an analogue of Lemma 1 for modules and the implication (i)⇒(ii) of Theorem 9.

Given $S \subset k\langle X\rangle$, put $A = k\langle X|S\rangle$. We can regard every left A-module ${}_AM$ as a $k\langle X\rangle$-module in a natural way: $fm := (f + Id(S))m$ for $f \in k\langle X\rangle$ and $m \in M$. Observe that ${}_AM$ is an epimorphic image of some free A-module. Assume now that ${}_AM = \mathrm{Mod}_A\langle Y|T\rangle = \mathrm{Mod}_A\langle Y\rangle/AT$, where $T \subset \mathrm{Mod}_A\langle Y\rangle$. Put

$$T_1 = \left\{\sum f_i y_i \in \mathrm{Mod}_{k\langle X\rangle}\langle Y\rangle | \sum (f_i + Id(S))y_i \in T\right\}$$

and $R = SX^*Y \cup T_1$. Then ${}_AM = \mod_{k\langle X\rangle}\langle Y|R\rangle$ as $k\langle X\rangle$-modules.

Theorem 10 *Given a submodule I of* $\mathrm{Mod}_{k\langle X\rangle}\langle Y\rangle$ *and a monomial ordering $<$ on X^*Y as above, there exists a unique reduced Gröbner–Shirshov basis S for I.*

Corollary 2 (Cohn) *Every left ideal I of $k\langle X\rangle$ is a free left $k\langle X\rangle$-module.*

Proof Take a reduced Gröbner–Shirshov basis S of I as a $k\langle X\rangle$-submodule of the cyclic $k\langle X\rangle$-module. Then I is a free left $k\langle X\rangle$-module with a $k\langle X\rangle$-basis S. □

As an application of the CD-lemma for modules, we give GS bases for the Verma modules over the Lie algebras of coefficients of free Lie conformal algebras. We find linear bases for these modules.

Let $\mathcal{B}$ be a set of symbols. Take the constant locality function $N : \mathcal{B}\times\mathcal{B}\to\mathbb{Z}_+$; that is, $N(a,b)\equiv N$ for all $a,\ b\in\mathcal{B}$. Put $X=\{b(n)|\, b\in\mathcal{B},\ n\in\mathbb{Z}\}$ and consider the Lie algebra $L=Lie(X|S)$ over a field k of characteristic 0 generated by X with the relations

$$S=\left\{\sum_s(-1)^s\binom{N}{s}[b(n-s)a(m+s)]=0|\, a,\ b\in\mathcal{B},\ m,n\in\mathbb{Z}\right\}.$$

For every $b\in\mathcal{B}$, put $\widetilde{b}=\sum_n b(n)z^{-n-1}\in L[[z,z^{-1}]]$. It is well-known that these elements generate a free Lie conformal algebra C with data $(\mathcal{B},N)$ (see [194]). Moreover, the coefficient algebra of C is just L.

Suppose that $\mathcal{B}$ is linearly ordered. Define an ordering on X as

$$a(m)<b(n)\Leftrightarrow m<n\ \text{ or } (m=n \text{ and } a<b).$$

We use the deg-lex ordering on X^*. It is clear that the leading term of each polynomial in S is $b(n)a(m)$ with

$$n-m>N \text{ or } (n-m=N \text{ and } (b>a \text{ or } (b=a \text{ and } N \text{ is odd}))).$$

The following lemma is essentially from [194].

Lemma 3 ([78]) *With the deg-lex ordering on X^*, the set S is a GS basis in $Lie(X)$.*

Corollary 3 ([78]) *A linear basis of the universal enveloping algebra $\mathcal{U}=\mathcal{U}(L)$ of L consists of the monomials*

$$a_1(n_1)a_2(n_2)\cdots a_k(n_k)$$

with $a_i\in\mathcal{B}$ and $n_i\in\mathbb{Z}$ such that for every $1\le i<k$ we have

$$n_i-n_{i+1}\le\begin{cases}N-1 & \text{if } a_i>a_{i+1} \text{ or } (a_i=a_{i+1} \text{ and } N \text{ is odd})\\ N & \text{otherwise.}\end{cases}$$

An L-module M is called *restricted* if for all $a\in C$ and $v\in M$ there is some integer T such that $a(n)v=0$ for $n\ge T$.

An L-module M is called a *highest weight module* whenever it is generated over L by a single element $m\in M$ satisfying $L_+m=0$, where L_+ is the subspace of L generated by $\{a(n)|a\in C, n\ge 0\}$. In this case m is called a *highest weight vector*.

Let us now construct a universal highest weight module V over L, which is often called the *Verma module*. Take the trivial 1-dimensional L_+-module kI_v generated by I_v; hence, $a(n)I_v = 0$ for all $a \in \mathcal{B}$, $n \geq 0$. Clearly,

$$V = Ind_{L_+}^{L} kI_v = \mathcal{U}(L) \otimes_{\mathcal{U}(L_+)} kI_v \cong \mathcal{U}(L)/\mathcal{U}(L)L_+.$$

Then V has the structure of the highest weight module over L with the action given by multiplication on $\mathcal{U}(L)/\mathcal{U}(L)L_+$ and a highest weight vector $I \in \mathcal{U}(L)$. In addition, $V = \mathcal{U}(L)/\mathcal{U}(L)L_+$ is the universal enveloping vertex algebra of C and the embedding $\varphi : C \to V$ is given by $a \mapsto a(-1)I$ (see also [194]).

Theorem 11 ([78]) *With the above notions, a linear basis of V consists of the elements*

$$a_1(n_1)a_2(n_2)\cdots a_k(n_k),\ a_i \in \mathcal{B},\ n_i \in \mathbb{Z}$$

satisfying the condition in Corollary 3 and $n_k < 0$.

Proof Clearly, as $k\langle X\rangle$-modules, we have

$$_{\mathcal{U}}V = {}_{\mathcal{U}}(\mathcal{U}(L)/\mathcal{U}(L)L_+) = \mathrm{Mod}_{k\langle X\rangle}\langle I \mid S^{(-)}X^*I,\ a(n)I,\ n \geq 0\rangle =_{k\langle X\rangle} \langle I \mid S'\rangle,$$

where $S' = \{S^{(-)}X^*I,\ a(n)I,\ n \geq 0\}$. In order to show that S' is a Gröbner–Shirshov basis, we only need to verify that $w = b(n)a(m)I$, where $m \geq 0$. Take

$$f = \sum_{s}(-1)^s \binom{n}{s}(b(n-s)a(m+s) - a(m+s)b(n-s))I \quad \text{and} \quad g = a(m)I.$$

Then $(f, g)_w = f - b(n)a(m)I \equiv 0 \mod (S', w)$ since $n - m \geq N$, $m + s \geq 0$, $n - s \geq 0$, and $0 \leq s \leq N$. It follows that S' is a Gröbner–Shirshov basis. Now, the result follows from the CD-lemma for modules. □

2.4 Composition-Diamond lemma for categories

Denote by X an oriented multi-graph. A path

$$a_n \to a_{n-1} \to \cdots \to a_1 \to a_0, \quad n \geq 0,$$

in X with edges $x_n, \ldots, x_2, x_1$ is a partial word $u = x_1x_2\cdots x_n$ on X with source a_n and target a_0. Denote by $C(X)$ the free category generated by X (the set of all partial words (paths) on X with partial multiplication, the free 'partial path monoid' on X). A well-ordering on $C(X)$ is called monomial whenever it is compatible with partial multiplication.

A polynomial $f \in kC(X)$ is a linear combination of partial words with the same source and target. Then $kC(X)$ is the *partial path algebra* on X (the free associative partial path algebra generated by X).

Given $S \subset kC(X)$, denote by $Id(S)$ the minimal subset of $kC(X)$ that includes S and is closed under the partial operations of addition and multiplication. The elements of $Id(S)$ are of the form $\sum \alpha_i a_i s_i b_i$ with $\alpha_i \in k$, $a_i, b_i \in C(X)$, and $s_i \in S$, and all S-words have the same source and target.

Both inclusion and intersection compositions are possible.

With these differences, the statement and proof of the CD-lemma are the same as for the free associative algebra.

Theorem 12 ([36], the CD-lemma for categories) *Consider a nonempty set $S \subset kC(X)$ of monic polynomials and a monomial ordering $<$ on $C(X)$. Denote by $Id(S)$ the ideal of $kC(X)$ generated by S. The following statements are equivalent:*

(i) *The set S is a Gröbner–Shirshov basis in $kC(X)$.*
(ii) *$f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in C(X)$.*
(iii) *the set $Irr(S) = \{u \in C(X) | u \neq a\bar{s}b \ a, b \in C(X),\ s \in S\}$ is a linear basis for $kC(X)/Id(S)$, which is denoted by $kC(X|S)$.*

Outline of the proof.

Define $w = \mathrm{lcm}(u, v),\ u, v \in C(X)$ and the general composition $(f, g)_w$ for $f, g \in kC(X)$ and $w = \mathrm{lcm}(\bar{f}, \bar{g})$ by the same formulas as above. Under the conditions of the analogue of Lemma 1, we again have $a_1 s_1 b_1 - a_2 s_2 b_2 = c(s_1, s_2)_w d \equiv 0 \mod (S, w)$, where $w = \mathrm{lcm}(\bar{s}_1, \bar{s}_2)$ and $c, d \in C(X)$. This implies the analogue of Lemma 1 and the main assertion (i)⇒(ii) of Theorem 12.

Let us present some applications of CD-lemma for categories.

For each non-negative integer p, denote by $[p]$ the set $\{0, 1, 2, \ldots, p\}$ of integers in their usual ordering. A (weakly) monotonic map $\mu : [q] \to [p]$ is a function from $[q]$ to $[p]$ such that $i \leq j$ implies $\mu(i) \leq \mu(j)$. The objects $[p]$ with weakly monotonic maps as morphisms constitute the category Δ called the simplex category. It is convenient to use two special families of monotonic maps,

$$\varepsilon_q^i : [q-1] \to [q], \quad \eta_q^i : [q+1] \to [q]$$

defined for $i = 0, 1, \ldots q$ (and for $q > 0$ in the case of ε^i) by

$$\varepsilon_q^i(j) = \begin{cases} j & \text{if } i > j, \\ j+1 & \text{if } i \leq j, \end{cases}$$
$$\eta_q^i(j) = \begin{cases} j & \text{if } i \geq j, \\ j-1 & \text{if } j > i. \end{cases}$$

Take the oriented multi-graph $X = (V(X), E(X))$ with

$$V(X) = \{[p] \mid p \in Z^+ \cup \{0\}\},$$
$$E(X) = \{\varepsilon_p^i : [p-1] \to [p],\ \eta_q^j : [q+1] \to [q] \mid p > 0, 0 \leq i \leq p, 0 \leq j \leq q\}.$$

Consider the relation $S \subseteq C(X) \times C(X)$ consisting of:

$$
\begin{aligned}
f_{q+1,q} &: \ \varepsilon^i_{q+1}\varepsilon^{j-1}_q = \varepsilon^j_{q+1}\varepsilon^i_q \quad \text{for } j > i;\\
g_{q,q+1} &: \ \eta^j_q\eta^i_{q+1} = \eta^i_q\eta^{j+1}_{q+1} \quad \text{for } j \geq i;\\
h_{q-1,q} &: \ \eta^j_{q-1}\varepsilon^i_q = \begin{cases} \varepsilon^i_{q-1}\eta^{j-1}_{q-2} & \text{for } j > i,\\ 1_{q-1} & \text{for } i = j \text{ or } i = j+1,\\ \varepsilon^{i-1}_{q-1}\eta^j_{q-2} & \text{for } i > j+1. \end{cases}
\end{aligned}
$$

This yields a presentation $\Delta = C(X|S)$ of the simplex category Δ.

Order now $C(X)$ as follows.

Firstly, for $\eta^i_p, \eta^j_q \in \{\eta^i_p | p \geq 0, 0 \leq i \leq p\}$ put $\eta^i_p > \eta^j_q$ iff $p > q$ or ($p = q$ and $i < j$).

Secondly, for

$$u = \eta^{i_1}_{p_1}\eta^{i_2}_{p_2}\cdots\eta^{i_n}_{p_n} \in \{\eta^i_p | p \geq 0, 0 \leq i \leq p\}^*$$

(these are all possible words on $\{\eta^i_p | p \geq 0, 0 \leq i \leq p\}$, including the empty word 1_v, where $v \in Ob(X)$), define

$$\mathrm{wt}(u) = (n, \eta^{i_n}_{p_n}, \eta^{i_{n-1}}_{p_{n-1}}, \cdots, \eta^{i_1}_{p_1}).$$

Then, for $u, v \in \{\eta^i_p | p \geq 0, 0 \leq i \leq p\}^*$ put $u > v$ iff $\mathrm{wt}(u) > \mathrm{wt}(v)$ lexicographically.

Thirdly, for $\varepsilon^i_p, \varepsilon^j_q \in \{\varepsilon^i_p, | p \in Z^+, 0 \leq i \leq p\}$, put $\varepsilon^i_p > \varepsilon^j_q$ iff $p > q$ or ($p = q$ and $i < j$).

Finally, for $u = v_0\varepsilon^{i_1}_{p_1}v_1\varepsilon^{i_2}_{p_2}\cdots\varepsilon^{i_n}_{p_n}v_n \in C(X)$, where $n \geq 0$, and $v_j \in \{\eta^i_p | p \geq 0, 0 \leq i \leq p\}^*$ put $\mathrm{wt}(u) = (n, v_0, v_1, \ldots, v_n, \varepsilon^{i_1}_{p_1}, \ldots, \varepsilon^{i_n}_{p_n})$. Then for every $u, v \in C(X)$,

$$u \succ_1 v \Leftrightarrow \mathrm{wt}(u) > \mathrm{wt}(v) \quad \text{lexicographically.}$$

It is easy to check that $\succ_1$ is a monomial ordering on $C(X)$. Then we have

Theorem 13 ([36]) *For X and S defined above, with the ordering $\succ_1$ on $C(X)$, the set S is a Gröbner–Shirshov basis for the simplex partial path algebra $kC(X|S)$.*

Corollary 4 ([157]) *Every morphism $\mu : [q] \to [p]$ of the simplex category has a unique expression of the form*

$$\varepsilon^{i_1}_p \ldots \varepsilon^{i_m}_{p-m+1}\eta^{j_1}_{q-n} \ldots \eta^{j_n}_{q-1}$$

with $p \geq i_1 > \cdots > i_m \geq 0$, $0 \leq j_1 < \cdots < j_n < q$, and $q - n + m = p$.

The cyclic category is defined by generators and relations as follows, see [104]. Take the oriented (multi) graph $Y = (V(Y), E(Y))$ with $V(Y) = \{[p] \mid p \in Z^+ \cup \{0\}\}$ and

$$E(Y) = \{\varepsilon_p^i : [p-1] \to [p],\ \eta_q^j : [q+1] \to [q],\ t_q : [q] \\ \to [q] \mid p > 0, 0 \le i \le p, 0 \le j \le q\}.$$

Consider the relation $S \subseteq C(Y) \times C(Y)$ consisting of:

$$\begin{aligned}
f_{q+1,q} &: \ \varepsilon_{q+1}^i \varepsilon_q^{j-1} = \varepsilon_{q+1}^j \varepsilon_q^i \quad \text{for } j > i;\\
g_{q,q+1} &: \ \eta_q^j \eta_{q+1}^i = \eta_q^i \eta_{q+1}^{j+1} \quad \text{for } j \ge i;\\
h_{q-1,q} &: \ \eta_{q-1}^j \varepsilon_q^i = \begin{cases} \varepsilon_{q-1}^i \eta_{q-2}^{j-1} & \text{for } j > i,\\ 1_{q-1} & \text{for } i = j \text{ or } i = j+1,\\ \varepsilon_{q-1}^{i-1} \eta_{q-2}^j & \text{for } i > j+1,\end{cases}\\
\rho_1 &: \quad t_q \varepsilon_q^i = \varepsilon_q^{i-1} t_{q-1} \quad \text{for } i = 1, \dots, q;\\
\rho_2 &: \quad t_q \eta_q^i = \eta_q^{i-1} t_{q+1} \quad \text{for } i = 1, \dots, q;\\
\rho_3 &: \quad t_q^{q+1} = 1_q.
\end{aligned}$$

The category $C(Y|S)$ is called the cyclic category and denoted by Λ.

Define an ordering on $C(Y)$ as follows.

Firstly, for $t_p^i, t_q^j \in \{t_q | q \ge 0\}^*$ put $(t_p)^i > (t_q)^j$ iff $i > j$ or ($i = j$ and $p > q$).

Secondly, for $\eta_p^i, \eta_q^j \in \{\eta_p^i | p \ge 0, 0 \le i \le p\}$ put $\eta_p^i > \eta_q^j$ iff $p > q$ or ($p = q$ and $i < j$).

Thirdly, for

$$u = w_0 \eta_{p_1}^{i_1} w_1 \eta_{p_2}^{i_2} \cdots w_{n-1} \eta_{p_n}^{i_n} w_n \in \{t_q, \eta_p^i | q, p \ge 0, 0 \le i \le p\}^*,$$

where $w_i \in \{t_q | q \ge 0\}^*$, put

$$\mathrm{wt}(u) = (n, w_0, w_1, \dots, w_n, \eta_{p_n}^{i_n}, \eta_{p_{n-1}}^{i_{n-1}}, \dots, \eta_{p_1}^{i_1}).$$

Then for every $u, v \in \{t_q, \eta_p^i | q, p \ge 0, 0 \le i \le p\}^*$ put $u > v$ iff $\mathrm{wt}(u) > \mathrm{wt}(v)$ lexicographically.

Fourthly, for $\varepsilon_p^i, \varepsilon_q^j \in \{\varepsilon_p^i, | p \in Z^+, 0 \le i \le p\}$, $\varepsilon_p^i > \varepsilon_q^j$ iff $p > q$ or ($p = q$ and $i < j$).

