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On n-layered $QTAG$-modules

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Abstract A module M over an associative ring with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. There are many fascinating concepts related to these modules. Here we introduce the notion of n-layered $QTAG$-modules and discuss some interesting properties of these modules. We show that a $QTAG$-module M is n-layered if and only if M/N is an n-layered module, whenever N is a finitely generated submodule of M and $n \geq 1$ is an integer.

Keywords QTAG-module · ω-elongation · Totally projective modules · $(\omega + k)$-projective modules

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1 Introduction and preliminaries

The study of QTAG-modules was initiated by Singh [8]. Mehdi et al. [4] worked a lot on these modules. They studied different notions and structures of QTAG-modules and developed the theory of these modules by introducing several notions, investigated some interesting properties and characterized them. Yet there is much to explore.

Throughout this paper, all rings are associative with unity and modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\{d(\frac{yR}{xR}) \mid y \in M,\ x \in yR$ and y uniform$\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A submodule N of a QTAG-module M is height finite, if the heights of the elements of N take only finitely many values. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height.

A submodule $N \subset M$ is nice [3, Definition 2.3] in M, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ, i.e. every coset of M modulo N may be represented by an element of the same height.

A family of nice submodules $\mathcal{N}$ of submodules of M is called a nice system in M if

(i) $0 \in \mathcal{N}$;
(ii) If $\{N_i\}_{i \in I}$ is any subset of $\mathcal{N}$, then $\Sigma_I N_i \in \mathcal{N}$;
(iii) Given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated [4].

A h-reduced $QTAG$-module M is called totally projective if it has a nice system.

For a $QTAG$-module M, there is a chain of submodules $M^0 \supset M^1 \supset M^2 \cdots \supset M^\tau = 0$, for some ordinal τ. $M^{\sigma+1} = (M^\sigma)^1$, where M^σ is the σth-Ulm submodule of M. A fully invariant submodule $L \subset M$ is a large submodule of M, if $L + B = M$ for every basic submodule B of M. Several results which hold for TAG-modules also hold good for $QTAG$-modules [8]. Notations and terminology are follows from [1,2].

2 *n*-Layered $QTAG$-modules and its properties

Recall that a $QTAG$-module M is $(\omega + 1)$-projective if there exists submodule $N \subset H^1(M)$ such that M/N is a direct sum of uniserial modules and a $QTAG$ module M is $(\omega + k)$-projective if there exists a submodule $N \subset H^k(M)$ such that M/N is a direct sum of uniserial modules [4].

Let σ be a limit ordinal such that $\sigma = \omega + \beta$. A $QTAG$-module M is called σ-projective, if there exists a submodule $N \subset H^\beta(M)$ such that M/N is a direct sum of uniserial modules. A QTAG-module M is totally projective, if and only if $M/H_\sigma(M)$ is σ-projective for every ordinal σ.

A $QTAG$-module is an ω-elongation of a totally projective $QTAG$-module by a $(\omega+k)$-projective $QTAG$-module if and only if $H_\omega(M)$ is totally projective and $M/H_\omega(M)$ is $(\omega+k)$-projective.

A $QTAG$-module M is a *strong ω-elongation* of a totally projective module by a $(\omega+n)$-projective module if $H_\omega(M)$ is totally projective and there exists $N\subseteq H^n(M)$ such that $\frac{M}{N+H_\omega(M)}$ is a direct sum of uniserial modules [5].

Referring to our criterion from [7], M is a Σ-module or layered module if $\mathrm{Soc}(M)=\bigcup_{k<\omega}M_k$, where $M_k\subseteq M_{k+1}\subseteq \mathrm{Soc}(M)$ and for every k, $M_k\cap H_k(M)=\mathrm{Soc}(H_\omega(M))$.

In [5], it was shown that any $(\omega+1)$-projective σ-module is a direct sum of countable modules of length atmost $(\omega+1)$. Moreover, we extended this assertion to the so called strong ω-elongations. It was established that any strong ω-elongation of a totally projective module by a $(\omega+1)$-projective module is a Σ-module precisely when it is totally projective.

