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Loewy decomposition of linear differential equations

Fritz Schwarz

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Abstract This paper explains the developments on factorization and decomposition of linear differential equations in the last two decades. The results are applied for developing solution procedures for these differential equations. Although the subject is more than 100 years old, it has been rediscovered as recently as about 20 years ago. A fundamental ingredient has been the easy availability of symbolic computation systems to accomplish the extensive calculations usually involved in applications; to this end the interactive website http://www.alltypes.de has been provided. Although originally only developed for ordinary equations, it has been extended to large classes of partial equations as well. In the first part Loewy's results for ordinary equations are outlined. Thereafter those results of differential algebra are summarized that are required for extending Loewy's theory to partial equations. In the remaining part a fairly complete discussion of second- and some third-order partial differential equations in the plane is given; it is shown that Loewy's result remains essentially true for these equations. Finally, several open problems and possible extensions are discussed.

Keywords Linear differential equations · Factorization · Loewy decomposition

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F. Schwarz (✉)
Fraunhofer Gesellschaft, Institut SCAI, 53754 Sankt Augustin, Germany
e-mail: fritz.schwarz@scai.fraunhofer.de

1 Introduction

Solving differential equations has always been a central topic in pure as well as in applied mathematics. The efforts for developing systematic solution procedures go back to the middle of the nineteenth century. Guiding principles were often the methods that had been developed before for solving algebraic equations. A well-known example is Lie's symmetry analysis that was inspired by the success of Galois' theory. Its primary area of application are nonlinear differential equations [43,51].

For linear equations another important principle is the decomposition into constituents of lower order as is well known from algebraic equations. Originally it has been applied to linear ordinary differential equations (ode's) by Beke [3] and Schlesinger [48]. Based on their work, Loewy [39] took up the subject and developed a complete theory of factoring an ordinary equation of any order. It culminates in a theorem that provides a unique decomposition of any equation into so-called completely reducible components of highest possible order; it turned out to be a fundamental prerequisite for solving an equation.

Soon after Loewy's theory had become known, several efforts were started to extend it to linear partial differential equations (pde's) [4,41]. However, for reasons that will become clear later in this article, they were only of limited success. Actually the whole field of factorization fell into oblivion for almost a century. One possible reason is that applications to any concrete problem require huge amounts of analytical computations that hardly can be performed by pencil and paper. As a result, for a long time the best possible proceeding for solving linear differential equations consisted of searching through collections of solved examples as e.g. the books by Kamke [28] or Polyanin [46]. This is rather unsatisfactory and obviously a better approach is desirable in compliance with the following guidelines.

First of all, it has to be made precise what is meant by a solution to a given differential equation. In this article, numerical or graphical solutions are excluded as well as any kind of series expansions. The goal is to find closed form solutions in a well defined function field, e.g. an elementary or Liouvillian extension of the coefficient field of the given equation; the latter is usually the field of rational functions in the independent variables. A solution is called the *general solution* if it contains undetermined elements such that any other solution may be obtained from it by specialization. For ordinary equations of order n this means to find n functions such that the general solution is a linear combination with n constant coefficients. For partial equations the answer is more involved; the undetermined elements may be functions depending on one or more arguments. It is part of the problem to ascertain the number of undetermined functions and its arguments.

After it has been settled what it means to solve an equation the question remains how to find the solution, if there is any. In this process any trial-and-error methods are excluded, i.e. the outcome should be algorithms that guarantee to find any solution that may exist; if the algorithm terminates without returning a solution it should be equivalent to a proof that there is none. In other words, the desired algorithms should be decision procedures for the existence of solutions under consideration.

It turns out that for partial differential equations it is not always possible to find an algorithm for the various steps of the solution procedure; e.g. at present there does not exist a general procedure for determining a Laplace divisor of any order. In such a situation the question arises whether an algorithm for the problem under consideration may exist at all. If it could be proved that an algorithm does not exist the problem would be undecidable like for example solving a general diophantine equation. The best possible answer then is to identify classes of decidable problems. Beyond that, it would be highly desirable to identify the boundary between problems that are algorithmically solvable and those for which an algorithm cannot exist.

Efforts along these lines started about 20 years ago, fostered by the advent of symbolic computation systems. At the beginning various aspects of factoring linear ode's were considered, e.g. algorithmic and complexity issues by Schwarz [49] and Grigoriev [18]; quite a few publications on the subject followed, e.g. by Bronstein [5] and van Hoeij [22].

The next natural step was to consider special systems of linear pde's with the property that its general solution contains only constants like linear ode's. In a series of publications Li, Schwarz and Tsarev [36–38] considered such systems of pde's in the plane and showed that a theory similar as for the ordinary case may be developed.

General decomposition problems for linear pde's were first considered by Grigoriev and Schwarz [19] and continued later on [20,21]; see also Tsarev [57].

The subject of this article are factorizations and the corresponding Loewy decompositions for individual equations, either ordinary or partial in two independent variables. It will become clear that a Loewy decomposition provides the most complete information on the solutions of the respective equation.

Section 2 outlines Loewy's theory for ordinary differential equations of any order. Equations of order two or three are discussed in more detail; the close relation between a decomposition and a fundamental system is shown and illustrated by examples.

Section 3 summarizes some concepts from differential algebra. They are required in later parts dealing with partial differential equations.

Section 4 deals with factorizations and decompositions of partial differential operators of order two and three in the plane.

Section 5 applies the results of the preceding section for solving differential equations corresponding to the respective operators.

Section 6. The results of the preceding sections are summarized and its limitations are pointed out. Various possible extensions are discussed.

Appendix. A short outline of the solutions of ordinary and partial Riccati equations is given; they are required in various solution procedures described in preceding sections.

The website http://www.alltypes.de has been provided for applying the theory described in this article to concrete problems; a short description of it may be found in [52]. A more detailed discussion of the subject of this article may be found in the book [53].

2 Ordinary differential equations

It is assumed that the reader is familiar with the classical literature on ordinary differential equations as covered e.g. in the books by Ince [24] or Kamke [27]; more details on factorization of linear ode's may be found in Chapter 4 of the book by van der Put and Singer [58], or in Chapter 2 of the book by Schwarz [51].

Let $D \equiv \frac{d}{dx}$ denote the derivative w.r.t. the variable x. A differential operator of order n is a polynomial of the form

$$L \equiv D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n \tag{2.1}$$

where the coefficients $a_i, i = 1, \ldots, n$ are from some function field, the *base field* of L. Usually it is the field of rational functions in the variable x, i.e. $a_i \in \mathbb{Q}(x)$. If y is an indeterminate with $\frac{dy}{dx} \neq 0$, Ly becomes a differential polynomial, and $Ly = 0$ is the differential equation corresponding to L. Sometimes the notation $y' \equiv \frac{dy}{dx}$ is applied.

The equation $Ly = 0$ allows the *trivial solution* $y \equiv 0$. The general solution contains n constants $C_1, \ldots, C_n$. Due to its linearity, these constants appear in the form $y = C_1 y_1 + \cdots + C_n y_n$. The y_k are linearly independent over the field of constants; they form a so-called *fundamental system* and generate a n-dimensional vector space.

2.1 Factoring linear ode's

An operator L of order n is called *reducible* if it may be represented as the product of two operators L_1 and L_2, both of order lower than n; then one writes $L = L_1 L_2$, i.e. juxtaposition means the operator product; it is defined by the rule $Da_i = a_i D + a_i'$. It is non-commutative, i.e. in general $L_1 L_2 \neq L_2 L_1$. Consequently left and right factors have to be distinguished. If $L = L_1 L_2$, the left factor L_1 is called the *exact quotient* of L by L_2, and L_2 is said to divide L from the right; it is a *right divisor* or simply *divisor* of L.

By default, the coefficient domain of the factors is assumed to be the base field of L, possibly extended by some algebraic numbers, i.e. $\bar{\mathbb{Q}}(x)$ is allowed; sometimes it is also called the base field. An operator or an equation is called *irreducible* if such a decomposition is not possible without enlarging the base field. If the coefficients of the factors may be from an extension of the base field it has to be specified explicitly. Very much like in commutative algebra, any operator L may be represented as product of first-order factors if coefficients from a universal field are admitted for them.

For any two operators L_1 and L_2 the *least common left multiple* $Lclm(L_1, L_2)$ is the operator of lowest order such that both L_1 and L_2 divide it from the right. The *greatest common right divisor* $Gcrd(L_1, L_2)$ is the operator of highest order that divides both L_1 and L_2 from the right. Two operators L_1 and L_2 are called *relatively prime* if there is no operator of positive order dividing both on the right. If an operator may be represented as $Lclm$ of irreducible operators it is called *completely reducible*.

Due to the non-commutativity of the product of differential operators another new phenomenon occurs in comparison to algebraic polynomials, i.e. the factorization of differential operators is not unique as the following example due to Landau [33] shows.

Example 2.1 [33] Consider $L \equiv D^2 - \frac{2}{x} D + \frac{2}{x^2}$. Two possible factorizations are $L = \left(D - \frac{1}{x}\right)\left(D - \frac{1}{x}\right) = \left(D - \frac{1}{x}\right)^2$ and $L = \left(D - \frac{1}{x(1+x)}\right)\left(D - \frac{1+2x}{x(1+x)}\right)$.

More generally, the factorization

$$\left(D - \frac{1}{x(1+ax)}\right)\left(D - \frac{1+2ax}{x(1+ax)}\right)$$

is valid with a constant parameter a. On the other hand, L may be represented as

$$L = Lclm\left(D - \frac{1}{x} - \frac{1}{C_1 + x}, D - \frac{1}{x} - \frac{1}{C_2 + x}\right)$$

with $C_1 \neq C_2$.

A systematic scheme for obtaining a *unique* decomposition of a differential operator of any order into lower order components has been given by Loewy [39]; see also Ore [44]. At first the irreducible right factors beginning with lowest order are determined, e.g. by applying Lemma 2.4 below. The $Lclm$ of these factors is the completely reducible right factor of highest order; by construction it is uniquely determined by the given operator. If its order equals n, the order of the given differential operator, the operator is called completely reducible and the procedure terminates. According to this definition, Landau's operator in the above example is completely reducible.

If an operator is not completely reducible, the $Lclm$ of its irreducible right factors is divided out and the same procedure is repeated with the exact quotient. Due to the lowering of order in each step, this proceeding terminates after a finite number of iterations and the desired decomposition is obtained. Based on these considerations, Loewy [39] obtained the following fundamental result.

Theorem 2.2 *Let $D = \frac{d}{dx}$ be a derivative and $a_i \in \mathbb{Q}(x)$. A differential operator*

$$L \equiv D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

of order n may be written uniquely as the product of completely reducible factors $L_k^{(d_k)}$ of maximal order d_k over $\mathbb{Q}(x)$ in the form

$$L = L_m^{(d_m)} L_{m-1}^{(d_{m-1})} \ldots L_1^{(d_1)}$$

with $d_1 + \cdots + d_m = n$. The factors $L_k^{(d_k)}$ are unique.

The decomposition determined in the above theorem is called the *Loewy decomposition* of L. The completely reducible factors $L_k^{(d_k)}$ are called the *Loewy factors* of L, the rightmost of them is simply called the *Loewy factor*.

Loewy's decomposition may be refined if the completely reducible components are split into irreducible factors as shown next.

Corollary 2.3 *Any factor $L_k^{(d_k)}$, $k = 1, \ldots, m$ in Theorem 2.2 may be written as*

$$L_k^{(d_k)} = Lclm(l_{j_1}^{(e_1)}, l_{j_2}^{(e_2)}, \ldots, l_{j_k}^{(e_k)})$$

with $e_1 + e_2 + \cdots + e_k = d_k$; $l_{j_i}^{(e_i)}$ *for* $i = 1, \ldots, k$, *denotes an irreducible operator of order* e_i *over* $\mathbb{Q}(x)$.

Loewy's fundamental result described in Theorem 2.2 and the preceding corollary provide a detailed description of the function spaces containing the solution of a reducible linear differential equation. More general field extensions associated to irreducible equations are studied by differential Galois theory. Good introductions to the latter are the above quoted book by van der Put and Singer and the lecture by Magid [40].

Next the question comes up how to obtain a factorization for any given equation or operator; for order two and three the answer is as follows.

Lemma 2.4 *Determining the right irreducible factors of an ordinary operator up to order three with rational function coefficients amounts to finding rational solutions of Riccati equations; as usual* $D \equiv \frac{d}{dx} ='$.

(i) *A second order operator* $D^2 + AD + B$, $A, B \in \mathbb{Q}(x)$ *has a right factor* $D + a$ *with* $a \in \mathbb{Q}(x)$ *if* a *is a rational solution of*

$$a' - a^2 + Aa - B = 0.$$

(ii) *A third order operator* $D^3 + AD^2 + BD + C$, $A, B, C \in \mathbb{Q}(x)$ *has a right factor* $D + a$ *with* $a \in \mathbb{Q}(x)$ *if* a *is a rational solution of*

$$a'' - 3aa' + a^3 + A(a' - a^2) + Ba - C = 0.$$

It has a right factor $D^2 + bD + c$, $b, c \in \mathbb{Q}(x)$, *if* b *is a rational solution of*

$$\begin{aligned} &b'' - 3bb' + b^3 + 2A(b' - b^2) + (A' + A^2 + B)b - B' - AB + C = 0;\\ &\textit{then}\quad c = -(b' - b^2 + Ab - B). \end{aligned}$$

Proof Dividing the given second-order operator by $D + a$ and requiring that this division be exact yields immediately the given constraint. The same is true if the given third order operator is divided by $D + a$. Dividing the given third-order operator by $D + bD + c$ yields a system comprising two equations that may easily be simplified to the above conditions. □

This lemma reduces the problem of determining right factors in the base field for second- and third-order operators to finding rational solutions of Riccati equations; a short outline is given in the Appendix.

For operators of fixed order the possible Loewy decompositions, differing by the number and the order of factors, may be listed explicitly; some of the factors may contain parameters. Each alternative is called a *type of Loewy decomposition*. The complete answers for the most important cases $n = 2$ and $n = 3$ are detailed in the following corollaries to the above theorem.

Corollary 2.5 *Let* L *be a second-order operator. Its possible Loewy decompositions are denoted by* $\mathcal{L}_0^2, \ldots \mathcal{L}_3^2$, *they may be described as follows;* $l^{(i)}$ *and* $l_j^{(i)}$ *are irreducible operators of order* i; C *is a constant;*

$$\mathcal{L}_1^2 : L = l_2^{(1)} l_1^{(1)};\ \mathcal{L}_2^2 : L = Lclm(l_2^{(1)}, l_1^{(1)});\ \mathcal{L}_3^2 : L = Lclm(l^{(1)}(C)).$$

An irreducible second-order operator is defined to have decomposition type $\mathcal{L}_0^2$. The decompositions $\mathcal{L}_0^2$, $\mathcal{L}_2^2$ and $\mathcal{L}_3^2$ are completely reducible; for decomposition type $\mathcal{L}_1^2$ the unique first-order factors are the Loewy factors.

Proof According to Lemma 2.4, case (i), the possible factors are determined by the solutions of a Riccati equation; they are completely classified in the Appendix. The general solution may be rational; there may be two non-equivalent or a single rational solution; or there may be no rational solution at all; these alternatives correspond to $\mathcal{L}_3^2$, $\mathcal{L}_2^2$, $\mathcal{L}_1^2$ or $\mathcal{L}_0^2$ respectively. □

By definition, a Loewy factor comprises *all* irreducible right factors, either corresponding to special rational solutions or those involving a constant. As a consequence, the decomposition type $\mathcal{L}_1^2$ implies that a decomposition of type $\mathcal{L}_2^2$ or $\mathcal{L}_3^2$ does *not* exist, i.e. there is only a single first-order right factor. Similarly, decomposition type $\mathcal{L}_2^2$ excludes type $\mathcal{L}_3^2$. A factor containing a parameter corresponds to a factorization that is not unique; any special value for C generates a special irreducible factor. Because the originally given operator has order 2, two different special values must be chosen in order to represent it in the form $Lclm(l^{(1)}(C_1), l^{(1)}(C_2))$, $C_1 \neq C_2$.

For operators of order 3 there are 12 types of Loewy decompositions.

Corollary 2.6 *Let L be a third-order operator. Its possible Loewy decompositions are denoted by $\mathcal{L}_0^3, \ldots \mathcal{L}_{11}^3$, they may be described as follows; $l_j^{(i)}$ is an irreducible operator of order i; $L_j^{(i)}$ is a Loewy factor of order i as defined above; C, C_1 and C_2 are constants;*

$$\begin{aligned}
&\mathcal{L}_1^3 : L = l^{(2)} l^{(1)};\quad \mathcal{L}_2^3 : L = l_1^{(1)} l_2^{(1)} l_3^{(1)};\quad \mathcal{L}_3^3 : L = Lclm(l_1^{(1)}, l_2^{(1)}) l_3^{(1)};\\
&\mathcal{L}_4^3 : L = Lclm(l_1^{(1)}(C)) l_2^{(1)};\quad \mathcal{L}_5^3 : L = l^{(1)} l^{(2)};\quad \mathcal{L}_6^3 : L = l_1^{(1)} Lclm(l_2^{(1)}, l_3^{(1)});\\
&\mathcal{L}_7^3 : L = l_1^{(1)} Lclm(l_2^{(1)}(C));\quad \mathcal{L}_8^3 : L = Lclm(l_1^{(1)}, l_2^{(1)}, l_3^{(1)});\\
&\mathcal{L}_9^3 : L = Lclm(l^{(2)}, l^{(1)});\quad \mathcal{L}_{10}^3 : L = Lclm(l_1^{(1)}(C), l_2^{(1)});\\
&\mathcal{L}_{11}^3 : L = Lclm(l^{(1)}(C_1, C_2)).
\end{aligned}$$

An irreducible third-order operator is defined to have decomposition type $\mathcal{L}_0^3$. The decompositions $\mathcal{L}_0^3$, and $\mathcal{L}_8^3$ through $\mathcal{L}_{11}^3$ are completely reducible; the decomposition types $\mathcal{L}_1^3$, $\mathcal{L}_3^3$ and $\mathcal{L}_4^3$ have the structure $L = L_2^{(2)} L_1^{(1)}$; the decomposition types $\mathcal{L}_5^3$, $\mathcal{L}_6^3$ and $\mathcal{L}_7^3$ have the structure $L = L_2^{(1)} L_1^{(2)}$.

Proof According to Lemma 2.4, case (ii), the possible factors are determined by the solutions of a second-order Riccati equation. They are completely classified in the Appendix. The general solution may be rational involving two constants; there may be a rational solution involving a single constant and in addition a special rational solution; there may be only a rational solution involving a single constant; or there may be three, two or a single special rational solution, or nor rational solution at all. These alternatives may easily be correlated with the various decomposition types given above. □

2.2 Solving linear homogeneous ode's

There remains to be discussed how the solution procedure for a linear ode with a nontrivial Loewy decomposition is simplified. The general procedure that applies to reducible equations of any order is described first.

Proposition 2.7 *Let a linear differential operator P of order n factor into $P = QR$ with R of order m and Q of order $n - m$. Further let $y_1, \dots, y_m$ be a fundamental system for $R(y) = 0$, and $\bar{y}_1, \dots, \bar{y}_{n-m}$ a fundamental system for $Q(y) = 0$. Then a fundamental system for $P(y) = 0$ is given by the union of $y_1, \dots, y_m$ and special solutions of $R(y) = \bar{y}_i$ for $i = 1, \dots, n - m$.*

Proof From $Ry_i = 0$ it follows that $Py_i = QRy_i = 0$, i.e. y_i belongs to a fundamental system of $Py = 0$ if this is true for $Ry = 0$; this proves the first part. Furthermore, from $Ry_i = \bar{y}_i$ and $Q\bar{y}_i = 0$ it follows that $Py_i = QRy_i = 0$; this proves the second part. □

This proceeding will be applied for solving reducible second- and third-order equations. The following two corollaries are obtained by straightforward application of the above Proposition 2.7. In some cases a solution of an inhomogeneous second-order linear ode is required.