Finally, for $u = v_0 \varepsilon_{p_1}^{i_1} v_1 \varepsilon_{p_2}^{i_2} \cdots \varepsilon_{p_n}^{i_n} v_n \in C(Y)$ and $v_j \in \{t_q, \eta_p^i | q, p \ge 0, 0 \le i \le p\}^*$ define $\mathrm{wt}(u) = (n, v_0, v_1, \dots, v_n, \varepsilon_{p_1}^{i_1}, \dots, \varepsilon_{p_n}^{i_n})$.

Then for every $u, v \in C(Y)$ put $u \succ_2 v \Leftrightarrow \mathrm{wt}(u) < \mathrm{wt}(v)$ lexicographically.

It is also easy to verify that $\succ_2$ is a monomial ordering on $C(Y)$ which extends $\succ_1$. Then we have

Theorem 14 ([36]) *Consider Y and S defined as the above. Put $\rho_4: \ t_q\varepsilon_q^0 = \varepsilon_q^q$ and $\rho_5: \ t_q\eta_q^0 = \eta_q^q t_{q+1}^2$. Then*

(1) *With the ordering $\succ_2$ on $C(Y)$, the set $S \cup \{\rho_4, \rho_5\}$ is a Gröbner–Shirshov basis for the cyclic category $C(Y|S)$.*
(2) *Every morphism $\mu : [q] \to [p]$ of the cyclic category $\Lambda = C(Y|S)$ has a unique expression of the form*

$$\varepsilon_p^{i_1} \dots \varepsilon_{p-m+1}^{i_m} \eta_{q-n}^{j_1} \dots \eta_{q-1}^{j_n} t_q^k$$

with $p \geq i_1 > \dots > i_m \geq 0$, $0 \leq j_1 < \dots < j_n < q$, $0 \leq k \leq q$, and $q - n + m = p$.

2.5 Composition-Diamond lemma for associative algebras over commutative algebras

Given two well-ordered sets X and Y, put

$$N = [X]Y^* = \{u = u^X u^Y | u^X \in [X] \ and \ u^Y \in Y^*\}$$

and denote by kN the k-space spanned by N. Define the multiplication of words as

$$u = u^X u^Y, v = v^X v^Y \in N \Rightarrow uv = u^X v^X u^Y v^Y \in N.$$

This makes kN an algebra isomorphic to the tensor product $k[X] \otimes k\langle Y \rangle$, called a'double free associative algebra'. It is a free object in the category of all associative algebras over all commutative algebras (over k): every associative algebra $_K A$ over a commutative algebra K is isomorphic to $k[X] \otimes k\langle Y \rangle / Id(S)$ as a k-algebra and a $k[X]$-algebra.

Choose a monomial ordering $>$ on N. The following definitions of compositions and the GS basis are taken from [170].

Take two monic polynomials f and g in $k[X] \otimes k\langle Y \rangle$ and denote by L the least common multiple of $\bar{f}^X$ and $\bar{g}^X$.

1. Inclusion. Assume that $\bar{g}^Y$ is a subword of $\bar{f}^Y$, say, $\bar{f}^Y = c\bar{g}^Y d$ for some $c, d \in Y^*$. If $\bar{f}^Y = \bar{g}^Y$ then $\bar{f}^X \geq \bar{g}^X$ and if $\bar{g}^Y = 1$ then we set $c = 1$. Put $w = L\bar{f}^Y = Lc\bar{g}^Y d$. Define the composition $C_1(f, g, c)_w = \frac{L}{\bar{f}^X} f - \frac{L}{\bar{g}^X} cgd$.
2. Overlap. Assume that a non-empty beginning of $\bar{g}^Y$ is a non-empty ending of $\bar{f}^Y$, say, $\bar{f}^Y = cc_0$, $\bar{g}^Y = c_0 d$, and $\bar{f}^Y d = c\bar{g}^Y$ for some $c, d, c_0 \in Y^*$ and $c_0 \neq 1$. Put $w = L\bar{f}^Y d = Lc\bar{g}^Y$. Define the composition $C_2(f, g, c_0)_w = \frac{L}{\bar{f}^X} fd - \frac{L}{\bar{g}^X} cg$.
3. External. Take a (possibly empty) associative word $c_0 \in Y^*$. In the case that the greatest common divisor of $\bar{f}^X$ and $\bar{g}^X$ is non-empty and both $\bar{f}^Y$ and $\bar{g}^Y$ are non-empty, put $w = L\bar{f}^Y c_0 \bar{g}^Y$ and define the composition $C_3(f, g, c_0)_w = \frac{L}{\bar{f}^X} f c_0 \bar{g}^Y - \frac{L}{\bar{g}^X} \bar{f}^Y c_0 g$.

A monic subset S of $k[X] \otimes k\langle Y\rangle$ is called a GS basis whenever all compositions of elements of S, say $(f, g)_w$, are trivial modulo (S, w):

$$(f, g)_w = \sum_i \alpha_i a_i s_i b_i,$$

where $a_i, b_i \in N$, $s_i \in S$, $\alpha_i \in k$, and $a_i \bar{s_i} b_i < w$ for all i.

Theorem 15 (Mikhalev and Zolotykh [170,228], the CD-lemma for associative algebras over commutative algebras) *Consider a monic subset $S \subseteq k[X] \otimes k\langle Y\rangle$ and a monomial ordering $<$ on N. The following statements are equivalent:*

- (i) *The set S is a Gröbner–Shirshov basis in $k[X] \otimes k\langle Y\rangle$.*
- (ii) *For every element $f \in Id(S)$, the monomial $\bar{f}$ contains $\bar{s}$ as its subword for some $s \in S$.*
- (iii) *The set $Irr(S) = \{w \in N | w \neq a\bar{s}b,\ a, b \in N,\ s \in S\}$ is a linear basis for the quotient $k[X] \otimes k\langle Y\rangle$.*

Outline of the proof. For

$$w = \mathrm{lcm}(u, v) = \mathrm{lcm}(u^X, v^X)\,\mathrm{lcm}(u^Y, v^Y)$$

the general composition is

$$(s_1, s_2)_w = (\mathrm{lcm}(u^X, v^X)/u^X)w|_{u\mapsto s_1} - (\mathrm{lcm}(u^X, v^X)/v^X)w|_{v\mapsto s_2},$$

where $s_1, s_2 \in k[X]\langle Y\rangle$ are k-monic with $u = \bar{s}_1$ and $v = \bar{s}_2$. Moreover, $(s_1, s_2)_w \equiv 0 \mod (\{s_1, s_2\}, w)$ whenever $w = u^X v^X u^Y c^Y v^Y$ with $c^Y \in Y^*$, that is, w is a trivial least common multiple relative to both X-words and Y-words. This implies the analog of Lemma 1 and the claim (i)⇒(ii) in Theorem 15.

We apply this lemma in Sect. 4.3.

2.6 PBW-theorem for Lie algebras

Consider a Lie algebra $(L, [\,])$ over a field k with a well-ordered linear basis $X = \{x_i | i \in I\}$ and multiplication table $S = \{[x_i x_j] = [|x_i x_j|] | i > j,\ i, j \in I\}$, where for every $i, j \in I$ we write $[|x_i x_j|] = \Sigma_t \alpha_{ij}^t x_t$ with $\alpha_{ij}^t \in k$. Then $U(L) = k\langle X | S^{(-)}\rangle$ is called the universal enveloping associative algebra of L, where $S^{(-)} = \{x_i x_j - x_j x_i = [|x_i x_j|] | i > j,\ i, j \in I\}$.

Theorem 16 (PBW Theorem) *In the above notation and with the deg-lex ordering on X^*, the set $S^{(-)}$ is a Gröbner–Shirshov basis of $k\langle X\rangle$. Then by the CD-lemma for associative algebras, the set $Irr(S^{(-)})$ consists of the elements*

$$x_{i_1} \dots x_{i_n} \text{ with } i_1 \leq \dots \leq i_n,\ i_1, \dots, i_n \in I,\ n \geq 0,$$

and constitutes a linear basis of $U(L)$.

Theorem 17 (The PBW Theorem in Shirshov's form) *Consider $L = Lie(X|S)$ with $S \subset Lie(X) \subset k\langle X\rangle$ and $U(L) = k\langle X|S^{(-)}\rangle$. The following statements are equivalent.*

(i) *For the deg-lex ordering, S is a GS basis of $Lie(X)$.*
(ii) *For the deg-lex ordering, $S^{(-)}$ is a GS basis of $k\langle X\rangle$.*
(iii) *A linear basis of $U(L)$ consists of the words $u = u_1 \cdots u_n$, where $u_1 \preceq \cdots \preceq u_n$ in the lex ordering, $n \geq 0$, and every u_i is an $S^{(-)}$-irreducible associative Lyndon–Shirshov word in X.*
(iv) *A linear basis of L is the set of all S-irreducible Lyndon–Shirshov Lie monomials $[u]$ in X.*
(v) *A linear basis of $U(L)$ consists of the polynomials $u = [u_1] \cdots [u_n]$, where $u_1 \preceq \cdots \preceq u_n$ in the lex ordering, $n \geq 0$, and every $[u_i]$ is an S-irreducible non-associative Lyndon–Shirshov word in X.*

The PBW theorem, Theorem 33, the CD-lemmas for associative and Lie algebras, Shirshov's factorization theorem, and property (**VIII**) of Sect. 4.2 imply that every LS-subword of u is a subword of some u_i.

Makar–Limanov gave [158] an interesting form of the PBW theorem for a finite dimensional Lie algebra.

2.7 Drinfeld–Jimbo algebra $U_q(A)$, Kac–Moody enveloping algebra $U(A)$, and the PBW basis of $U_q(A_N)$

Take an integral symmetrizable $N \times N$ Cartan matrix $A = (a_{ij})$. Hence, $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, and there exists a diagonal matrix D with diagonal entries d_i, which are nonzero integers, such that the product DA is symmetric. Fix a nonzero element q of k with $q^{4d_i} \neq 1$ for all i. Then the Drinfeld–Jimbo quantum enveloping algebra is

$$U_q(A) = k\langle X \cup H \cup Y | S^+ \cup K \cup T \cup S^-\rangle,$$

where

$$\begin{aligned}
X &= \{x_i\}, \;\; H = \{h_i^{\pm 1}\}, \;\; Y = \{y_i\},\\
S^+ &= \left\{ \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \binom{1-a_{ij}}{\nu}_t x_i^{1-a_{ij}-\nu} x_j x_i^\nu, \text{ where } i \neq j,\; t = q^{2d_i} \right\},\\
S^- &= \left\{ \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \binom{1-a_{ij}}{\nu}_t y_i^{1-a_{ij}-\nu} y_j y_i^\nu, \text{ where } i \neq j,\; t = q^{2d_i} \right\},\\
K &= \{h_i h_j - h_j h_i, h_i h_i^{-1} - 1, h_i^{-1} h_i - 1, x_j h_i^{\pm 1} - q^{\mp 1} d_i a_{ij} h^{\pm 1} x_j,\\
&\qquad h_i^{\pm 1} y_j - q^{\mp 1} y_j h^{\pm 1}\},\\
T &= \left\{ x_i y_j - y_j x_i - \delta_{ij} \frac{h_i^2 - h_i^{-2}}{q^{2d_i} - q^{-2d_i}} \right\},
\end{aligned}$$

and

$$\binom{m}{n}_t = \begin{cases} \prod_{i=1}^{n} \frac{t^{m-i+1} - t^{i-m-1}}{t^i - t^{-i}} & \text{(for } m > n > 0\text{)}, \\ 1 & \text{(for } n = 0 \text{ or } m = n\text{)}. \end{cases}$$

Theorem 18 ([55]) *For every symmetrizable Cartan matrix A, with the deg-lex ordering on* $\{X \cup H \cup Y\}^*$, *the set* $S^{+c} \cup T \cup K \cup S^{-c}$ *is a Gröbner–Shirshov basis of the Drinfeld–Jimbo algebra* $U_q(A)$, *where* S^{+c} *and* S^{-c} *are the Shirshov completions of* S^+ *and* S^-.

Corollary 5 (Rosso [195], Yamane [220]) *For every symmetrizable Cartan matrix A we have the triangular decomposition*

$$U_q(A) = U_q^+(A) \otimes k[H] \otimes U_q^-(A)$$

with $U_q^+(A) = k\langle X | S^+ \rangle$ *and* $U_q^-(A) = k\langle Y | S^- \rangle$.

Similar results are valid for the Kac–Moody Lie algebras $g(A)$ and their universal enveloping algebras

$$U(A) = k\langle X \cup H \cup Y | S^+ \cup H \cup K \cup S^- \rangle,$$

where $S^+,\ S^-$ are the same as for $U_q(A)$,

$$K = \{h_i h_j - h_j h_i,\ x_j h_i - h_i x_j + d_i a_{ij} x_i, h_i y_i - y_i h_i + d_i a_{ij} y_j\},$$

and $T = \{x_i y_j - y_j x_i - \delta_{ij} h_i\}$.

Theorem 19 ([55]) *For every symmetrizable Cartan matrix A, the set* $S^{+c} \cup T \cup K \cup S^{-c}$ *is a Gröbner–Shirshov basis of the universal enveloping algebra* $U(A)$ *of the Kac–Moody Lie algebra* $g(A)$.

The PBW theorem in Shirshov's form implies

Corollary 6 (Kac [117]) *For every symmetrizable Cartan matrix A, we have the triangular decomposition*

$$U(A) = U^+(A) \otimes k[H] \otimes U^-(A), \quad g(A) = g^+(A) \oplus k[H] \oplus g^-(A).$$

Poroshenko [179,180] found GS bases for the Kac–Moody algebras of types $\widetilde{A_n}$, $\widetilde{B_n}$, $\widetilde{C_n}$, and $\widetilde{D_n}$. He used the available linear bases of the algebras [117].

Consider now

$$A = A_N = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

and assume that $q^8 \neq 1$. Introduce new variables, defined by Jimbo (see [220]), which generate $U_q(A_N)$:

$$\widetilde{X} = \{x_{ij}, 1 \leq i < j \leq N+1\},$$

where

$$x_{ij} = \begin{cases} x_i & j = i+1, \\ qx_{i,j-1}x_{j-1,j} - q^{-1}x_{j-1,j}x_{i,j-1} & j > i+1. \end{cases}$$

Order the set $\widetilde{X}$ as follows: $x_{mn} > x_{ij} \iff (m,n) >_{lex} (i,j)$. Recall from Yamane [220] the notation

$$C_1 = \{((i,j),(m,n))|i = m < j < n\},\ C_2 = \{((i,j),(m,n))|i < m < n < j\},$$
$$C_3 = \{((i,j),(m,n))|i < m < j = n\},\ C_4 = \{((i,j),(m,n))|i < m < j < n\},$$
$$C_5 = \{((i,j),(m,n))|i < j = m < n\},\ C_6 = \{((i,j),(m,n))|i < j < m < n\}.$$

Consider the set $\widetilde{S}^+$ consisting of Jimbo's relations:

$$\begin{array}{ll} x_{mn}x_{ij} - q^{-2}x_{ij}x_{mn} & ((i,j),(m,n)) \in C_1 \cup C_3, \\ x_{mn}x_{ij} - x_{ij}x_{mn} & ((i,j),(m,n)) \in C_2 \cup C_6, \\ x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2 - q^{-2})x_{in}x_{mj} & ((i,j),(m,n)) \in C_4, \\ x_{mn}x_{ij} - q^2x_{ij}x_{mn} + qx_{in} & ((i,j),(m,n)) \in C_5. \end{array}$$

It is easy to see that $U_q^+(A_N) = k\langle \widetilde{X}|\widetilde{S^+}\rangle$.

A direct proof [86] shows that $\widetilde{S}^+$ is a GS basis for $k\langle \widetilde{X}|\widetilde{S^+}\rangle = U_q^+(A_N)$ [55]. The proof is different from the argument of Bokut and Malcolmson [55]. This yields

Theorem 20 ([55]) *In the above notation and with the deg-lex ordering on $\{\widetilde{X} \cup H \cup \widetilde{Y}\}^*$, the set $\widetilde{S}^+ \cup T \cup K \cup \widetilde{S}^-$ is a Gröbner–Shirshov basis of*

$$U_q(A_N) = k\langle \widetilde{X} \cup H \cup \widetilde{Y}|\widetilde{S}^+ \cup T \cup K \cup \widetilde{S}^-\rangle.$$

Corollary 7 ([195,220]) *For $q^8 \neq 1$, a linear basis of $U_q(A_n)$ consists of*

$$y_{m_1n_1} \cdots y_{m_ln_l}h_1^{s_1} \cdots h_N^{s_N}x_{i_1j_1} \cdots x_{i_kj_k}$$

with $(m_1,n_1) \leq \cdots \leq (m_l,n_l)$, $(i_1,j_1) \leq \cdots \leq (i_k,j_k)$, $k,l \geq 0$ and $s_t \in \mathbb{Z}$.

3 Gröbner–Shirshov bases for groups and semigroups

In this section we apply the method of GS bases for braid groups in different sets of generators, Chinese monoids, free inverse semigroups, and plactic monoids in two sets of generators (row words and column words).

Given a set X consider $S \subseteq X^* \times X^*$ the congruence $\rho(S)$ on X^* generated by S, the quotient semigroup

$$A = \mathrm{sgp}\langle X|S\rangle = X^*/\rho(S),$$

and the semigroup algebra $k(X^*/\rho(S))$. Identifying the set $\{u = v|(u, v) \in S\}$ with S, it is easy to see that

$$\sigma : k\langle X|S\rangle \to k(X^*/\rho(S)), \quad \sum \alpha_i u_i + Id(S) \mapsto \sum \alpha_i \overline{u_i}$$

is an algebra isomorphism.