That is why it naturally comes under what additional conditions on the module structure this type of results hold for every $n\in\mathbb{N}$. To achieve this goal we state the following new concept, which is a generalization of the corresponding one for Σ-module.

Definition 1 A $QTAG$-module M is said to be *n-layered module* if for some $n<\omega$, $H^n(M)=\bigcup_{k<\omega}M_k$, $M_k\subseteq M_{k+1}\subseteq H^n(M)$ and for all $k\geq 1$, $M_k\cap H_k(M)=H^n(H_\omega(M))$.

Remark 1 Equivalently, we may say that M is a n-layered module if and only if $H^n(M)=\bigcup N_k$, $N_k\subseteq N_{k+1}\subseteq H^n(M)$ and for every $k\geq 1$, $N_k\cap H_k(M)\subseteq H_\omega(M)$.

Also, $N_k\subseteq N_k+H^n(H_\omega(M))$ implies that $H^n(M)=\bigcup(N_k+H^n(H_\omega(M)))$ and $(N_k+H^n(H_\omega(M)))\cap H_k(M)=H^n(H_\omega(M))+(N_k\cap H_k(M))=H^n(H_\omega(M))$. Therefore $M_k=N_k+H^n(H\omega(M))$ and $N_k\cap H_k(M)\subseteq H_\omega(M)$, equivalently $N_k\cap H_k(M)=H^n(H_\omega(M))$.

Remark 2 Every layered module is 1-layered module and vice-versa. Since $H^n(M)\subseteq H^m(M)$, for $n\leq m$, every m-layered module is a n-layered module.

Now we investigate some properties of n-layered modules.

Lemma 1 *For $n\geq 1$, h-pure submodules of n-layered modules are n-layered modules. Moreover, the submodules of n-layered modules with the same first Ulm submodules are n-layered.*

Proof Let M be a n-layered $QTAG$-module such that $H^n(M)=\bigcup_{j<\omega}M_j$, $M_j\subseteq M_{j+1}\subseteq H^n(M)$ and $M_j\cap H_j(M)\subseteq H_\omega(M)$. Now for any h-pure submodule N of M, $H^n(N)=\bigcup_{j<\omega}N_j$, where $N_j=M_j\cap N$ and

$$N_j\cap H_j(N)\subseteq N\cap H_\omega(M)=H_\omega(N)$$

and the result follows.

If K is an arbitrary submodule of M such that $H_\omega(K) = H_\omega(M)$, then $H^n(K) = \bigcup_{j<\omega} K_j$, where $K_j = M_j \cap K$ and

$$K_j \cap H_j(K) \subseteq K \cap H_\omega(M) = H_\omega(K)$$

and we are done.

Lemma 2 *Let N be submodule of a h-reduced module M and $n \geq 1$. Then M is n-layered if and only if N is n-layered, where N is a $H_{\omega+n-1}(M)$-high submodule of M.*

Proof For any ordinal α, $H_\alpha(M)$-high submodules of M are h-pure in M. Now if N is a $H_{\omega+n-1}(M)$-high submodule of M, it is h-pure in M and by Lemma 1, if M is n-layered, then N is also n-layered.

We have "Let M be a h-reduced QTAG-module and let N be a $H_{\alpha+k}$-high submodule of M with α a limit ordinal and $k \geq 1$. Then $H^n(M) = H^n(N) \oplus H^n(K)$ for $n > k$ and any complementary summand K of a maximal summand of $H_\alpha(M)$ bounded by k" [4].