Corollary 2.8 *Let L be a second-order differential operator, $D \equiv \frac{d}{dx}$, y a differential indeterminate, and $a_i \in \mathbb{Q}(x)$. Define $\varepsilon_i(x) \equiv \exp\left(-\int a_i dx\right)$ for $i = 1, 2$ and $\varepsilon(x, C) \equiv \exp\left(-\int a(C) dx\right)$, C is a parameter; the barred quantities $\bar{C}$ and $\bar{\bar{C}}$ are arbitrary numbers, $\bar{C} \neq \bar{\bar{C}}$. For the three nontrivial decompositions of Corollary 2.5 the following elements y_1 and y_2 of a fundamental system are obtained.*

$$\mathcal{L}_1^2 : Ly = (D+a_2)(D+a_1)y=0; \quad y_1=\varepsilon_1(x), \; y_2=\varepsilon_1(x)\int \frac{\varepsilon_2(x)}{\varepsilon_1(x)}\, dx.$$

$$\mathcal{L}_2^2 : Ly = Lclm(D + a_2, D + a_1)y = 0; \quad y_i = \varepsilon_i(x); \quad a_1 \text{ is not equivalent to } a_2.$$

$$\mathcal{L}_3^2 : Ly = Lclm(D + a(C))y = 0; \quad y_1 = \varepsilon(x, \bar{C}), \; y_2 = \varepsilon(x, \bar{\bar{C}}).$$

Here two rational functions $p, q \in \mathbb{Q}(x)$ are called *equivalent* if there exists another rational function $r \in \mathbb{Q}(x)$ such that $p - q = \frac{r'}{r}$ holds.

For differential equations of order three there are four decomposition types into first-order factors with no constants involved. Fundamental systems for them may be obtained as follows.

Corollary 2.9 *Let L be a third-order differential operator, $D \equiv \frac{d}{dx}$, y a differential indeterminate, and $a_i \in \mathbb{Q}(x)$. Define $\varepsilon_i(x) \equiv \exp\left(-\int a_i dx\right)$ for $i = 1, 2, 3$. For the four nontrivial decompositions of Corollary 2.6 involving only first-order factors without parameters the following elements y_i, $i = 1, 2, 3$, of a fundamental system are obtained.*

$\mathcal{L}_2^3 : Ly = (D + a_3)(D+a_2)(D+a_1)y = 0;\ \ y_1 = \varepsilon_1(x),\ \ y_2 = \varepsilon_1(x)\int \frac{\varepsilon_2(x)}{\varepsilon_1(x)}\,dx,$

$$y_3 = \varepsilon_1(x)\left(\int \frac{\varepsilon_3(x)}{\varepsilon_2(x)}\,dx \int \frac{\varepsilon_2(x)}{\varepsilon_1(x)}\,dx - \int \frac{\varepsilon_3(x)}{\varepsilon_2(x)} \int \frac{\varepsilon_2(x)}{\varepsilon_1(x)}\,dx\,dx\right).$$

$\mathcal{L}_3^3 : Ly = Lclm(D + a_3, D + a_2)(D + a_1)y = 0;$

$$y_1 = \varepsilon_1(x),\ \ y_2 = \varepsilon_1(x)\int \frac{\varepsilon_2(x)}{\varepsilon_1(x)}\,dx,\ \ y_3 = \varepsilon_1(x)\int \frac{\varepsilon_3(x)}{\varepsilon_1(x)}\,dx.$$

$\mathcal{L}_6^3 : Ly = (D + a_3)Lclm(D + a_2, D + a_1)y = 0;\quad a_1 \neq a_2,\ \ y_i = \varepsilon_i(x)$ *for* $i = 1, 2;$

$$y_3 = \varepsilon_1(x)\int \frac{\varepsilon_3(x)}{\varepsilon_1(x)}\frac{dx}{a_2 - a_1} - \varepsilon_2(x)\int \frac{\varepsilon_3(x)}{\varepsilon_2(x)}\frac{dx}{a_2 - a_1}.$$

$\mathcal{L}_8^3 : Ly = Lclm(D + a_3, D + a_2, D + a_1)y = 0,\ \ y_i = \varepsilon_i(x),\quad i = 1, 2, 3.$

The proof of these corollaries is a straightforward application of Proposition 2.7. The following examples show how they may be applied for solving linear ode's.

Example 2.10 Equation 2.201 from Kamke's collection has the $\mathcal{L}_2^2$ decomposition

$$y'' + \left(2 + \frac{1}{x}\right)y' - \frac{4}{x^2}y$$
$$= Lclm\left(D + \frac{2}{x} - \frac{2x - 2}{x^2 - 2x + \frac{3}{2}},\ D + 2 + \frac{2}{x} - \frac{1}{x + \frac{3}{2}}\right)y = 0.$$

The coefficients $a_1 = 2 + \frac{2}{x} - \frac{1}{x + \frac{3}{2}}$ and $a_2 = \frac{2}{x} - \frac{2x - 2}{x^2 - 2x + \frac{3}{2}}$ are rational solutions of $a' - a^2 + \left(2 + \frac{1}{x}\right) + \frac{4}{x^2} = 0$ corresponding to case (i) of Lemma 2.4; they yield the fundamental system

$$y_1 = \frac{2}{3} - \frac{4}{3x} + \frac{1}{x^2},\ y_2 = \frac{2}{x} + \frac{3}{x^2}e^{-2x}.$$

Example 2.11 Equation 3.73 from Kamke's collection has the $\mathcal{L}_2^3$ decomposition

$$\mathcal{L}_2^3 : y''' - \left(\frac{2}{x+1} + \frac{2}{x}\right)y'' + \left(\frac{6}{x} + \frac{4}{x^2} - \frac{6}{x+1}\right)y' + \left(\frac{8}{x} - \frac{8}{x^2} - \frac{4}{x^3} - \frac{8}{x+1}\right)y$$
$$= \left(D - \frac{2}{x+1} - \frac{1}{x}\right)\left(D - \frac{1}{x}\right)\left(D - \frac{2}{x}\right)y = 0.$$

It has the coefficients $a_1 = -\frac{2}{x}, a_2 = -\frac{1}{x}$ and $a_3 = \frac{1}{x} - \frac{2}{x+1}$; they yield $\varepsilon_1 = x^2, \varepsilon_2 = x$ and $\varepsilon_3 = \frac{1}{x}(x+1)^2$. If they are substituted in the above expressions the fundamental system

$$y_1 = x^2, \quad y_2 = x^2 \log x \quad \text{and} \quad y_3 = x + x^3 + x^2 \log(x)^2$$

is obtained.

Example 2.12 The $\mathcal{L}_6^3$ decomposition

$$\mathcal{L}_6^3 : y''' - y'' - \left(\frac{1}{x-2} - \frac{1}{x}\right) y' + \left(\frac{1}{x-2} - \frac{1}{x}\right) y$$
$$= \left(D + \frac{1}{x-2} + \frac{1}{x}\right) Lclm \left(D - 1, D - \frac{2}{x}\right) y = 0$$

of equation 3.37 of Kamke's collection yields the coefficients $a_1 = -\frac{2}{x}$, $a_2 = -1$ and $a_3 = \frac{1}{x} + \frac{1}{x-2}$. If they are substituted in the above expressions, the fundamental system

$$y_1 = x^2, \quad y_2 = e^x, \quad \text{and} \quad y_3 = \frac{x(x^2-2)}{4(x-2)} + \frac{x^2}{4} \log \frac{x-2}{x} + e^x \int e^{-x} \frac{dx}{(x-2)^2}$$

is obtained.

In addition to the decompositions of the above Corollary 2.9 there are four decompositions into first-order factors involving one or two parameters. They are considered next.

Corollary 2.13 *Let L be a third-order differential operator, $D \equiv \frac{d}{dx}$, y a differential indeterminate, and $a_i \in \mathbb{Q}(x)$. Define $\varepsilon_i(x) \equiv \exp\left(-\int a_i\, dx\right)$ for $i = 1, 2, 3$, $\varepsilon(x, C) \equiv \exp\left(-\int a(C)\, dx\right)$ and $\varepsilon(x, C_1, C_2) = \exp\left(-\int a(C_1, C_2)\, dx\right)$; C, C_1 and C_2 are parameters; the barred quantities $\bar{C}$, $\bar{C}_i$ etc are numbers; for each case they are pairwise different from each other. For the four decompositions of Corollary 2.6 involving first-order factors and parameters the following elements y_i, $i = 1, 2, 3$, of a fundamental system are obtained.*

$$\mathcal{L}_4^3 : Ly = Lclm\,(D + a(C))\,(D + a_1)y = 0;$$
$$y_1 = \varepsilon_1(x),\ y_2 = \varepsilon_1(x) \int \frac{x, \varepsilon(x, \bar{C})}{\varepsilon_1(x)}\, dx,\ y_3 = \varepsilon_1(x) \int \frac{\varepsilon(x, \bar{\bar{C}})}{\varepsilon_1(x)}\, dx;$$
$$\mathcal{L}_7^3 : Ly = (D + a_3) Lclm\,(D + a(C))\,y = 0; \quad y_1 = \varepsilon(x, \bar{C}),\ y_2 = \varepsilon(x, \bar{\bar{C}}),$$
$$y_3 = \varepsilon(\bar{C}, x) \int \frac{\varepsilon_3(x)}{\varepsilon(\bar{C}, x)} \frac{dx}{a(\bar{\bar{C}}) - a(\bar{C})} - \varepsilon(\bar{\bar{C}}, x) \int \frac{\varepsilon_3(x)}{\varepsilon(\bar{\bar{C}}, x)} \frac{dx}{a(\bar{\bar{C}}) - a(\bar{C})}$$
$$\mathcal{L}_{10}^3 : Ly = Lclm(D + a(C), D + a_1)y = 0;$$
$$y_1 = \varepsilon_1(x),\ y_2 = \varepsilon(x, \bar{C}),\ y_3 = \varepsilon(x, \bar{\bar{C}}).$$
$$\mathcal{L}_{11}^3 : Ly = Lclm(D + a(C_1, C_2))y = 0,$$
$$y_1 = \varepsilon(x, \bar{C}_1, \bar{C}_2),\ y_2 = \varepsilon(x, \bar{\bar{C}}_1, \bar{\bar{C}}_2),\ y_2 = \varepsilon(x, \bar{\bar{\bar{C}}}_1, \bar{\bar{\bar{C}}}_2).$$

Again the proof follows immediately from Proposition 2.7 and is omitted. The following examples show some applications.

Example 2.14 The equation with the type $\mathcal{L}_4^3$ decomposition

$$y''' - \frac{4x^2-1}{x^3}y'' + \frac{6x^2-10}{x^4}y' + \frac{30}{x^5}y$$
$$= Lclm\left(D - \frac{2}{x} - \frac{1}{x+C}\right)\left(D + \frac{1}{x^3}\right)y = 0$$

leads to $\varepsilon_1(x) = \exp\left(\frac{1}{2x^2}\right)$ and $\varepsilon(C,x) = (C+x)x^2$. This yields the fundamental system

$$y_1 = \exp\left(\frac{1}{2x^2}\right),\ y_2 = \exp\left(\frac{1}{2x^2}\right)\int \exp\left(-\frac{1}{2x^2}\right)x^3\,dx,$$
$$y_3 = \exp\left(\frac{1}{2x^2}\right)\int \exp\left(-\frac{1}{2x^2}\right)x^2\,dx.$$

Example 2.15 An equation with type $\mathcal{L}_{10}^3$ decomposition is

$$y''' - \frac{x^3-3x+3}{x(x^2+x-1)} - \frac{x+3}{x^2+x-1}y' + \frac{x+3}{x(x^2+x-1)}y$$
$$= Lclm\left(D + \frac{1}{x} - \frac{2x}{x^2+C}, D-1\right)y = 0$$

with fundamental system $y_1 = e^x$, $y_2 = x$ and $y_3 = \frac{1}{x}$; y_2 and y_3 satisfy $y_2 - x^2y_3 = 0$ whereas y_1 is linearly independent of y_2 and y_3 over the base field.

The results of this section show that factorization provides an algorithmic scheme for solving linear ode's. Whenever an equation of order two or three factorizes according to one of the types defined above the elements of a fundamental system may be given explicitly, i.e. factorization is equivalent to solving it. If an equation is irreducible it may occur that its Galois group is nontrivial. In these cases algebraic solution may exist; otherwise there may exist special function solutions, see [6] or [51], page 39.

The website http://www.alltypes.de provides an interactive userinterface for applying the above results to concrete problems.

3 Rings of partial differential operators

In the ring of ordinary differential operators all ideals are principal. Hence, the relation between an individual operator and the ideal that is generated by it is straightforward. The situation is different in rings of partial differential operators where in general ideals may have any number of generators, and only a Janet basis provides a unique representation. Therefore a more algebraic language is appropriate for dealing with partial differential operators and the ideals or modules they generate. Some basic concepts of this subject are introduced in this section; general references are the books by Kolchin [31], Kaplansky [29] or van der Put and Singer [58], or the article by Buium and Cassidy [8].

3.1 Basic differential algebra

A field $\mathcal{F}$ is called a *differential field* if it is equipped with a *derivation operator*. An operator δ on a field $\mathcal{F}$ is called a derivation operator if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for all elements $a, b \in \mathcal{F}$. A field with a single derivation operator is called an *ordinary differential field*; if there is a finite set Δ containing several commuting derivation operators the field is called a *partial differential field*.

In this article rings of differential operators with derivatives $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$ with coefficients from some differential field are considered. Its elements have the form $\sum_{i,j} r_{i,j}(x,y)\partial_x^i \partial_y^j$; almost all coefficients $r_{i,j}$ are zero. The coefficient field is called the *base field*. If constructive and algorithmic methods are the main issue it is $\mathbb{Q}(x,y)$. However, in some places this is too restrictive and a suitable extension $\mathcal{F}$ of it may be allowed. The respective ring of differential operators is denoted by $\mathcal{D} = \mathbb{Q}(x,y)[\partial_x, \partial_y]$ or $\mathcal{D} = \mathcal{F}[\partial_x, \partial_y]$; if not mentioned explicitly, the exact meaning will be clear from the context.

The ring $\mathcal{D}$ is non-commutative, $\partial_x a = a\partial_x + \frac{\partial a}{\partial x}$ and similarly for the other variables; a is from the base field.

For an operator $L = \sum_{i+j \le n} r_{i,j}(x,y)\partial_x^i \partial_y^j$ of order n the *symbol of* L is the homogeneous algebraic polynomial $symb(L) \equiv \sum_{i+j=n} r_{i,j}(x,y)X^i Y^j$, X and Y algebraic indeterminates.

Let I be a left ideal which is generated by elements $l_i \in \mathcal{D}$, $i = 1, \dots, p$. Then one writes $I = \langle l_1, \dots, l_p \rangle$. Because right ideals are not considered in this article, sometimes I is simply called an ideal. In this article the term ideal means always differential ideal.

A m-dimensional left *vector module* $\mathcal{D}^m$ over $\mathcal{D}$ has elements $(l_1, \dots, l_m)$, $l_i \in \mathcal{D}$ for all i. The sum of two elements of $\mathcal{D}^m$ is defined by componentwise addition; multiplication with a ring element l by $l(l_1, \dots, l_m) = (ll_1, \dots, ll_m)$.

The relation between left ideals in $\mathcal{D}$ or submodules of $\mathcal{D}^m$ on the one hand, and systems of linear pde's on the other is established as follows. Let $(z_1, \dots, z_m)^T$ be an m-dimensional column vector of differential indeterminates such that $\partial_x z_i \ne 0$ and $\partial_y z_i \ne 0$. Then the product

$$(l_1, \dots, l_m)(z_1, \dots, z_m)^T = l_1 z_1 + l_2 z_2 + \dots + l_m z_m \tag{3.1}$$

defines a linear differential polynomial in the z_i that may be considered as the left hand side of a partial differential equation; $z_1, \dots, z_m$ are called the *dependent variables* or *functions*, depending on the *independent variables* x and y.

A $N \times m$ matrix $\{c_{i,j}\}$, $i = 1, \dots, N$, $j = 1, \dots, m$, $c_{i,j} \in \mathcal{D}$, defines a system of N linear homogeneous pde's

$$c_{i,1} z_1 + \dots + c_{i,m} z_m = 0, \quad i = 1, \dots, N. \tag{3.2}$$

The $i-th$ equation of (3.2) corresponds to the vector

$$(c_{i,1}, c_{i,2}, \ldots, c_{i,m}) \in \mathcal{D}^m \quad \text{for } i = 1, \ldots, N. \tag{3.3}$$

This correspondence between the elements of $\mathcal{D}^m$, the differential polynomials (3.1), and its corresponding pde (3.2) allows to turn from one representation to the other whenever it is appropriate.

For $m = 1$ this relation becomes more direct. The elements $l_i \in \mathcal{D}$ are simply applied to a single differential indeterminate z. In this way the ideal $I = \langle l_1, l_2, \ldots\rangle$ corresponds to the system of pde's $l_1 z = 0, l_2 z = 0, \ldots$ for the single function z. Sometimes the abbreviated notation $Iz = 0$ is applied for the latter.

3.2 Janet bases of ideals and modules

The generators of an ideal are highly non-unique; its members may be transformed in infinitely many ways by taking linear combinations of them or its derivatives without changing the ideal. This ambiguity makes it difficult to decide membership in an ideal or to recognize whether two sets of generators represent the same ideal. Furthermore, it is not clear what the solutions of the corresponding system of pde's are. The same remarks apply to the vector-modules introduced above.

This was the starting point for Maurice Janet [25] early in the 20th century to introduce a normal form for systems of linear pde's that has been baptized *Janet basis* in [50]. They are the differential analog to Gröbner bases of commutative algebra, originally introduced by Bruno Buchberger [7], see also the interesting article by Gjunter [47]; therefore they are also called *differential Gröbner basis*. Good introductions to the subject may be found in the articles by Oaku [42], Castro-Jiménez and Moreno-Frías [10], Plesken and Robertz [45] or Chapter 2 of Schwarz [51].

In order to generate a Janet basis, a ranking of derivatives must be defined. It is a total ordering such that for any derivatives δ, δ_1 and δ_2, and any derivation operator θ obeys $\delta \preceq \theta\delta$, and $\delta_1 \preceq \delta_2 \rightarrow \delta\delta_1 \preceq \delta\delta_2$. In this article lexicographic term orderings lex and graded lexicographic term orderings $grlex$ are applied. For partial derivatives of a single function their definition is analogous to the monomial orderings in commutative algebra. If $x \succ y$ is defined, derivatives up to order three in lex order are arranged like

$$\partial_{xxx} \succ \partial_{xxy} \succ \partial_{xx} \succ \partial_{xyy} \succ \partial_{xy} \succ \partial_x \succ \partial_{yyy} \succ \partial_{yy} \succ \partial_y \succ 1, \tag{3.4}$$

and in $grlex$ ordering

$$\partial_{xxx} \succ \partial_{xxy} \succ \partial_{xyy} \succ \partial_{yyy} \succ \partial_{xx} \succ \partial_{xy} \succ \partial_{yy} \succ \partial_x \succ \partial_y \succ 1. \tag{3.5}$$

For modules these orderings have to be generalized appropriately, e.g. the orderings TOP or POT of Adams and Loustaunau [1] may be applied.

The following convention will always be obeyed. In an individual operator or differential polynomial the terms are arranged decreasingly from left to right, i.e. the first term contains the highest derivative. A collection of such objects like the generators of an ideal or a module is arranged such that the leading terms do not increase. In particular, if the leading terms are pairwise different they will decrease from left to right, and from top to bottom. If the term order is not explicitly given it is assumed to be $grlex$ with $x \succ y$.

The most distinctive feature of a Janet basis is the fact that it contains all algebraic consequences for the derivatives in the ideal generated by its members explicitly. This is achieved by two basic operations, *reductions* and adding *integrability conditions*; the latter correspond to the S-pairs in commutative algebra.

An operator l_1 may be reduced w.r.t. another operator l_2 if the leading derivative of l_2 or a derivative thereof occurs in l_1. If this is true, its occurrence in l_1 may be removed by replacing it by the negative reductum of l_2 or its appropriate derivative. This process may be repeated until no further reduction is possible. This process will always terminate because in each step the derivatives in l_1 are lowered. The following example shows a single-step reduction of two operators.

