Differentiability with respect to initial functions of solutions to nonlinear hyperbolic functional differential systems

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Abstract A generalized Cauchy problem for nonlinear hyperbolic functional differential systems is considered. A theorem on the existence of weak solutions is proved. The initial problem is transformed into a system of functional integral equations for an unknown function and for their partial derivatives with respect to spatial variables. The existence of solutions of this system is proved by using a method of successive approximations. It is shown a result on the differentiability of solutions with respect to initial functions. This is the main result of the paper.

Keywords Functional differential equations · Generalized Cauchy problem · Volterra condition · Differentiability of solutions

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1 Introduction

For any metric spaces U and V we denote by $C(U, V)$ the class of all continuous functions from U into V. We will use vectorial inequalities if the same inequalities hold between corresponding components. Suppose that $M \in C([0, a], \mathbb{R}^n_+)$, $a > 0$, $\mathbb{R}_+ = [0, +\infty)$, M is nondecreasing and $M(0) = 0_{[n]}$ where $0_{[n]} = (0, \ldots, 0) \in \mathbb{R}^n$. Let E be the Haar pyramid

$$E = \{(t, x) \in \mathbb{R}^{1+n} : t \in [0, a], \ -b + M(t) \le x \le b - M(t)\},$$

where $b \in \mathbb{R}^n$ and $b > M(a)$. Suppose that $b_0 \in R_+$ and $M^-, M^+ \in C([-b_0, 0], R^n)$, $M^- = (M^-_1, \ldots, M^-_n)$, $M^+ = (M^+_1, \ldots, M^+_n)$ and $M^-(0) = -b$, $M^+(0) = b$ and

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$M^-(t) < M^+(t)$ for $t \in [-b_0, 0]$. Set $E_0 = \{(t, x) \in \mathbb{R}^{1+n} : t \in [-b_0, 0], M^-(t) \le x \le M^+(t)\}$. For $(t, x) \in E$ we define

$$D[t, x] = \{(\tau, y) \in \mathbb{R}^{1+n} : \tau \le 0, \ (t + \tau, x + y) \in E_0 \cup E\}.$$

Then $D[t, x] = D_0[t, x] \cup D_\star[t, x]$ where

$$D_0[t, x] = \{(\tau, y) \in \mathbb{R}^{1+n} : -b_0 - t \le \tau \le -t, \ -x + M^-(t + \tau) \le y \le -x$$
$$+ M^+(t + \tau)\},$$
$$D_\star[t, x] = \{(\tau, y) : -t \le \tau \le 0, \ -b - x + M(\tau + t) \le y \le b - x - M(\tau + t)\}.$$

Set $b^- = (b_1^-, \dots, b_n^-)$, $b^+ = (b_1^+, \dots, b_n^+)$ where

$$b_i^- = \min\{M_i^-(t) : t \in [-b_0, 0]\}, \quad b_i^+ = \max\{M_i^+(t) : t \in [-b_0, 0]\}, \qquad 1 \le i \le n,$$

and $B = [-b_0 - a, 0] \times [-b + b^-, b + b^+]$. Then $D[t, x] \subset B$ for $(t, x) \in E$. Denote by $\mathbb{N}$ the set of natural numbers. Let $\mathbb{S}$ be the class of all sequences $p = \{p_k\}_{k \in \mathbb{N}}$ where $p_k \in \mathbb{R}$ for $k \in \mathbb{N}$. Write

$$E_{0.k} = (E_0 \cup E) \cap ([-b_0, a_k] \times \mathbb{R}^n), \quad k \in \mathbb{N},$$

where $a_k \ge 0$ for $k \in \mathbb{N}$ and $\kappa = \sup\{a_k : k \in \mathbb{N}\} < a$. For a function $z \colon E_0 \cup E \to \mathbb{S}$, $z = \{z_k\}_{k \in \mathbb{N}}$, and for a point $(t, x) \in E$ we define $z_{(t,x)} \colon D[t, x] \to \mathbb{S}, z_{(t,x)} = \{(z_k)_{(t,x)}\}_{k \in \mathbb{N}}$, by

$$(z_k)_{(t,x)}(\tau, y) = z_k(t + \tau, x + y), \quad (\tau, y) \in D[t, x], \ k \in \mathbb{N}.$$

Then $z_{(t,x)}$ is the restriction of z to the set $(E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n)$ and this restriction is shifted to the set $D[t, x]$.