Now for the converse, we have $H^n(M) = H^n(N) \oplus H^n(K)$, where K is a $H_{n-1}(H_\omega(M))$-high submodule of $H_\omega(M)$. Since $H^n(N) = \bigcup_{j<\omega} N_j$, $N_j \subseteq N_{j+1} \subseteq H^n(N)$ and $N_j \cap H_j(N) \subseteq H_\omega(N)$, by defining $M_j = N_j \oplus H^n(K)$, we have $H^n(M) = \bigcup_{j<\omega} M_j$. Since N is h-pure in M and $K \subseteq H_\omega(M)$, $M_j \cap H_j(M) = H^n(K) + (N_j \cap H_j(M)) = H^n(K) + (N_j \cap H_j(N)) \subseteq H_\omega(M)$.

Proposition 1 *For $n \geq 1$, all Σ-modules with h-divisible first Ulm submodule are n-layered modules.*

Proof Let M be a Σ-module such that $H_\omega(M)$ is h-divisible. Since h-divisible submodules are direct summands, we have $M = H_\omega(M) \oplus N$, where N is contained in a high submodule of M, hence N is a direct sum of uniserial submodules. Again $\frac{M}{H_\omega(M)} \simeq N$ and we are done.

Proposition 2 *Direct sums of n-layered modules are n-layered modules.*

Proof Let M be a direct sum of n-layered modules such that $M = \bigoplus_{j\in J} N_j$. Here N_j's are n-layered modules. Therefore $H^n(N_j) = \bigcup_{i<\omega} N_{ij}$, $N_{ij} \subseteq N_{(i+1)j} \subseteq N_j$ and $N_{ij} \cap H_i(N_j) \subseteq H_\omega(N_j)$ for $i < \omega$, $j \in J$.

Furthermore, $H^n(M) = \bigoplus_{j\in J} H^n(N_j) = \bigoplus_{j\in J}(\bigcup_{i<\omega} N_{ij}) = \bigcup_{i<\omega}(\bigoplus_{j\in J} N_{ij}) = \bigcup_{i<\omega} M_i$, where $M_i = \bigoplus_{j\in J} N_{ij}$ and

$$\begin{aligned} M_i \cap H_i(M) &= \left(\bigoplus_{j\in I} N_{ij}\right) \cap \left(\bigoplus_{j\in I} H_i(N_j)\right) \\ &= \bigoplus_{j\in I}(N_{ij} \cap H_i(N_j)) \\ &\subseteq \bigoplus_{j\in I} H_\omega(N_j) \end{aligned}$$

$$= H_\omega\left(\bigoplus_{j\in I} N_j\right)$$
$$= H_\omega(M)$$

and the result follows.

Proposition 3 *For $k \geq 1$, M is a n-layered module if and only if $H_k(M)$ is n-layered.*

Proof If M is a n-layered module, then $H^n(M) = \bigcup_{j<\omega} N_j$, $N_j \subseteq N_{j+1} \subseteq H^n(M)$ and $N_j \cap H_j(M) \subseteq H_\omega(M)$, for every $j < \omega$. Therefore $H^n(H_k(M)) = \bigcup_{i<\omega} T_j$, where $T_j = N_j \cap H_k(M)$ and

$$\begin{aligned} T_j \cap H_{k+j}(M) &= T_j \cap H_j(H_k(M)) \\ &\subseteq N_j \cap H_j(M) \\ &\subseteq H_\omega(M) \\ &= H_\omega(H_k(M)). \end{aligned}$$

Thus $H_k(M)$ is also n-layered.

For the converse, suppose $H_k(M)$ is n-layered. If $k{=}1$, then $H_1(M)$ is n-layered and we have $H^n(H_1(M)) = \bigcup_{j<\omega} T_j$, $T_j \subseteq T_{j+1} \subseteq H^n(H_1(M))$ and $T_j \cap H_{j+1}(M) \subseteq H_\omega(M)$. Let $S = \{x \mid x \in H^n(M), x \notin H^n(H_1(M)\}$. Then $H^n(M) = S \cup H^n(H_1(M))$. Define $K_j \subseteq M$ such that $K_j \cap H_1(M) = \phi$ and $H^n(M \backslash H_1(M)) = \bigcup_{j<\omega} K_j$ such that $\langle K_j\rangle \cap H_1(M) \subseteq T_j$. This implies that $H^n(M) = \bigcup(T_j + \langle K_j\rangle)$, and

$$\begin{aligned} (T_j + \langle K_j\rangle) \cap H_{j+1}(M) &\subseteq (T_j + \langle K_j\rangle) \cap H_1(M) \\ &= T_j + (\langle K_j\rangle \cap H_1(M)) \\ &= T_j. \end{aligned}$$