Example 3.1 Let two operators l_1 and l_2 be given.

$$l_1 \equiv \partial_{xy} - \frac{x^2}{y^2}\partial_x - \frac{x-y}{y^2}, l_2 \equiv \partial_x + \frac{1}{y}\partial_y + x.$$

The derivatives ∂_{xy} and ∂_x may be removed from l_1 with the result

$$\begin{aligned} Reduce(l_1, l_2) &= -\frac{1}{y}\partial_{yy} + \frac{1}{y^2}\partial_y - x\partial_y + \frac{x^2}{y^2}\left(\frac{1}{y}\partial_y + x\right) - \frac{x-y}{y^2} \\ &= -\frac{1}{y}\left(\partial_{yy} + \frac{1}{y^2}(xy^3 - x^2 - y)\partial_y - \frac{1}{y}(x^3 - x + y)\right). \end{aligned}$$

There are no further reductions possible.

If a system of operators or differential polynomials is given, various reductions may be possible between pairs of its members. If all of them have been performed such that no further reduction is possible, the system is called *autoreduced*.

For an autoreduced system the integrability conditions have to be investigated. They arise if the same leading derivative occurs in two different members of the system or its derivatives. Upon subtraction, possibly after multiplication with suitable factors from the base field, the difference does not contain it any more. If it does not vanish after reduction w.r.t. the remaining members of the system, it is called an *integrability condition* that has to be added to the system. The following example shows this process.

Example 3.2 Consider the ideal

$$I = \left\langle l_1 \equiv \partial_{xx} - \frac{1}{x}\partial_x - \frac{y}{x(x+y)}\partial_y, l_2 \equiv \partial_{xy} + \frac{1}{x+y}\partial_y, l_3 \equiv \partial_{yy} + \frac{1}{x+y}\partial_y \right\rangle$$

in *grlex* term order with $x \succ y$. Its generators are autoreduced. If the integrability condition

$$l_{1,y} = l_{2,x} - l_{2,y} = \frac{y+2x}{x(x+y)}\partial_{xy} + \frac{y}{x(x+y)}\partial_{yy}$$

is reduced w.r.t. to I, the new generator ∂_y is obtained. Adding it to the generators and performing all possible reductions, the given ideal is represented as $I = \langle \partial_{xx} - \frac{1}{x}\partial_x, \partial_y \rangle$. Its generators are autoreduced and the single integrability condition is satisfied.

It may be shown that for any given system of operators or differential polynomials and a fixed ranking autoreduction and adding integrability conditions always terminates with a unique result. Due to its fundamental importance a special term is introduced for it.

Definition 3.3 (*Janet basis*) For a given ranking an autoreduced system of differential operators is called a Janet basis if all integrability conditions reduce to zero.

If a system of operators or differential polynomials forms a Janet basis, it is a unique representation for the ideal or module it generates. The proof and many details may be found in the references given above.

Due to its importance the following notation will be applied from now on. If the generators of an ideal or module are assured to be a Janet basis they are enclosed by a pair of $\langle\!\langle \ldots \rangle\!\rangle$. In general, if the Janet basis property is not known, the usual notation $\langle \ldots \rangle$ will be applied. Therefore in the preceding example the result may be written as $I = \langle\!\langle \partial_{xx} - \frac{1}{x}\partial_x, \partial_y \rangle\!\rangle$. By definition, a single element l is a Janet basis, i.e. $\langle l \rangle = \langle\!\langle l \rangle\!\rangle$ is always valid. A system of operators or pde's with the property that all integrability conditions are satisfied is called *coherent*.

3.3 General properties of ideals and modules

Just like in commutative algebra, the generators of an ideal in a ring of differential operators obey certain relations which are known as *syzygies*. Let a set of generators be $f = \{f_1, \ldots, f_p\}$ where $f_i \in \mathcal{D}$ for all i. Syzygies of f are relations of the form

$$d_{k,1} f_1 + \cdots + d_{k,p} f_p = 0$$

where $d_{k,i} \in \mathcal{D}, i = 1, \ldots p, k = 1, 2, \ldots$. The $(d_{k,1}, \ldots, d_{k,p})$ may be considered as elements of the module $\mathcal{D}^p$. The totality of syzygies generates a submodule.

Example 3.4 Consider the ideal $\langle f_1 \equiv \partial_x + a, f_2 \equiv \partial_y + b \rangle$ with the constraint $a_y = b_x$. The coherence condition for $\partial_x - a - f_1 = 0$ and $\partial_y + b - f_2 = 0$ yields $a\partial_y + a_y - f_{1,y} - b\partial_x - b_x + f_{2,x} = 0$. Reduction w.r.t. to the given generators and some simplification yields the single syzygy $(\partial_y + b) f_1 - (\partial_x + a) f_2 = 0$.

Example 3.5 Consider the ideal

$$\left\langle\!\left\langle f_1 \equiv \partial_{xx} + \frac{4}{x}\partial_x + \frac{2}{x^2},\ f_2 \equiv \partial_{xy} + \frac{1}{x}\partial_y,\ f_3 \equiv \partial_{yy} + \frac{1}{y}\partial_y - \frac{x}{y^2}\partial_x - \frac{2}{y^2}\right\rangle\!\right\rangle.$$

The integrability condition for $\partial_{xx} + \frac{4}{x}\partial_x + \frac{2}{x^2} - f_1 = 0$ and $\partial_{xy} + \frac{1}{x}\partial_y - f_2 = 0$ yields upon reduction and simplification $f_{1,y} + f_{2,x} - \frac{3}{x} f_2 = 0$. Similarly from the last two elements $f_1 - \frac{y^2}{x} f_{2,y} - \frac{y}{x} f_2 + \frac{y^2}{x} f_{3,x} + \frac{y^2}{x^2} f_3 = 0$ is obtained. Autoreduction of these two equations yields the following two syzygies as the final answer

$$\left(\partial_{yy} + \frac{3}{y}\partial_y - \frac{x}{y^2}\partial_x - \frac{2}{y^2}\right) f_2 - \left(\partial_{xy} + \frac{1}{x}\partial_y + \frac{2}{y}\partial_x + \frac{2}{xy}\right) f_3 = 0,$$

$$f_1 - \left(\frac{y^2}{x}\partial_y + \frac{y}{x}\right) f_2 + \left(\frac{y^2}{x}\partial_x + \frac{y^2}{x^2}\right) f_3 = 0.$$

Consider $I \subseteq \mathcal{D}$, and denote by I_n the intersection of I with the $\mathcal{F}$-linear space of all derivatives of order not higher than n. Then according to Kolchin [31], see also Buium and Cassidy [8], and Pankratiev et al. [32], the Hilbert–Kolchin polynomial of I is defined by

$$H_I(n) \equiv \binom{n+k}{k} - \dim I_n; \tag{3.6}$$

k is the number of variables. The first term equals the number of all derivatives of order not higher than n. Consequently, for sufficiently large n the value of $H_I(n)$ counts the number of derivatives of order not higher than n which is not in the ideal generated by the leading derivatives of the generators of I. The degree $deg(H_I)$ of H_I is called the *differential type* of I ([31], page 130; [8], page 602). Its leading coefficient $lc(H_I)$ is called the *typical differential dimension* of I, ibid. If $I, J \subseteq \mathcal{D}$ are two ideals, Sit [55], Theorem 4.1, has shown the important equality

$$lc(H_{I+J}) + lc(H_{I\cap J}) = lc(H_I) + lc(H_J) \tag{3.7}$$

for its typical differential dimensions.

According to Kolchin [30], $deg(H_I)$ and $lc(H_I)$) are differential birational invariants; their importance justifies the introduction of a special term for these quantities.

Definition 3.6 The pair $(deg(H_I), lc(H_I))$ for an ideal I is called the differential dimension of I, denoted by d_I.

For the solutions of the differential equations attached to any ideal or module of differential operators, these quantities have an important meaning [8,30].

Theorem 3.7 [30] *The differential type denotes the largest number of arguments occurring in any undetermined function of the general solution. The typical differential dimension means the number of functions depending on this maximal number of arguments.*

Apparently the differential dimension describes somehow the *"size"* of the solution space. In this terminology the differential dimension $(0, m)$ corresponds to a system of pde's with a finite-dimensional solution space over the field of constants of dimension m. This discussion shows that the differential dimension is the proper generalization of the dimension of a solution space to general systems of linear pde's. In the language of differential forms this problem has been considered in full generality by Cartan [9], Chapter IV; see also [54].

Example 3.8 For the ideal $I = \langle\!\langle \partial_{xx} - \frac{1}{x}\partial_x, \partial_y \rangle\!\rangle$ only the two derivatives 1 and ∂_x are not contained in the ideal generated by the leading derivatives. It follows that $H_I = 2$ and $d_I = (0, 2)$.

Example 3.9 Let the principal ideal $I = \langle \partial_x + a\partial_y + b \rangle$ be given. There are $\frac{1}{2}(n^2 + n)$ derivatives of order not higher than n containing at least a single derivative ∂_x. Therefore $H_I = n + 1$ and $d_I = (1, 1)$.

Example 3.10 Consider the ideal $I = \langle\!\langle \partial_{xxx}, \partial_{xxy} \rangle\!\rangle$. The number of derivatives which are multiples of either leading term is $\frac{1}{2}(n-2)(n+1)$. Therefore $H_I = 2n + 2$ and $d_I = (1, 2)$.

Given any ideal I it may occur that it is properly contained in some larger ideal J with coefficients in the base field of I; then J is called a *divisor* of I. If the divisor J has the same differential type as I the latter is called *reducible*; if such a divisor does not exist it is called *irreducible*. If a divisor ideal of the same differential type does not exist even if a universal differential field is allowed for its coefficients, I is called *absolutely irreducible*. According to this definition an ideal may be irreducible, yet it may have divisors of lower differential type as the following example shows.

Example 3.11 Consider the operator L defined by $L \equiv \partial_{xx} + \frac{2}{x}\partial_x + \frac{y}{x^2}\partial_y - \frac{1}{x^2}$ of differential dimension $(1, 2)$, i.e. its differential type is 1. The principal ideal $\langle L \rangle$ has the two divisors $l_1 = \langle\!\langle \partial_x + \frac{1}{x}, \partial_y - \frac{1}{y} \rangle\!\rangle$ and $l_2 = \langle\!\langle \partial_x, \partial_y - \frac{1}{y} \rangle\!\rangle$, both of differential type 0; $l_1 z = 0$ has the solution $z = \frac{y}{x}$, $l_2 z = 0$ has the solution $z = y$. Both are also solutions of $Lz = 0$. It can be shown that L is irreducible according to the above definition, and even absolutely irreducible.

The *greatest common right divisor (Gcrd)* or *sum* of two ideals I and J is the smallest ideal with the property that both I and J are contained in it. If they have the representation

$$I \equiv \langle f_1, \ldots, f_p \rangle \quad \text{and} \quad J \equiv \langle g_1, \ldots, g_q \rangle,$$

$f_i, g_j \in \mathcal{D}$ for all i and j, the sum is generated by the union of the generators of I and J (Cox et al. [11,12], page 191). The solution space of the equations corresponding to $Gcrd(I, J)$ is the intersection of the solution spaces of its arguments.

The *least common left multiple (Lclm)* or *left intersection* of two ideals I and J is the largest ideal with the property that it is contained both in I and J. The solution space of $Lclm(I, J)z = 0$ is the smallest space containing the solution spaces of its arguments.[1]

Example 3.12 Consider the ideals $I = \langle\!\langle \partial_{yyy} + \frac{3}{y}\partial_{yy}, \partial_x + \frac{y}{x}\partial_y \rangle\!\rangle$ and

$$J = \left\langle\!\!\left\langle \partial_{xx} + \frac{1}{x}\partial_x - \frac{1}{x^2}, \partial_{xy} + \frac{1}{x}\partial_y + \frac{1}{y}\partial_x + \frac{1}{xy}, \partial_{yy} + \frac{1}{y}\partial_y - \frac{1}{y^2} \right\rangle\!\!\right\rangle.$$

According to the above definitions the $Gcrd$ and the $Lclm$ are

$$Gcrd(I, J) = \left\langle\!\!\left\langle \partial_{yy} + \frac{1}{y}\partial_y - \frac{1}{y^2}, \partial_x + \frac{y}{x}\partial_y \right\rangle\!\!\right\rangle,$$

$$Lclm(I, J) = \left\langle\!\!\left\langle \partial_{yyy} + \frac{3}{y}\partial_{yy}, \partial_{xx} - \frac{y^2}{x^2}\partial_{yy} + \frac{1}{x}\partial_x - \frac{y}{x^2}\partial_y, \partial_{xy} + \frac{y}{x}\partial_{yy} + \frac{1}{y}\partial_x + \frac{2}{x}\partial_y \right\rangle\!\!\right\rangle;$$

$lc(H_I) = lc(H_J) = 3, lc(H_{I+J}) = 2$ and $lc(H_{I\cap J}) = 4$ in accordance with Sit's relation (3.7). In terms of solution spaces this result may be understood as follows. For $Iz = 0$ a basis of the solution space is $\left\{1, \frac{x}{y}, \frac{y}{x}\right\}$, and for $Jz = 0$ a basis is $\left\{\frac{1}{xy}, \frac{x}{y}, \frac{y}{x}\right\}$. A basis for their two-dimensional intersection space is $\left\{\frac{x}{y}, \frac{y}{x}\right\}$, it is the solution space of $Gcrd(I, J)z = 0$.

Example 3.13 Consider the two ideals

$$I = \left\langle\!\!\left\langle \partial_x + \frac{1}{x}, \partial_y + \frac{1}{y} \right\rangle\!\!\right\rangle \quad \text{and} \quad J = \left\langle\!\!\left\langle \partial_x + \frac{1}{x+y}, \partial_y + \frac{1}{x+y} \right\rangle\!\!\right\rangle.$$

Their one-dimensional solution spaces are generated by $\left\{\frac{1}{xy}\right\}$ and $\left\{\frac{1}{x+y}\right\}$ respectively. It follows that $Gcrd(I, J) = \langle 1 \rangle$ and

$$Lclm(I, J) = \left\langle\!\!\left\langle \partial_{yy} + \frac{2x+4y}{xy+y^2}\partial_y + \frac{2}{xy+y^2}, \partial_x - \frac{y^2}{x^2}\partial_y + \frac{x-y}{x^2} \right\rangle\!\!\right\rangle.$$

A basis for the solution space of $Lclm(I, J)z = 0$ is $\left\{\frac{1}{xy}, \frac{1}{x+y}\right\}$.

For ordinary differential operators the exact quotient has been defined above. Because all ideals of ordinary differential operators are principal, it is obtained by the usual division scheme. This is different in rings of partial differential operators and a proper generalization of the exact quotient is required. Let $I \equiv \langle f_1, \ldots, f_p \rangle \in \mathcal{D}$ and $J \equiv \langle g_1, \ldots, g_q \rangle \in \mathcal{D}$ be such that $I \subseteq J$, i.e. J is a *divisor* of I. Their *exact quotient* is generated by

$$\{(e_{i,1}, \ldots, e_{i,q}) \in \mathcal{D}^q | e_{i,1}g_1 + \cdots + e_{i,q}g_q = f_i, i = 1, \ldots, p\}.$$

[1] Some authors define it as the lowest element w.r.t. the term order in the ideal defined above.

The *exact quotient module* $Exquo(I, J)$ is generated by

$$\{h = (h_1, \ldots, h_q) \in \mathcal{D}^q | h_1 g_1 + \cdots + h_q g_q \in I\}.$$

It generalizes the syzygy module of J; the latter is obtained for the special choice $I = 0$. If the elements of the exact quotient module are arranged as rows of a matrix with q columns, and the generators of J as elements of a q-dimensional vector, they satisfy

$$I = Exquo(I, J)J. \tag{3.8}$$

This defines the juxtaposition of $Exquo(I, J)$ and J in terms of matrix multiplication; it generalizes the product representation $L = L_1 L_2$ of an operator L. In general, the result at the right hand side of (3.8) has to be transformed into a Janet basis in order to obtain the original generators $f_1, \ldots, f_p$.

Example 3.14 [37,38] Consider the following ideal in $grlex$; $x \succ y$ term order $I = \left\langle\!\left\langle \partial_{yy} - \frac{xy-1}{y}\partial_y - \frac{x}{y}, \partial_x - \frac{y}{x}\partial_y \right\rangle\!\right\rangle$ and its divisor $J = \langle\!\langle \partial_x - y, \partial_y - x \rangle\!\rangle$. Division yields $\left\langle\!\left\langle \left(0, \partial_y + \frac{1}{y}\right), \left(1, -\frac{y}{x}\right) \right\rangle\!\right\rangle$. There is the single syzygy $(\partial_y - x, -\partial_x + y)$ of J (compare Example 3.4). From these three generators the Janet basis for the exact quotient module is obtained in the form

$$Exquo(I, J) = \left\langle\!\left\langle (0, \partial_x), \left(0, \partial_y + \frac{1}{y}\right), \left(1, -\frac{y}{x}\right) \right\rangle\!\right\rangle.$$

Example 3.15 Consider the ideal $I = \left\langle\!\left\langle \partial_{xx} + \frac{1}{x}\partial_x, \partial_{xy}, \partial_{yy} + \frac{1}{y}\partial_y \right\rangle\!\right\rangle$ in $grlex, x \succ y$ term order. There is a single maximal divisor $J = \langle\!\langle \partial_x, \partial_y \rangle\!\rangle$. It yields the exact quotient module $Exquo(I, J) = \left\langle\!\left\langle \left(\partial_x + \frac{1}{x}, 0\right), (\partial_y, 0), (0, \partial_x), \left(0, \partial_y + \frac{1}{y}\right) \right\rangle\!\right\rangle$. It may be represented as the intersection of two maximal modules of order 1, i.e.

$$\begin{aligned} Exquo(I, J) = Lclm \Bigg(&\left\langle\!\left\langle (1, 0), (0, \partial_x) \left(0, \partial_y + \frac{1}{y}\right) \right\rangle\!\right\rangle, \\ &\left\langle\!\left\langle \left(\frac{x}{y}, 1\right), \left(\partial_x + \frac{1}{x}, 0\right), (\partial_y, 0) \right\rangle\!\right\rangle \Bigg). \end{aligned}$$

By analogy with the well known *Landau symbol* of asymptotic analysis, the following notation will frequently be applied. Whenever in an expression terms of order lower than some fixed term τ are not relevant, they are collectively denoted by $o(\tau)$. This will frequently occur in lex term orderings where τ denotes the highest term involving a particular variable.

Another short hand notation concerns the generators of ideals or modules of differential operators. If only the number of generators and its leading derivatives are of interest, the abbreviated notation $\langle \ldots \rangle_{LT}$ will be used. For example, if an ideal of differential operators is generated by two elements with leading derivatives ∂_{xx} and ∂_{xy}, it is denoted by $\langle \partial_{xx}, \partial_{xy} \rangle_{LT}$. A principal ideal that is generated by a single generator with highest derivative ∂_{xxx} is abbreviated by $\langle \partial_{xxx} \rangle_{LT}$.

3.4 Laplace divisors $\mathbb{L}_{x^m}(L)$ and $\mathbb{L}_{y^n}(L)$

The origin of these ideals goes back to Laplace who introduced an iterative solution scheme for equations with leading derivative z_{xy}. Later on it was realized that this procedure is essentially equivalent to determining a so called *involutive system*. In more modern language this comes down to constructing a Janet basis for an ideal that is generated by the operator corresponding to the originally given equation, and an ordinary operator of fixed order involving exclusively derivatives w.r.t. x or y. Thus this important concept may be generalized to large classes of operators with a mixed leading derivative. Due to its origin the following definition is suggested.

Definition 3.16 Let L be a partial differential operator in the plane; define

$$\mathfrak{l}_m \equiv \partial_{x^m} + a_{m-1}\partial_{x^{m-1}} + \cdots + a_1\partial_x + a_0 \tag{3.9}$$

and

$$\mathfrak{k}_n \equiv \partial_{y^n} + b_{n-1}\partial_{y^{n-1}} + \cdots + b_1\partial_y + b_0 \tag{3.10}$$

be ordinary differential operators w.r.t. x or y; $a_i, b_i \in \mathbb{Q}(x, y)$ for all i; m and n are natural numbers not less than 2. Assume the coefficients a_i, $i = 0, \ldots, m-1$ are such that L and $\mathfrak{l}_m$ form a Janet basis. If m is the smallest integer with this property then $\mathbb{L}_{x^m}(L) \equiv \langle\langle L, \mathfrak{l}_m\rangle\rangle$ is called a Laplace divisor of L. Similarly, if b_j, $j = 0, \ldots, n-1$ are such that L and $\mathfrak{k}_n$ form a Janet basis and n is minimal, then $\mathbb{L}_{y^n}(L) \equiv \langle\langle L, \mathfrak{k}_n\rangle\rangle$ is called a Laplace divisor of L. Both Laplace divisors have differential dimension $(1, 1)$.