The Shirshov completion S^c of S consists of semigroup relations, $S^c = \{u_i - v_i,\ i \in I\}$. Then $Irr(S^c)$ is a linear basis of $k\langle X|S\rangle$, and so $\sigma(Irr(S^c))$ is a linear basis of $k(X^*/\rho(S))$. This shows that $Irr(S^c)$ consists precisely of the normal forms of the elements of the semigroup $\mathrm{sgp}\langle X|S\rangle$.

Therefore, in order to find the normal forms of the semigroup $\mathrm{sgp}\langle X|S\rangle$, it suffices to find a GS basis S^c in $k\langle X|S\rangle$. In particular, consider a group $G = gp\langle X|S\rangle$, where $S = \{(u_i, v_i) \in F(X) \times F(X)|i \in I\}$ and $F(X)$ is the free group on a set X. Then G has a presentation

$$G = \mathrm{sgp}\langle X \cup X^{-1}|S, x^{\varepsilon}x^{-\varepsilon} = 1, \varepsilon = \pm 1, x \in X\rangle, \quad X \cap X^{-1} = \emptyset$$

as a semigroup.

3.1 Gröbner–Shirshov bases for braid groups

Consider the Artin braid group B_n of type $\mathbf{A}_{n-1}$ (Artin [5]). We have

$$B_n = gp\langle \sigma_1, \ldots, \sigma_n \mid \sigma_j\sigma_i = \sigma_i\sigma_j\ (j-1>i),\ \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i,\ 1 \le i \le n-1\rangle.$$

3.1.1 Braid groups in the Artin–Burau generators

Assume that $X = Y\dot{\cup}Z$ with Y^* and Z well-ordered and that the ordering on Y^* is monomial. Then every word in X has the form $u = u_0 z_1 u_1 \cdots z_k u_k$, where $k \ge 0$, $u_i \in Y^*$, and $z_i \in Z$. Define the inverse weight of the word $u \in X^*$ as

$$\mathrm{inwt}(u) = (k, u_k, z_k, \ldots, u_1, z_1, u_0)$$

and the inverse weight lexicographic ordering as

$$u > v \Leftrightarrow \mathrm{inwt}(u) > \mathrm{inwt}(v).$$

Call this ordering the inverse tower ordering for short. Clearly, it is a monomial ordering on X^*.

When $X = Y\dot{\cup}Z$, $Y = T\dot{\cup}U$, and Y^* is endowed with the inverse tower ordering, define the inverse tower ordering on X^* with respect to the presentation $X = (T\dot{\cup}U)\dot{\cup}Z$. In general, for

$$X = (\cdots(X^{(n)}\dot{\cup}X^{(n-1)})\dot{\cup}\cdots)\dot{\cup}X^{(0)}$$

with $X^{(n)}$-words equipped with a monomial ordering we can define the inverse tower ordering of X-words.

Introduce a new set of generators for the braid group B_n, called the Artin–Burau generators. Put

$$s_{i,i+1} = \sigma_i^2, \quad s_{i,j+1} = \sigma_j \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_j^{-1}, \quad 1 \le i < j \le n-1;$$
$$\sigma_{i,j+1} = \sigma_i^{-1} \cdots \sigma_j^{-1}, \quad 1 \le i \le j \le n-1; \quad \sigma_{ii} = 1, \quad \{a, b\} = b^{-1}ab.$$

Form the sets

$$S_j = \{s_{i,j}, s_{i,j}^{-1}, 1 \le i, j < n\} \text{ and } \Sigma^{-1} = \{\sigma_1^{-1}, \cdots \sigma_{n-1}^{-1}\}.$$

Then the set

$$S = S_n \cup S_{n-1} \cup \cdots \cup S_2 \cup \Sigma^{-1}$$

generates B_n as a semigroup.

Order now the alphabet S as

$$S_n < S_{n-1} < \cdots < S_2 < \Sigma^{-1},$$

and

$$s_{1,j}^{-1} < s_{1,j} < s_{2,j}^{-1} < \cdots < s_{j-1,j}, \quad \sigma_1^{-1} < \sigma_2^{-1} < \cdots \sigma_{n-1}^{-1}.$$

Order S_n-words by the *deg-inlex* ordering; that is, first compare words by length and then by the inverse lexicographic ordering starting from their last letters. Then we use the inverse tower ordering of S-words.

Lemma 4 (Artin [6], Markov [160]) *The following Artin–Markov relations hold in the braid group B_n:*

$$\sigma_k^{-1}s_{i,j}^{\delta} = s_{i,j}^{\delta}\sigma_k^{-1} \textit{for } k \ne i-1, i, j-1, j, \tag{1}$$
$$\sigma_i^{-1}s_{i,i+1}^{\delta} = s_{i,i+1}^{\delta}\sigma_1^{-1}, \tag{2}$$
$$\sigma_{i-1}^{-1}s_{i,j}^{\delta} = s_{i-1,j}^{\delta}\sigma_{i-1}^{-1}, \tag{3}$$
$$\sigma_i^{-1}s_{i,j}^{\delta} = \{s_{i+1,j}^{\delta}, s_{i,i+1}\}\sigma_i^{-1}, \tag{4}$$
$$\sigma_{j-1}^{-1}s_{i,j}^{\delta} = s_{i,j-1}^{\delta}\sigma_{j-1}^{-1}, \tag{5}$$
$$\sigma_j^{-1}s_{i,j}^{\delta} = \{s_{i,j+1}^{\delta}, s_{j,j+1}\}\sigma_j^{-1}, \tag{6}$$

where $\delta = \pm 1$*;*

$$s_{j,k}^{-1}s_{k,l}^{\varepsilon} = \{s_{k,l}^{\varepsilon}, s_{j,l}^{-1}\}s_{j,k}^{-1}, \tag{7}$$
$$s_{j,k}s_{k,l}^{\varepsilon} = \{s_{k,l}^{\varepsilon}, s_{j,l}s_{k,l}\}s_{j,k}, \tag{8}$$
$$s_{j,k}^{-1}s_{j,l}^{\varepsilon} = \{s_{j,l}^{\varepsilon}, s_{k,l}^{-1}s_{j,l}^{-1}\}s_{j,k}^{-1}, \tag{9}$$
$$s_{j,k}s_{j,l}^{\varepsilon} = \{s_{j,l}^{\varepsilon}, s_{k,l}\}s_{j,k}, \tag{10}$$
$$s_{i,k}^{-1}s_{j,l}^{\varepsilon} = \{s_{j,l}^{\varepsilon}, s_{k,l}s_{i,l}s_{k,l}^{-1}s_{i,l}^{-1}\}s_{i,k}^{-1}, \tag{11}$$
$$s_{i,k}s_{j,l}^{\varepsilon} = \{s_{j,l}^{\varepsilon}, s_{i,l}^{-1}s_{k,l}^{-1}s_{i,l}s_{k,l}\}s_{i,k}, \tag{12}$$

where $i < j < k < l$ *and* $\varepsilon = \pm 1$*;*

$$s_{i,k}^{\delta}s_{j,l}^{\varepsilon} = s_{j,l}^{\varepsilon}s_{i,k}^{\delta}, \tag{13}$$
$$\sigma_j^{-1}\sigma_k^{-1} = \sigma_k^{-1}\sigma_j^{-1} \quad \textit{for } j < k-1 \tag{14}$$
$$\sigma_{j,j+1}\sigma_{k,j+1} = \sigma_{k,j+1}\sigma_{j-1,j} \quad \textit{for } j < k, \tag{15}$$
$$\sigma_i^{-2} = s_{i,i+l}^{-1}, \tag{16}$$
$$s_{i,j}^{\pm 1}s_{i,j}^{\mp 1} = 1, \tag{17}$$

where $j < i < k < l$ *or* $i < k < j < l$*, and* $\varepsilon, \delta = \pm 1$.

Theorem 21 ([25]) *The Artin–Markov relations* (1)–(13) *form a Gröbner–Shirshov basis of the braid group* B_n *in terms of the Artin–Burau generators with respect to the inverse tower ordering of words.*

It is claimed in [25] that some compositions are trivial. Processing all compositions explicitly, [82] supported the claim.

Corollary 8 (Markov–Ivanovskii [6]) *The following words are normal forms of the braid group* B_n*:*

$$f_n f_{n-1} \dots f_2 \sigma_{i_n n}\sigma_{i_{n-1}n-1} \dots \sigma_{i_2 2},$$

where all f_j *for* $2 \le j \le n$ *are free irreducible words in* $\{s_{ij},\ i < j\}$.

3.1.2 Braid groups in the Artin–Garside generators

The Artin–Garside generators of the braid group B_{n+1} are σ_i, $1 \le i \le n$, $\triangle$, $\triangle^{-1}$ (Garside [103] 1969), where $\triangle = \Lambda_1 \cdots \Lambda_n$ with $\Lambda_i = \sigma_1 \cdots \sigma_i$.

Putting $\triangle^{-1} < \triangle < \sigma_1 < \cdots < \sigma_n$, order $\{\triangle^{-1}, \triangle, \sigma_1, \dots, \sigma_n\}^*$ by the deg-lex ordering.

Denote by $V(j,i)$, $W(j,i), \dots$ for $j \le i$ positive words in the letters $\sigma_j, \sigma_{j+1}, \dots, \sigma_i$, assuming that $V(i+1,i) = 1$, $W(i+1,i) = 1, \dots$.

Given $V = V(1,i)$, for $1 \le k \le n-i$ denote by $V^{(k)}$ the result of shifting the indices of all letters in V by k: $\sigma_1 \mapsto \sigma_{k+1}, \dots, \sigma_i \mapsto \sigma_{k+i}$, and put $V' = V^{(1)}$. Define $\sigma_{ij} = \sigma_i\sigma_{i-1}\dots\sigma_j$ for $j \le i-1$, while $\sigma_{ii} = \sigma_i$ and $\sigma_{ii+1} = 1$.

Theorem 22 ([23,47]) *A Gröbner–Shirshov basis S of B_{n+1} in the Artin–Garside generators consists of the following relations:*

$$\begin{aligned}
&\sigma_{i+1}\sigma_i V(1, i-1)W(j,i)\sigma_{i+1j} = \sigma_i\sigma_{i+1}\sigma_i V(1, i-1)\sigma_{ij} W(j,i)',\\
&\sigma_s\sigma_k = \sigma_k\sigma_s \quad \text{for } s-k \geq 2,\\
&\sigma_1 V_1\sigma_2\sigma_1 V_2 \cdots V_{n-1}\sigma_n \cdots \sigma_1 = \triangle V_1^{(n-1)} V_2^{(n-2)} \cdots V'_{(n-1)},\\
&\sigma_l\triangle = \triangle\sigma_{n-l+1} \quad \text{for } 1 \leq l \leq n,\\
&\sigma_l\triangle^{-1} = \triangle^{-1}\sigma_{n-l+1} \quad \text{for } 1 \leq l \leq n,\\
&\triangle\triangle^{-1} = 1,\ \triangle^{-1}\triangle = 1,
\end{aligned}$$

where $1 \leq i \leq n-1$ and $1 \leq j \leq i+1$; moreover, $W(j,i)$ begins with σ_i unless it is empty, and $V_i = V_i(1,i)$.

There are corollaries.

Corollary 9 *The S-irreducible normal form of each word of B_{n+1} coincides with its Garside normal form [103].*

Corollary 10 (Garside [103]) *The semigroup B^+_{n+1} of positive braids can be embedded into a group.*

3.1.3 Braid groups in the Birman–Ko–Lee generators

Recall that the Birman–Ko–Lee generators σ_{ts} of the braid group B_n are

$$\sigma_{ts} = (\sigma_{t-1}\sigma_{t-2}\ldots\sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1})$$

and we have the presentation

$$\begin{aligned}
B_n = gp\langle \sigma_{ts},\ n \geq t > s \geq 1 | &\sigma_{ts}\sigma_{rq} = \sigma_{rq}\sigma_{ts},\ (t-r)(t-q)(s-r)(s-q) > 0,\\
&\sigma_{ts}\sigma_{sr} = \sigma_{tr}\sigma_{ts} = \sigma_{sr}\sigma_{tr}, n \geq t > s > r \geq 1\rangle.
\end{aligned}$$

Denote by δ the Garside word, $\delta = \sigma_{nn-1}\sigma_{n-1n-2}\cdots\sigma_{21}$.

Define the order as $\delta^{-1} < \delta < \sigma_{ts} < \sigma_{rq}$ iff $(t,s) < (r,q)$ lexicographically. Use the deg-lex ordering on $\{\delta^{-1}, \delta, \sigma_{ts},\ n \geq t > s \geq 1\}^*$.

Instead of σ_{ij}, we write simply (i,j) or (j,i). We also set

$$(t_m, t_{m-1}, \ldots, t_1) = (t_m, t_{m-1})(t_{m-1}, t_{m-2})\ldots(t_2, t_1),$$

where $t_j \neq t_{j+1},\ 1 \leq j \leq m-1$. In this notation, we can write the defining relations of B_n as

$$\begin{aligned}
(t_3, t_2, t_1) = (t_2, t_1, t_3) = (t_1, t_3, t_2) \quad &\text{for } t_3 > t_2 > t_1,\\
(k,l)(i,j) = (i,j)(k,l) \quad &\text{for } k > l,\ i > j,\ k > i,
\end{aligned}$$

where either $k > i > j > l$ or $k > l > i > j$.

Denote by $V_{[t_2,t_1]}$, where $n \geq t_2 > t_1 \geq 1$, a positive word in (k,l) satisfying $t_2 \geq k > l \geq t_1$. We can use any capital Latin letter with indices instead of V, and appropriate indices (for instance, t_3 and t_0 with $t_3 > t_0$) instead of t_2 and t_1. Use also the following equalities in B_n:

$$V_{[t_2-1,t_1]}(t_2,t_1) = (t_2,t_1)V'_{[t_2-1,t_1]}$$

for $t_2 > t_1$, where $V'_{[t_2-1,t_1]} = (V_{[t_2-1,t_1]})|_{(k,l)\mapsto(k,l),\text{ if } l\neq t_1;\ (k,t_1)\mapsto(t_2,k)}$;

$$W_{[t_2-1,t_1]}(t_1,t_0) = (t_1,t_0)W^{\star}_{[t_2-1,t_1]}$$

for $t_2 > t_1 > t_0$, where $W^{\star}_{[t_2-1,t_1]} = (W_{[t_2-1,t_1]})|_{(k,l)\mapsto(k,l),\text{ if } l\neq t_1;\ (k,t_1)\mapsto(k,t_0)}$.

Theorem 23 ([24]) *A Gröbner–Shirshov basis of the braid group B_n in the Birman–Ko–Lee generators consists of the following relations:*

$$\begin{aligned}
&(k,l)(i,j) = (i,j)(k,l) \quad \textit{for } k > l > i > j,\\
&(k,l)V_{[j-1,1]}(i,j) = (i,j)(k,l)V_{[j-1,1]} \quad \textit{for } k > i > j > l,\\
&(t_3,t_2)(t_2,t_1) = (t_2,t_1)(t_3,t_1),\\
&(t_3,t_1)V_{[t_2-1,1]}(t_3,t_2) = (t_2,t_1)(t_3,t_1)V_{[t_2-1,1]},\\
&(t,s)V_{[t_2-1,1]}(t_2,t_1)W_{[t_3-1,t_1]}(t_3,t_1) = (t_3,t_2)(t,s)V_{[t_2-1,1]}(t_2,t_1)W'_{[t_3-1,t_1]},\\
&(t_3,s)V_{[t_2-1,1]}(t_2,t_1)W_{[t_3-1,t_1]}(t_3,t_1) = (t_2,s)(t_3,s)V_{[t_2-1,1]}(t_2,t_1)W'_{[t_3-1,t_1]},\\
&(2,1)V_{2[2,1]}(3,1)\ldots V_{n-1[n-1,1]}(n,1) = \delta V'_{2[2,1]}\ldots V'_{n-1[n-1,1]},\\
&(t,s)\delta = \delta(t+1,s+1),\quad (t,s)\delta^{-1} = \delta^{-1}(t-1,s-1) \textit{ with } t\pm 1,\ s\pm 1 \pmod{n},\\
&\delta\delta^{-1} = 1,\ \delta^{-1}\delta = 1,
\end{aligned}$$

where $V_{[k,l]}$ means, as above, a word in (i,j) satisfying $k \geq i > j \geq l$, $t > t_3$, and $t_2 > s$.

There are two corollaries.

Corollary 11 (Birman et al. [13]) *The semigroup B_n^+ of positive braids in the Birman–Ko–Lee generators embeds into a group.*

Corollary 12 (Birman et al. [13]) *The S-irreducible normal form of a word in B_n in the Birman–Ko–Lee generators coincides with the Birman–Ko–Lee–Garside normal form $\delta^k A$, where $A \in B_n^+$.*

3.1.4 Braid groups in the Adjan–Thurston generators

The symmetric group S_{n+1} has the presentation

$$S_{n+1} = gp\langle s_1,\ldots,s_n \mid s_i^2 = 1, s_js_i = s_is_j\ (j-1 > i), s_{i+1}s_is_{i+1} = s_is_{i+1}s_i\rangle.$$

Bokut and Shiao [58] found the normal form for S_{n+1} in the following statement: the set $N = \{s_{1i_1}s_{2i_2}\cdots s_{ni_n} \mid i_j \le j+1\}$ is a Gröbner–Shirshov normal form for S_{n+1} in the generators $s_i = (i, i+1)$ relative to the deg-lex ordering, where $s_{ji} = s_j s_{j-1}\cdots s_i$ for $j \ge i$ and $s_{jj+1} = 1$.

Take $\alpha \in S_{n+1}$ with the normal form $\overline{\alpha} = s_{1i_1}s_{2i_2}\cdots s_{ni_n} \in N$. Define the length of α as $|\overline{\alpha}| = l(s_{1i_1}s_{2i_2}\cdots s_{ni_n})$ and write $\alpha \perp \beta$ whenever $|\overline{\alpha\beta}| = |\overline{\alpha}| + |\overline{\beta}|$. Moreover, every $\overline{\alpha} \in N$ has a unique expression $\overline{\alpha} = s_{l_1 i_{l_1}} s_{l_2 i_{l_2}} \cdots s_{l_t i_{l_t}}$ with all $s_{l_j i_{l_j}} \ne 1$. The number t is called the *breadth* of α.