Therefore $(T_j + \langle K_j\rangle) \cap H_{j+1}(M) \subseteq T_j \cap H_{j+1}(M) \subseteq H_\omega(M)$ and the result follows.

Proposition 4 *A $QTAG$-module M is n-layered module if and only if its large submodule L is n-layered.*

Proof For a large submodule L of M, $H_\omega(L) = H_\omega(M)$ [6]. Therefore by Lemma 1, L is n-layered whenever M is n-layered.

Conversely, suppose L is n-layered such that $L = \sum_{k<\omega} H^k(H_{m_k}(M))$, where $m_1 \leq m_2 \leq \cdots \leq m_k$ is a monotonically increasing sequence of positive integers. Now $H^n(L) = Soc(H_{m_1}(M) + \cdots + H^n(H_{m_n}(M))$ therefore

$$H^n(H_{m_n}(M)) \subseteq H^n(L) \subseteq H^n(H_{m_1}(M)).$$

Also $H^n(L) = \bigcup_{j<\omega} L_j$, $L_j \subseteq L_{j+1} \subseteq H^n(L)$ and $L_j \cap H_j(L) \subseteq H_\omega(L)$ and $H^n(H_{m_n}(M)) = \bigcup_{j<\omega} N_j$, where $N_j = L_j \cap H^n(H_{m_n}(M)) = L_j \cap (H_{m_n}(M))$. Again $N_j \cap H_{t_j}(M) \subseteq N_j \cap H_j(L) \subseteq H_\omega(L) = H_\omega(M)$, for some $t_j \geq \max(j, m_n)$ with $H^n(H_{t_j}(M)) \subseteq H^n(H_j(L))$, as $H_j(L)$ is also a large submodule of M. Now $H_{m_n}(M)$ is n-layered module and by Proposition 3, M is also n-layered module.

Proposition 5 *Let N be a submodule of M such that M/N is bounded. Then M is n-layered module if and only if N is n-layered module.*

Proof Since M/N is bounded, then there exists an integer k such that $H_k(M/N) = 0$ or $H_k(M) \subseteq N$. Therefore $H_\omega(H_k(M)) = H_\omega(M) = H_\omega(N)$ and by Lemma 1, if M is n-layered then N is also n-layered.

Conversely, if N is a n-layered module then by Lemma 1, $H_k(M) \subseteq N$ is also n-layered. Therefore by Proposition 3, M is also n-layered.

Proposition 6 *Let N be a height-finite, submodule of M. If M/N is n-layered, then M is n-layered.*

Proof Since M/N is n-layered, $H^n(M/N) = \bigcup_{j<\omega}(K_j/N) = (\bigcup K_j)/N$, where $K_j \subseteq K_{j+1} \subseteq M$ and $\left(\frac{K_j}{N}\right) \cap H_j\left(\frac{M}{N}\right) \subseteq H_\omega\left(\frac{M}{N}\right)$. Now N is height-finite, therefore nice in M and $K_j \cap H_j(M) \subseteq H_\omega(M) + N$. There exists a positive integer $t_j \geq j$ such that $N \cap H_{t_j}(M) \subseteq H_\omega(M)$. Also

$$\begin{aligned} K_j \cap H_{t_j}(M) &\subseteq (H_\omega(M) + N) \cap H_{t_j}(M) \\ &= H_\omega(M) + (N \cap H_{t_j}(M)) \\ &= H_\omega(M). \end{aligned}$$

Now $\left(\frac{H^n(M)+N}{N}\right) \subseteq H^n(M/N)$ and $H^n(M) \subseteq \bigcup_{j<\omega} K_j$. Thus $H^n(M) = \bigcup_{j<\omega} T_j$, where $T_j = K_j \cap H^n(M) = H^n(K_j)$ and the result follows.