The possible existence of a Laplace divisor for special operators of order two or three is investigated next.

Proposition 3.17 *Let the second-order partial differential operator*

$$L \equiv \partial_{xy} + A_1\partial_x + A_2\partial_y + A_3 \tag{3.11}$$

be given with $A_i \in \mathbb{Q}(x, y)$ for all i; m and n are natural numbers not less than 2.

(i) If A_1, A_2 and A_3 satisfy a single differential polynomial, there exists a Laplace divisor $\mathbb{L}_{x^m}(L) = \langle\langle L, \mathfrak{l}_m\rangle\rangle$.
(ii) If A_1, A_2 and A_3 satisfy a single differential polynomial there exists a Laplace divisor $\mathbb{L}_{y^n}(L) = \langle\langle L, \mathfrak{k}_n\rangle\rangle$.
(iii) If there are two Laplace divisors $\mathbb{L}_{x^m}(L)$ and $\mathbb{L}_{y^n}(L)$, the operator L is completely reducible; L may be represented as $\langle L\rangle = Lclm\big(\mathbb{L}_{x^m}(L), \mathbb{L}_{y^n}(L)\big)$.

The proof involves rather lengthy calculations; essentially the integrability conditions between (3.11) and (3.9) or (3.10) respectively have to be determined and reduced w.r.t. themselves. At the end, the two differential constraints mentioned in the above proposition for case (i) and case (ii) are obtained; if they are satisfied a Laplace divisor of the respective order does exist and may be determined algorithmically.

Example 3.18 The operator $L \equiv \partial_{xy} + xy\partial_x - 2y$ has been considered by Imschenetzky [23]. For $m = 3$ in (3.9 the coefficients are $a_0 = a_1 = a_2 = 0$; hence there is the divisor $\mathbb{L}_{x^3}(L) = \langle\!\langle \partial_{xxx}, \partial_{xy} + xy\partial_x - 2y \rangle\!\rangle$.

In general, no upper bound for the degrees m and n in the above proposition for a possible Laplace divisor is known. In special cases however such a bound may be obtained, e.g. it may be shown that a divisor $\mathbb{L}_{y^k}(L)$ does not exist for any value of k in the preceding example.

3.5 The ideals $\mathbb{J}_{xxx}$ and $\mathbb{J}_{xxy}$

Among the ideals involving only derivatives of order not higher than three there occur two ideals as generic intersection of first-order operators that are of particular importance; they are denoted by

$$\mathbb{J}_{xxx} \equiv \langle \partial_{xxx}, \partial_{xxy} \rangle_{LT} \quad \text{and} \quad \mathbb{J}_{xxy} \equiv \langle \partial_{xxy}, \partial_{xyy} \rangle_{LT}.$$

The subscripts of $\mathbb{J}$ denote the highest derivative occurring in the respective ideal. Both are generated by two third-order operators forming a Janet basis; their differential dimension is $(1, 2)$. For later use some of their properties are investigated next.

Lemma 3.19 *The ideal*

$$\begin{aligned}\mathbb{J}_{xxx} \equiv \langle L_1 &\equiv \partial_{xxx} + p_1\partial_{xyy} + p_2\partial_{yyy} + p_3\partial_{xx} + p_4\partial_{xy} + p_5\partial_{yy} + p_6\partial_x \\ &+ p_7\partial_y + p_8, L_2 \equiv \partial_{xxy} + q_1\partial_{xyy} + q_2\partial_{yyy} + q_3\partial_{xx} + q_4\partial_{xy} + q_5\partial_{yy} \\ &+ q_6\partial_x + q_7\partial_y + q_8 \rangle\end{aligned}$$

is coherent if the coefficients of its generators obey the conditions

$$\begin{aligned}
&p_1 - q_2 + q_1^2 = 0,\ p_2 + q_2q_1 = 0,\\
&q_{2,y} - q_{1,x} - q_{1,y}q_1 + p_4 - p_3q_1 - q_5 + 2q_4q_1 + q_3q_2 - 2q_3q_1^2 = 0,\\
&q_{2,x} + q_{1,y}q_2 - p_5 + p_3q_2 - q_5q_1 - q_4q_2 + 2q_3q_2q_1 = 0,\\
&p_{3,y} - q_{3,x} + q_{3,y}q_1 - q_6 + q_4q_3 - q_3^2q_1 = 0,\\
&p_{4,y} - q_{4,x} + q_{4,y}q_1 + p_6 + p_4q_3 - p_3q_4 - q_7 + q_6q_1 + q_4^2 - q_4q_3q_1 = 0,\\
&p_{5,y} - q_{5,x} + q_{5,y}q_1 + p_7 + p_5q_3 - p_3q_5 + q_7q_1 + q_5q_4 - q_5q_3q_1 = 0,\\
&p_{6,y} - q_{6,x} + q_{6,y}q_1 + p_6q_3 - p_3q_6 - q_8 + q_6q_4 - q_6q_3q_1 = 0,\\
&p_{7,y} - q_{7,x} + q_{7,y}q_1 + p_8 + p_7q_3 - p_3q_7 + q_8q_1 + q_7q_4 - q_7q_3q_1 = 0,\\
&p_{8,y} - q_{8,x} + q_{8,y}q_1 + p_8q_3 - p_3q_8 + q_8q_4 - q_8q_3q_1 = 0.
\end{aligned}$$

A grlex term order with $p_i \succ q_j$ for all i and j, $q_i \succ q_j$ and $p_i \succ p_j$ for $i < j$ is applied. There is a single syzygy between the generators.

$$L_{1,y} + q_3L_1 - L_{2,x} + q_1L_{2,y} - (p_3 + q_1q_3 - q_4)L_2 = 0. \tag{3.12}$$

These ten conditions are obtained by generating the integrability conditions between the operators L_1 and L_2 and reducing them w.r.t. L_1 and L_2. The syzygy is obtained by the procedure described at the beginning of Sect. 3.3. An example is given next.

Example 3.20 The ideal

$$\begin{aligned}&\langle\!\langle L_1 \equiv \partial_{xxx} - \partial_{xyy} + 2x\partial_{xy} - 3(3x^2+1)\partial_x + 4(2x^2+1)\partial_y - 8x^3 - 24x,\\ &L_2 \equiv \partial_{xxy} + \partial_{xyy} + x\partial_{xx} - (x^2+1)\partial_x - 4(2x^2+1)\partial_y - 8x^3\rangle\!\rangle\end{aligned}$$

is generated by a Janet basis because its coefficients satisfy the constraints of the preceeding lemma. The single syzygy $L_{1,y} + xL_1 = L_{2,x} + xL_2$ is particularly simple in this case.

Lemma 3.21 *The ideal*

$$\begin{aligned}\mathbb{J}_{xxy} \equiv \langle &K_1 \equiv \partial_{xxy} + p_1\partial_{yyy} + p_2\partial_{xx} + p_3\partial_{xy} + p_4\partial_{yy} + p_5\partial_x + p_6\partial_y + p_7,\\ &K_2 \equiv \partial_{xyy} + q_1\partial_{yyy} + q_2\partial_{xx} + q_3\partial_{xy} + q_4\partial_{yy} + q_5\partial_x + q_6\partial_y + q_7\rangle\end{aligned}$$

is coherent if the coefficients of its generators obey the conditions

$$\begin{aligned}&q_2 = 0,\ p_1 + q_1^2 = 0,\\ &p_{7,y} - q_{7,x} + q_{7,y}q_1 - p_2p_7 - p_3q_7 - p_7q_1q_2 + p_7q_3 - q_1q_3q_7 + q_4q_7 = 0,\\ &p_{6,y} - q_{6,x} + q_{6,y}q_1 - p_2p_6 - p_3q_6 - p_6q_1q_2\\ &\quad + p_6q_3 + p_7 - q_1q_3q_6 + q_1q_7 + q_4q_6 = 0,\\ &p_{5,y} - q_{5,x} + q_{5,y}q_1 - p_2p_5 - p_3q_5 - p_5q_1q_2\\ &\quad + p_5q_3 - q_1q_3q_5 + q_4q_5 - q_7 = 0,\\ &p_{4,y} - q_{4,x} + q_{4,y}q_1 - p_2p_4 - p_3q_4 - p_4q_1q_2\\ &\quad + p_4q_3 + p_6 - q_1q_3q_4 + q_1q_6 + q_4^2 = 0,\\ &p_{3,y} - q_{3,x} + q_{3,y}q_1 - p_2p_3 - p_3q_1q_2 + p_5 - q_1q_3^2 + q_1q_5 + q_3q_4 - q_6 = 0,\\ &p_{2,y} - q_{2,x} + q_{2,y}q_1 - p_2^2 - p_2q_1q_2\\ &\quad + p_2q_3 - p_3q_2 - q_1q_2q_3 + q_2q_4 - q_5 = 0,\\ &p_{1,y} - q_{1,x} + q_{1,y}q_1 - p_1p_2 - p_1q_1q_2\\ &\quad + p_1q_3 - p_3q_1 + p_4 - q_1^2q_3 + 2q_1q_4 = 0.\end{aligned}$$

A $grlex$ term order with $p_i \succ q_j$ for all i and j, $q_i \succ q_j$ and $p_i \succ p_j$ for $i < j$ is applied. There is a single syzygy between the generators.

$$K_{1,y} - (p_2 + q_1q_2 - q_3)K_1 - K_{2,x} + q_1K_{2,y} - (p_3 + q_1q_3 - q_4)K_2 = 0. \tag{3.13}$$

Similar remarks apply as above following Lemma 3.19.

3.6 Lattice structure of ideals in $\mathbb{Q}(x, y)[\mathcal{D}x, \mathcal{D}y]$

In any ring, commutative or not, its ideals form a lattice if the join operation is defined as the sum of ideals, and the meet operation as its intersection. In order to understand the structure of this lattice, these two operations have to be studied in detail. The basics of lattice theory required for this purpose may be found in the books by Grätzer [17] or Davey and Priestley [13].

The first result deals with a special case that guarantees the existence of a principal intersection ideal of first order operators.

Proposition 3.22 *Let L be a partial differential operator in x and y with leading term ∂_{x^n}, and let $l_i \equiv \partial_x + a_i\partial_y + b_i$, $i = 1, \ldots, n$, $a_i \neq a_j$ for $i \neq j$, be n right divisors of L. Then the intersection ideal generated by the l_i is principal and is generated by L.*

Proof Let $I_i = \langle l_i \rangle$ for $1 \leq i \leq n$ and $I = I_1 \cap \ldots \cap I_n$ be the intersection ideal. For any $P \in I$, $symb(P)$ is divided by $\prod_{1 \leq i \leq n}(\partial_x + a_i\partial_y)$, considered as algebraic polynomial in ∂_x and ∂_y; therefore $ord_x(P) \geq n$. On the other hand, according to Sit's relation (3.7) on page 34, for the typical differential dimension $dim(I) \leq n$ is valid. Hence if $I = \langle L \rangle$ is principal then $ord_x(L) = dim(I) = n$. Conversely let $P \in I$ and divide P by L with remainder, i.e. $P = QL + R$. Then $ord_x(R) < n$, therefore $R = 0$. Thus $I = \langle L \rangle$. □

The intersection ideals generated by two first-order operators in the plane are described in detail now.

Theorem 3.23 *Let the ideals $I_i = \langle \partial_x + a_i\partial_y + b_i \rangle$ for $i = 1, 2$ with $I_1 \neq I_2$ be given. Both ideals have differential dimension $(1, 1)$. There are three different cases for their intersection $I_1 \cap I_2$, all are of differential dimension $(1, 2)$.*

(i) *If $a_1 \neq a_2$ and $\left(\frac{b_1 - b_2}{a_1 - a_2}\right)_x = \left(\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}\right)_y$ then*
$I_1 \cap I_2 = \langle \partial_{xx} \rangle_{LT}$ *and* $I_1 + I_2 = \left\langle \partial_x + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}, \partial_y + \frac{b_1 - b_2}{a_1 - a_2} \right\rangle$.

(ii) *If $a_1 \neq a_2$ and $\left(\frac{b_1 - b_2}{a_1 - a_2}\right)_x \neq \left(\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}\right)_y$ then*
$I_1 \cap I_2 = \mathbb{J}_{xxx}$ *and* $I_1 + I_2 = \langle 1 \rangle$.

(iii) *If $a_1 = a_2 = a$ and $b_1 \neq b_2$ then*
$I_1 \cap I_2 = \langle \partial_{xx} \rangle_{LT}$ *and* $I_1 + I_2 = \langle 1 \rangle$.

Case (ii) is the generic case for the intersection of two ideals I_1 and I_2.

Proof The proof follows closely Grigoriev and Schwarz [19]. In accordance with Cox, Little and O'Shea [11,12], Theorem 11 on page 186, an auxiliary parameter u is introduced and the operators $u(\partial_x + a_1\partial_y + b_1)$ and $(1 - u)(\partial_x + a_2\partial_y + b_2)$ are considered. In order to compute generators for the intersection ideal, a Janet basis with u as the highest variable has to be generated. To this end, computationally it is more convenient to find the Janet basis with respect to the differential indeterminate z and a new indeterminate $w = uz$ with $w \succ z$ in a lexicographic term ordering. The intersection ideal is obtained from the expressions not involving w; the sum ideal is obtained by substituting $z = 0$. This yields the differential polynomials

$$w_x + a_1 w_y + b_1 w \quad \text{and} \quad w_x + a_2 w_y + b_2 w - z_x - a_2 z_y - b_2 z. \tag{3.14}$$

If $a_1 \neq a_2$ autoreduction leads to

$$\begin{aligned} &w_x + \frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} w - \frac{a_1}{a_1 - a_2}(z_x + a_2 z_y + b_2 z) \text{ and} \\ &w_y + \frac{b_1 - b_2}{a_1 - a_2} w + \frac{1}{a_1 - a_2}(z_x + a_2 z_y + b_2 z). \end{aligned} \tag{3.15}$$

Defining $U \equiv z_x + a_2 z_y + b_2 z$, the integrability condition between these two elements has the form

$$\left[\left(\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2}\right)_y - \left(\frac{b_1 - b_2}{a_1 - a_2}\right)_x\right] w - \frac{1}{a_1 - a_2} U_x - \frac{a_1}{a_1 - a_2} U_y$$
$$- \left[\left(\frac{1}{a_1 - a_2}\right)_x + \left(\frac{a_1}{a_1 - a_2}\right)_y + \frac{b_1}{a_1 - a_2}\right] U = 0.$$

If the coefficient of w vanishes, the remaining expression has the leading term z_{xx} and is the lowest element of a Janet basis; it corresponds to an intersection ideal $\mathbb{J}_{xxx}$. The sum ideal is obtained from (3.15). This is case (i).

If the coefficient of w does not vanish, this expression may be applied to eliminate w in (3.15). It yields two expressions with leading derivatives u_{xxx} and u_{xxy} respectively; they correspond to an intersection ideal $\mathbb{J}_{xxx}$. The sum ideal is trivial. This is case (ii).

Finally, if $a_1 = a_2 = a$, autoreduction of (3.14) yields two expressions of the type $w + o(z_x)$ and $o(z_{xx})$ respectively; they correspond to an intersection ideal $\langle \partial_{xx} \rangle_{LT}$ and a trivial sum ideal. This is case (iii).

Case (ii) is the generic case because it does not involve any constraints for the coefficients of the generators of I_1 and I_2. □

The two subsequent examples are applications of this theorem. In general it is difficult to find first-order operators generating an intersection ideal of moderate coefficient size; in particular this applies to the generic case.

Example 3.24 Consider the two ideals

$$I_1 = \langle \partial_x + 1 \rangle \quad \text{and} \quad I_2 = \langle \partial_x + (y+1)\partial_y \rangle,$$

both of differential dimension (1, 1). The condition for case (i) of the above theorem is satisfied. Consequently

$$\begin{aligned} Lclm(I_1, I_2) &= \langle\!\langle \partial_{xx} + (y+1)\partial_{xy} + \partial_x + (y+1)\partial_y \rangle\!\rangle, \\ Gcrd(I_1, I_2) &= \left\langle\!\!\left\langle \partial_x + 1, \partial_y - \frac{1}{y+1} \right\rangle\!\!\right\rangle; \end{aligned}$$

their differential dimension is (1, 2) and (0, 1) respectively.

Example 3.25 The two ideals $I_1 = \langle \partial_x + 1 \rangle$ and $I_2 = \langle \partial_x + x\partial_y \rangle$, both of differential dimension $(1, 1)$, do not satisfy the condition of case (i) of the above theorem; furthermore $a_1 \neq a_2$. Therefore by case (ii) the intersection ideal is

$$Lclm(I_1, I_2) = \Big\langle\!\Big\langle \partial_{xxx} - x^2\partial_{xyy} + 3\partial_{xx} + (2x+3)\partial_{xy} - x^2\partial_{yy} + 2\partial_x + (2x+3)\partial_y,$$
$$\partial_{xxy} + x\partial_{xyy} - \frac{1}{x}\partial_{xy} + x\partial_{yy} - \frac{1}{x}\partial_x - \left(1 + \frac{1}{x}\right)\partial_y \Big\rangle\!\Big\rangle$$

of differential dimension $(1, 2)$; $Gcrd(I_1, I_2) = \langle 1 \rangle$.

The above Theorem 3.23 does not cover the case that the two operators generating I_1 and I_2 have different leading derivatives; it is considered next.

Theorem 3.26 *Let the ideals $I_1 = \langle \partial_x + a_1\partial_y + b_1 \rangle$ and $I_2 = \langle \partial_y + b_2 \rangle$ be given, $I_1 \neq I_2$. There are two different cases for their intersection $I_1 \cap I_2$.*

(i) *If $(b_1 - a_1 b_2)_y = b_{2,x}$ then*
$I_1 \cap I_2 = \langle \partial_{xy} \rangle_{LT}$ and $I_1 + I_2 = \langle \partial_x + b_1 - a_1 b_2, \partial_y + b_2 \rangle$.

(ii) *If the preceding case does not apply then*
$I_1 \cap I_2 = \mathbb{J}_{xxy}$ and $I_1 + I_2 = \langle 1 \rangle$.

The proof is similar as for Theorem 3.23 and is therefore omitted.

4 Decomposing partial differential operators in the plane

This section deals with a genuine extension of Loewy's theory. The ideals under consideration have differential type greater than zero. This means the corresponding differential equations have a general solution involving not only constants but undetermined functions of varying numbers of arguments. Loewy's results are applied to individual linear pde's of second and third order in the plane with coordinates x and y, and the principal ideals generated by the corresponding operators. Second-order equations have been considered extensively in the literature of the nineteenth century [14,16,23,34,35]. Like in the classical theory, equations with leading derivatives ∂_{xx} or ∂_{xy} are distinguished.

4.1 Operators with leading derivative ∂_{xx}

At first the generic second-order operator with leading derivative ∂_{xx} is considered. It is not assumed that any coefficient of a lower derivative vanishes. As usual factorizations in the base field $\mathbb{Q}(x, y)$, possibly extended by some algebraic numbers, are considered.

Proposition 4.1 *Let the second-order partial differential operator*

$$L \equiv \partial_{xx} + A_1\partial_{xy} + A_2\partial_{yy} + A_3\partial_x + A_4\partial_y + A_5 \tag{4.1}$$

be given with $A_i \in \mathbb{Q}(x, y)$ *for all* i. *Its first order factors* $\partial_x + a\partial_y + b$ *with* $a, b \in \mathbb{Q}(x, y)$ *are determined by the roots* a_1 *and* a_2 *of* $a^2 - A_1 a + A_2 = 0$. *The following alternatives may occur.*

(i) *If* $a_1 \neq a_2$ *are two different rational solutions, and* b_1 *and* b_2 *are determined by* (4.5), *a factor* $l_i = \partial_x + a_i\partial_y + b_i$ *exists if the pair* a_i, b_i *satisfies* (4.6). *If there are two factors* l_1 *and* l_2, *the operator* (4.1) *is completely reducible with the representation* $L = Lclm(l_1, l_2)$.