Now put

$$B'_{n+1} = gp\langle r(\overline{\alpha}),\ \alpha \in S_{n+1}\backslash\{1\} \mid r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}),\ \alpha \perp \beta\rangle,$$

where $r(\overline{\alpha})$ stands for a letter with index $\overline{\alpha}$.

Then for the braid group with n generators we have $B_{n+1} \cong B'_{n+1}$. Indeed, define

$$\theta\colon B_{n+1} \to B'_{n+1}, \quad \sigma_i \mapsto r(s_i),$$
$$\theta'\colon B'_{n+1} \to B_{n+1}, \quad r(\overline{\alpha}) \mapsto \overline{\alpha}|_{s_i \mapsto \sigma_i}.$$

These mappings are homomorphism satisfying $\theta\theta' = l_{B'_{n+1}}$ and $\theta'\theta = l_{B_{n+1}}$. Hence,

$$B_{n+1} = gp\langle r(\overline{\alpha}),\ \alpha \in S_{n+1}\backslash\{1\} \mid r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}),\ \alpha \perp \beta\rangle.$$

Put $X = \{r(\overline{\alpha}),\ \alpha \in S_{n+1}\backslash\{1\}\}$. These generators of B_{n+1} are called the Adjan–Thurston generators.

Then the positive braid semigroup generated by X is

$$B^{+}_{n+1} = \mathrm{sgp}\langle X \mid r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}),\ \alpha \perp \beta\rangle.$$

Assume that $s_1 < s_2 < \cdots < s_n$. Define $r(\overline{\alpha}) < r(\overline{\beta})$ if and only if $|\overline{\alpha}| > |\overline{\beta}|$ or $|\overline{\alpha}| = |\overline{\beta}|$ and $\overline{\alpha} <_{lex} \overline{\beta}$. Clearly, this is a well-ordering on X. We will use the deg-lex ordering on X^*.

Theorem 24 ([89]) *The Gröbner–Shirshov basis of B^{+}_{n+1} in the Adjan–Thurston generator X relative to the deg-lex ordering on X^* consists of the relations*

$$r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta}) \ \ \text{for } \alpha \perp \beta; \qquad r(\overline{\alpha})r(\overline{\beta\gamma}) = r(\overline{\alpha\beta})r(\overline{\gamma}) \ \ \text{for } \alpha \perp \beta \perp \gamma.$$

Theorem 25 ([89]) *The Gröbner–Shirshov basis of B_{n+1} in the Adjan–Thurston generator X with respect to the deg-lex ordering on X^* consists of the relations*

(1) $r(\overline{\alpha})r(\overline{\beta}) = r(\overline{\alpha\beta})$ *for* $\alpha \perp \beta$,
(2) $r(\overline{\alpha})r(\overline{\beta\gamma}) = r(\overline{\alpha\beta})r(\overline{\gamma})$ *for* $\alpha \perp \beta \perp \gamma$,
(3) $r(\overline{\alpha})\Delta^{\varepsilon} = \Delta^{\varepsilon} r(\overline{\alpha}')$ *for* $\overline{\alpha}' = \overline{\alpha}|_{s_i \mapsto s_{n+1-i}}$,
(4) $r(\overline{\alpha\beta})r(\overline{\gamma\mu}) = \Delta r(\overline{\alpha}')r(\overline{\mu})$ *for* $\alpha \perp \beta \perp \gamma \perp \mu$ *with* $r(\overline{\beta\gamma}) = \Delta$,
(5) $\Delta^{\varepsilon}\Delta^{-\varepsilon}$=*1*.

Corollary 13 (Adjan–Thurston) *The normal forms for B_{n+1} are $\Delta^k r(\overline{\alpha_1})\cdots r(\overline{\alpha_s})$ for $k\in\mathbb{Z}$, where $r(\overline{\alpha_1})\cdots r(\overline{\alpha_s})$ is minimal in the deg-lex ordering.*

3.2 Gröbner–Shirshov basis for the Chinese monoid

The Chinese monoid $CH(X,<)$ over a well-ordered set $(X,<)$ has the presentation $CH(X)=\mathrm{sgp}\langle X|S\rangle$, where $X=\{x_i|i\in I\}$ and S consists of the relations

$$x_ix_jx_k=x_ix_kx_j=x_jx_ix_k \quad \text{for } i>j>k,$$
$$x_ix_jx_j=x_jx_ix_j,\ x_ix_ix_j=x_ix_jx_i \quad \text{for } i>j.$$

Theorem 26 ([85]) *With the deg-lex ordering on X^*, the following relations* (1)–(5) *constitute a Gröbner–Shirshov basis of the Chinese monoid $CH(X)$:*

(1) $x_ix_jx_k-x_jx_ix_k$,
(2) $x_ix_kx_j-x_jx_ix_k$,
(3) $x_ix_jx_j-x_jx_ix_j$,
(4) $x_ix_ix_j-x_ix_jx_i$,
(5) $x_ix_jx_ix_k-x_ix_kx_ix_j$,

where $x_i,x_j,x_k\in X$ and $i>j>k$.

Denote by Λ the set consistsing of the words on X of the form $u_n=w_1w_2\cdots w_n$ with $n\geq 0$, where

$$\begin{aligned}
w_1&=x_1^{t_{11}}\\
w_2&=(x_2x_1)^{t_{21}}x_2^{t_{22}}\\
w_3&=(x_3x_1)^{t_{31}}(x_3x_2)^{t_{32}}x_3^{t_{33}}\\
&\cdots\\
w_n&=(x_nx_1)^{t_{n1}}(x_nx_2)^{t_{n2}}\cdots(x_nx_{n-1})^{t_{n(n-1)}}x_n^{t_{nn}}
\end{aligned}$$

for $x_i\in X$ with $x_1<x_2<\cdots<x_n$, and all exponents are non-negative.

Corollary 14 ([71]) *This Λ is a set of normal forms of elements of the Chinese monoid $CH(X)$.*

3.3 Gröbner–Shirshov basis for free inverse semigroup

Consider a semigroup S. An element $s\in S$ is called an inverse of $t\in S$ whenever $sts=s$ and $tst=t$. An inverse semigroup is a semigroup in which every element t has a unique inverse, denoted by t^{-1}.

Given a set X, put $X^{-1}=\{x^{-1}|x\in X\}$. On assuming that $X\cap X^{-1}=\varnothing$, denote $X\cup X^{-1}$ by Y. Define the formal inverses of the elements of Y^* as

$$1^{-1}=1,\ (x^{-1})^{-1}=x\ (x\in X),$$
$$(y_1y_2\cdots y_n)^{-1}=y_n^{-1}\cdots y_2^{-1}y_1^{-1}\ (y_1,\ y_2,\ldots,\ y_n\in Y).$$

It is well known that

$$\mathcal{FI}(X) = \mathrm{sgp}\langle Y \mid aa^{-1}a = a,\ aa^{-1}bb^{-1} = bb^{-1}aa^{-1},\ a, b \in Y^*\rangle$$

is the free inverse semigroup (with identity) generated by X.

Introduce the notions of a formal idempotent, a (prime) canonical idempotent, and an ordered (prime) canonical idempotent in Y^*. Assume that $<$ is a well-ordering on Y.

(i) The empty word 1 is an idempotent.
(ii) If h is an idempotent and $x \in Y$ then $x^{-1}hx$ is both an idempotent and a prime idempotent.
(iii) If $e_1, e_2, \ldots, e_m$, where $m > 1$, are prime idempotents then $e = e_1e_2\cdots e_m$ is an idempotent.
(iv) An idempotent $w \in Y^*$ is called *canonical* whenever w avoids subwords of the form $x^{-1}exfx^{-1}$, where $x \in Y$, both e and f are idempotents.
(v) A canonical idempotent $w \in Y^*$ is called *ordered* if every subword $e = e_1e_2\cdots e_m$ of w with $m > 2$ and e_i being idempotents satisfies $\mathrm{fir}(e_1) < \mathrm{fir}(e_2) < \cdots < \mathrm{fir}(e_m)$, where $\mathrm{fir}(u)$ is the first letter of $u \in Y^*$.

Theorem 27 ([44]) *Denote by S the subset of $k\langle Y\rangle$ consisting two kinds of polynomials:*

- *$ef - fe$, where e and f are ordered prime canonical idempotents with $ef > fe$;*
- *$x^{-1}e'xf'x^{-1} - f'x^{-1}e'$, where $x \in Y$, $x^{-1}e'x$, and $xf'x^{-1}$ are ordered prime canonical idempotents.*

Then, with the deg-lex ordering on Y^*, the set S is a Gröber–Shirshov basis of the free inverse semigroup $\mathrm{sgp}\langle Y|S\rangle$.

Theorem 28 ([44]) *The normal forms of elements of the free inverse semigroup* $\mathrm{sgp}\langle Y|S\rangle$ *are*

$$u_0e_1u_1\cdots e_mu_m \in Y^*,$$

where $m \geq 0$, $u_1, \ldots, u_{m-1} \neq 1$ and $u_0u_1\cdots u_m$ avoids subwords of the form yy^{-1} for $y \in Y$, while $e_1, \ldots, e_m$ are ordered canonical idempotents such that the first (respectively last) letter of e_i, for $1 \leq i \leq m$ is not equal to the first (respectively last) letter of u_i (respectively u_{i-1}).

The above normal form is analogous to the semi-normal forms of Poliakova and Schein [176], 2005.

3.4 Approaches to plactic monoids via Gröbner–Shirshov bases in row and column generators

Consider the set $X = \{x_1, \ldots, x_n\}$ of n elements with the ordering $x_1 < \cdots < x_n$. Schützenberger called $P_n = \mathrm{sgp}\langle X|T\rangle$ a plactic monoid (see also Lothaire [153], Chapter 5), where T consists of the Knuth relations

$$\begin{aligned} x_i x_k x_j &= x_k x_i x_j \quad \text{for } x_i \le x_j < x_k, \\ x_j x_i x_k &= x_j x_k x_i \quad \text{for } x_i < x_j \le x_k. \end{aligned}$$

A nondecreasing word $R \in X^*$ is called a row and a strictly decreasing word $C \in X^*$ is called a column; for example, $x_1x_1x_3x_5x_5x_5x_6$ is a row and $x_6x_4x_2x_1$ is a column.

For two rows $R, S \in A^*$ say that R dominates S whenever $|R| \le |S|$ and every letter of R is greater than the corresponding letter of S, where $|R|$ is the length of R.

A (semistandard) Young tableau on A (see [152]) is a word $w = R_1R_2\cdots R_t$ in U^* such that R_i dominates R_{i+1} for all $i = 1, \dots, t-1$. For example,

$$x_4x_5x_5x_6 \cdot x_2x_2x_3x_3x_5x_7 \cdot x_1x_1x_1x_2x_4x_4x_4$$

is a Young tableau.

Cain et al. [69] use the Schensted–Knuth normal form (the set of (semistandard) Young tableaux) to prove that the multiplication table of column words, $uv = u'v'$, forms a finite GS basis of the finitely generated plactic monoid. Here the Young tableaux $u'v'$ is the output of the column Schensted algorithm applied to uv, but $u'v'$ is not made explicit.

In this section we give new explicit formulas for the multiplication tables of row and column words. In addition, we give independent proofs that the resulting sets of relations are GS bases in row and column generators respectively. This yields two new approaches to plactic monoids via their GS bases.

3.4.1 Plactic monoids in the row generators

Consider the plactic monoid $P_n = \mathrm{sgp}\langle X|T\rangle$, where $X = \{1, 2, \dots, n\}$ with $1 < 2 < \cdots < n$. Denote by $\mathbb{N}$ the set of non-negative integers. It is convenient to express the rows $R \in X^*$ as $R = (r_1, r_2, \dots, r_n)$, where r_i for $i = 1, 2, \dots, n$ is the number of occurrences of the letter i. For example, $R = 111225 = (3, 2, 0, 0, 1, 0, \dots, 0)$.

Denote by U the set of all rows in X^* and order U^* as follows. Given $R = (r_1, r_2, \dots, r_n) \in U$, define the length $|R| = r_1 + \cdots + r_n$ of R in X^*.

Firstly, order U: for every $R, S \in U$, put $R < S$ if and only if $|R| < |S|$ or $|R| = |S|$ and $(r_1, r_2, \dots, r_n) > (s_1, s_2, \dots, s_n)$ lexicographically. Clearly, this is a well-ordering on U. Then, use the deg-lex ordering on U^*.

Lemma 5 ([29]) *Take* $\Phi = (\phi_1, \dots, \phi_n) \in U$. *For* $1 \le p \le n$ *put*

$$\Phi_p = \sum_{i=1}^{p} \phi_i,$$

where ϕ_i *(*w_i*,* z_i*,* w_i'*, and* z_i'*, see below) stands for a lowercase symbol, and* Φ_p *(* W_p*,* Z_p*,* W_p'*, and* Z_p'*, see below) for the corresponding uppercase symbol. Take*

$W = (w_1, w_2, \ldots, w_n)$ and $Z = (z_1, z_2, \ldots, z_n)$ in U. Put $W' = (w'_1, w'_2, \ldots, w'_n)$ and $Z' = (z'_1, z'_2, \ldots, z'_n)$, where

$$w'_1 = 0, \quad w'_p = \min(Z_{p-1} - W'_{p-1}, w_p), \quad z'_q = w_q + z_q - w'_q \tag{*}$$

for $n \geq p \geq 2$ and $n \geq q \geq 1$.

Then $W \cdot Z = W' \cdot Z'$ in $P_n = \mathrm{sgp}\langle X|T\rangle$ and $W' \cdot Z'$ is a Young tableau on X, which could have only one row, that is, $Z' = (0, 0, \ldots, 0)$. Moreover,

$$P_n = \mathrm{sgp}\langle X|T\rangle \cong \mathrm{sgp}\langle U|\Gamma\rangle,$$

where $\Gamma = \{W \cdot Z = W' \cdot Z' \mid W, Z \in V\}$.

We should emphasize that $(*)$ gives explicitly the product of two rows obtained by the Schensted row algorithm.

Jointly with our students Weiping Chen and Jing Li we proved [29], independently of Knuth's normal form theorem [137], that Γ is a GS basis of the plactic monoid algebra in row generators with respect to the deg-lex ordering. In particular, this yields a new proof of Knuth's theorem.

3.4.2 Plactic monoids in the column generators

Consider the plactic monoid $P_n = \mathrm{sgp}\langle X|T\rangle$, where $X = \{1, 2, \ldots, n\}$ with $1 < 2 < \cdots < n$. Every Young tableaux is a product of columns. For example,

$$4{,}556 \cdot 223{,}357 \cdot 1{,}112{,}444 = (421)(521)(531)(632)(54)(74)(4)$$

is a Young tableau.

Given a column $C \in X^*$, denote by c_i the number of occurrences of the letter i in C. Then $c_i \in \{0, 1\}$ for $i = 1, 2, \ldots, n$. We write $C = (c_1; c_2; \ldots; c_n)$. For example, $C = 6{,}421 = (1; 1; 0; 1; 0; 1; 0; \ldots; 0)$.

Put $V = \{C \mid C \text{ is a column in } X^*\}$. For $R = (r_1; r_2; \ldots; r_n) \in V$ define $\mathrm{wt}(R) = (|R|, r_1, \ldots, r_n)$. Order V as follows: for $R, S \in V$, put $R < S$ if and only if $\mathrm{wt}(R) > \mathrm{wt}(S)$ lexicographically. Then, use the deg-lex ordering on V^*.

For $\Phi = (\phi_1; \ldots; \phi_n) \in V$, put $\Phi_p = \sum_{i=1}^{p} \phi_i$, $1 \leq p \leq n$, where ϕ stands for some lowercase symbol defined above and Φ stands for the corresponding uppercase symbol.

Lemma 6 ([29]) *Take $W = (w_1; w_2; \ldots; w_n)$, $Z = (z_1; z_2; \ldots; z_n) \in V$. Define $W' = (w'_1; w'_2; \ldots; w'_n)$ and $Z' = (z'_1; z'_2; \ldots; z'_n)$, where*

$$z'_1 = \min(w_1, z_1), \quad z'_p = \min(W_p - Z'_{p-1}, z_p), \quad w'_q = w_q + z_q - z'_q \tag{**}$$

for $n \geq p \geq 2$ and $n \geq q \geq 1$. Then $W', Z' \in V$ and $W \cdot Z = W' \cdot Z'$ in $P_n = \mathrm{sgp}\langle X|T\rangle$, and $W' \cdot Z'$ is a Young tableau on X. Moreover,

$$P_n = \mathrm{sgp}\langle X|T\rangle \cong \mathrm{sgp}\langle V|\Lambda\rangle,$$

where $\Lambda = \{W \cdot Z = W' \cdot Z' \mid W, Z \in V\}$.

Equation $(**)$ gives explicitly the product of two columns obtained by the Schensted column algorithm.

Jointly with our students Weiping Chen and Jing Li we proved [29], independently of Knuth's normal form theorem [137], that Λ is a GS basis of the plactic monoid algebra in column generators with respect to the deg-lex ordering. In particular, this yields another new proof of Knuth's theorem. Previously Cain, Gray, and Malheiro [69] established the same result using Knuth's theorem, and they did not find Λ explicitly.

Remark All results of [29] are valid for every plactic monoid, not necessarily finitely generated.

4 Gröbner–Shirshov bases for Lie algebras

In this section we first give a different approach to the LS basis and the Hall basis of a free Lie algebra by using Shirshov's CD-lemma for anti-commutative algebras. Then, using the LS basis, we construct the classical theory of GS bases for Lie algebras over a field. Finally, we mention GS bases for Lie algebras over a commutative algebra and give some applications.