Remark 3 Let N be a height-finite submodule of M. If M/N is a Σ-module, then M is also a Σ-module.

Proposition 7 *Let N be a submodule of M.*

(i) *if $N \cap H_n(M) = H_n(N)$ and N is finitely generated or $N \subseteq H_\omega(M)$ and M is n-layered, then M/N is also n-layered;*
(ii) *if $N \subseteq H^k(M)$, for some $k \geq 1$ and either N is finitely generated or $N \subseteq H_\omega(M)$ and M is $(n+k)$-layered, then M/N is also n-layered.*

Proof (i) If M is n-layered, then $H^n(M) = \bigcup_{j<\omega} M_j$, $M_j \subseteq M_{j+1}$ and $M_j \cap H_j(M) \subseteq H_\omega(M)$. Now, $H^n\left(\frac{M}{N}\right) = \left(\frac{H^n(M)+N}{N}\right) = \bigcup_{j<\omega}\left(\frac{M_j+N}{N}\right)$. Therefore, $\left(\frac{M_j+N}{N}\right) \cap H_j\left(\frac{M}{N}\right) = \frac{[N+((M_j+N)\cap H_j(M))]}{N}$.

When $N \subseteq H_j(M)$, for every positive integer j, then

$$(M_j + N) \cap H_j(M) \subseteq N + (M_j \cap H_j(M)) \subseteq N + H_\omega(N).$$

Since $\left(\frac{N+H_\omega(M)}{N}\right) \subseteq H_\omega\left(\frac{M}{N}\right)$, the result follows.

When N is finitely generated, then there exists an integer $t_j \geq j$ such that $(M_j + N) \cap H_{t_j}(M) \subseteq H_\omega(M)$. Therefore $\left(\frac{M_j+N}{N}\right) \cap H_{t_j}\left(\frac{M}{N}\right) \subseteq \left(\frac{N+H_\omega(M)}{N}\right) = H_\omega\left(\frac{M}{N}\right)$ and we are done.

If $N \subseteq H_\omega(M)$ and we have $H^n\left(\frac{M}{N}\right) = \bigcup_{j<\omega}\left(\frac{M_j}{N}\right)$, $M_j \subseteq M_{j+1} \subseteq M$, where $(M_j/N) \cap H_j(M/N) \subseteq H_\omega(M/N) = H_\omega(M)/N$. Therefore $M_j \cap H_j(M) \subseteq H_\omega(M)$. Since $\left(\frac{H^n(M)+N}{N}\right) \subseteq H^n\left(\frac{M}{N}\right)$, $H^n(M) = \bigcup_{j<\omega} H^n(M_j)$ and the result follows.

(ii) Since $H^n\left(\frac{M}{N}\right) \subseteq \left(\frac{H^{n+k}(M)+N}{N}\right)$, we are through.

Proposition 8 *If for some ordinal α, $M/H_\alpha(M)$ is n-layered, then M is n-layered.*

Proof We have $H^n(M/H_\alpha(M)) = \bigcup_{j<\omega}(M_j/H_\alpha(M))$, $M_j \subseteq M_{j+1} \subseteq M$, $M_j \cap H_j(M) \subseteq H_\omega(M)$ for every $j < \omega$. Now

$$\left(\frac{H^n(M) + H_\alpha(M)}{H_\alpha(M)}\right) \subseteq H^n\left(\frac{M}{H_\alpha(M)}\right)$$

therefore $H^n(M) \subseteq \bigcup_{j<\omega} M_j$. If we put $T_j = H^n(M_j)$, then $H^n(M) = \bigcup_{j<\omega} T_j$. But $T_j \cap H_j(M) \subseteq M_j \cap H_j(M) \subseteq H_\omega(M)$ and we are done.