(ii) *If* $a_1 = a_2 = a$ *is a double root and*

$$A_{1,x} + \frac{1}{2}A_1 A_{1,y} + A_1 A_3 = 2A_4, \tag{4.2}$$

the factorization depends on the rational solutions of the partial Riccati equation

$$b_x + \frac{1}{2}A_1 b_y - b^2 + A_3 b = A_5. \tag{4.3}$$

(a) *A right factor* $l(\Phi) \equiv \partial_x + \frac{1}{2}A_1\partial_y + R(x, y, \Phi(\varphi))$ *exists if* (4.3) *has a rational general solution* $R(x, y, \Phi(\varphi))$; $\varphi(x, y)$ *is a rational first integral of* $\frac{dy}{dx} = \frac{1}{2}A_1(x, y)$; Φ *is an undetermined function.* L *is completely reducible with the representation* $L = Lclm(l(\Phi_1), l(\Phi_2))$ *for any two choices* Φ_1 *and* Φ_2 *such that* $\Phi_1 \neq \Phi_2$.

(b) *A right factor* $l \equiv \partial_x + \frac{1}{2}A_1\partial_y + r(x, y)$ *exists if* (4.3) *has the single rational solution* $r(x, y)$.

(c) *Two right factors* $l_i \equiv \partial_x + \frac{1}{2}A_1\partial_y + r_i(x, y)$ *exist if* (4.3) *has the rational solutions* $r_1(x, y)$ *and* $r_2(x, y)$. *The operator* (4.1) *is completely reducible, it may be represented as* $L = Lclm(l_2, l_1)$.

Proof Dividing the operator (4.1) by $\partial_x + a\partial_y + b$, the condition that this division be exact leads to the following set of equations between the coefficients.

$$a^2 - A_1 a + A_2 = 0, \tag{4.4}$$

$$a_x + (A_1 - a)a_y + A_3 a + (A_1 - 2a)b = A_4, \tag{4.5}$$

$$b_x + (A_1 - a)b_y - b^2 + A_3 b = A_5. \tag{4.6}$$

The first equation determines the coefficient a.

Case (i) Assume that (4.4) has two simple rational roots a_1 and a_2. Then $a_1 \neq a_2$ and $a_i \neq \frac{1}{2}A_1$ for $i = 1, 2$; the second equation (4.5) determines rational values of b_1 and b_2. The third equation (4.6) is a constraint. Those pairs a_i, b_i which satisfy it lead to a factor. There may be none, a single one l_1, or two factors l_1 and l_2. In the latter case, by Proposition 3.22 L is given by $L = Lclm(l_2, l_1)$.

Case (ii) If $a_1 = a_2 = \frac{1}{2}A_1$ is a twofold root A_2 is given by $A_2 = \frac{1}{4}A_1^2$. The coefficient of b in Eq. (4.5) vanishes, it becomes the constraint (4.2). If it is not satisfied, factors cannot exist in any field extension. If it is satisfied, b is determined by (4.3) which is obtained from (4.6) by simplification. Depending on the type of its rational solutions (see Appendix), the three subcases (a), (b) or (c) occur. If there are two factors, by case (iii) of Theorem 3.23 their intersection equals L. □

In order to apply this result for solving any given differential equation involving the operator (4.1) the question arises whether its first-order factors may be determined algorithmically. The subsequent corollary provides the answer for factors with coefficients either in the base field or a universal field extension.

Corollary 4.2 *In general, first-order right factors of* (4.1) *in the base field cannot be determined algorithmically. The following cases are distinguished.*

(i) *Separable symbol polynomial. Any factor may be determined.*
(ii) *Double root of symbol polynomial. In general it is not possible to determine the right factors in the base field.*

The existence of factors in a universal field, i.e. absolute irreducibility, may always be decided.

Proof In the separable case , solving Eq. (4.4) and testing condition (4.6) requires only differentiations and arithmetic in the base field or in a quadratic function field; this can always be performed. In the non-separable case, testing condition (4.2) requires only arithmetic and differentiations in the base field. If it is not satisfied, factors cannot exist in any field extension. If it is satisfied, factors are determined by the solutions of the partial Riccati equation (4.3). However, in general no algorithm is known at present for determining the rational solutions of (4.3) as shown in the appendix. □

In particular this result means that factors of an equation that is irreducible but not absolutely irreducible in general may not be determined, although its existence is assured.

Applying Proposition 4.1, Loewy's Theorem 2.2 may be generalized to operators of the form (4.1) as follows.

Theorem 4.3 *Let the differential operator L be defined by*

$$L \equiv \partial_{xx} + A_1\partial_{xy} + A_2\partial_{yy} + A_3\partial_x + A_4\partial_y + A_5 \tag{4.7}$$

such that $A_i \in \mathbb{Q}(x, y)$ for all i. Let $l_i \equiv \partial_x + a_i\partial_y + b_i$ for $i = 1$ and $i = 2$, and $l(\Phi) \equiv \partial_x + a\partial_y + b(\Phi)$ be first-order operators with $a_i, b_i, a \in \mathbb{Q}(x, y)$; Φ is an undetermined function of a single argument. Then L has a Loewy decomposition according to one of the following types.

$$\mathcal{L}^1_{xx} : L = l_2 l_1; \quad \mathcal{L}^2_{xx} : L = Lclm(l_2, l_1); \quad \mathcal{L}^3_{xx} : L = Lclm(l(\Phi)). \tag{4.8}$$

If L does not have any first-order factor in the base field, its decomposition type is defined to be $\mathcal{L}^0_{xx}$. Decompositions $\mathcal{L}^0_{xx}$, $\mathcal{L}^2_{xx}$ and $\mathcal{L}^3_{xx}$ are completely reducible. For decomposition $\mathcal{L}^1_{xx}$ the first-order right factor is a Loewy divisor.

Proof It is based on Proposition 4.1. In the separable case (i) there are either two first order factors with a principal intersection corresponding to decomposition type

$\mathcal{L}_{xx}^2$; or a single first-order factor corresponding to type $\mathcal{L}_{xx}^1$; or no factor at all corresponding to type $\mathcal{L}_{xx}^0$. In case (ii), depending on the rational solutions of the Riccati equation (4.6), a similar distinction as in case (i) occurs. In addition there may be a factor containing an undetermined function yielding decomposition type $\mathcal{L}_{xx}^3$. $\square$

The subsequent examples are applications of the above theorem. Later in this article it will be shown how the solutions of the corresponding pde's are obtained from these decompositions.

Example 4.4 [15] Forsyth [15], vol. VI, page 16, considered the differential equation $Lz = 0$ where

$$L \equiv \partial_{xx} - \partial_{yy} + \frac{4}{x+y}\partial_x.$$

The rational solutions of $a^2 - 1 = 0$ are $a_1 = 1, a_2 = -1$, i.e. case (i) of Proposition 4.1 applies. From (4.5) it follows that $b_{1,2} = \frac{2}{x+y}$. Only $a_2 = -1, b_2 = \frac{2}{x+y}$ satisfy (4.6). There is a single right factor leading to the type $\mathcal{L}_{xx}^1$ decomposition

$$L = \left(\partial_x + \partial_y + \frac{2}{x+y}\right)\left(\partial_x - \partial_y + \frac{2}{x+y}\right);$$

continued in Example 5.2.

The next example shows that complete reducibility may occur for non-separable operators.

Example 4.5 [41] Let the operator

$$L \equiv \partial_{xx} + \frac{2y}{x}\partial_{xy} + \frac{y^2}{x^2}\partial_{yy} + \frac{1}{x}\partial_x + \frac{y}{x^2}\partial_y - \frac{1}{x^2}$$

be given. Because $\frac{1}{4}A_1^2 - A_2 = 0$, case (ii) of Proposition 4.1 applies. It yields $a = \frac{y}{x}$ and leads to the equation $b_x + \frac{y}{x}b_y - b^2 + \frac{1}{x}b + \frac{1}{x^2} = 0$ for b with general solution $b = \frac{1}{x}\frac{1 + x^2\Phi(\varphi)}{1 - x^2\Phi(\varphi)}$ where $\varphi = \frac{y}{x}$; Φ is an undetermined function of its argument. Thus the given second-order operator has an infinite number of first-order right factors of the form $l(\Phi) \equiv \partial_x + \frac{y}{x}\partial_y + \frac{1}{x}\frac{1 + x^2\Phi(\varphi)}{1 - x^2\Phi(\varphi)}$ which are parametrized by Φ; the decomposition type is $\mathcal{L}_{xx}^3$; completed in Example 5.3.

4.2 Operators with leading derivative ∂_{xy}

If an operator does not contain a derivative ∂_{xx} but ∂_{yy} does occur, permuting the variables x and y leads to an operator of the form (4.1) such that the above theorem may be applied. If there is neither a derivative ∂_{xx} or ∂_{yy}, possible factors must obviously be of the form $\partial_x + a$ or $\partial_y + b$; the same is true for the arguments of a representation as intersection due to Theorem 3.26. Consequently the possible factorizations may be described as follows.

Proposition 4.6 *Let the second-order operator*

$$L \equiv \partial_{xy} + A_1\partial_x + A_2\partial_y + A_3 \tag{4.9}$$

be given with $A_i \in \mathbb{Q}(x, y)$ for all i. The following factorizations may occur.

(i) *If $A_3 - A_1A_2 = A_{2,y}$ then $L = (\partial_y + A_1)(\partial_x + A_2)$.*
(ii) *If $A_3 - A_1A_2 = A_{1,x}$ then $L = (\partial_x + A_2)(\partial_y + A_1)$.*
(iii) *If $A_3 - A_1A_2 = A_{1,x}$ and $A_{1,x} = A_{2,y}$ there are two right factors; then $L = Lclm(\partial_x + A_2, \partial_y + A_1)$.*
(iv) *There may exist a Laplace divisor $\mathbb{L}_{y^n}(L)$ for $n \geq 2$.*
(v) *There may exist a Laplace divisor $\mathbb{L}_{x^m}(L)$ for $m \geq 2$.*
(vi) *There may exist both Laplace divisors $\mathbb{L}_{x^m}(L)$ and $\mathbb{L}_{y^n}(L)$. In this case L is completely reducible; L is the left intersection of two Laplace divisors.*

Proof Dividing the operator (4.9) by $\partial_x + a\partial_y + b$, the condition that this division be exact leads to the following set of equations between the coefficients

$$a = 0,\ A_2 - A_1a - a_y - b = 0,\ A_3 - A_1b - b_y = 0$$

with the solution $a = 0$, $b = A_2$ and the constraint $A_3 - A_1A_2 = A_{2,y}$. Dividing out the right factor $\partial_x + A_2$ yields the left factor $\partial_y + A_1$. This is case (i). Dividing (4.9) by $\partial_y + c$, the condition that this division be exact leads to $c = A_1$ and the constraint $A_3 - A_1A_2 = A_{1,x}$. This is case (ii). Finally, if the conditions for cases (i) and (ii) are satisfied simultaneously, a simple calculation shows that L is the left intersection of its right factors. This is case (iii).

The possible existence of the Laplace divisors in cases to is a consequence of Proposition 3.17. □

Case (iv), $n = 1$ and case (v), $m = 1$ are covered by case (i), (ii) and (iii). The corresponding ideals are maximal and principal because they are generated by $\partial_y + a_1$ and $\partial_x + b_1$ respectively. The term *factorization* applies in these cases in the proper sense because the obvious analogy to ordinary differential operators where all ideals are principal.

The following corollary describes to what extent the factorizations described above may be determined algorithmically.

Corollary 4.7 *The coefficients of any first-order factor or Laplace divisor of fixed order are in the base field; they may be determined algorithmically. However, a bound for the order of a Laplace divisor is not known.*

Proof For the first-order factors in cases (i), (ii) and (iii) this is obvious. For any Laplace divisor of fixed order this follows from Proposition 3.17. □

It should be emphasized that according to this corollary finding a Laplace divisor in general is not algorithmic. To this end, an upper bound for the order of a possible divisor would be required; at present such a bound is not known. However, in special cases it may be possible to prove the non-existence of any Laplace divisor.

Applying the preceding results, Loewy decompositions of (4.9) involving first-order principal factors may be described as follows.

Theorem 4.8 *Let the differential operator L be defined by*

$$L \equiv \partial_{xy} + A_1\partial_x + A_2\partial_y + A_3 \tag{4.10}$$

with $A_i \in \mathbb{Q}(x, y)$ for all i; $l \equiv \partial_x + A_2$ and $k \equiv \partial_y + A_1$ are first-order operators. L has Loewy decompositions involving first-order principal divisors according to one of the following types.

$$\mathcal{L}^1_{xy} : L = kl;\ \mathcal{L}^2_{xy} : L = lk;\ \mathcal{L}^3_{xy} : L = Lclm(k, l).$$

The decomposition of type $\mathcal{L}^3_{xy}$ is completely reducible; the first-order factors in decompositions $\mathcal{L}^1_{xy}$ and $\mathcal{L}^2_{xy}$ are Loewy divisors.

Proof It is based on Proposition 4.6. If the conditions for case (i) or (ii) or are satisfied, the decomposition type is $\mathcal{L}^1_{xy}$ or $\mathcal{L}^2_{xy}$ respectively; if both are satisfied the decomposition type $\mathcal{L}^3_{xy}$ is obtained. □

Example 4.9 The operator

$$L \equiv \partial_{xy} + (x + y)\partial_x + \left(y + \frac{1}{x}\right)\partial_y + xy + y^2 + 2 + \frac{y}{x}$$

obeys the conditions of case (iii) of Proposition 4.6. Therefore the type $\mathcal{L}^3_{xy}$ decomposition $L = Lclm(\partial_x + y + \frac{1}{x}, \partial_y + x + y)$ is obtained. Continued in Example 5.5.

Loewy decompositions of (4.9) involving non-principal divisors, possibly in addition to principal ones, are considered next.

Theorem 4.10 *Let the differential operator L be defined by*

$$L \equiv \partial_{xy} + A_1\partial_x + A_2\partial_y + A_3 \tag{4.11}$$

with $A_i \in \mathbb{Q}(x, y)$ for all i. $\mathbb{L}_{x^m}(L)$ and $\mathbb{L}_{y^n}(L)$ as well as $\mathfrak{l}_m$ and $\mathfrak{k}_n$ are defined in Definition 3.17; furthermore $l \equiv \partial_x + a, k \equiv \partial_y + b, a, b \in \mathbb{Q}(x, y)$. L has Loewy decompositions involving Laplace divisors according to one of the following types; m and n obey $m, n \geq 2$.

$$\mathcal{L}_{xy}^4 : L = Lclm(\mathbb{L}_{x^m}(L), \mathbb{L}_{y^n}(L));$$
$$\mathcal{L}_{xy}^5 : L = Exquo\,(L, \mathbb{L}_{x^m}(L))\,\mathbb{L}_{x^m}(L) = \begin{pmatrix} 1 & 0 \\ 0 & \partial_y + A_1 \end{pmatrix} \begin{pmatrix} L \\ \mathfrak{l}_m \end{pmatrix};$$
$$\mathcal{L}_{xy}^6 : L = Exquo\,\big(L, \mathbb{L}_{y^n}(L)\big)\,\mathbb{L}_{y^n}(L) = \begin{pmatrix} 1 & 0 \\ 0 & \partial_x + A_2 \end{pmatrix} \begin{pmatrix} L \\ \mathfrak{k}_n \end{pmatrix};$$
$$\mathcal{L}_{xy}^7 : L = Lclm\,(k, \mathbb{L}_{x^m}(L))\,; \quad \mathcal{L}_{xy}^8 : L = Lclm\,\big(l, \mathbb{L}_{y^n}(L)\big)\,.$$

If L does not have a first order right factor and it may be shown that a Laplace divisor does not exist its decomposition type is defined to be $\mathcal{L}_{xy}^0$. The decompositions $\mathcal{L}_{xy}^0$, $\mathcal{L}_{xy}^4$, $\mathcal{L}_{xy}^7$ and $\mathcal{L}_{xy}^8$ are completely reducible. The Laplace divisors and the exact quotients in decompositions $\mathcal{L}_{xy}^5$ and $\mathcal{L}_{xy}^6$ are Loewy divisors.

Proof Decomposition types $\mathcal{L}_{xy}^4$, $\mathcal{L}_{xy}^7$ and $\mathcal{L}_{xy}^8$ are completely reducible with the obvious representation given above. For decomposition type $\mathcal{L}_{xy}^5$, dividing L by $\mathbb{L}_{x^m}(L)$ yields the exact quotient $(1, 0)$. The single syzygy of $\mathbb{L}_{x^m}(L)$ yields upon reduction w.r.t. $(1, 0)$ the generators $(1, 0)$ and $(0, \partial_y + A_1)$. The calculation for decomposition type $\mathcal{L}_{xy}^6$ is similar. □

The following example taken from Forsyth shows how complete reducibility has its straightforward generalization although if there are Laplace divisors involved.

Example 4.11 The operator

$$L \equiv \partial_{xy} + \frac{2}{x-y}\partial_x - \frac{2}{x-y}\partial_y - \frac{4}{(x-y)^2}$$

has been considered in [15], vol. VI, page 80, Ex. 5 (iii). By Proposition 4.6, a first-order factor does not exist. However, by Proposition 3.17 there exist Laplace divisors

$$\mathbb{L}_{x^2}(L) \equiv \left\langle\!\!\left\langle \partial_{xx} - \frac{2}{x-y}\partial_x + \frac{2}{(x-y)^2}, L \right\rangle\!\!\right\rangle$$

and

$$\mathbb{L}_{y^2}(L) \equiv \left\langle\!\!\left\langle L, \partial_{yy} + \frac{2}{x-y}\partial_y + \frac{2}{(x-y)^2} \right\rangle\!\!\right\rangle,$$

each of differential dimension $(1, 1)$. The ideal generated by L has the representation $\langle L\rangle = Lclm(\mathbb{L}_{x^2}(L), \mathbb{L}_{y^2}(L))$, i.e. it is completely reducible; its decomposition type is $\mathcal{L}_{xy}^4$. Continued in Example 5.3.

The next example due to Imschenetzky is not completely reducible because it has been shown before to allow a single Laplace divisor.

Example 4.12 Imschenetzky's operator $L = \partial_{xy} + xy\partial_x - 2y$ has been considered already in Example 3.18. Using these results the decomposition

$$L = \begin{pmatrix} 1 & 0 \\ 0 & \partial_y + xy \end{pmatrix} \begin{pmatrix} \partial_{xy} + xy\partial_x - 2y \\ \partial_{xxx} \end{pmatrix}$$

of type $\mathcal{L}^5_{xy}$ is obtained. Continued in Example 5.8.

4.3 Operators with leading derivative ∂_{xxx}

Similar to second order operators considered in the preceding section, third order operators are characterized in the first place by their leading derivative. If invariance under permutations is taken into account, three cases with leading derivative ∂_{xxx}, ∂_{xxy} or ∂_{xyy} are distinguished. The corresponding ideals are of differential dimension $(1, 3)$.

In this article only the first case is considered. It is particularly interesting for historical reasons because an operator of this kind was the first third order partial differential operator for which factorizations were considered in Blumberg's thesis [4]; it is discussed in detail in Example 4.19.

Proposition 4.13 *Let the third order partial differential operator*

$$\begin{aligned} L \equiv\ & \partial_{xxx} + A_1\partial_{xxy} + A_2\partial_{xyy} + A_3\partial_{yyy} \\ & + A_4\partial_{xx} + A_5\partial_{xy} + A_6\partial_{yy} + A_7\partial_x + A_8\partial_y + A_9 \end{aligned} \tag{4.12}$$

be given with $A_i \in \mathbb{Q}(x, y)$ for all i. Any first order right factor $\partial_x + a\partial_y + b$ with $a, b \in \mathbb{Q}(x, y)$ is essentially determined by the roots a_1, a_2 and a_3 of the equation $a^3 - A_1a^2 + A_2a - A_3 = 0$. The following alternatives may occur.