Suppose the $\phi_0 : [0, a] \to \mathbb{R}$ and $\phi : E \to \mathbb{R}^n$, $\phi = (\phi_1, \dots, \phi_n)$ are given functions. The requirements on ϕ_0 and ϕ are that $0 \le \phi_0(t) \le t$ and $(\phi_0(t), \phi(t, x)) \in E$ for $(t, x) \in E$. Write $\varphi(t, x) = (\phi_0(t), \phi(t, x))$ on E. Let l^∞ be the class of all sequences $p = \{p_k\}_{k \in \mathbb{N}}$ such that $\|p\|_\infty = \sup\{|p_k| : k \in \mathbb{N}\} < \infty$. Set $\Omega = E \times C(B, l^\infty) \times C(B, l^\infty) \times \mathbb{R}^n$ and suppose that

$$F \colon \Omega \to \mathbb{S}, \ F = \{F_k\}_{k \in \mathbb{N}}, \ \text{and} \ \psi = \{\psi_k\}_{k \in \mathbb{N}}, \ \psi_k \colon E_{0.k} \to \mathbb{R} \ \text{for } k \in \mathbb{N},$$

are given functions. We will say that F satisfies condition (V) if for each $(t, x, q) \in E \times \mathbb{R}^n$ and for $v, \tilde{v}, w, \tilde{w} \in C(B, l^\infty)$ such that $v(\tau, y) = \tilde{v}(\tau, y)$ for $(\tau, y) \in D[t, x]$ and $w(\tau, y) = \tilde{w}(\tau, y)$ for $(\tau, y) \in D[\varphi(t, x)]$ we have $F(t, x, v, w, q) = F(t, x, \tilde{v}, \tilde{w}, q)$. Note that the condition (V) means that the values of F at the point $(t, x, v, w, q) \in \Omega$ depends on (t, x, q) and on the restriction of v and w to the sets $D[t, x]$ and $D[\varphi(t, x)]$ only.

Let us denote by $z = \{z_k\}_{k \in \mathbb{N}}$ an unknown function of the variables (t, x), $x = (x_1, \dots, x_n)$. We consider the system of functional differential equations

$$\partial_t z_k(t, x) = F_k(t, x, z_{(t,x)}, z_{\varphi(t,x)}, \partial_x z_k(t, x)), \quad k \in \mathbb{N}, \tag{1}$$

with the initial conditions

$$z_k(t, x) = \psi_k(t, x) \text{ on } E_{0.k}, \quad k \in \mathbb{N}, \tag{2}$$

where $\partial_x z_k = (\partial_{x_1} z_k, \dots, \partial_{x_n} z_k)$, $1 \le i \le k$. We assume that F satisfies the condition (V).
Write

$$E_t = (E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n), \quad S_t = [-b + M(t), b - M(t)], \qquad t \in [0, a],$$
$$I_{c.k}[x] = \{t \in [a_k, c] : -b + M(t) \le x \le b - M(t)\}, \quad x \in [-b, b], \ k \in \mathbb{N},$$

where $\kappa < c \le a$. We consider weak solutions of initial problems. A function $\tilde{z} : E_c \to \mathbb{S}$, $\tilde{z} = \{\tilde{z}_k\}_{k \in \mathbb{N}}$, where $\kappa < c \le a$, is a solution to (1), (2) provides

(i) $\tilde{z}_{(t,x)} \in C(B, l^\infty)$ for $(t, x) \in E_c$, $0 \le t \le c$ and $\partial_x \tilde{z}_k$ exist and they are continuous on $E \cap ([a_k, c] \times \mathbb{R}^n)$ for $k \in \mathbb{N}$,

(ii) for each $k \in \mathbb{N}$ and $x \in [-b, b]$ the function $\tilde{z}_k(\,\cdot\,, x) : I_{c.k}[x] \to \mathbb{R}$ is absolutely continuous,

(iii) for each $x \in [-b, b]$ and for $k \in \mathbb{N}$, the k-th equation in (1) is satisfied for almost all $t \in I_{c.k}[x]$ and conditions (2) hold.

System (1) with initial conditions (2) is called a generalized Cauchy problem. If $a_k = 0$ for $k \in \mathbb{N}$ then (1), (2) reduces to the classical Cauchy problem. The following question is considered in the paper. We prove that under natural assumptions on given functions there exists exactly one solution to (1), (2) defined on E_c and we give an estimate of $c \in (\kappa, a]$. Let us denote by $\mathbb{X}$ the class of all $\psi = \{\psi_k\}_{k \in \mathbb{N}}$, $\psi_k : E_{0.k} \to \mathbb{R}$ for $k \in \mathbb{N}$, such that there exists exactly one solution $\Xi[\psi] : E_c \to l^\infty$ to (1), (2). We give a construction of the space $\mathbb{X}$. We prove that there is $\mathbb{Y} \subset \mathbb{X}$ such that for each $\psi \in \mathbb{Y}$ there exists the Fréchet derivative $\partial \Xi[\psi]$ of Ξ at ψ. Moreover, if $\psi \in \mathbb{Y}$ and $\pi \in \mathbb{X}$ and $\tilde{z} = \partial \Xi[\psi]\pi$ then $\tilde{z}$ is a solution of an integral functional system generated by (1), (2).