Now we are in the state to prove our main result which motivated this article.

Theorem 1 *The $QTAG$-module M is a n-layered module which is a strong ω-elongation of a totally projective module by a $(\omega + n)$-projective module if and only if M is a totally projective module.*

Proof Since M is a strong ω-elongation, $H_\omega(M)$ is totally projective and there exists a submodule $N \subseteq H^n(M)$ such that $\frac{M}{N+H_\omega(M)}$ is a direct sum of uniserial modules and $\frac{M}{N+H_\omega(M)} \simeq \frac{M/H_\omega(M)}{(N+H_\omega(M))/H_\omega(M)}$. Now by the definition of n-layered modules, $H^n(M) = \bigcup_{j<\omega} M_j$, $M_j \subseteq M_{j+1} \subseteq H^n(M)$ and $M_j \cap H_j(M) = H^n(H_\omega(M))$, for every j. Since $N \subseteq H^n(M)$, $N = \bigcup_{j<\omega} N_j$ where $N_j = N \cap M_j$ and $\left(\frac{N+H_\omega(M)}{H_\omega(M)}\right) = \bigcup_{j<\omega}\left(\frac{(N_j+H_\omega(M))}{H_\omega(M)}\right)$.

Now,

$$\begin{aligned}\left(\frac{(N_j + H_\omega(M))}{H_\omega(M)}\right) \cap H_j\left(\frac{M}{H_\omega(M)}\right) &= \frac{(N_j + H_\omega(M))}{H_\omega(M)} \cap \left(\frac{H_j(M)}{H_\omega(M)}\right),\\ &= \frac{[(N_j + H_\omega(M)) \cap H_j(M)]}{H_\omega(M)},\\ &= \frac{(H_\omega(M) + (N_j \cap H_j(M)))}{H_\omega(M)},\\ &= 0.\end{aligned}$$

Therefore $M/H_\omega(M)$ is a direct sum of uniserial modules and M is totally projective.

We have shown that if N is a finite submodule of M such that $N \cap H_n(M) = H_n(N)$, then M is an n-layered module if and only if M/N is an n-layered module. Moreover, in [7] we showed that M is Σ-module if and only if M/N is a Σ-module.

We generalize this assertion to n-layered modules for an arbitrary natural number n. For doing this, we need following technical lemmas:

Lemma 3 *Let N be a finitely generated submodule of M. Then for an integer $n \geq 1$,*

$$H^n(M/N) = \frac{H^n(M) + K}{N}$$

where K is a finitely generated submodule of M with $H_n(K) \subseteq N \subseteq K$.

Proof Let $x + N \in H^k(M/N)$ for some $x \in M$ such that there exists $y \in N$ with $d\left(\frac{xR}{yR}\right) = n$. We may express $N \cap H_n(M) = \sum_{i=1}^m x_i R$ for some $m \in Z^+$ and put $K = N + \sum y_i R$, where $d\left(\frac{y_i R}{x_i R}\right) = n$. If $y_k \in M$ such that $k \neq 1, \ldots, m$ and $x_k \in N$ such that $d\left(\frac{y_k R}{x_k R}\right) = n$, then $x_k R = x_i R$ for some $i \in \{1, 2, \ldots, m\}$. Therefore $y_k R \subseteq y_i R + H^n(M) \subseteq K + H^n(M)$. The converse is trivial and the result follows.