- *(i) If $a_i \neq a_j$ for $i \neq j$ are three pairwise different rational roots and the corresponding b_i are determined by (4.20), each pair a_i, b_i satisfying (4.21) and (4.22) yields a factor $l_i = \partial_x + a_i\partial_y + b_i$. If there are three factors, the operator is completely reducible with the representation $L = Lclm(l_1, l_2, l_3)$; if there are two factors, their intersection may or may not be principal according to Theorem 3.23; there may be a single factor or no factor at all.*
- *(ii) If $a_1 = a_2$ is a twofold rational root and $a_3 \neq a_1$ a simple one, the following factors may exist. For $a = a_3 \in \mathbb{Q}(x, y)$, the value of $b = b_3$ is determined by (4.20); if the pair (a_3, b_3) satisfies (4.21) and (4.22), there is a factor $\partial_x + a_3\partial_y + b_3$.*
  For the double root $a = a_1 = a_2$, a necessary condition for a factor to exist is

$$(A_1 - 3a)a_x + (3a^2 - 3A_1a + 2A_2)a_y - A_4a^2 + A_5a = A_6. \tag{4.13}$$

  The type of solutions for $b = b_1 = b_2$ of the system comprising (4.21) and (4.22) determines the possible factors. The following alternatives may occur; r and r_i are undetermined functions of the respective arguments.

$$\partial_x + a_1\partial_y + r(x, y, c_1, c_2), \quad c_1 \textit{ and } c_2 \textit{ constants};$$
$$\partial_x + a_1\partial_y + r(x, y, c), \quad c \textit{ constant};$$
$$\partial_x + a_1\partial_y + r_i(x, y), \quad i = 1 \textit{ or } i = 1, 2;$$

(iii) *If* $a_1 = a_2 = a_3 = \frac{1}{3}A_1$ *is a threefold rational solution, the condition*

$$A_1^2 A_4 - 3A_1A_5 + 9A_6 = 0 \tag{4.14}$$

must be valid in order for a factor to exist. The following subcases may occur.

(a) *If the coefficient of* b *in*

$$\left(A_{1,x} + \frac{1}{3}A_1A_{1,y} + \frac{2}{3}A_1A_4 - A_5\right)b = \frac{1}{3}A_{1,xx} + \frac{2}{9}A_1A_{1,xy} + \frac{1}{27}A_1^2A_{1,yy}$$
$$-\frac{2}{9}A_{1,x}A_{1,y} + \frac{1}{3}A_4A_{1,x} - \frac{2}{27}A_1A_{1,y}^2 - \frac{1}{9}A_1A_4A_{1,y} + \frac{1}{3}A_5A_{1,y}$$
$$+\frac{1}{3}A_1A_7 - A_8 = 0 \tag{4.15}$$

does not vanish, b *may be determined uniquely from this equation. A factor does exist if the constraint*

$$b_{xx} + \frac{2}{3}A_1b_{xy} + \frac{1}{9}A_1^2b_{yy} - 3bb_x + A_4b_x - A_1bb_y$$
$$-\left(\frac{2}{3}A_{1,x} + \frac{2}{9}A_1A_{1,y} + \frac{1}{3}A_1A_4 - A_5\right)b_y + b^3 - A_4b^2 + A_7b - A_9 = 0. \tag{4.16}$$

is satisfied.

(b) *If the coefficient of* b *in* (4.15) *vanishes and the conditions*

$$A_{1,x} + \frac{1}{3}A_1A_{1,y} + \frac{2}{3}A_1A_4 - A_5 = 0,$$
$$A_{1,xx} + \frac{2}{3}A_1A_{1,xy} + \frac{1}{9}A_1^2A_{1,yy} - \frac{2}{3}A_{1,x}A_{1,y} + A_4A_{1,x} - \frac{2}{9}A_1A_{1,y}^2$$
$$-\frac{1}{3}A_1A_4A_{1,y} + A_5A_{1,y} + A_1A_7 - 3A_8 = 0 \tag{4.17}$$

are valid, then b *has to be determined from*

$$b_{xx} + \frac{2}{3}A_1b_{xy} + \frac{1}{9}A_1^2b_{yy} - 3bb_x + A_4b_x - A_1bb_y$$
$$+\frac{1}{3}\left(A_{1,x} + \frac{1}{3}A_1A_{1,y} + A_1A_4\right)b_y + b^3 - A_4b^2 + A_7b - A_9 = 0. \tag{4.18}$$

Proof Dividing the operator (4.12) by $\partial_x + a\partial_y + b$, the requirement that this division be exact leads to the following set of equations between the coefficients.

$$a^3 - A_1a^2 + A_2a - A_3 = 0, \tag{4.19}$$

$$\begin{aligned}&(A_1 - 3a)a_x + (3a^2 - 3A_1a + 2A_2)a_y - A_4a^2 + A_5a\\ &\quad +(3a^2 - 2A_1a + A_2)b = A_6,\end{aligned} \tag{4.20}$$

$$\begin{aligned}&(A_1 - 3a)b_x + (3a^2 - 3A_1a + 2A_2)b_y - (A_1 - 3a)b^2\\ &\quad +(A_5 - 2A_4a - 2A_1a_y + 3aa_y - 3a_x)b\\ &\quad +a_{xx} + (A_1 - a)a_{xy} + (a^2 - A_1a + A_2)a_{yy}\\ &\quad -2a_xa_y + A_4a_x - (A_1 - a)a_y^2 - (A_4a - A_5)a_y + A_7a - A_8 = 0,\end{aligned} \tag{4.21}$$

$$\begin{aligned}&b_{xx} + (A_1 - a)b_{xy} + (a^2 - A_1a + A_2)b_{yy}\\ &\quad -(2a_x + (A_1 - a)a_y + A_4a - A_5)b_y\\ &\quad +(A_4 - 3b)b_x + (3a - 2A_1)bb_y + b^3 - A_4b^2 + A_7b - A_9 = 0.\end{aligned} \tag{4.22}$$

The algebraic equation (4.19) determines a. The following discussion is organized by the type of its roots.

Case (i). Assume at first that (4.19) has three simple roots a_1, a_2 and a_3. None of them may be rational, there may be a single rational solution, or all three roots may be rational. For none of these roots the coefficient of b in (4.20) does vanish; this follows because it is the derivative of the left hand side of (4.19) that does not vanish for simple roots. Therefore for each a_i, equation (4.20) determines the corresponding value b_i. For those pairs a_i, b_i which satisfy the constraints (4.21) and (4.22), a factor $l_i \equiv \partial_x + a_i\partial_y + b_i$ exists. If there are three right factors, by Proposition 3.22 L is completely reducible, it may be represented as $L = Lclm(l_1, l_2, l_3)$.

Case (ii). Assume now (4.19) has a twofold rational root $a_1 = a_2$ and a simple one $a_3 \neq a_1$. This is assured if its coefficients satisfy

$$A_1^2A_2^2 - 4A_2^3 + 18A_1A_2A_3 - 27A_3^2 - 4A_1^3A_3 = 0 \tag{4.23}$$

and $A_1^2 - 3A_2 \neq 0$. It follows that

$$a_1 = a_2 = \frac{1}{2}\frac{A_1A_2 - 9A_3}{A_1^2 - 3A_2}, \quad a_3 = \frac{A_1^3 - 4A_1A_2 + 9A_3}{A_1^2 - 3A_2}.$$

For the root a_3, the coefficient b_3 follows from (4.20). The existence of a factor corresponding to a_3 and b_3 depends on whether these values satisfy the constraints (4.21) and (4.22).

The double root $a_1 = a_2$ is one of the roots of $3a^2 - 2A_1a + A_2 = 0$. Thus the coefficient of b in (4.20) vanishes for this value of a; the remaining part of (4.20) becomes the constraint (4.13). If it is not obeyed, a factor originating from a_1 does not exist. If it is obeyed, the corresponding value for b has to be determined from the system comprising (4.21) and (4.22) with $a = a_1$. Because $A_1 \neq 3a_1$, reducing (4.22) w.r.t. (4.21) yields a system of the type

$$b_x + o(b_y) = 0, b_{yy} + o(b_y) = 0 \tag{4.24}$$

if lex term order with $x \succ y$ is applied. If this autoreduced system forms a Janet basis, the Riccati equation (4.21) has to be solved applying Lemma 6.5. If its rational solution contains an undetermined function, it has to be adjusted such that it satisfies the second equation (4.24). Any rational solution without undetermined elements is only retained if it satisfies this equation. In any case, the final result may be a rational function $r(x, y, c_1, c_2)$ involving two constants; it may be a rational function $r(x, y, c)$ involving a single constant; or there may be one or two rational solutions $r_i(x, y)$ containing no constant; or there may be no rational solution at all.

If the autoreduced system is not a Janet basis, its integrability conditions have to be included and autoreduction has to be applied again; possibly this procedure has to be repeated several times. It cannot be described for generic coefficients $A_1, \ldots, A_9$ because the resulting expressions become too large. The final result may be a system of the type $b_x + o(b) = 0$, $b_y + o(b) = 0$ the general solution of which contains a constant; it may be an algebraic equation for b with one or two solutions; or it may turn out to be inconsistent. The respective solutions are subsumed among those described above.

Case (*iii*). Finally assume there is a threefold solution $a_1 = a_2 = a_3 = \frac{1}{3}A_1$ of (4.19). This is assured if $A_2 = \frac{1}{3}A_1^2$ and $A_3 = \frac{1}{27}A_1^3$. The coefficient of b in (4.20) vanishes again; the remaining part becomes the constraint (4.14). Equation (4.21) simplifies to (4.15); if the coefficient of b does not vanish, b may be determined from it. In order for a factor to exist, in addition (4.16) must be satisfied which originates from (4.22). This is subcase (a). In the exceptional case that the coefficient of b in (4.15) vanishes, it reduces to constraints (4.17); b has to be determined from (4.18) which is obtained from (4.22) by simplification. This is subcase (b). □

In order to apply the above proposition for solving concrete problems the question arises to what extent the various factors may be determined algorithmically. The answer may be summarized as follows.

Corollary 4.14 *Any factor corresponding to a simple root of the symbol polynomial of the operator* (4.13) *may be determined algorithmically. In general this is not possible for factors corresponding to a double or triple root. Absolute irreducibility may always be decided.*

Proof If there are three simple roots a_i, $i = 1, 2, 3$, the b_i, may be determined from the algebraic system (4.19) and (4.20); the constraints (4.21) and (4.22) require only arithmetic and differentiations in the base field. These operations may always be performed. The same arguments apply for the simple root in case (ii). For the double root in case (ii) the corresponding value of b has to be determined from (4.24); it may lead to a partial Riccati equation, an ordinary Riccati equation, an algebraic equation or turn out to be inconsistent. For the first alternative, rational solutions may not be determined in general whereas this is possible in the remaining cases. If there is a threefold root of (4.19) it may occur that b has to be determined from equation (4.18); in general there is no solution algorithm available for solving it. □

After the possible factorizations of an operator (4.12) have been determined, a listing of its various decomposition types involving first-order principal factors may be set up as follows.

Theorem 4.15 *Let the differential operator L be given by*

$$L \equiv \partial_{xxx} + A_1\partial_{xxy} + A_2\partial_{xyy} + A_3\partial_{yyy} \\ +A_4\partial_{xx} + A_5\partial_{xy} + A_6\partial_{yy} + A_7\partial_x + A_8\partial_y + A_9 \tag{4.25}$$

with $A_1, \ldots, A_9 \in \mathbb{Q}(x, y)$. *Moreover let* $l_i \equiv \partial_x + a_i\partial_y + b_i$ *for* $i = 1, 2, 3$ *and* $l(\Phi) \equiv \partial_x + a\partial_y + b(\Phi)$ *be first order operators with* $a_i, b_i, a \in \mathbb{Q}(x, y)$; Φ *is an undetermined function of a single argument. Then L has the following Loewy decomposition types involving first-order principal divisors.*

$$\mathcal{L}^1_{xxx} : L = l_3l_2l_1; \quad \mathcal{L}^2_{xxx} : L = Lclm(l_3, l_2)l_1; \quad \mathcal{L}^3_{xxx} : L = Lclm(l(\Phi))l_1;$$
$$\mathcal{L}^4_{xxx} : L = l_3Lclm(l_2, l_1); \quad \mathcal{L}^5_{xxx} : L = l_3Lclm(l(\Phi));$$
$$\mathcal{L}^6_{xxx} : L = Lclm(l_3, l_2, l_1); \quad \mathcal{L}^7_{xxx} : L = Lclm(l(\Phi), l_1).$$

If none of these alternatives applies, and a decomposition of type $\mathcal{L}^8_{xxx}$ *defined below does not exist either, the decomposition type is defined to be* $\mathcal{L}^0_{xxx}$.

Proof It is based on Proposition 4.13. In the separable case (i) there may be three first-order factors with a principal intersection, this yields type $\mathcal{L}^6_{xxx}$. If there are two factors with a principal intersection they lead to a type $\mathcal{L}^4_{xxx}$ decomposition. If there is a single right factor it is divided out and a second-order left factor with leading derivative ∂_{xx} is obtained. It may be decomposed according to Theorem 4.3; if it is not irreducible it may yield the type $\mathcal{L}^1_{xxx}$, $\mathcal{L}^2_{xxx}$ or $\mathcal{L}^3_{xxx}$ respectively decomposition.

If in case (ii) only a single factor is allowed, the same reasoning as above leads to a decomposition of type $\mathcal{L}^1_{xxx}$ or $\mathcal{L}^2_{xxx}$.

In case (iii), subcase (a), a single factor may exist leading again to type $\mathcal{L}^1_{xxx}$ or type $\mathcal{L}^2_{xxx}$ decompositions as above. □

It can be shown that each decomposition type of the above theorem actually does exist. Two of them are illustrated in the following examples. The equations and its solutions corresponding to these operators are discussed in the next section.

Example 4.16 The operator

$$L \equiv \partial_{xxx} + y\partial_{xxy} - \left(1 - \frac{1}{x}\right)\partial_{xx} - \left(y - \frac{y}{x}\right)\partial_{xy} - \left(\frac{1}{x} + \frac{1}{x^2}\right)\partial_x - \frac{y}{x}\partial_y + \frac{1}{x^2}$$

has the symbol equation $a^2(a - y) = 0$; therefore case (ii) of Proposition 4.13 applies. There is a double root $a_1 = a_2 = 0$ and a simple root $a_3 = y$. The latter yields $b_3 = 0$. Because the pair a_3, b_3 violates constraints (4.21) and (4.22) it does not yield a factor. On the other hand, the double root leads to the equation $b_x - b^2 - \left(1 - \frac{1}{x}\right)b + \frac{1}{x} = 0$ for b. Its single rational solution is $b = -1$; it yields the factor $\partial_x - 1$. Dividing it out the operator $\partial_{xx} + y\partial_{xy} + \frac{1}{x}\partial_x + \frac{y}{x}\partial_y - \frac{1}{x^2}$ is obtained. By Proposition 4.1 it has the right factor $\partial_x + \frac{1}{x}$. Altogether the decomposition

$$L = (\partial_x + y\partial_y)\left(\partial_x + \frac{1}{x}\right)(\partial_x - 1)$$

of type $\mathcal{L}^1_{xxx}$ follows. Continued in Example 5.10.

Example 4.17 For the operator

$$L \equiv \partial_{xxx} + (y+1)\partial_{xxy} + \left(1 - \frac{1}{x}\right)\partial_{xx} + \left(1 - \frac{1}{x}\right)(y+1)\partial_{xy}$$
$$-\frac{1}{x}\partial_x - \frac{1}{x}(y+1)\partial_y$$

Eq. (4.19) reads $a^2(a - y - 1) = 0$ with double root $a_1 = a_2 = 0$ and simple root $a_3 = y+1$, consequently case (ii) applies with $a_1 = 0$, $b_1 = 1$ and $a_3 = y+1$, $b_3 = 0$. The corresponding first order factors yield the divisor as the principal intersection

$$Lclm(\partial_x + 1, \partial_x + (y+1)\partial_y) = \partial_{xx} + (y+1)\partial_{xy} + \partial_x + (y+1)\partial_y. \quad (4.26)$$

Therefore L has the decomposition

$$L = \left(\partial_x - \frac{1}{x}\right) Lclm(\partial_x + 1, \partial_x + (y+1)\partial_y)$$

of type $\mathcal{L}^4_{xxx}$; continued in Example 5.11.

In addition to the decompositions described in Theorem 4.15 there exists a decomposition type involving a non-principal right divisor; it occurs when there are two first-order right factors with *non-principal* intersection ideal $\mathbb{J}_{xxx}$ introduced in Lemma 3.19.

Theorem 4.18 *Assume the differential operator*

$$L \equiv \partial_{xxx} + A_1\partial_{xxy} + A_2\partial_{xyy} + A_3\partial_{yyy}$$
$$+A_4\partial_{xx} + A_5\partial_{xy} + A_6\partial_{yy} + A_7\partial_x + A_8\partial_y + A_9 \quad (4.27)$$

has two first-order right factors $l_i \equiv \partial_x + a_i\partial_y + b_i$; $A_1, \ldots, A_9, a_i, b_i \in \mathbb{Q}(x, y)$. Assume further that l_1 and l_2 have the non-principal left intersection ideal $\mathbb{J}_{xxx}$. Then L may be decomposed as

$$\mathcal{L}^8_{xxx} : L = Exquo(\langle L\rangle, \mathbb{J}_{xxx})\mathbb{J}_{xxx}$$
$$= \begin{pmatrix} 1, & A_1 \\ 0, & \partial_x + (A_1 - q_1)\partial_y + A_{1,y} + q_3A_1 + p_3 + q_1q_3 - q_4 \end{pmatrix}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}.$$

The coefficients p_i and q_i are defined in Lemma 3.19.

Proof According to Theorem 3.23 the ideal $\mathbb{J}_{xxx}$ is generated by two third-order operators $L_1 = \partial_{xxx} + o(\partial_{xyy})$ and $L_2 = \partial_{xxy} + o(\partial_{xyy})$. Hence an operator L with two generic factors l_1 and l_2 is contained in this ideal and has the form $L = L_1 + A_1 L_2$, i.e. the exact quotient in this basis is $(1, A_1)$. The exact quotient module is the sum of this quotient and the syzygy of $\mathbb{J}_{xxx}$ given in Lemma 3.19. □

In the following example the operator introduced in Blumberg's dissertation [4] is discussed in detail; originally it has been suggested by Landau to him. This operator is the generic case for operators that are not completely reducible allowing only two first-order right factors as may be seen from Theorem 3.23.

Example 4.19 [4] In his dissertation Blumberg [4] considered the third order operator

$$L \equiv \partial_{xxx} + x\partial_{xxy} + 2\partial_{xx} + 2(x+1)\partial_{xy} + \partial_x + (x+2)\partial_y \tag{4.28}$$

generating a principal ideal of differential dimension $(1, 3)$. He gave its factorizations

$$L = \begin{cases} \big(\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y\big)(\partial_x + 1), \\ (\partial_{xx} + 2\partial_x + 1)(\partial_x + x\partial_y). \end{cases} \tag{4.29}$$

This result may be obtained by Proposition 4.13 as follows. Equation (4.19) is $a^3 - xa^2 = a^2(a - x) = 0$ with the double root $a_1 = a_2 = 0$, and the simple root $a_3 = x$. The latter yields $b_3 = 0$. For the double root $a_1 = 0$, the system (4.21) and (4.22) has the form

$$b_x - b^2 + \left(2 + \frac{2}{x}\right) b - 1 - \frac{2}{x} = 0,$$
$$b_{xx} + xb_{xy} - 3bb_x + 2b_x - 2xbb_y + 2(x+1)b_y + b^3 - 2b^2 + b = 0.$$

It yields the Janet basis $b - 1 = 0$, i.e. $b = 1$ and the factor $l_1 \equiv \partial_x + 1$. Because a_3 and b_3 satisfy (4.21) and (4.22), there is a second factor $l_2 \equiv \partial_x + x\partial_y$, i.e. case (iv) of Theorem 4.13 applies.