4.1 Lyndon–Shirshov basis and Lyndon–Shirshov words in anti-commutative algebras

A linear space A equipped with a bilinear product $x \cdot y$ is called an anti-commutative algebra if it satisfies the identity $x^2 = 0$, and so $x \cdot y = -y \cdot x$ for every $x, y \in A$.

Take a well-ordered set X and denote by X^{**} the set of all non-associative words. Define three orderings $\succ_{lex}$, $>_{deg-lex}$, and $>_{n-deg-lex}$ (non-associative deg-lex) on X^{**}. For $(u), (v) \in X^{**}$ put

- $(u) = ((u_1)(u_2)) \succ_{lex} (v) = ((v_1)(v_2))$ (here (u_2) or (v_2) is empty when $|(u)| = 1$ or $|(v)| = 1$) iff one of the following holds:
  - (a) $u_1u_2 > v_1v_2$ in the lex ordering;
  - (b) $u_1u_2 = v_1v_2$ and $(u_1) \succ_{lex} (v_1)$;
  - (c) $u_1u_2 = v_1v_2$, $(u_1) = (v_1)$, and $(u_2) \succ_{lex} (v_2)$;
- $(u) = ((u_1)(u_2)) >_{deg-lex} (v) = ((v_1)(v_2))$ iff one of the following holds:
  - (a) $u_1u_2 > v_1v_2$ in the deg-lex ordering;
  - (b) $u_1u_2 = v_1v_2$ and $(u_1) >_{deg-lex} (v_1)$;
  - (c) $u_1u_2 = v_1v_2$, $(u_1) = (v_1)$, and $(u_2) >_{deg-lex} (v_2)$;
- $(u) >_{n-deg-lex} (v)$ iff one of the following holds:
  - (a) $|(u)| > |(v)|$;
  - (b) if $|(u)| = |(v)|$, $(u) = ((u_1)(u_2))$, and $(v) = ((v_1)(v_2))$ then $(u_1) >_{n-deg-lex} (v_1)$ or $((u_1) = (v_1)$ and $(u_2) >_{n-deg-lex} (v_2))$.

Define regular words $(u) \in X^{**}$ by induction on $|(u)|$:

(i) $x_i \in X$ is a regular word.

(ii) $(u) = ((u_1)(u_2))$ is regular if both (u_1) and (u_2) are regular and $(u_1) \succ_{lex} (u_2)$.

Denote (u) by $[u]$ whenever (u) is regular.

The set $N(X)$ of all regular words on X constitutes a linear basis of the free anti-commutative algebra $AC(X)$ on X.

The following result gives an alternative approach to the definition of LS words as the radicals of associative supports u of the normal words $[u]$.

Theorem 29 ([37]) *Suppose that $[u]$ is a regular word of the anti-commutative algebra $AC(X)$. Then $u = v^m$, where v is a Lyndon–Shirshov word in X and $m \geq 1$. Moreover, the set of associative supports of the words in $N(X)$ includes the set of all Lyndon–Shirshov words in X.*

Fix an ordering $>_{deg-lex}$ on X^{**} and choose monic polynomials f and g in $AC(X)$. If there exist $a, b \in X^*$ such that $[w] = [\bar{f}] = [a[\bar{g}]b]$ then the inclusion composition of f and g is defined as $(f, g)_{[w]} = f - [a[g]b]$.

A monic subset S of $AC(X)$ is called a GS basis in $AC(X)$ if every inclusion composition $(f, g)_{[w]}$ in S is trivial modulo $(S, [w])$.

Theorem 30 (Shirshov's CD-lemma for anti-commutative algebras, cf. [206]) *Consider a nonempty set $S \subset AC(X)$ of monic polynomials with the ordering $>_{deg-lex}$ on X^{**}. The following statements are equivalent:*

(i) *The set S is a Gröbner–Shirshov basis in $AC(X)$.*
(ii) *If $f \in Id(S)$ then $[\bar{f}] = [a[\bar{s}]b]$ for some $s \in S$ and $a, b \in X^*$, where $[asb]$ is a normal S-word.*
(iii) *The set*

$$Irr(S) = \{[u] \in N(X) | \, [u] \neq [a[\bar{s}]b] \, a, b \in X^*, \; s \in S \\ and \, [asb] is \, a \, normal \, S{-}word\}$$

is a linear basis of the algebra $AC(X|S) = AC(X)/Id(S)$.

Define the subset S_1 the free anti-commutative algebra $AC(X)$ as

$$S_1 = \{([u][v])[w] - ([u][w])[v] - [u]([v][w]) \mid \\ [u], [v], [w] \in N(X) \text{ and } [u] \succ_{lex} [v] \succ_{lex} [w]\}.$$

It is easy to prove that the free Lie algebra admits a presentation as an anti-commutative algebra: $Lie(X) = AC(X)/Id(S_1)$.

The next result gives an alternating approach to the definition of the LS basis of a free Lie algebra $Lie(X)$ as a set of irreducible non-associative words for an anti-commutative GS basis in $AC(X)$.

Theorem 31 ([37]) *Under the ordering $>_{deg-lex}$, the subset S_1 of $AC(X)$ is an anti-commutative Gröbner–Shirshov basis in $AC(X)$. Then $Irr(S_1)$ is the set of all non-associative LS words in X. So, the LS monomials constitute a linear basis of the free Lie algebra $Lie(X)$.*

Theorem 32 ([34]) *Define S_2 by analogy with S_1, but using $>_{n-deg-lex}$ instead of $\succ_{lex}$. Then with the ordering $>_{n-deg-lex}$ the subset S_2 of $AC(X)$ is also an anti-commutative GS basis. The set $Irr(S_2)$ amounts to the set of all Hall words in X and forms a linear basis of a free Lie algebra $Lie(X)$.*

4.2 Composition-Diamond lemma for Lie algebras over a field

We start with some concepts and results from the literature concerning the theory of GS bases for the free Lie algebra $Lie(X)$ generated by X over a field k.

Take a well-ordered set $X = \{x_i | i \in I\}$ with $x_i > x_t$ whenever $i > t$, for all $i, t \in I$. Given $u = x_{i_1} x_{i_2} \cdots x_{i_m} \in X^*$, define the length (or degree) of u to be m and denote it by $|u| = m$ or $deg(u) = m$, put $\mathrm{fir}(u) = x_{i_1}$, and introduce

$$x_\beta = \min(u) = \min\{x_{i_1}, x_{i_2}, \cdots, x_{i_m}\},$$
$$X'(u) = \{x_i^j = x_i \underbrace{x_\beta \cdots x_\beta}_{j} | i > \beta, j \geq 0\}.$$

Order the new alphabet $X'(u)$ as follows:

$$x_{i_1}^{j_1} > x_{i_2}^{j_2} \Leftrightarrow i_1 > i_2 \text{ or } i_1 = i_2 \text{ and } j_2 > j_1.$$

Assuming that

$$u = x_{r_1} \underbrace{x_\beta \cdots x_\beta}_{m_1} \cdots x_{r_t} \underbrace{x_\beta \cdots x_\beta}_{m_t},$$

where $r_i > \beta$, define the Shirshov elimination

$$u' = x_{r_1}^{m_1} \cdots x_{r_t}^{m_t} \in (X'(u))^*.$$

We use two linear orderings on X^*:

(i) the lex ordering (or lex-antideg ordering): $1 \succ v$ if $v \neq 1$ and, by induction, if $u = x_i u_1$ and $v = x_j v_1$ then $u \succ v$ if and only if $x_i > x_j$ or $x_i = x_j$ and $u_1 \succ v_1$;
(ii) the deg-lex ordering: $u > v$ if $|u| > |v|$ or $|u| = |v|$ and $u \succ v$.

Remark In commutative algebras, the lex ordering is understood to be the lex-deg ordering with the condition $v > 1$ for $v \neq 1$.

We cite some useful properties of ALSWs and NLSWs (see below) following Shirshov [203,204,207], see also [209]. Property (**X**) was given by Shirshov [204] and Chen et al. [72]. Property (**VIII**) was implicitly used in Shirshov [207], see also Chibrikov [94].

We regard $Lie(X)$ as the Lie subalgebra of the free associative algebra $k\langle X\rangle$ generated by X with the Lie bracket $[u, v] = uv - vu$. Below we prove that $Lie(X)$ is the free Lie algebra generated by X for every commutative ring k (Shirshov [203]). For a field, this follows from the PBW theorem because the free Lie algebra $Lie(X) = Lie(X|\emptyset)$ has the universal enveloping associative algebra $k\langle X\rangle = k\langle X|\emptyset\rangle$.

Given $f \in k\langle X\rangle$, denote by $\bar{f}$ the leading word of f with respect to the deg-lex ordering and write $f = \alpha_{\bar{f}}\bar{f} - r_f$ with $\alpha_{\bar{f}} \in k$.

Definition 3 ([156,203]) Refer to $w \in X^*\backslash\{1\}$ as an associative Lyndon–Shirshov word, or ALSW for short, whenever

$$(\forall u, v \in X^*, u, v \neq 1)\; w = uv \Rightarrow w > vu.$$

Denote the set of all ALSWs on X by $ALSW(X)$.

Associative Lyndon–Shirshov words enjoy the following properties (Lyndon [156], Chen et al. [72], Shirshov [203,204]).

(**I**) Put $x_\beta = \min(uv)$. If $\mathrm{fir}(u) \neq x_\beta$ and $\mathrm{fir}(v) \neq x_\beta$ then

$$u \succ v \text{ (in the lex ordering on } X^*) \Leftrightarrow u' \succ v' \text{ (in the lex ordering on } (X'_{uv})^*).$$

(**II**) (Shirshov's key property of ALSWs) A word u is an ALSW in X^* if and only if u' is an ALSW in $(X'(u))^*$.

Properties (**I**) and (**II**) enable us to prove the properties of ALSWs and NLSWs (see below) by induction on length.

(**III**) (down-to-up bracketing) $u \in ALSW(X) \Leftrightarrow (\exists k)\; |u^{(k)}|_{(X(u))^{(k)}} = 1$, where $u^{(k)} = (u')^{(k-1)}$ and $(X(u))^{(k)} = (X'(u))^{(k-1)}$. In the process $u \to u' \to u'' \to \cdots$ we use the algorithm of joining the minimal letters of $u, u' \dots$ to the previous words.

(**IV**) If $u, v \in ALSW(X)$ then $uv \in ALSW(X) \Leftrightarrow u \succ v$.

(**V**) $w \in ALSW(X) \Leftrightarrow$ (for every $u, v \in X^*\backslash\{1\}$ and $w = uv \Rightarrow w \succ v$).

(**VI**) If $w \in ALSW(X)$ then an arbitrary proper prefix of w cannot be a suffix of w and $wx_\beta \in ALSW(X)$ if $x_\beta = \min(w)$.

(**VII**) (Shirshov's factorization theorem) Every associative word w can be uniquely represented as $w = c_1c_2\dots c_n$, where $c_1, \dots, c_n \in ALSW(X)$ and $c_1 \preceq c_2 \preceq \cdots \preceq c_n$.

Actually, if we apply to w the algorithm of joining the minimal letter to the previous word using the Lie product, $w \to w' \to w'' \to \cdots$, then after finitely many steps we obtain $w^{(k)} = [c_1][c_2]\dots[c_n]$, with $c_1 \preceq c_2 \preceq \cdots \preceq c_n$, and $w = c_1c_2\dots c_n$ would be the required factorization (see an example in the Introduction).

(**VIII**) If an associative word w is represented as in (**VII**) and v is a LS subword of w then v is a subword of one of the words $c_1, c_2, \dots, c_n$.

(**IX**) If u_1u_2 and u_2u_3 are ALSWs then so is $u_1u_2u_3$ provided that $u_2 \neq 1$.

(**X**) If $w = uv$ is an ALSW and v is its longest proper ALSW ending, then u is an ALSW as well (Chen et al. [72], Shirshov [204]).

Definition 4 (*down-to-up bracketing of ALSW, Shirshov* [203]) For an ALSW w, there is the down-to-up bracketing $w \to w' \to w'' \to \cdots \to w^{(k)} = [w]$, where each time we join the minimal letter of the previous word using Lie multiplication. To be more precise, we use the induction $[w] = [w']_{x_i^j \mapsto [[x_i x_\beta]\cdots x_\beta]}$.

Definition 5 (*up-to-down bracketing of ALSW, Shirshov* [204], Chen et al. [72]) For an ALSW w, we define the up-to-down Lie bracketing $[[w]]$ by the induction $[[w]] = [[[u]][[v]]]$, where $w = uv$ as in (**X**).

(**XI**) If $w \in ALSW(X)$ then $[w] = [[w]]$.

(**XII**) Shirshov's definition of a NLSW (non-associative LS word) (w) below is the same as $[w]$ and $[[w]]$; that is, $(w) = [w] = [[w]]$. Chen et al. [72] used $[[w]]$.

Definition 6 (*Shirshov* [203]) A non-associative word (w) in X is a NLSW if

(i) w is an ALSW;
(ii) if $(w) = ((u)(v))$ then both (u) and (v) are NLSWs (then (**IV**) implies that $u \succ v$);
(iii) if $(w) = (((u_1)(u_2))(v))$ then $u_2 \preceq v$.

Denote the set of all NLSWs on X by $NLSW(X)$.

(**XIII**) If $u \in ALSW(X)$ and $[u] \in NLSW(X)$ then $\overline{[u]} = u$ in $k\langle X\rangle$.

(**XIV**) The set $NLSW(X)$ is linearly independent in $Lie(X) \subset k\langle X\rangle$ for every commutative ring k.

(**XV**) $NLSW(X)$ is a set of linear generators in every Lie algebra generated by X over an arbitrary commutative ring k.

(**XVI**) $Lie(X) \subset k\langle X\rangle$ is the free Lie algebra over the commutative ring k with the k-basis $NLSW(X)$.

(**XVII**) (Shirshov's special bracketing [203]) Consider $w = aub$ with $w, u \in ALSW(X)$. Then

(i) $[w] = [a[uc]d]$, where $b = cd$ and possibly $c = 1$.
(ii) Express c in the form $c = c_1c_2\dots c_n$, where $c_1, \dots, c_n \in ALSW(X)$ and $c_1 \preceq c_2 \preceq \dots \preceq c_n$. Replacing $[uc]$ by $[\dots[[u][c_1]]\dots[c_n]]$, we obtain the word

$$[w]_u = [a[\dots[[[u][c_1]][c_2]]\dots[c_n]]d]$$

which is called the Shirshov special bracketing of w relative to u.
(iii) $[w]_u = a[u]b + \sum_i \alpha_i a_i[u]b_i$ in $k\langle X\rangle$ with $\alpha_i \in k$ and $a_i, b_i \in X^*$ satisfying $a_iub_i < aub$, and hence $\overline{[w]_u} = w$.

Outline of the proof. Put $x_\beta = \min(w)$. Then $w' = a'(ux_\beta^m)'(b_1)'$ in $(X(w)')^*$, where $b = x_\beta^m b_1$ and ux_β^m is an ALSW. Claim (i) follows from (**II**) by induction on length. The same applies to claim (iii).

(**XVIII**) (Shirshov's Lie elimination of the leading word) Take two monic Lie polynomials f and s with $\bar{f} = a\bar{s}b$ for some $a, b \in X^*$. Then $f_1 = f - [asb]_{\bar{s}}$ is a Lie polynomial with smaller leading word, and so $\bar{f_1} < \bar{f}$.

(**XIX**) (Shirshov's double special bracketing) Assume that $w = aubvc$ with $w, u, v \in ALSW(X)$. Then there exists a bracketing $[w]_{u,v}$ such that $[w]_{u,v} = [a[u]b[v]c]_{u,v}$ and $\overline{[w]_{u,v}} = w$.

More precisely, $[w]_{u,v} = [a[up]_u q[vr]_v s]$ if $[w] = [a[up]q[vr]s]$, and

$$[w]_{u,v} = [a[\dots[\dots[[u][c_1]]\dots[c_i]_v]\dots[c_n]]p]$$

if $[w] = [a[uc]p]$, where $c = c_1 \dots c_n$ is the Shirshov factorization of c and v is a subword of c_i. In both cases $[w]_{u,v} = a[u]b[v]d + \sum \alpha_i a_i[u]b_i[v]d_i$ in $k\langle X\rangle$, where $a_i u b_i v d_i < w$.

(**XX**) (Shirshov's algorithm for recognizing Lie polynomials, cf. the Dynkin–Specht–Wever and Friedrich algorithms). Take $s \in Lie(X) \subset k\langle X\rangle$. Then $\bar{s}$ is an ALSW and $s_1 = s - \alpha_{\bar{s}}[\bar{s}]$ is a Lie polynomial with a smaller maximal word (in the deg-lex ordering), $\bar{s}_1 < \bar{s}$, where $s = \alpha_{\bar{s}}[\bar{s}] + \dots$. Then $s_2 = s_1 - \alpha_{\bar{s}_1}[\bar{s}_1]$, $\overline{s_2} < \overline{s_1}$. Consequently, $s \in Lie(X)$ if and only if after finitely many steps we obtain

$$s_{m+1} = s - \alpha_{\bar{s}}[\bar{s}] - \alpha_{\bar{s}_1}[\bar{s}_1] - \cdots - \alpha_{\bar{s}_m}[\bar{s}_m] = 0.$$

Here k can be an arbitrary commutative ring.

Definition 7 Consider $S \subset Lie(X)$ with all $s \in S$ monic. Take $a, b \in X^*$ and $s \in S$. If $a\bar{s}b$ is an ALSW then we call $[asb]_{\bar{s}} = [a\bar{s}b]_{\bar{s}}|_{[\bar{s}]\mapsto s}$ a special normal S-word (or a special normal s-word), where $[a\bar{s}b]_{\bar{s}}$ is defined in (**XVII**) (ii). A Lie S-word (asb) is called a normal S-word whenever $\overline{(asb)} = a\bar{s}b$. Every special normal s-word is a normal s-word by (**XVII**) (iii).