Lemma 4 *Let K be a h-finite submodule of M having only finite heights in M. If N is a finitely generated submodule of M then $N + K$ is also h-finite assuming finite heights only.*

Proof Since the elements of K assumes only finite number of finite heights, $K \cap H_k(M) \subseteq M^1$, for some $k \geq 1$. Now N is finitely generated submodule and we may express N as $\sum_{i=1}^m x_i R$. Consider the submodule N' of N where $N' = \sum_{i=1}^t x_i R$ such that $x_i + y_i \in H_{n_i}(M)$ but $x_i + y_i \notin H_{n_{i+1}}(M)$ with $n_i > k$, $\forall\, i = 1, 2, \ldots, t$ for some $y_i \in K$. Therefore for each $y \in K$ we have $y + x_i = y - y_i + y_i + x_i \notin H_{n_i+1}(M)$, otherwise $y - y_i + y_i + x_i \in H_{n_i}(M)$ implying that $y - y_i \in H_k(M)$ and $y - y_i \in M^1$. Therefore $y_i + x_i \in H_{n_i+1}(M)$ which is a contradiction whenever $1 \leq i \leq t$. If we put $n = \max\{n_1 + 1, \ldots, n_t + 1\}$, $y + x_i \notin H_n(M)$. Since $y + x_j \notin H_n(M)$ for $t + 1 \leq j \leq n$, we are done.

Now we are ready to prove our main result:

Theorem 2 *For each natural number n, a $QTAG$-module M is n-layered if and only if M/N is n-layered, where N is a finitely generated submodule of M.*

Proof Suppose that M is an n-layered QTAG-module, then $H^n(M) = \bigcup_{i<\omega} M_i$, $M_i \subseteq M_{i+1} \subseteq H^n(M)$ and, for all $i < \omega$ $M_i \cap H_i(M) \subseteq M^1$. By Lemma 3 we may write $H^n(M/N) = (H^n(M) + K)/N$, for some finitely generated submodule K of M containing N. Furthermore, $H^n(M/N) = \bigcup_{i<\omega}((M_i + K)/N)$ and by Lemma 4, we calculate that

$$\left(\frac{M_i+K}{N}\right)\bigcap H_{t_i}\left(\frac{M}{N}\right)=\left(\frac{M_i+K}{N}\right)\bigcap\left(\frac{H_{t_i}(M)+N}{N}\right)$$
$$=\frac{(M_i+K)\cap(H_{t_i}(M)+N)}{N}$$
$$=\frac{(M_i+K)\cap H_{t_i}(M)+N}{N}\subseteq\frac{M^1+N}{N}\subseteq\left(\frac{M}{N}\right)^1$$

for every i and some natural number $t_i\geq i$, implying that M/N is an n-layered module.

For reverse implication, suppose that M/N is an n-layered module. Now write $H^n(M/N)=\bigcup_{i<\omega}\left(\frac{T_i}{N}\right)$, $T_i\subseteq T_{i+1}\subseteq M$ and for all $i<\omega$,

$$\left(\frac{T_i}{N}\right)\bigcap H_i\left(\frac{M}{N}\right)=\left(\frac{M}{N}\right)^1.$$

Since N is finitely generated it is nice in M. Now we may say $\frac{H^n(M)+N}{N}\subseteq H^n\left(\frac{M}{N}\right)$, $\bigcup_{i<\omega}\left(\frac{T_i}{N}\right)=\frac{\bigcup_{i<\omega}T_i}{N}$ and $\left(\frac{M}{N}\right)^1=\frac{M^1+N}{N}$. Therefore $H^n(M)=\bigcup_{i<\omega}H^n(T_i)$ and $\left(\frac{T_i}{N}\right)\cap\left(\frac{H_i(M)+N}{N}\right)=\frac{M^1+N}{N}$.
Therefore

$$\frac{(T_i\cap H_i(M)+N)}{N}=\frac{M^1+N}{N}\quad\text{and}$$
$$\frac{N+(T_i\cap H_i(M))}{N}=\frac{M^1+N}{N}$$
$$\Rightarrow T_i\cap H_i(M)\subseteq M^1+N.$$

Since N is finitely generated so there exists $m\in\mathbb{N}$ such that $N\cap H_m(M)\subseteq M^1$, therefore

$$T_i\cap H_{t_i}(M)\subseteq(M^1+N)\cap H_m(M)$$
$$\subseteq M^1+N\cap H_m(M)=M^1,$$

for every i and $t_i=m+i$, implying that M is also n-layered.

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