The second order left factor in the first line at the right hand side of (4.29) is absolutely irreducible, whereas the second order factor in the second line is the left intersection of two first order factors, i.e. (4.29) may be further decomposed into irreducibles as

$$L = \begin{cases} (\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y)(l_1 = \partial_x + 1), \\ Lclm(\partial_x + 1, \partial_x + 1 - \frac{1}{x})(l_2 = \partial_x + x\partial_y). \end{cases} \tag{4.30}$$

The intersection ideal of l_1 and l_2 is not principal, by Theorem 3.23 it is

$$\begin{aligned} &Lclm(l_2, l_1) \\ &= \Big\langle\!\Big\langle L_1 \equiv \partial_{xxx} - x^2\partial_{xyy} + 3\partial_{xx} + (2x+3)\partial_{xy} - x^2\partial_{yy} + 2\partial_x + (2x+3)\partial_y, \\ &\quad L_2 \equiv \partial_{xxy} + x\partial_{xyy} - \frac{1}{x}\partial_{xx} - \frac{1}{x}\partial_{xy} + x\partial_{yy} - \frac{1}{x}\partial_x - \left(1 + \frac{1}{x}\right)\partial_y \Big\rangle\!\Big\rangle \end{aligned} \tag{4.31}$$

with differential dimension $(1, 2)$; therefore the type $\mathcal{L}^8_{xxx}$ decomposition of (4.28) is

$$L = \begin{pmatrix} 1 & x \\ 0 & \partial_x + 1 + \frac{1}{x} \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix};$$

it follows that $L = L_1 + xL_2$. L_1 as well as L_2 have the divisors l_1 and l_2. Completed in Example 5.13.

5 Solving partial differential equations

The results of the preceding sections are applied now for solving differential equations of second or third order for an unknown function $z(x, y)$. At first some general properties of the solutions are discussed.

For linear ode's, or systems of linear pde's with a finite dimensional solution space, the undetermined elements in the general solution are constants. For general pde's the undetermined elements are described by Theorem 3.7, due to Kolchin. The equations considered in this section allow a decomposition into first-order principal divisors, or certain non-principal divisors. It will be shown that the solution of such equations with differential dimension $(1, n)$ has the form

$$z(x, y) = z_1(x, y, F_1(\varphi_1(x, y))) + \cdots + z_n(x, y, F_n(\varphi_n)(x, y)); \tag{5.1}$$

each z_i is a sum of terms containing an undetermined function F_i depending on an argument $\varphi_i(x, y)$. Collectively the $\{z_1, \ldots, z_n\}$ are called a *differential fundamental system* or simply *fundamental system*. For the decomposition types considered in this section fundamental systems with the following properties will occur.

- Each $F_i(\varphi_i(x, y))$, or derivatives or integrals thereof, occurs linearly in the corresponding z_i.
- The arguments $\varphi_i(x, y)$ of the undetermined functions are determined by the coefficients of the given equation.

It turns out that the detailed structure of the solutions is essentially determined by the decomposition type of the equation.

5.1 Equations with leading derivative z_{xx}

Reducible equations with leading derivative ∂_{xx} are considered first. Because there are only principal divisors the answer is similar to ordinary second-order equations.

Proposition 5.1 *Let a reducible second-order equation*

$$Lz \equiv (\partial_{xx} + A_1\partial_{xy} + A_2\partial_{yy} + A_3\partial_x + A_4\partial_y + A_5)z = 0$$

be given with $A_1, \ldots, A_5 \in \mathbb{Q}(x, y)$. *Define* $l_i \equiv \partial_x + a_i\partial_y + b_i$, $a_i, b_i \in \mathbb{Q}(x, y)$ *for* $i = 1, 2$; $\varphi_i(x, y) = const$ *is a rational first integral of* $\frac{dy}{dx} = a_i(x, y)$; $\bar{y} \equiv \varphi_i(x, y)$

and the inverse $y = \psi_i(x, \bar{y})$; both φ_i and ψ_i are assumed to exist. Furthermore define

$$\mathcal{E}_i(x, y) \equiv \exp\left(-\int b_i(x, y)\big|_{y=\psi_i(x,\bar{y})}\, dx\right)\Big|_{\bar{y}=\varphi_i(x,y)} \tag{5.2}$$

for $i = 1, 2$. A differential fundamental system has the following structure for the various decompositions into first-order components.

$$\mathcal{L}_{xx}^1 : \begin{cases} z_1(x, y) = \mathcal{E}_1(x, y) F_1(\varphi_1), \\ z_2(x, y) = \mathcal{E}_1(x, y) \int \frac{\mathcal{E}_2(x,y)}{\mathcal{E}_1(x,y)} F_2\left(\varphi_2(x, y)\right)\big|_{y=\psi_1(x,\bar{y})}\, dx\Big|_{\bar{y}=\varphi_1(x,y)}; \end{cases}$$

$$\mathcal{L}_{xx}^2 : z_i(x, y) = \mathcal{E}_i(x, y) F_i(\varphi_i(x, y)), \quad i = 1, 2;$$

$$\mathcal{L}_{xx}^3 : z_i(x, y) = \mathcal{E}_i(x, y) F_i(\varphi(x, y)), \quad i = 1, 2.$$

The F_i are undetermined functions of a single argument; φ, φ_1 and φ_2 are rational in all arguments; ψ_1 is assumed to exist. In general $\varphi_1 \neq \varphi_2$, they are determined by the coefficients A_1, A_2 and A_3 of the given equation.

Proof It is based on Theorem 4.3 and Lemma 6.3. For a decomposition $L = l_2 l_1$ of type $\mathcal{L}_{xx}^2$, Eq. (6.5) applied to the factor l_1 yields the above given $z_1(x, y)$. The left factor equation $l_2 w = 0$ yields $w = \mathcal{E}_2(x, y) F_2(\varphi_2)$. Taking it as inhomogeneity for the right factor equation, by (6.4) the given expression for $z_2(x, y)$ is obtained.

For a decomposition $L = Lclm(l_2, l_1)$ of type $\mathcal{L}_{xx}^2$, similar arguments as for the right factor l_1 in the preceding decomposition lead to the above solutions $z_1(x, y)$ and $z_2(x, y)$. It may occur that $\varphi_1 = \varphi_2$; this is always true if the decomposition originates from case (ii) of Proposition 4.1 instead of case (i).

Finally in a decomposition $L = Lclm(l(\Phi))$ of type $\mathcal{L}_{xx}^3$, two special independent functions Φ_1 and Φ_2 in the operator $l(\Phi)$ may be chosen. Both first-order operators obtained in this way have the same coefficient of ∂_y; as a consequence the arguments of the undetermined functions are the same, i.e. $\varphi_1 = \varphi_2 = \varphi$. □

The following taxonomy may be seen from this result. Whenever an operator is not completely reducible, undetermined functions occur under an integral sign, in general with shifted arguments. If an undetermined function occurs in the decomposition of the operator, the undetermined functions have the same arguments in both members of a differential fundamental system. The following examples show how the above proposition may be applied to concrete problems.

Example 5.2 Forsyth's operator considered in Example 4.4 has a type $\mathcal{L}_{xx}^1$ decomposition leading to the equation

$$Lz \equiv l_2 l_1 z = \left(\partial_x + \partial_y + \frac{2}{x+y}\right)\left(\partial_x - \partial_y + \frac{2}{x+y}\right) z = 0.$$

It follows that

$$\varphi_1(x,y)=x+y,\quad \psi_1(x,y)=\bar y-x,\quad \mathcal{E}_1(x,y)=\exp\left(\frac{2y}{x+y}\right),$$
$$\varphi_2(x,y)=x-y,\quad \psi_2(x,y)=x-\bar y,\quad \mathcal{E}_2(x,y)=-\frac{1}{x+y}.$$

Consequently

$$z_1(x,y)=\exp\left(\frac{2y}{x+y}\right)F(x+y),$$
$$z_2(x,y)=\frac{1}{x+y}\exp\left(\frac{2y}{x+y}\right)\int\exp\left(\frac{2x-\bar y}{\bar y}\right)G(2x-\bar y)\,dx\Big|_{\bar y=x+y}.$$

F and G are undetermined functions.

Example 5.3 Miller' operator has been considered in Example 4.5. Its type $\mathcal{L}_{xx}^3$ decomposition yields the factor

$$l(\Phi)=\partial_x+\frac{y}{x}\partial_y+\frac{1}{x}\frac{1+x^2\Phi(\varphi)}{1-x^2\Phi(\varphi)}$$

where $\varphi=\frac{y}{x}$ and Φ is an undetermined function. It follows that

$$\varphi(x,y)=\frac{y}{x},\ \psi(x,\bar y)=x\bar y,\ \mathcal{E}(x,y)=\Phi(\varphi)x-\frac{1}{x}.$$

Choosing $\Phi=0$ and $\Phi\to\infty$ the solutions

$$z_1(x,y)=xF_1(\varphi)\quad\text{and}\quad z_2(x,y)=\frac{1}{x}F_2(\varphi)$$

are obtained; F_1 and F_2 are undetermined functions.

5.2 Equations with leading derivative z_{xy}

Solutions of reducible equations with mixed leading derivative ∂_{xy} and principal divisors are considered next.

Proposition 5.4 *Let a reducible second-order equation*

$$Lz\equiv(\partial_{xy}+A_1\partial_x+A_2\partial_y+A_3)z=0$$

be given with $A_i\in\mathbb{Q}(x,y)$, $i=1,2,3$. *Define* $l\equiv\partial_x+b$, $k\equiv\partial_y+c$; $b,c\in\mathbb{Q}(x,y)$,

$$\varepsilon_l(x,y)\equiv\exp\left(-\int b(x,y)\,dx\right)\quad\textit{and}\quad\varepsilon_k(x,y)\equiv\exp\left(-\int c(x,y)\,dy\right).$$

A differential fundamental system has the following structure for decompositions into principal divisors.

$$\mathcal{L}^1_{xy}: z_1(x,y) = F(y)\varepsilon_l(x,y),\, z_2(x,y) = \varepsilon_l \int \frac{\varepsilon_k(x,y)}{\varepsilon_l(x,y)} G(x)\,dx;$$
$$\mathcal{L}^2_{xy}: z_1(x,y) = F(x)\varepsilon_k(x,y),\, z_2(x,y) = \varepsilon_k \int \frac{\varepsilon_l(x,y)}{\varepsilon_k(x,y)} G(y)\,dy;$$
$$\mathcal{L}^3_{xy}: z_1(x,y) = F(y)\varepsilon_l(x,y),\, z_2(x,y) = G(x)\varepsilon_k(x,y).$$

F and G are undetermined functions of a single argument.

Proof It is based on Theorem 4.8 and Lemma 6.1; the notation is the same as in this theorem. For decomposition type $\mathcal{L}^1_{xy}$, Eq. (6.5) applied to the factor l yields the above solution $z_1(x,y)$. The left factor equation $w_y + xw = 0$ has the solution $w = G(x)\varepsilon_k(x,y)$. The second solution $z_2(x,y)$ then follows from $z_x + bz = G(x)\varepsilon_k(x,y)$. Interchanging k and l yields the result for decomposition type $\mathcal{L}^2_{xy}$. The two first-order right factors of decomposition type $\mathcal{L}^3_{xy}$ yield the given expressions for both $z_1(x,y)$ and $z_2(x,y)$ as it is true for $z_1(x,y)$ in the previous case. □

Example 5.5 The two arguments of the type $\mathcal{L}^3_{xy}$ decomposition of Example 4.9 yield the two solutions

$$z_1(x,y) = \exp\left(-xy - \frac{1}{2}y^2\right)F(x) \quad \text{and} \quad z_2(x,y) = \exp(-xy)\frac{1}{x}G(y);$$

F and G are undetermined functions.

According to Proposition 4.6 there are five more decompositions with non-principal divisors.

Proposition 5.6 *With the same notation as in Proposition 5.4 a differential fundamental system for the various decomposition types involving non-principal divisors may be described as follows.*

$$\mathcal{L}^4_{xy}: z_1(x,y) = \sum_{i=0}^{m-1} f_i(x,y)F^{(i)}(y), \quad z_2(x,y) = \sum_{i=0}^{n-1} g_i(x,y)G^{(i)}(x);$$
$$\mathcal{L}^5_{xy}: z_1(x,y) = \sum_{i=0}^{m-1} f_i(x,y)F^{(i)}(y), \quad z_2(x,y) = \sum_{i=0}^{m-1} g_i(x,y)\int h_i(x,y)G(x)\,dx;$$
$$\mathcal{L}^6_{xy}: z_1(x,y) = \sum_{i=0}^{m-1} f_i(x,y)F^{(i)}(x), \quad z_2(x,y) = \sum_{i=0}^{m} g_i(x,y)\int h_i(x,y)G(y)\,dy.$$

$$\mathcal{L}^7_{xy}: z_1(x,y) = F(x)\varepsilon_k(x,y), \quad z_2(x,y) = \sum_{i=0}^{m-1} f_i(x,y)F^{(i)}(y),$$

$$\mathcal{L}^8_{xy}: z_1(x,y) = F(y)\varepsilon_l(x,y), \quad z_2(x,y) = \sum_{i=0}^{m-1} g_i(x,y)F^{(i)}(x),$$

F and G are undetermined functions of a single argument; f, g and h are Liouvillian over the base field; they are determined by the coefficients A_1, A_2 and A_3 of the given equation.

Proof Let $\mathbb{L}_{x^m}(L)$ be a Laplace divisor as defined in Proposition 4.6. The linear ode $\mathfrak{l}_m z = 0$ has the general solution $z = C_1 f_1(x,y) + \cdots + C_m f_m(x,y)$. The C_i are constants w.r.t. x; they are undetermined functions of y. This expression for z must also satisfy the equation $Lz = 0$. Because the operators L and $\mathfrak{l}_m$ combined are a Janet basis generating an ideal of differential dimension $(1,1)$, by Kolchin's Theorem 3.7 it must be possible to express $C_1, \ldots, C_m$ in terms of a single function $F(y)$ and its derivatives $F', F'', \ldots, F^{(m-1)}$. This yields the first sum of the solution for decomposition type $\mathcal{L}^4_{xy}$. For the second Laplace divisor $\mathbb{L}_{y^n}(L)$ the same steps with x and y interchanged yield the second sum.

If there is a single Laplace divisor $\mathbb{L}_{x^m}(L)$ as in decomposition type $\mathcal{L}^5_{xy}$, the solution $z_1(x,y)$ is the same as above. In order to obtain the second solution, according to Theorem 4.10 define $w_1 \equiv Lz$ and $w_2 = \mathfrak{l}_m z$; then the quotient equations are $w_1 = 0$, $w_{2,y} + A_1 w_2 = 0$. The solution of the latter leads to $\mathfrak{l}_m = G(x)\exp\left(-\int A_1 dy\right)$, G an undetermined function. This is an inhomogeneous linear ode; a special solution leads to $z_2(x,y)$ as given above; the $g_i(x,y)$ and $h_i(x,y)$ are determined by the coefficients of $\mathfrak{l}_m$. The discussion for decomposition type $\mathcal{L}^6_{xy}$ is similar. □

The preceding propositions subsume all results that are known for a linear pde with leading derivative ∂_{xy} from the classical literature back in the nineteenth century under the general principle of determining divisors of various types; there is no heuristics involved whatsoever, and the selection of possible divisors is complete. These results will be illustrated now by several examples. The first example taken from Forsyth shows how complete reducibility has its straightforward generalization if there are no principal divisors.

Example 5.7 For Forsyth's equation with type $\mathcal{L}^4_{xy}$ decomposition considered in Example 4.11 the sum ideal is

$$J = \left\langle\!\!\left\langle \mathfrak{l}_2 \equiv \partial_{xx} - \frac{2}{x-y}\partial_x + \frac{2}{(x-y)^2}, L, \mathfrak{k}_2 \equiv \partial_{yy} + \frac{2}{x-y}\partial_y + \frac{2}{(x-y)^2} \right\rangle\!\!\right\rangle;$$

the corresponding system of equations $Jz = 0$ has a three-dimensional solution space; a basis is $\{x-y, (x-y)^2, xy(x-y)\}$. The general solution of $\mathfrak{l}_2 z = 0$ is $z = C_1(x-y) + C_2 x(x-y)$ where C_1 and C_2 are undetermined functions of y. Substitution into $Lz = 0$ yields the constraint $C_{1,y} + yC_{2,y} - C_2 = 0$ with the solution

$C_1 = 2F(y) - yF'(y)$ and $C_2 = F'(y)$. Thus

$$z_1(x, y) = 2(x - y)F(y) + (x - y)^2 F'(y)$$

is obtained. The equation $\mathfrak{k}_2 z = 0$ has the solution $C_1(y - x) + C_2 y(y - x)$ where $C_{1,2}$ are undetermined functions of x. By a similar procedure as above there follows

$$z_2(x, y) = 2(y - x)G(x) + (y - x)^2 G'(x)$$

Example 5.8 A Laplace divisor has been determined for Imschenetzky's equation $Lz = (\partial_{xy} + xy\partial_x - 2y)z = 0$ in Example 3.18; it yields a $\mathcal{L}^5_{xy}$ type decomposition. The equation $\partial_{xxx} z = 0$ has the general solution $C_1 + C_2 x + C_3 x^2$ where the C_i, $i = 1, 2, 3$ are constants w.r.t. x. Substituting it into $Lz = 0$ and equating the coefficients of x to zero leads to the system $C_{2,y} - 2yC_1 = 0$, $C_{3,y} - \frac{1}{2} y C_2 = 0$. The C_i may be represented as

$$C_1 = \frac{1}{y^2} F'' - \frac{1}{y^3} F', C_2 = \frac{2}{y} F', C_3 = F;$$

F is an undetermined function of y, $F' \equiv \dfrac{dF}{dy}$. It yields the solution

$$z_1(x, y) = x^2 F(y) + \frac{2xy^2 - 1}{y^3} F'(y) + \frac{1}{y^2} F''(y). \tag{5.3}$$

From the decomposition

$$Lz = \begin{pmatrix} 1 & 0 \\ 0 & \partial_y + xy \end{pmatrix} \begin{pmatrix} w_1 \equiv z_{xy} + xyz_x - 2yz \\ w_2 \equiv z_{xxx} \end{pmatrix}$$

the equations $w_1 = 0$, $w_{2,y} + xyw_2 = 0$ are obtained with the solution $w_1 = 0$, $w_2 = G(x) \exp\left(-\frac{1}{2}xy^2\right)$. The resulting equation $z_{xxx} = G(x) \exp\left(-\frac{1}{2}xy^2\right)$ yields the second member

$$\begin{aligned} z_2(x, y) = \frac{1}{2} \int G(x) \exp\left(-\frac{1}{2}xy^2\right) x^2 \, dx \\ -x \int G(x) \exp\left(-\frac{1}{2}xy^2\right) x \, dx + \frac{1}{2} x^2 \int G(x) \exp\left(-\frac{1}{2}xy^2\right) dx \end{aligned} \tag{5.4}$$

of a fundamental system.

5.3 Equations with leading derivative z_{xxx}

The operators corresponding to the third-order equations considered in this section generate ideals of differential dimension $(1, 3)$. Therefore, by Kolchin's Theorem 3.7, these equations have a differential fundamental system containing three undetermined functions of a single argument. The remarks on the structure of the solutions of linear pde's on page 57 apply here as well.

As opposed to second-order equations, third-order equations have virtually never been treated in the literature before. Equations corresponding to decompositions involving only principal divisors are considered first.