For $f,\ g \in S$ there are two kinds of Lie compositions:

(i) If $w = \bar{f} = a\bar{g}b$ for some $a, b \in X^*$ then the polynomial $\langle f, g\rangle_w = f - [agb]_{\bar{g}}$ is called the inclusion composition of f and g with respect to w.
(ii) If w is a word satisfying $w = \bar{f}b = a\bar{g}$ for some $a, b \in X^*$ with $deg(\bar{f}) + deg(\bar{g}) > deg(w)$ then the polynomial $\langle f, g\rangle_w = [fb]_{\bar{f}} - [ag]_{\bar{g}}$ is called the intersection composition of f and g with respect to w, and w is an ALSW by (**IX**).

Given a Lie polynomial h and $w \in X^*$, say that h is trivial modulo (S, w) and write $h \equiv_{Lie} 0\, mod(S, w)$ whenever $h = \sum_i \alpha_i(a_i s_i b_i)$, where each $\alpha_i \in k$, $(a_i s_i b_i)$ is a normal S-word and $a_i\overline{s_i}b_i < w$.

A set S is called a GS basis in $Lie(X)$ if every composition $(f, g)_w$ of polynomials f and g in S is trivial modulo S and w.

(**XXI**) If $s \in Lie(X)$ is monic and (asb) is a normal S-word then $(asb) = asb + \sum_i \alpha_i a_i s b_i$, where $a_i\bar{s}b_i < a\bar{s}b$.

A proof of (**XXI**) follows from the CD-lemma for associative algebras since $\{s\}$ is an associative GS basis by (**IV**).

(**XXII**) Given two monic Lie polynomials f and g, we have

$$\langle f, g\rangle_w - (f, g)_w \equiv_{ass} 0 \quad \mathrm{mod}\ (\{f, g\}, w).$$

Proof If $\langle f, g\rangle_w$ and $(f, g)_w$ are intersection compositions, where $w = \bar{f}b = a\bar{g}$, then (**XIII**) and (**XVII**) yield

$$\langle f, g\rangle_w = [fb]_{\bar{f}} - [ag]_{\bar{g}} = fb + \sum_{I_1} \alpha_i a_i f b_i - ag - \sum_{I_2} \beta_j a_j g b_j,$$

where $a_i\bar{f}b_i, a_j\bar{g}b_j < \bar{f}b = a\bar{g} = w$. Hence,

$$\langle f, g\rangle_w - (f, g)_w \equiv_{ass} 0 \mod (\{f, g\}, w).$$

In the case of inclusion compositions we arrive at the same conclusion. □

Theorem 33 (PBW Theorem in Shirshov's form [56,57], see Theorem 17) *A nonempty set $S \subset Lie(X) \subset k\langle X\rangle$ of monic Lie polynomials is a Gröbner–Shirshov basis in $Lie(X)$ if and only if S is a Gröbner–Shirshov basis in $k\langle X\rangle$.*

Proof Observe that, by definition, for any $f, g \in S$ the composition lies in $Lie(X)$ if and only if it lies $k\langle X\rangle$.

Assume that S is a GS basis in $Lie(X)$. Then we can express every composition $\langle f, g\rangle_w$ as $\langle f, g\rangle_w = \sum_{I_1} \alpha_i(a_i s_i b_i)$, where $(a_i s_i b_i)$ are normal S-words and $a_i\bar{s_i}b_i < w$. By (**XXI**), we have $\langle f, g\rangle_w = \sum_{I_2} \beta_j c_j s_j d_j$ with $c_j\bar{s_j}d_j < w$. Therefore, (**XXII**) yields $(f, g)_w \equiv_{ass} 0 \mod (S, w)$. Thus, S is a GS basis in $k\langle X\rangle$.

Conversely, assume that S is a GS basis in $k\langle X\rangle$. Then the CD-lemma for associative algebras implies that $\overline{\langle f, g\rangle_w} = a\bar{s}b < w$ for some $a, b \in X^*$ and $s \in S$. Then $h = \langle f, g\rangle_w - \alpha[asb]_{\bar{s}} \in Id_{ass}(S)$ is a Lie polynomial and $\bar{h} < \overline{\langle f, g\rangle_w}$. Induction on $\overline{\langle f, g\rangle_w}$ yields $\langle f, g\rangle_w \equiv_{Lie} 0 \mod (S, w)$. □

Theorem 34 (The CD-lemma for Lie algebras over a field) *Consider a nonempty set $S \subset Lie(X) \subset k\langle X\rangle$ of monic Lie polynomials and denote by $Id(S)$ the ideal of $Lie(X)$ generated by S. The following statements are equivalent:*

(i) *The set S is a Gröbner–Shirshov basis in $Lie(X)$.*
(ii) *If $f \in Id(S)$ then $\bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$.*
(iii) *The set*

$$Irr(S) = \{[u] \in NLSW(X) \mid u \neq a\bar{s}b,\ s \in S,\ a, b \in X^*\}$$

is a linear basis for $Lie(X|S)$.

Proof (i)⇒(ii). Denote by $Id_{ass}(S)$ and $Id_{Lie}(S)$ the ideals of $k\langle X\rangle$ and $Lie(X)$ generated by S respectively. Since $Id_{Lie}(S) \subseteq Id_{ass}(S)$, Theorem 33 and the CD-lemma for associative algebras imply the claim.

(ii)⇒(iii). Suppose that $\sum \alpha_i[u_i] = 0$ in $Lie(X|S)$ with $[u_i] \in Irr(S)$ and $u_1 > u_2 > \cdots$, that is, $\sum \alpha_i[u_i] \in Id_{Lie}(S)$. Then all α_i must vanish. Otherwise we may assume that $\alpha_1 \neq 0$. Then $\overline{\sum \alpha_i[u_i]} = u_1$ and (ii) implies that $[u_1] \notin Irr(S)$, which is a contradiction. On the other hand, by the next property (**XXIII**), $Irr(S)$ generates $Lie(X|S)$ as a linear space.

(iii)⇒(i). This part follows from (**XXIII**). □

The next property is similar to Lemma 2.

(**XXIII**) Given $S \subset Lie(X)$, we can express every $f \in Lie(X)$ as

$$f = \sum \alpha_i[u_i] + \sum \beta_j[a_j s_j b_j]_{\bar{s_j}}$$

with $\alpha_i, \beta_j \in k$, $[u_i] \in Irr(S)$ satisfying $\overline{[u_i]} \leq \bar{f}$, and $[a_j s_j b_j]_{\bar{s_j}}$ are special normal S-word satisfying $\overline{[a_j s_j b_j]_{\bar{s_j}}} \leq \bar{f}$.

(**XXIV**) Given a normal s-word (asb), take $w = a\bar{s}b$. Then $(asb) \equiv [asb]_{\bar{s}} \mod (s, w)$. It follows that $h \equiv_{Lie} 0 \mod (S, w)$ is equivalent to $h = \sum_i \alpha_i [a_i s_i b_i]_{\bar{s_i}}$, where $[a_i s_i b_i]_{\bar{s_i}}$ are special normal S-words with $a_i \bar{s_i} b_i < w$.

Proof Observe that for every monic Lie polynomial s, the set $\{s\}$ is a GS basis in $Lie(X)$. Then (**XVIII**) and the CD-lemma for Lie algebras yield $(asb) \equiv [asb]_{\bar{s}} \mod (s, w)$. □

Summary of the proof of Theorem 34.

Given two ALSWs u and v, define the ALSW-lcm(u, v) (or lcm(u, v) for short) as follows:

$$
\begin{aligned}
w = \mathrm{lcm}(u, v) \in \{&aucvb \text{ (an ALSW)}, a, b, c \in X^* \text{ (a trivial lcm)};\\
&u = avb, a, b \in X^* \text{ (an inclusion lcm)};\\
&ub = av, a, b \in X^*, deg(ub) < deg(u) + deg(v) \text{ (an intersection lcm)}\}.
\end{aligned}
$$

Denote by $[w]_{u,v}$ the Shirshov double special bracketing of w in the case that w is a trivial lcm(u, v), by $[w]_u$ and $[w]_v$ the Shrishov special bracketings of w if w is an inclusion or intersection lcm respectively. Then we can define a general Lie composition for monic Lie polynomials f and g with $\bar{f} = u$ and $\bar{g} = v$ as

$$(f, g)_w = [w]_{u,v}|_{[u] \mapsto f} - [w]_{u,v}|_{[v] \mapsto g}$$

if w is a trivial lcm(u, v) (it is $0 \mod (\{f, g\}, w)$), and

$$(f, g)_w = [w]_u|_{[u] \mapsto f} - [w]_v|_{[v] \mapsto g}$$

if w is an inclusion or intersection lcm(u, v).

If $S \subset Lie(X) \subset k\langle X\rangle$ is a Lie GS basis then S is an associative GS basis. This follows from property (**XVII**) (iii) and justifies the claim (i)⇒(ii) of Theorem 34.

Shirshov's original proof of (i)⇒(ii) in Theorem 34, (see [207,209]), rests on an analogue of Lemma 1 for Lie algebras.

Lemma 7 ([207,209]) *If* $(a_1 s_1 b_1), (a_2 s_2 b_2)$ *are normal* S*-words with equal leading associative words,* $w = a_1 \bar{s_1} b_1 = a_2 \bar{s_2} b_2$*, then they are equal* $\mod (S, w)$*, that is,* $(a_1 s_1 b_1) - (a_2 s_2 b_2) \equiv 0 \mod (S, w)$.

Outline of the proof. We have $w_1 = cwd$ and $w = \mathrm{lcm}(\bar{s_1}, \bar{s_2})$. Shirshov's (double) special bracketing lemma yields

$$[w_1]_w = [c[[w]d_1]d_2] = c[w]d + \sum \alpha_i a_i [w] b_i$$

with $a_i w b_i < w_1$. The ALSW w includes $u = \bar{s_1}$ and $v = \bar{s_2}$ as subwords, and so there is a bracketing $\{w\} \in \{[w]_{u,v}, [w]_u, [w]_v\}$ such that

$$[a_1 s_1 b_1] = [c\{w\}|_{[u] \mapsto s_1} d], \quad [a_2 s_2 b_2] = [c\{w\}|_{[v] \mapsto s_2} d]$$

are normal s_1- and s_2- words with the same leading associative word w_1. Then

$$[a_1 s_1 b_1] - [a_2 s_2 b_2] = [c(s_1, s_2)_w d] \equiv 0 \mod (S, w_1).$$

Now it is enough to prove that two normal Lie s-words with the same leading associative words, say w_1, are equal $\mod (s, w_1)$:

$$f = (asb) - [asb] \equiv_{Lie} 0 \mod (s, w_1) \quad \text{provided that } \bar{f} < w_1.$$

Since $f \in Id_{ass}(s)$, we have $\bar{f} = c_1 \bar{s} d_1$ by the CD-lemma for associative algebras with one Lie polynomial relation s. Then $f - \alpha[c_1 s d_1]_{\bar{s}}$ is a Lie polynomial with the leading associative word smaller than w_1. Induction on w_1 finishes the proof.

4.2.1 Gröbner–Shirshov basis for the Drinfeld–Kohno Lie algebra

In this section we give a GS basis for the Drinfeld–Kohno Lie algebra $\mathbf{L}_n$.

Definition 8 Fix an integer $n > 2$. The Drinfeld–Kohno Lie algebra $\mathbf{L}_n$ over $\mathbb{Z}$ is defined by generators $t_{ij} = t_{ji}$ for distinct indices $1 \leq i, j \leq n-1$ satisfying the relations $[t_{ij} t_{kl}] = 0$ and $[t_{ij}(t_{ik} + t_{jk})] = 0$ for distinct i, j, k, and l.

Therefore, we have the presentation $\mathbf{L}_n = Lie_{\mathbb{Z}}(T|S)$, where $T = \{t_{ij} | \, 1 \leq i < j \leq n-1\}$ and S consists of the following relations:

$$[t_{ij} t_{kl}] = 0 \quad \text{if } k < i < j, \ k < l, \ l \neq i, \ j; \tag{18}$$

$$[t_{jk} t_{ij}] + [t_{ik} t_{ij}] = 0 \quad \text{if } i < j < k; \tag{19}$$

$$[t_{jk} t_{ik}] - [t_{ik} t_{ij}] = 0 \quad \text{if} i < j < k. \tag{20}$$

Order T by setting $t_{ij} < t_{kl}$ if either $i < k$ or $i = k$ and $j < l$. Let $<$ be the deg-lex ordering on T^*.

Theorem 35 ([80]) *With $S = \{(18), (19), (20)\}$ as before and the deg-lex ordering $<$ on T^*, the set S is a Gröbner–Shirshov basis of $\mathbf{L}_n$.*

Corollary 15 *The Drinfeld–Kohno Lie algebra $\mathbf{L}_n$ is a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis $\cup_{i=1}^{n-2} NLSW(T_i)$, where $T_i = \{t_{ij} \mid i < j \leq n-1\}$ for $i = 1, \ldots, n-2$.*

Corollary 16 ([100]) *The Drinfeld–Kohno Lie algebra $\mathbf{L}_n$ is an iterated semidirect product of free Lie algebras A_i generated by $T_i = \{t_{ij} \mid i < j \leq n-1\}$, for $i = 1, \ldots, n-2$.*

4.2.2 Kukin's example of a Lie algebra with undecidable word problem

Markov [161], Post [182], Turing [211], Novikov [173], and Boone [60] constructed finitely presented semigroups and groups with undecidable word problem. For groups this also follows from Higman's theorem [115] asserting that every recursively presented group embeds into a finitely presented group. A weak analogue of Higman's theorem for Lie algebras was proved in [21], which was enough for the existence of a finitely presented Lie algebra with undecidable word problem. In this section we give Kukin's construction [142] of a Lie algebra A_P for every semigroup P such that if P has undecidable word problem then so does A_P.

Given a semigroup $P = \mathrm{sgp}\langle x, y | u_i = v_i,\ i \in I\rangle$, consider the Lie algebra

$$A_P = Lie(x, \hat{x}, y, \hat{y}, z | S)$$

with S consisting of the relations

(1) $[\hat{x}x] = 0,\ [\hat{x}y] = 0,\ [\hat{y}x] = 0,\ [\hat{y}y] = 0$;
(2) $[\hat{x}z] = -[zx],\ [\hat{y}z] = -[zy]$;
(3) $\lfloor zu_i \rfloor = \lfloor zv_i \rfloor,\ i \in I$.

Here, $\lfloor zu \rfloor$ stands for the left normed bracketing.

Put $\hat{x} > \hat{y} > z > x > y$ and denote by $>$ the deg-lex ordering on the set $\{\hat{x}, \hat{y}, x, y, z\}^*$. Denote by ρ the congruence on $\{x, y\}^*$ generated by $\{(u_i, v_i),\ i \in I\}$. Put

(3′) $\lfloor zu \rfloor = \lfloor zv \rfloor,\ (u, v) \in \rho$ with $u > v$.

Lemma 8 ([80]) *In this notation, the set* $S_1 = \{(1), (2), (3')\}$ *is a GS basis in* $Lie(\hat{x}, \hat{y}, x, y, z)$.

Proof For every $u \in \{x, y\}^*$, we can show that $\overline{\lfloor zu \rfloor} = zu$ by induction on $|u|$. All possible compositions in S_1 are the intersection compositions of (2) and (3′), and the inclusion compositions of (3′) and (3′).

For $(2) \wedge (3')$, we take $f = [\hat{x}z] + [zx]$ and $g = \lfloor zu \rfloor - \lfloor zv \rfloor$. Therefore, $w = \hat{x}zu$ with $(u, v) \in \rho$ and $u > v$. We have

$$\begin{aligned}
\langle [\hat{x}z] + [zx], \lfloor zu \rfloor - \lfloor zv \rfloor \rangle_w &= [fu]_{\bar{f}} - [\hat{x}g]_{\bar{g}} \\
&\equiv \lfloor ([\hat{x}z] + [zx])u \rfloor - [\hat{x}(\lfloor zu \rfloor - \lfloor zv \rfloor)] \\
&\equiv \lfloor zxu \rfloor + \lfloor \hat{x}zv \rfloor \equiv \lfloor zxu \rfloor - \lfloor zxv \rfloor \equiv 0 \quad \mathrm{mod}\ (S_1, w).
\end{aligned}$$

For $(3') \wedge (3')$, we use $w = zu_1 = zu_2e$, where $e \in \{x, y\}^*$ and $(u_i, v_i) \in \rho$ with $u_i > v_i$ for $i = 1, 2$. We have

$$\begin{aligned}
&\langle \lfloor zu_1 \rfloor - \lfloor zv_1 \rfloor, \lfloor zu_2 \rfloor - \lfloor zv_2 \rfloor \rangle_w \equiv (\lfloor zu_1 \rfloor - \lfloor zv_1 \rfloor) - \lfloor (\lfloor zu_2 \rfloor - \lfloor zv_2 \rfloor) e \rfloor \\
&\equiv \lfloor \lfloor zv_2 \rfloor e \rfloor - \lfloor zv_1 \rfloor \equiv \lfloor zv_2 e \rfloor - \lfloor zv_1 \rfloor \equiv 0 \quad \mathrm{mod}\ (S_1, w).
\end{aligned}$$

Thus, $S_1 = \{(1), (2), (3')\}$ is a GS basis in $Lie(\hat{x}, \hat{y}, x, y, z)$. □

Corollary 17 (Kukin [142]) *For* $u, v \in \{x, y\}^*$ *we have*

$$u = v \textit{ in the semigroup } P \Leftrightarrow \lfloor zu \rfloor = \lfloor zv \rfloor \textit{in the Lie algebra } A_P.$$

Proof Assume that $u = v$ in the semigroup P. Without loss of generality we may assume that $u = au_1b$ and $v = av_1b$ for some $a, b \in \{x, y\}^*$ and $(u_1, v_1) \in \rho$. For every $r \in \{x, y\}$ relations (1) yield $[\hat{x}r] = 0$; consequently, $\lfloor zxc\rfloor = \lfloor [z\hat{x}]c\rfloor = [\lfloor zc\rfloor\hat{x}]$ and $\lfloor zyc\rfloor = [\lfloor zc\rfloor\hat{y}]$ for every $c \in \{x, y\}^*$. This implies that in A_P we have

$$\begin{aligned}\lfloor zu\rfloor &= \lfloor zau_1b\rfloor = \lfloor\lfloor zau_1\rfloor b\rfloor = \lfloor\lfloor zu_1 \widehat{\overleftarrow{a}}\rfloor b\rfloor = \lfloor zu_1 \widehat{\overleftarrow{a}} b\rfloor = \lfloor zv_1 \widehat{\overleftarrow{a}} b\rfloor \\ &= \lfloor zav_1b\rfloor = \lfloor zv\rfloor,\end{aligned}$$

where for every $x_{i_1}x_{i_2}\cdots x_{i_n} \in \{x, y\}^*$ we put

$$\overleftarrow{x_{i_1}x_{i_2}\cdots x_{i_n}} := x_{i_n}x_{i_{n-1}}\cdots x_{i_1}, \quad \widehat{x_{i_1}x_{i_2}\cdots x_{i_n}} := \widehat{x_{i_1}}\widehat{x_{i_2}}\cdots\widehat{x_{i_n}}.$$

Moreover, (3′) holds in A_P.