Proposition 5.9 *Let a third-order equation*

$$
\begin{aligned}
Lz \equiv (&\partial_{xxx} + A_1\partial_{xxy} + A_2\partial_{xyy} + A_3\partial_{yyy}\\
&+A_4\partial_{xx} + A_5\partial_{xy} + A_6\partial_{yy} + A_7\partial_x + A_8\partial_y + A_9)z = 0
\end{aligned}
$$

be given with $A_1, \ldots, A_9 \in \mathbb{Q}(x, y)$. *Define* $l_i \equiv \partial_x + a_i\partial_y + b_i$, $a_i, b_i \in \mathbb{Q}(x, y)$ *for* $i = 1, 2, 3$; $\varphi_i(x, y) = const$ *is a rational first integral of* $\frac{dy}{dx} = a_i(x, y)$; $\bar{y} \equiv \varphi_i(x, y)$ *and the inverse* $y = \psi_i(x, \bar{y})$; *both* φ_i *and* ψ_i *are assumed to exist;* F_1, F_2 *and* F_3 *are undetermined functions of a single argument. Furthermore let*

$$
\mathcal{E}_i(x, y) \equiv \exp\left(-\int b_i(x, y)|_{y=\psi_i(x,\bar{y})}\, dx\right)\Big|_{\bar{y}=\varphi_i(x,y)} \tag{5.5}
$$

for $i = 1, 2, 3$. *For decomposition types* $\mathcal{L}^1_{xxx}, \ldots, \mathcal{L}^7_{xxx}$ *involving only principal divisors a differential fundamental system has the following structure.*

$$
\mathcal{L}^1_{xxx}: \begin{cases}
z_1 = \mathcal{E}_1(x, y)F_1(\varphi_1),\\
z_2 = \mathcal{E}_1(x, y)\displaystyle\int \frac{\mathcal{E}_2(x, y)}{\mathcal{E}_1(x, y)}F_2(\varphi_2(x, y))\Big|_{y=\psi_1(x,\bar{y}}\, dx\Big|_{\bar{y}=\varphi_1(x,y)},\\
z_3 = \mathcal{E}_1(x, y)\displaystyle\int \frac{r(x, y)}{\mathcal{E}_1(x, y)}\Big|_{y=\psi_1(x,\bar{y})}\, dx\Big|_{\bar{y}=\varphi_1(x,y)},\\
r(x, y) = \mathcal{E}_2(x, y)\displaystyle\int \frac{\mathcal{E}_3(x, y)}{\mathcal{E}_2(x, y)}F_3(\varphi_3(x, y))\Big|_{y=\psi_2(x,\bar{y})}\, dx\Big|_{\bar{y}=\varphi_2(x,y)};
\end{cases}
$$

$$
\mathcal{L}^2_{xxx}: \begin{cases}
z_1 = \mathcal{E}_1(x, y)F_1(\varphi_1(x, y)),\\
z_i = \mathcal{E}_1(x, y)\displaystyle\int \frac{\mathcal{E}_i(x, y)}{\mathcal{E}_1(x, y)}F_i(\varphi_i(x, y))\Big|_{y=\psi_1(x,\bar{y})}\, dx\Big|_{\bar{y}=\varphi_1(x,y)}, \quad i = 2, 3;
\end{cases}
$$

$\mathcal{L}^3_{xxx}$: *The same as preceding case except that* $\varphi_2 = \varphi_3 = \varphi$, $\psi_2 = \psi_3 = \psi$;

$$
\mathcal{L}^4_{xxx}: \begin{cases}
z_i = \mathcal{E}_i(x, y)F_i(\varphi_i(x, y)), \quad i = 1, 2,\\
z_3 = \mathcal{E}_1(x, y)\displaystyle\int \frac{r(x, y)}{\mathcal{E}_1(x, y)}\Big|_{y=\psi_1(x,\bar{y})}\, dx\Big|_{\bar{y}=\varphi_1(x,y)}\\
-\mathcal{E}_2(x, y)\displaystyle\int \frac{r(x, y)}{\mathcal{E}_2(x, y)}\Big|_{y=\psi_2(x,\bar{y})}\, dx\Big|_{\bar{y}=\varphi_2(x,y)},\\
r(x, y) = r_0\displaystyle\int \frac{\mathcal{E}_3(x, y)}{a_2 - a_1}\frac{F_3(\varphi_3)}{r_0}\, dy, r_0 = \exp\left(-\int \frac{b_2 - b_1}{a_2 - a_1}\, dy\right);
\end{cases}
$$

$\mathcal{L}^5_{xxx}: \begin{cases} \textit{The same as preceding case except that } r(x,y) = \dfrac{\mathcal{E}_3(x,y)}{b_2-b_1}F_3(\varphi_3) \\ \textit{and } \varphi_1 = \varphi_2, \psi_1 = \psi_2; \end{cases}$

$\mathcal{L}^6_{xxx}: z_i = \mathcal{E}_i(x,y)F_i(\varphi_i(x,y)), \quad i = 1,2,3;$

$\mathcal{L}^7_{xxx}$: *The same as preceding case except that* $\varphi_2 = \varphi_3$ *and* $\psi_2 = \psi_3$.

F_i, $\quad i = 1,2,3$ *are undetermined functions of a single argument*; $f, g, h, \varphi_i, \psi_i, \varphi$ *and* ψ *are determined by the coefficients* $A_1, \ldots, A_9$ *of the given equation.*

The proof is similar as for Proposition 5.1 and is omitted. In the subsequent examples the results given in the above proposition are applied for determining the solutions of the corresponding equations. The reader is encouraged to verify them by substitution.

Example 5.10 The three factors of the type $\mathcal{L}^1_{xxx}$ decomposition in Example 4.16 yield $\varphi_1 = \varphi_2 = y, \varphi_3 = ye^{-x}, \psi_1 = \psi_2 = \bar{y}$ and $\psi_3 = \bar{y}e^x$. Furthermore, $\mathcal{E}_1 = e^x, \mathcal{E}_2 = \frac{1}{x}$ and $\mathcal{E}_3 = 1$. Substituting these values into the expressions given in Proposition 5.9 leads to

$$z_1(x,y) = F(y)e^x, \; z_2(x,y) = G(y)Ei(-x)e^x,$$
$$z_3(x,y) = e^x \int \frac{e^{-x}}{x} \int xH(ye^{-x})\,dx\,dx.$$

F, G and H are undetermined functions.

Example 5.11 The two first-order right factors of the type $\mathcal{L}^4_{xxx}$ decomposition in Example 4.17 yield

$$z_1 = F(y)e^{-x}, \; z_2 = G((y+1)e^{-x}).$$

The first-order left factor leads to the equation $w_x - \frac{1}{x}w = 0$ with the solution $w = xH(y)$; H is an undetermined function. Taking it as inhomogeneity of the second-order equation corresponding to the right factors yields

$$z_{xx} + (y+1)z_{xy} + z_x + (y+1)z_y = xH(y). \tag{5.6}$$

It can be shown that the desired special solution of (5.6) satisfies $z_{3,x} + (y+1)z_{3,y} = (x-1)H(y)$. The result is

$$z_3(x,y) = \int (x-1)H(y)\big|_{y=\psi(x,\bar{y})}\,dx\big|_{\bar{y}=\varphi(x,y)}; \tag{5.7}$$

it follows that $\varphi(x,y) = \log(y+1) - x$ and $\psi(x,\bar{y}) = \exp(\bar{y}+x) - 1$. F, G and H are undetermined functions.

According to Theorem 4.18 on page 55 there is one more decomposition of operators with leading derivative ∂_{xxx} involving a non-principal divisor. The subsequent proposition shows how it may be applied for solving the corresponding equation.

Proposition 5.12 *Let a third-order equation*

$$\begin{aligned}Lz \equiv (&\partial_{xxx} + A_1\partial_{xxy} + A_2\partial_{xyy} + A_3\partial_{yyy}\\ &+A_4\partial_{xx} + A_5\partial_{xy} + A_6\partial_{yy} + A_7\partial_x + A_8\partial_y + A_9)z = 0\end{aligned}$$

be given with $A_1, \ldots, A_9 \in \mathbb{Q}(x, y)$; *assume it has two first-order right factors* $l_i \equiv \partial_x + a_1\partial_y + b_i$, $i = 1, 2$ *generating a non-principal divisor* $\mathbb{J}_{xxx} = \langle L_1, L_2\rangle = Lclm(l_1, l_2)$. *A fundamental system may be obtained as follows.*

$$\mathcal{L}^8_{xxx} : \begin{cases} z_i(x, y) = \mathcal{E}_i(x, y)F_i(\varphi_i(x, y)), \quad i = 1, 2;\\ z_3(x, y) \textit{ is a special solution of } L_1z = w_1, L_2z = w_2,\\ w_1 \textit{ and } w_2 \textit{ are given by (5.9) below.}\end{cases}$$

Proof The first two members z_i follow immediately as solutions of $l_i z = 0$. In order to obtain the third member $z_3(x, y)$ of a fundamental system the exact quotient module

$$\begin{aligned}&Exquo(\langle L\rangle, \langle L_1, L_2\rangle) = \langle(1, A_1), (\partial_y + q_3, -\partial_x + q_1\partial_y - p_3 - q_1q_3 + q_4)\rangle\\ &\quad = \langle(1, A_1), (0, \partial_x + (A_1 - q_1)\partial_y + A_{1,y} + q_3A_1 + p_3 + q_1q_3 - q_4)\rangle\end{aligned}$$

is constructed. The first generator of the module at the right hand side in the first line follows from the division, i.e. from $L = L_1 + A_1L_2$; the second generator represents the single syzygy (3.12) given in Lemma 3.19. In the last line the generators have been transformed into a a Janet basis. Introducing the new differential indeterminates w_1 and w_2 the equations

$$\begin{aligned}&w_1 + A_1w_2 = 0,\\ &w_{2,x} + (A_1 - q_1)w_{2,y} + (A_{1,y} + q_3A_1 + p_3 + q_1q_3 - q_4)w_2 = 0\end{aligned} \tag{5.8}$$

are obtained. According to Corollary 6.4 on page 69 its solutions are

$$\begin{aligned}&w_1(x, y) = -A_1w_2(x, y),\\ &w_2(x, y) = \Phi(\varphi)\\ &\qquad \times \exp\left(-\int \left(A_{1,y} + q_3A_1 + p_3 + q_1q_3 - q_4\right)\Big|_{y=\psi(x,\bar{y})} dx\right)\Big|_{\bar{y}=\varphi(x,y)};\end{aligned} \tag{5.9}$$

here $\varphi(x, y)$ is a first integral of $\frac{dy}{dx} = A_1 - q_1$, $\bar{y} = \varphi(x, y)$ and $y = \psi(x, \bar{y})$. Then $z_3(x, y)$ is a special solution of $L_1z = w_1$, $L_2z = w_2$. □

It is not meaningful to describe $z_3(x, y)$ more explicitly as in the preceding proposition because several alternatives may occur due to the special structure of the problem at hand. This will become clear in the next example.

Example 5.13 Blumberg's Example 4.19 has a type $\mathcal{L}^8_{xxx}$ decomposition. The two first-order factors yield $z_1(x, y) = F\left(y - \frac{1}{2}x^2\right)$ and $z_2(x, y) = G(y)e^{-x}$. Furthermore $A_1 = x$, $p_3 = 3$, $q_1 = x$ and $q_3 = q_4 = -\frac{1}{x}$; the system (5.8) becomes $w_1 + xw_2 = 0$ and $w_{2,x} + \left(1 + \frac{1}{x}\right) w_2 = 0$. Its solutions are $w_1 = -H(y)e^{-x}$ and $w_2 = H(y)\frac{1}{x}e^{-x}$; H is an undetermined function of y. Thus, $z_3(x, y)$ is a special solution of

$$L_1 z = -H(y)e^{-x}, \quad L_2 z = H(y)\frac{1}{x}e^{-x}.$$

It can be shown that it is

$$z_3(x, y) = \int xe^{-x} H\left(\bar{y} + \frac{1}{2}x^2\right) dx\Big|_{\bar{y}=y-\frac{1}{2}x^2}. \tag{5.10}$$

H is an undetermined function.

6 Summary and conclusions

The importance of decomposing a differential operator for finding solutions of the corresponding differential equation has become obvious in this article. To a large extent, decomposing an operator and solving the corresponding differential equation in closed form are different aspects of the same subject. In this way more complete and more systematic procedures for determining closed form solutions of linear differential equation are achieved than by any other method; in particular, most results known from the classical literature may be obtained in a systematic way, without heuristics or ad hoc methods.

The result is a fairly complete theory for a well defined class of equations, i.e. linear ordinary or partial differential equations with rational function coefficients. The limitations are also clearly indicated. For irreducible linear ode's with finite Galois group additional algebraic solutions may exist. For linear pde's a Galois theory does not seem to exists at the moment.

More severe limitations are the non-existence of algorithms for certain subproblems. In particular this is true for finding a bound for the existence of a Laplace divisor [2,26], and of an algorithm for determining rational first integrals of first-order ode's. A possible answer may be a proof that algorithms for these problems do not exist.

Extensions of the work described in this article are almost self-evident. In many applications linear pde's in three or even four independent variables occur, see e.g. the collection by Polyanin [46]. In particular this is true for the symmetry analysis of nonlinear differential equations because the so-called determining system of the symmetries is a linear homogeneous system of pde's. Therefore it would be highly desirable to extend the results described in this article to systems of pde's. The same methods apply to these more general problems, although the complete answer will be much more involved; therefore special subclasses of interesting problems should be identified and treated along these lines.

An extremely interesting generalization of the algebraic methods described in this article would be to generalize them to nonlinear equations, e.g. monic quasilinear equations. They are easier to handle than the general nonlinear case, and many practical problems are of this type. Good introductions into this subject including a useful list of references are the articles by Sit [56] and Tsarev [57].

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Appendix

Historically Riccati equations were the first non-linear ordinary differential equations that have been systematically studied. A good account of these efforts may be found in the book by Ince [24]. Originally they were of first order, linear in the first derivative, and quadratic in the dependent variable. Its importance arises from the fact that they occur as subproblems in many more advanced applications as shown in this article. Later on ordinary Riccati equations of higher order have been considered. *Partial Riccati equations* are introduced as a straightforward generalization of the ordinary ones. All derivatives are of first order and occur only linearly, whereas the dependent variables may occur quadratically.

Ordinary Riccati equations

In the subsequent lemma the following terminology is applied. Two rational functions $p, q \in \mathbb{Q}(x)$ are called equivalent if there exists another function $r \in \mathbb{Q}(x)$ such that $p - q = \frac{r'}{r}$ is valid, i.e. if p and q differ only by a logarithmic derivative of a rational function. This defines an equivalence relation on $\mathbb{Q}(x)$. A *special rational solution* does not contain a constant parameter.

Lemma 6.1 *If a first order Riccati equation* $z' + z^2 + az + b = 0$ *with* $a, b \in \mathbb{Q}(x)$ *has rational solutions, one of the following cases applies.*

(i) The general solution is rational and has the form

$$z = \frac{r'}{r + C} + p \tag{6.1}$$

where $p, r \in \bar{\mathbb{Q}}(x)$; $\bar{\mathbb{Q}}$ *is a suitable algebraic extension of* $\mathbb{Q}$, *and* C *is a constant.*

(ii) There is only one, or there are two inequivalent special rational solutions.

Analogous results for Riccati equations of second order are given next.

Lemma 6.2 *If a second order Riccati equation*

$$z'' + 3zz' + z^3 + a(z' + z^2) + bz + c = 0$$

with $a, b, c \in \mathbb{Q}(x)$ has rational solutions, one of the following cases applies.

(i) *The general solution is rational and has the form*

$$z = \frac{C_2 u' + v'}{C_1 + C_2 u + v} + p \tag{6.2}$$

where $p, u, v \in \bar{\mathbb{Q}}(x)$; $\bar{\mathbb{Q}}$ is a suitable algebraic extension of $\mathbb{Q}$, and C_1 and C_2 are constants.

(ii) *There is a single rational solution containing a constant, it has the form shown in Eq.* (6.1).

(iii) *There is a rational solution containing a single constant as in the preceding case, and in addition a single special rational solution.*

(iv) *There is only a single one, or there are two or three special rational solutions that are pairwise inequivalent.*

The proofs of Lemma 6.1 and 6.2 may be found in Chapter 2 of the book by Schwarz [51].

Partial Riccati equations

At first general first-order linear pde's in x and y are considered; they may be obtained as specializations of a Riccati pde if the quadratic term in the unknown function is missing.

Lemma 6.3 *Let the first-order linear pde $z_x + az_y + bz = c$ for $z(x, y)$ be given where $a, b, c \in \mathbb{Q}(x, y)$. Define $\varphi(x, y) = const$ to be a rational first integral of $\frac{dy}{dx} = a(x, y)$; assign $\bar{y} = \varphi(x, y)$ and the inverse $y = \psi(x, \bar{y})$ which is assumed to exist. Define*

$$\mathcal{E}(x, y) \equiv \exp\left(-\int b(x, y)|_{y=\psi(x,\bar{y})}\, dx\right)\Big|_{\bar{y}=\varphi(x,y)}. \tag{6.3}$$

The general solution $z = z_1 + z_0$ of the given first-order pde is

$$z_1(x, y) = \mathcal{E}(x, y)\Phi(\varphi),\, z_0 = \mathcal{E}(x, y)\int \frac{c(x, y)}{\mathcal{E}(x, y)}\Big|_{y=\psi(x,\bar{y})} dx\Big|_{\bar{y}=\varphi(x,y)}; \tag{6.4}$$

Proof Introducing a new variable $\bar{y} = \varphi(x, y)$ as defined above leads to the first-order ode $\bar{z}_x + \bar{b}(x, \bar{y})\bar{z} = \bar{c}(x, \bar{y})$. Upon substitution of $\bar{y}$ into its general solution, the solution (6.4) in the original variables is obtained. □

For $b = 0$ or $c = 0$ the expressions (6.4) simplify considerably as shown next.

Corollary 6.4 *With the same notations as in the preceding lemma, the homogeneous equation $z_x + az_y + bz = 0$ has the solution*

$$z_1(x, y) = \mathcal{E}(x, y)\Phi(\varphi). \tag{6.5}$$

The equation $z_x + az_y = c$ *has the solution* $z = z_1 + z_0$ *where*

$$z_1(x,y) = \Phi(\varphi),\ z_0(x,y) = \int c(x,y)|_{y=\psi(x,\bar{y})}\,dx\,\Big|_{\bar{y}=\varphi(x,y)} \tag{6.6}$$

with $\bar{y}$ *and* Φ *as defined in Lemma 6.3.*

It should be noticed that Lemma 6.3 in general does not allow solving a linear pde algorithmically. To this end, a rational first integral of $\frac{dy}{dx} = a(x,y)$ is required. For this problem an algorithm does not exist at present. The subject of the next lemma is the general first-order Riccati pde in x and y.

Lemma 6.5 *If the partial Riccati equation*

$$z_x + az_y + bz^2 + cz + d = 0 \tag{6.7}$$

where $a, b, c, d \in \mathbb{Q}(x,y)$ *has rational solutions, two cases may occur.*

(i) *The general solution is rational and has the form*

$$z = \frac{1}{a}\left(\frac{r_x(x,\bar{y})}{r(x,\bar{y}) + \Phi(\bar{y})} + p(x,\bar{y})\right)\Big|_{\bar{y}=\varphi(x,y)} \tag{6.8}$$

where $\varphi(x,y)$ *is a rational first integral of* $\frac{dy}{dx} = a(x,y)$, r *and* p *are rational functions of its arguments and* Φ *is an undetermined function.*

(ii) *There is a single rational solution, or there are two inequivalent rational solutions which do not contain undetermined elements.*

Proof Introducing the new dependent variable w by $z = \frac{w}{b}$, Eq. (6.7) is transformed into

$$w_x + aw_y + w^2 + \left(c - \frac{b_x}{b} - a\frac{b_y}{b}\right)w + bd = 0. \tag{6.9}$$

Assume that the first integral $\varphi(x,y) \equiv \bar{y}$ of $\frac{dy}{dx} = a(x,y)$ is rational and the inverse $y = \bar{\varphi}(x,\bar{y})$ exists. Replacing y by $\bar{y}$ leads to

$$\bar{w}_x + \bar{w}^2 + \left(\bar{c} - \frac{\bar{b}_x}{\bar{b}} - \bar{a}\frac{\bar{b}_y}{\bar{b}}\right)\bar{w} + \bar{b}\bar{d} = 0$$

where $\bar{w}(x,\bar{y}) \equiv w(x,y)|_{y=\bar{y}}$, $\bar{a}(x,\bar{y}) \equiv a(x,y)|_{y=\bar{y}}$ and similar for the other coefficients. This is an ordinary Riccati equation for $\bar{w}$ in x with parameter $\bar{y}$. If its general solution is rational it has the form $\frac{r_x}{r + \Phi(\bar{y})} + p$ where r and p are rational functions of x and $\bar{y}$. Backsubstitution of the original variables yields (6.8). If the general solution is not rational, one or two special rational solutions may exist leading to case *(ii)*. □

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