Suppose that $\lfloor zu\rfloor = \lfloor zv\rfloor$ in the Lie algebra A_P. Then both $\lfloor zu\rfloor$ and $\lfloor zv\rfloor$ have the same normal form in A_P. Since S_1 is a GS basis in A_P, we can reduce both $\lfloor zu\rfloor$ and $\lfloor zv\rfloor$ to the same normal form $\lfloor zc\rfloor$ for some $c \in \{x, y\}^*$ using only relations (3′). This implies that $u = c = v$ in P. □

By the corollary, if the semigroup P has undecidable word problem then so does the Lie algebra A_P.

4.3 Composition-Diamond lemma for Lie algebras over commutative algebras

For a well-ordered set $X = \{x_i | i \in I\}$, consider the free Lie algebra $Lie(X) \subset k\langle X\rangle$ with the Lie bracket $[u, v] = uv - vu$.

Given a well-ordered set $Y = \{y_j | j \in J\}$, the free commutative monoid $[Y]$ generated by Y is a linear basis of $k[Y]$. Regard

$$Lie_{k[Y]}(X) \cong k[Y] \otimes Lie(X)$$

as a Lie subalgebra of the free associative algebra $k[Y]\langle X\rangle \cong k[Y] \otimes k\langle X\rangle$ generated by X over the polynomial algebra $k[Y]$, equipped with the Lie bracket $[u, v] = uv - vu$. Then $NLSW(X)$ constitutes a $k[Y]$-basis of $Lie_{k[Y]}(X)$. Put $[Y]X^* = \{\beta t | \beta \in [Y],\ t \in X^*\}$. For $u = \beta t \in [Y]X^*$, put $u^X = t$ and $u^Y = \beta$.

Denote the deg-lex orderings on $[Y]$ and X^* by $>_Y$ and $>_X$. Define an ordering $>$ on $[Y]X^*$ as follows: for $u, v \in [Y]X^*$, put

$$u > v \text{ if } (u^X >_X v^X) \text{ or } (u^X = v^X \text{ and } u^Y >_Y v^Y).$$

We can express every element $f \in Lie_{k[Y]}(X)$ as $f = \sum \alpha_i\beta_i[u_i]$, where $\alpha_i \in k$, $\beta_i \in [Y]$, and $[u_i] \in NSLW(X)$.

Then $f = \sum \alpha_i\beta_i[u_i] = \sum g_j(Y)[u_j]$, where $g_j(Y) \in k[Y]$ are polynomials in the k-algebra $k[Y]\langle X\rangle$. The leading word $\bar{f}$ of f in $k[Y]\langle X\rangle$ is of the form β_1u_1 with $\beta_1 \in [Y]$ and $u_1 \in ALSW(X)$. The polynomial f is called monic (or k-monic) if the coefficient of $\bar{f}$ is equal to 1, that is, $\alpha_1 = 1$. The notion of $k[Y]$-monic polynomials is introduced similarly: $\alpha_1 = 1$ and $\beta_1 = 1$.

Recall that every ALSW w admits a unique bracketing such that $[w]$ is a NLSW.

Consider a monic subset $S \subset Lie_{k[Y]}(X)$. Given a non-associative word (u) on X with a fixed occurrence of some x_i and $s \in S$, call $(u)_{x_i \mapsto s}$ an S-word. Define $|u|$ to be the s-length of $(u)_{x_i \mapsto s}$. Every S-word is of the form (asb) with $a, b \in X^*$ and $s \in S$. If $a\bar{s}^X b \in ALSW(X)$ then we have the special bracketing $[a\bar{s}^X b]_{\bar{s}^X}$ of $a\bar{s}^X b$ relative to $\bar{s}^X$. Refer to $[asb]_{\bar{s}} = [a\bar{s}^X b]_{\bar{s}^X}|_{[\bar{s}^X] \mapsto s}$ as a special normal s-word (or special normal S-word).

An S-word $(u) = (asb)$ is a normal s-word, denoted by $\lfloor u \rfloor_s$, whenever $\overline{(asb)}^X = a\bar{s}^X b$. The following condition is sufficient.

(i) The s-length of (u) is 1, that is, $(u) = s$;
(ii) if $\lfloor u \rfloor_s$ is a normal S-word of s-length k and $[v] \in NLSW(X)$ satisfies $|v| = l$ then $[v]\lfloor u \rfloor_s$ whenever $v > \overline{\lfloor u \rfloor}_s^X$ and $\lfloor u \rfloor_s [v]$ whenever $v < \overline{\lfloor u \rfloor}_s^X$ are normal S-words of s-length $k + l$.

Take two monic polynomials f and g in $Lie_{k[Y]}(X)$ and put $L = \mathrm{lcm}(\bar{f}^Y, \bar{g}^Y)$. There are four kinds of compositions.

C_1: Inclusion composition. If $\bar{f}^X = a\bar{g}^X b$ for some $a, b \in X^*$, then

$$C_1\langle f, g\rangle_w = \frac{L}{\bar{f}^Y} f - \frac{L}{\bar{g}^Y}[agb]_{\bar{g}}, \quad \text{where } w = L\bar{f}^X = La\bar{g}^X b.$$

C_2: Intersection composition. If $\bar{f}^X = aa_0$ and $\bar{g}^X = a_0 b$ with $a, b, a_0 \neq 1$ then

$$C_2\langle f, g\rangle_w = \frac{L}{\bar{f}^Y}[fb]_{\bar{f}} - \frac{L}{\bar{g}^Y}[ag]_{\bar{g}}, \quad \text{where } w = L\bar{f}^X b = La\bar{g}^X.$$

C_3: External composition. If $gcd(\bar{f}^Y, \bar{g}^Y) \neq 1$ then for all $a, b, c \in X^*$ satisfying

$$w = La\bar{f}^X b\bar{g}^X c \in T_A = \{\beta t | \beta \in [Y],\ t \in ALSW(X)\}$$

we have

$$C_3\langle f, g\rangle_w = \frac{L}{\bar{f}^Y}[afb\bar{g}^X c]_{\bar{f}} - \frac{L}{\bar{g}^Y}[a\bar{f}^X bgc]_{\bar{g}}.$$

C_4: Multiplication composition. If $\bar{f}^Y \neq 1$ then for every special normal f-word $[afb]_{\bar{f}}$ with $a, b \in X^*$ we have

$$C_4\langle f\rangle_w = [a\bar{f}^X b][afb]_{\bar{f}}, \quad \text{where } w = a\bar{f}^X ba\bar{f}b.$$

Given a k-monic subset $S \subset Lie_{k[Y]}(X)$ and $w \in [Y]X^*$, which is not necessarily in T_A, an element $h \in Lie_{k[Y]}(X)$ is called trivial modulo (S, w) if h can be expressed as a $k[Y]$-linear combination of normal S-words with leading words smaller than w.

The set S is a Gröbner–Shirshov basis in $Lie_{k[Y]}(X)$ if all possible compositions in S are trivial.

Theorem 36 ([31], the CD-lemma for Lie algebras over commutative algebras) *Consider a nonempty set $S \subset Lie_{k[Y]}(X)$ of monic polynomials and denote by $Id(S)$ the ideal of $Lie_{k[Y]}(X)$ generated by S. The following statements are equivalent:*

(i) *The set S is a Gröbner–Shirshov basis in $Lie_{k[Y]}(X)$.*
(ii) *If $f \in Id(S)$ then $\bar{f} = a\bar{s}b \in T_A$ for some $s \in S$ and $a, b \in [Y]X^*$.*
(iii) *The set $Irr(S) = \{[u] \mid [u] \in T_N,\ u \neq a\bar{s}b,\ for\ s \in S\ and\ a, b \in [Y]X^*\}$ is a linear basis for $Lie_{\mathbf{k}[Y]}(X|S) = (Lie_{k[Y]}(X))/Id(S)$.*

Here $T_A = \{\beta t \mid \beta \in [Y],\ t \in ALSW(X)\}$ and $T_N = \{\beta[t] \mid \beta \in [Y],\ [t] \in NLSW(X)\}$.

Outline of the proof.

Take $u, v \in [Y]ALSW(X)$ and write $u = u^Y u^X$ and $v = v^Y v^X$. Define the ALSW-lcm(u, v) (or lcm(u, v) for short) as $w = w^Y w^X = \mathrm{lcm}(u^Y, v^Y)\,\mathrm{lcm}(u^X, v^X)$, where

$$\mathrm{lcm}(u^X, v^X) \in \{au^X cv^X b\ (an\ ALSW), a, b, c \in X^*;$$
$$u^X = av^X b, a, b \in X^*;\ u^X b = av^X, a, b \in X^*, deg(u^X b) < deg(u^X) + deg(v^X)\}.$$

Six lcm(u, v) are possible:

(i) (Y-trivial, X-trivial) (a trivial lcm(u, v));
(ii) (Y-trivial, X-inclusion);
(iii) (Y-trivial, X-intersection);
(iv) (Y-nontrivial, X-trivial);
(v) (Y-nontrivial, X-inclusion);
(vi) (Y-nontrivial, X-intersection).

In accordance with lcm(u, v), six general compositions are possible.

Denote by $[w^X]_{u^X, v^X}$ the Shirshov double special bracketing of w^X whenever w^X is a X-trivial lcm(u^X, v^X), by $[w^X]_{u^X}$ and $[w^X]_{v^X}$ the Shirshov special bracketings of w^X whenever w^X is a lcm of X-inclusion or X-intersection respectively.

Define general Lie compositions for k-monic Lie polynomials f and g with $\bar{f} = u$ and $\bar{g} = v$ as

$$(f, g)_w = (\mathrm{lcm}(u^Y, v^Y)/u^Y)[w^X]_{u^X, v^X}|_{[u] \mapsto f} - (\mathrm{lcm}(u^Y, v^Y)/v^Y)[w^X]_{u^X, v^X}|_{[v] \mapsto g},$$
$$(f, g)_w = (\mathrm{lcm}(u^Y, v^Y)/u^Y)[w^X]_u|_{[u] \mapsto f} - (\mathrm{lcm}(u^Y, v^Y)/v^Y)[w^X]_v|_{[v] \mapsto g}.$$

Lemma 9 ([31]) *The general composition $(f, g)_w$ of k-monic Lie polynomials f and g with $\bar{f} = u$ and $\bar{g} = v$, where w is a (Y-trivial, X-trivial)* lcm(u, v), *is* 0 mod $(\{f, g\}, w)$.

Proof By (**XIX**), we have

$$\begin{aligned}(f, g)_w &= v^Y[afb[v^X]d] - u^Y[a[u^X]bgd] = [afb[v]d] - [aubgd] \\ &= [afb([v] - g)d] - [a([u] - f)bgd] \equiv 0 \mod (\{f, g\}, w).\end{aligned}$$

The proof is complete. □

A Lie GS basis $S \subset Lie_{k[Y]}(X) \subset k[Y]\langle X\rangle$ need not be an associative GS basis because the PBW-theorem is not valid for Lie algebras over a commutative algebra (Shirshov [201]). Therefore, the argument for $Lie_k(X)$ above (see Sect. 4.2) fails for $Lie_{k[Y]}(X)$.

Moreover, Shirshov's original proof of the CD-lemma fails because the singleton $\{s\} \in Lie_{k[Y]}(X)$ is not a GS basis in general. The reason is that there exists a nontrivial composition $(s, s)_w$ of type (Y-nontrivial, X-trivial).

There is another obstacle. For $Lie_k(X)$, every s-word is a linear combination of normal s-words. For $Lie_{k[Y]}(X)$ this is not the case. Hence, we must use a multiplication composition $[u^X]f$ such that $\bar{f} = u = u^Y u^X$.

Lemma 10 ([31]) *If every multiplication composition $[\bar{s}^X]s$, $s \in S$, is trivial modulo $(S, w = [u^X]u)$, where $u = \bar{s}$, then every S-word is a linear combination of normal S-words.*

In our paper with Yongshan Chen [31], we use the following definition of triviality of a polynomial f modulo (S, w):

$$f \equiv 0 \mod (S, w) \Leftrightarrow f = \sum \alpha_i e_i^Y [a_i^X s_i b_i^X],$$

where $[a_i^X[\bar{s_i}^X]b_i^X]$ is the Shirshov special bracketing of the ALSW $a_i^X \bar{s_i}^X b_i^X$ with an ALSW $\bar{s_i}^X$.

The previous definition of triviality modulo (S, w) is equivalent to the usual definition by Lemma 11, which is key in the proof of the CD-lemma for Lie algebras over a commutative algebra.

Lemma 11 ([31]) *Given a monic set S with trivial multiplication compositions, take a normal s-word (asb) and a special normal s-word $[asb]$ with the same leading monomial $w = a\bar{s}b$. Then they are equal modulo (s, w).*

Lemmas 10 and 11 imply

Lemma 12 ([31]) *Given a monic set S with trivial multiplication compositions, every element of the ideal generated by S is a linear combination of special normal S-words.*

On the other hand, (**XVII**) and (**XIX**) imply the following analogue of Lemma 1 for $Lie_{k[Y]}(X)$.

Lemma 13 ([31]) *Given two k-monic special normal S-words $e_1^Y[{a_1}^X s_1 {b_1}^X]$ and $e_2^Y[{a_2}^X s_2 {b_2}^X]$ with the same leading associative word w_1, their difference is equal to $[a(s_1, s_2)_w b]$, where $w = \mathrm{lcm}(\bar{s_1}, \bar{s_2})$, $w_1 = awb$, and $[a(s_1, s_2)_w b] = [w_1]_w|_{[w]\mapsto (s_1,s_2)_w}$. Hence, if S is a GS basis then the previous special normal S-words are equal modulo (S, w_1).*

Now the claim (i)⇒(ii) of the CD-lemma for $Lie_{k[Y]}(X)$ follows.

For every Lie algebra $\mathcal{L} = Lie_K(X|S)$ over the commutative algebra $K = k[Y|R]$,

$$U(\mathcal{L}) = K\langle X|S^{(-)}\rangle = k[Y]\langle X|S^{(-)}, RX\rangle,$$

where $S^{(-)}$ is just S with all commutators $[uv]4$ replaced with $uv - vu$, is the universal enveloping associative algebra of $\mathcal{L}$.

A Lie algebra $\mathcal{L}$ over a commutative algebra K is called *special* whenever it embeds into its universal enveloping associative algebra. Otherwise it is called *non-special*.

Shirshov (1953) and Cartier (1958) gave classical examples of non-special Lie algebras over commutative algebras over $GF(2)$, justified using ad hoc methods. Cohn (1963) suggested another non-special Lie algebra over a commutative algebra over a field of positive characteristic.

Example 1 (Shirshov (1953)) Take $k = GF(2)$ and

$$K = k[y_i, i = 0, 1, 2, 3|y_0 y_i = y_i \ (i = 0, 1, 2, 3), \ \ y_i y_j = 0 \ (i, j \neq 0)].$$

Consider $\mathcal{L} = Lie_K(x_i, 1 \leq i \leq 13|S_1, S_2)$, where

$$\begin{aligned}
S_1 = \{&[x_2x_1] = x_{11}, \ [x_3x_1] = x_{13}, \ [x_3x_2] = x_{12}, \\
&[x_5x_3] = [x_6x_2] = [x_8x_1] = x_{10}, \ [x_ix_j] = 0\,(i > j)\}; \\
S_2 = \{&y_0x_i = x_i \ (i = 1, 2, \ldots, 13), \\
&y_1x_1 = x_4, y_1x_2 = x_5, y_1x_3 = x_6, y_1x_{12} = x_{10}, \\
&y_2x_1 = x_5, y_2x_2 = x_7, y_2x_3 = x_8, y_2x_{13} = x_{10}, \\
&y_3x_1 = x_6, y_3x_2 = x_8, y_3x_3 = x_9, y_3x_{11} = x_{10}, \\
&y_1x_k = 0\,(k = 4, 5, \ldots, 11, 13), \\
&y_2x_t = 0\,(t = 4, 5, \ldots, 12), \\
&y_3x_l = 0\,(l = 4, 5, \ldots, 10, 12, 13)\}.
\end{aligned}$$

Then $\mathcal{L} = Lie_K(X|S_1, S_2) = Lie_{k[Y]}(X|S_1, S_2, RX)$ and

$$S = S_1 \cup S_2 \cup RX \cup \{y_1x_2 = y_2x_1, \ y_1x_3 = y_3x_1, \ y_2x_3 = y_3x_2\}$$

is a GS basis in $Lie_{k[Y]}(X)$, which implies that x_{10} belongs to the linear basis of $\mathcal{L}$ by Theorem 36, that is, $x_{10} \neq 0$ in $\mathcal{L}$.

On the other hand, the universal enveloping algebra of $\mathcal{L}$ has the presentation

$$U_K(\mathcal{L}) = K\langle X|S_1^{(-)}, S_2\rangle \cong \mathbf{k}[Y]\langle X|S_1^{(-)}, S_2, RX\rangle.$$

However, the GS completion (see Mikhalev and Zolotykh [170]) of $S_1^{(-)} \cup S_2 \cup RX$ in $k[Y]\langle X\rangle$ is

$$S^C = S_1^{(-)} \cup S_2 \cup RX \cup \{y_1x_2 = y_2x_1, \ y_1x_3 = y_3x_1, \ y_2x_3 = y_3x_2, \ x_{10} = 0\}.$$

Thus, $\mathcal{L}$ is not special.

Example 2 (Cartier [70]) Take $k = GF(2)$ and

$$K = k[y_1, y_2, y_3 | y_i^2 = 0,\ i = 1, 2, 3].$$

Consider $\mathcal{L} = Lie_K(x_{ij}, 1 \le i \le j \le 3|S)$, where

$$S = \{[x_{ii}x_{jj}] = x_{ji}\ (i > j), [x_{ij}x_{kl}] = 0,\ y_3x_{33} = y_2x_{22} + y_1x_{11}\}.$$

Then $\mathcal{L}$ is not special over K.

Proof The set $S' = S \cup \{y_i^2 x_{kl} = 0\ (\forall i, k, l)\} \cup S_1$ is a GS basis in $Lie_{k[Y]}(X)$, where

$$S_1 = \{y_3x_{23} = y_1x_{12},\ y_3x_{13} = y_2x_{12},\ y_2x_{23} = y_1x_{13},\ y_3y_2x_{22} = y_3y_1x_{11},$$
$$y_3y_1x_{12} = 0,\ y_3y_2x_{12} = 0,\ y_3y_2y_1x_{11} = 0,\ y_2y_1x_{13} = 0\}.$$

Then, $y_2y_1x_{12} \in Irr(S')$ and so $y_2y_1x_{12} \ne 0$ in $\mathcal{L}$.

However, in

$$U_K(\mathcal{L}) = K\langle X|S^{(-)}\rangle \cong \mathbf{k}[Y]\langle X|S^{(-)}, y_i^2 x_{kl} = 0\ (\forall i, k, l)\rangle$$

we have

$$0 = y_3^2 x_{33}^2 = (y_2x_{22} + y_1x_{11})^2 = y_2^2 x_{22}^2 + y_1^2 x_{11}^2 + y_2y_1[x_{22}, x_{11}] = y_2y_1x_{12}.$$

Thus, $\mathcal{L} \not\hookrightarrow U_K(\mathcal{L})$.

Conjecture (Cohn [95]) Take the algebra $K = k[y_1, y_2, y_3|y_i^p = 0, i = 1, 2, 3]$ of truncated polynomials over a field k of characteristic $p > 0$. The algebra

$$\mathcal{L}_p = Lie_K(x_1, x_2, x_3 \mid y_3x_3 = y_2x_2 + y_1x_1),$$

called Cohn's Lie algebra, is not special.

In $U_K(\mathcal{L}_p)$ we have

$$0 = (y_3x_3)^p = (y_2x_2)^p + \Lambda_p(y_2x_2, y_1x_1) + (y_1x_1)^p = \Lambda_p(y_2x_2, y_1x_1),$$

where Λ_p is a Jacobson–Zassenhaus Lie polynomial. Cohn conjectured that $\Lambda_p(y_2x_2, y_1x_1) \ne 0$ in $\mathcal{L}_p$. To prove this, we must know a GS basis of $\mathcal{L}_p$ up to degree p in X. We found it for $p = 2, 3, 5$. For example, $\Lambda_2 = [y_2x_2, y_1x_1] = y_2y_1[x_2x_1]$ and a GS basis of $\mathcal{L}_2$ up to degree 2 in X is

$$y_3x_3 = y_2x_2 + y_1x_1,\ y_i^2 x_j = 0\ (1 \le i, j \le 3),\ y_3y_2x_2 = y_3y_1x_1,\ y_3y_2y_1x_1 = 0,$$
$$y_2[x_3x_2] = y_1[x_3x_1],\ y_3y_1[x_2x_1] = 0,\ y_2y_1[x_3x_1] = 0.$$

Therefore, $y_2y_1[x_2x_1] \in Irr(S^C)$.

Similar though much longer computations show that $\Lambda_3 \neq 0$ in $\mathcal{L}_3$ and $\Lambda_5 \neq 0$ in $\mathcal{L}_5$. Thus, we have

Theorem 37 ([31]) *Cohn's Lie algebras $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_5$ are non-special.*

Theorem 38 ([31]) *Given a commutative k-algebra $K = k[Y|R]$, if S is a Gröbner–Shirshov basis in $Lie_{k[Y]}(X)$ such that every $s \in S$ is $k[Y]$-monic then $\mathcal{L} = Lie_K(X|S)$ is special.*

Corollary 18 ([31]) *Every Lie K-algebra $L_K = Lie_K(X|f)$ with one monic defining relation $f = 0$ is special.*

Theorem 39 ([31]) *Suppose that S is a finite homogeneous subset of $Lie_k(X)$. Then the word problem of $Lie_K(X|S)$ is solvable for every finitely generated commutative k-algebra K.*

Theorem 40 ([31]) *Every finitely or countably generated Lie K-algebra embeds into a two-generated Lie K-algebra, where K is an arbitrary commutative k-algebra.*

5 Gröbner–Shirshov bases for Ω-algebras and operads

5.1 CD-lemmas for Ω-algebras

Some new CD-lemmas for Ω-algebras have appeared: for associative conformal algebras [45] and n-conformal algebras [43], for the tensor product of free algebras [30], for metabelian Lie algebras [75], for associative Ω-algebras [41], for color Lie superalgebras and Lie p-superalgebras [165,166], for Lie superalgebras [167], for associative differential algebras [76], for associative Rota–Baxter algebras [32], for L-algebras [33], for dialgebras [38], for pre-Lie algebras [35], for semirings [40], for commutative integro-differential algebras [102], for difference-differential modules and difference-differential dimension polynomials [225], for λ-differential associative Ω-algebras [185], for commutative associative Rota–Baxter algebras [186], for algebras with differential type operators [111].

Latyshev studied general versions of GS (or standard) bases [147,148].

Let us state the CD-lemma for pre-Lie algebras, see [35].

A non-associative algebra A is called a pre-Lie (or a right-symmetric) algebra if A satisfies the identity $(x, y, z) = (x, z, y)$ for the associator $(x, y, z) = (xy)z - x(yz)$. It is a Lie admissible algebra in the sense that $A^{(-)} = (A, [xy] = xy - yx)$ is a Lie algebra.

Take a well-ordered set $X = \{x_i | i \in I\}$. Order X^{**} by induction on the lengths of the words (u) and (v):

(i) When $|((u)(v))| = 2$ put $(u) = x_i > (v) = x_j$ if and only if $i > j$.
(ii) When $|((u)(v))| > 2$ put $(u) > (v)$ if and only if one of the following holds:
  (a) $|(u)| > |(v)|$;
  (b) if $|(u)| = |(v)|$ with $(u) = ((u_1)(u_2))$ and $(v) = ((v_1)(v_2))$ then $(u_1) > (v_1)$ or $(u_1) = (v_1)$ and $(u_2) > (v_2)$.

We now quote the definition of good words (see [198]) by induction on length:

(1) x is a good word for any $x \in X$;
(2) a non-associative word $((v)(w))$ is called a good word if
  (a) both (v) and (w) are good words and
  (b) if $(v) = ((v_1)(v_2))$ then $(v_2) \leq (w)$.

Denote (u) by $[u]$ whenever (u) is a good word.

Denote by W the set of all good words in the alphabet X and by $RS\langle X\rangle$ the free right-symmetric algebra over a field k generated by X. Then W forms a linear basis of $RS\langle X\rangle$, see [198]. Kozybaev et al. [141] proved that the deg-lex ordering on W is monomial.

Given a set $S \subset RS\langle X\rangle$ of monic polynomials and $s \in S$, an S-word $(u)_s$ is called a normal S-word whenever $(u)_{\bar{s}} = (a\bar{s}b)$ is a good word.

Take $f, g \in S$, $[w] \in W$, and $a, b \in X^*$. Then there are two kinds of compositions.

(i) If $\bar{f} = [a\bar{g}b]$ then $(f, g)_{\bar{f}} = f - [agb]$ is called the inclusion composition.
(ii) If $(\bar{f}[w])$ is not good then $f \cdot [w]$ is called the right multiplication composition.

Theorem 41 ([35], the CD-lemma for pre-Lie algebras) *Consider a nonempty set $S \subset RS\langle X\rangle$ of monic polynomials and the ordering $<$ defined above. The following statements are equivalent:*

(i) *The set S is a Gröbner–Shirshov basis in $RS\langle X\rangle$.*
(ii) *If $f \in Id(S)$ then $\bar{f} = [a\bar{s}b]$ for some $s \in S$ and $a, b \in X^*$, where $[asb]$ is a normal S-word.*
(iii) *The set $Irr(S) = \{[u] \in W | [u] \neq [a\bar{s}b]\, a, b \in X^*,\ s \in S$ and $[asb]$ is a normal S-word$\}$ is a linear basis of the algebra $RS\langle X|S\rangle = RS\langle X\rangle/Id(S)$.*

As an application, we have a GS basis for the universal enveloping pre-Lie algebra of a Lie algebra.

Theorem 42 ([35]) *Consider a Lie algebra $(\mathcal{L}, [\ ])$ with a well-ordered linear basis $X = \{e_i | i \in I\}$. Write $[e_i e_j] = \sum_m \alpha_{ij}^m e_m$ with $\alpha_{ij}^m \in k$. Denote $\sum_m \alpha_{ij}^m e_m$ by $\{e_i e_j\}$. Denote by*

$$U(\mathcal{L}) = RS\langle \{e_i\}_I | e_i e_j - e_j e_i = \{e_i e_j\},\ i, j \in I\rangle$$

the universal enveloping pre-Lie algebra of $\mathcal{L}$. The set

$$S = \{f_{ij} = e_i e_j - e_j e_i - \{e_i e_j\},\ i, j \in I \text{ and } i > j\}$$

is a Gröbner–Shirshov basis in $RS\langle X\rangle$.

Theorems 41 and 42 directly imply the following PBW theorem for Lie algebras and pre-Lie algebras.

Corollary 19 (Segal [198]) *A Lie algebra $\mathcal{L}$ embeds into its universal enveloping pre-Lie algebra $U(\mathcal{L})$ as a subalgebra of $U(\mathcal{L})^{(-)}$.*

Recently the CD-lemmas mentioned above and other combinatorial methods yielded many applications: for groups of Novikov–Boone type [119–121] (see also [16,17,77,118], for Coxeter groups [58,150], for center-by-metabelian Lie algebras [214], for free metanilpotent Lie algebras, Lie algebras and associative algebras [112,168,215,216], for Poisson algebras [159], for quantum Lie algebras and related problems [132,135], for PBW-bases [131,134,158], for extensions of groups and associative algebras [73,74], for (color) Lie (p)-superalgebras [9,48,91,92,105–107,169,227,228], for Hecke algebras and Specht modules [125], for representations of Ariki–Koike algebras [126], for the linear algebraic approach to GS bases [127], for HNN groups [87], for certain one-relator groups [88], for embeddings of algebras [39,83], for free partially commutative Lie algebras [84,181], for quantum groups of type D_n, E_6, and G_2 [174,189,221,222], for calculations of homogeneous GS bases [145], for Picard groups, Weyl groups, and Bruck–Reilly extensions of semigroups [7,128–130,139], for Akivis algebras and pre-Lie algebras [79], for free Sabinin algebras [93].

5.2 CD-lemma for operads

Following Dotsenko and Khoroshkin ([98], Proposition 3), linear bases for a symmetric operad and a shuffle operad are the same provided both of them are defined by the same generators and defining relations. It means that we need CD-lemma for shuffle operads only (and we define a GS basis for a symmetric operad as a GS basis of the corresponding shuffle operad).

We express the elements of the free shuffle operad using planar trees.

Put $\mathscr{V}=\bigcup_{n=1}^{\infty}\mathscr{V}_n$, where $\mathscr{V}_n=\{\delta_i^{(n)}|i\in I_n\}$ is the set of n-ary operations.

Call a planar tree with n leaves *decorated* whenever the leaves are labeled by $[n]=\{1,2,3,\ldots,n\}$ for $n\in\mathbb{N}$ and every vertex is labeled by an element of $\mathscr{V}$.

For an arrow in a decorated tree, let its value be the minimal value of the leaves of the subtree grafted to its end. A decorated tree is called a tree monomial whenever for each its internal vertex the values of the arrows beginning from it increase from the left to the right.

Denote by $\mathscr{F}_{\mathscr{V}}(n)$ the set of all tree monomials with n leaves and put $T=\cup_{n\geq 1}\mathscr{F}_{\mathscr{V}}(n)$. Given $\alpha=\alpha(x_1,\ldots,x_n)\in\mathscr{F}_{\mathscr{V}}(n)$ and $\beta\in\mathscr{F}_{\mathscr{V}}(m)$, define the shuffle composition $\alpha\circ_{i,\sigma}\beta$ as

$$\alpha(x_1,\ldots,x_{i-1},\beta(x_i,x_{\sigma(i+1)},\ldots,x_{\sigma(i+m-1)}),x_{\sigma(i+m)},\ldots,x_{\sigma(m+n-1)}),$$

which lies in $\mathscr{F}_{\mathscr{V}}(n+m-1)$, where $1\leq i\leq n$ and the bijection

$$\sigma:\{i+1,\ldots,m+n-1\}\to\{i+1,\ldots,m+n-1\}$$

is an $(m-1,n-i)$-shuffle, that is,

$$\sigma(i+1)<\sigma(i+2)<\cdots<\sigma(i+m-1),$$
$$\sigma(i+m)<\sigma(i+m+1)<\cdots<\sigma(n+m-1).$$

The set T is freely generated by $\mathscr{V}$ with the shuffle composition.

Denote by $\mathscr{F}_{\mathscr{V}} = kT$ the k-linear space spanned by T. This space with the shuffle compositions $\circ_{i,\sigma}$ is called the free shuffle operad.

Take a homogeneous subset S of $\mathscr{F}_{\mathscr{V}}$. For $s \in S$, define an S-word $u|_s$ as before.

A well ordering $>$ on T is called monomial (admissible) whenever

$$\alpha > \beta \Rightarrow u|_\alpha > u|_\beta \text{ for any } u \in T.$$

Assume that T is equipped with a monomial ordering. Then each S-word is a normal S-word.

For example, the following ordering $>$ on T is monomial, see Proposition 5 of [98].

Every $\alpha = \alpha(x_1, \ldots, x_n) \in \mathscr{F}_{\mathscr{V}}(n)$ has a unique expression

$$\alpha = (\mathrm{path}(1), \ldots, \mathrm{path}(n), [i_1 \ldots i_n]),$$

where $\mathrm{path}(r) \in \mathscr{V}^*$ for $1 \leq r \leq n$ is the unique path from the root to the leaf r and the permutation $[i_1 \ldots i_n]$ lists the labels of the leaves of the underlying tree in the order determined by the planar structure, from left to right. In this case define

$$\mathrm{wt}(\alpha) = (n, \mathrm{path}(1), \ldots, \mathrm{path}(n), [i_1 \ldots i_n]).$$

Assume that $\mathscr{V}$ is a well-ordered set and use the deg-lex ordering on $\mathscr{V}^*$. Take the order on the permutations in reverse lexicographic order: $i > j$ if and only if i is less than j as numbers.

Now, given $\alpha, \beta \in T$, define

$$\alpha > \beta \Leftrightarrow \mathrm{wt}(\alpha) > \mathrm{wt}(\beta) \quad \text{lexicographically}.$$

An element of $\mathscr{F}_{\mathscr{V}}$ is called *homogeneous* whenever all tree monomials occurring in this element with nonzero coefficients have the same arity degree (but not necessarily the same operation degree).

For two tree monomials α and β, say that α is divisible by β whenever there exists a subtree of the underlying tree of α for which the corresponding tree monomial α' is equal to α.

A tree monomial γ is called a common multiple of two tree monomials α and β whenever it is divisible by both α and β. A common multiple γ of two tree monomials α and β is called a least common multiple and denoted by $\gamma = \mathrm{lcm}(\alpha, \beta)$ whenever $|\alpha| + |\beta| > |\gamma|$, where $|\delta| = n$ for $\delta \in \mathscr{F}_{\mathscr{V}}(n)$.

Take two monic homogeneous elements f and g of $\mathscr{F}_{\mathscr{V}}$. If $\bar{f}$ and $\bar{g}$ have a least common multiple w then $(f, g)_w = w_{\bar{f} \mapsto f} - w_{\bar{g} \mapsto g}$.

Theorem 43 ([98], the CD-lemma for shuffle operads) *In the above notation, consider a nonempty set $S \subset \mathscr{F}_{\mathscr{V}}$ of monic homogeneous elements and a monomial ordering $<$ on T. The following statements are equivalent:*

(i) *The set S is a Gröbner–Shirshov basis in $\mathscr{F}_{\mathscr{V}}$.*

(ii) *If $f \in Id(S)$ then $\bar{f} = u|_{\bar{s}}$ for some S-word $u|_s$.*

(iii) *The set* $Irr(S) = \{u \in T | u \neq v|_{\bar{s}}$ *for all* S*-word* $v|_s\}$ *is a* k*-linear basis of* $\mathscr{F}_{\mathscr{V}}/Id(S)$.

As applications, the authors of [98] calculate Gröbner–Shirshov bases for some well-known operads: the operad Lie of Lie algebras, the operad As of associative algebras, and the operad PreLie of pre-Lie algebras.

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