Until now there have not been any results on the differentiability with respect to initial functions for solutions of nonlinear hyperbolic functional differential systems. Our theorems are new also in the case when (1), (2) reduces to a finite functional differential system.

In recent years, a number of papers concerning first order partial functional differential equations have been published. The following questions were considered: functional differential inequalities generated by initial or mixed problems and their applications [1,5,6,12], existence theory of classical or weak solutions of equations or finite systems with initial or initial boundary conditions [2–4,9,14,22] approximate solutions of functional differential problems [15–17,25]. Essential extensions of some ideas concerning generalized solution of Hamilton–Jacobi equations are given in [20,21] where viscosity solutions are considered.

Infinite systems of first order partial functional differential equations were first treated in [18,19]. The existence result presented in [18] is based on a method of successive approximations which was introduced by Ważewski [23] for systems without the functional dependence. Existence results for initial problems [11] and for mixed problems [8] related to infinite systems of nonlinear equations are obtained by a quasilinearization procedure and by construction of functional integral systems for unknown functions and for their derivatives with respect to spatial variables. This method was initiated in [7] for nonlinear systems without functional variables. Differential inequalities and suitable comparison results for infinite systems of hyperbolic functional differential inequalities are given in [13,19].

Information on applications of functional differential equations can be found in [12,24]. The monograph [10] contains results on differentiability with respect to initial functions for solutions of ordinary functional differential equations.

The paper is organized as follows. In Sect. 2 we transform the generalized Cauchy problem into a system of integral functional equations. This system is solved in Sect. 3 by the method of successive approximations. As a consequence we obtain a theorem on the existence of solution to (1), (2) an on continuous dependence of solutions on initial functions.

A theorem on the differentiability of solutions with respect to initial functions is presented in Sect. 4. It is the main result of the paper.

Two types of assumptions are needed in theorems on the existence of solutions to initial or initial boundary value problems related to hyperbolic functional differential systems. The first type conditions concern the bicharacteristics. The second type assumptions concern the

regularity of given functions. The authors of the papers [2–4,8,9,11,18,22] have assumed that the partial derivatives of given functions satisfy the Lipschitz condition with respect to all variables except for t. These conditions are global. Our assumptions on the regularity of given functions are more general. We assume that the partial derivatives of F satisfy the Lipschitz condition and suitable estimates are local with respect to all variables. It is clear that there are differential systems with deviated variables and differential integral systems such that local estimates hold and global inequalities are not satisfied.

Motivations for investigations of functional differential systems with two functional variables are given in Remark 2.4. We give examples of functional differential systems which can be derived from (1) be specializing the operator F.

Example 1.1 Suppose that $G : E \times l^\infty \times l^\infty \times \mathbb{R}^n \to \mathbb{S}$, $G = \{G_k\}_{k \in \mathbb{N}}$, is a given function and F is defined by

$$F(t, x, v, w, q) = G(t, x, v(0, 0_{[n]}), w(0, 0_{[n]}), q) \quad \text{on } \Omega. \tag{3}$$

Then (1) reduces to the system of differential equations with deviated variables

$$\partial_t z_k(t, x) = G_k(t, x, z(t, x), z(\varphi(t, x)), \partial_x z_k(t, x)), \quad k \in \mathbb{N}. \tag{4}$$

Example 1.2 Suppose that $b_0 > 0$ and that there is $\bar{M} \in \mathbb{R}^n_+$ such that $M(t) = \bar{M}t$ for $t \in [0, a]$. Then E is the classical Haar pyramid

$$E = \{(t, x) \in \mathbb{R}^{1+n} : t \in [0, a], -b + \bar{M}t \le x \le b - \bar{M}t\}. \tag{5}$$

Set $M^-(t) = -b + \bar{M}t$ and $M^+(t) = b - \bar{M}t$ for $t \in [-b_0, 0]$. Then

$$E_0 = \{(t, x) \in \mathbb{R}^{1+n} : t \in [-b_0, 0], -b + \bar{M}t \le x \le b - \bar{M}t\}. \tag{6}$$

Suppose that $0 < \nu < \mu \le b_0$ and $0_{[n]} < h \le \bar{M}\nu$. For the above G we put

$$F(t, x, v, w, q) = G\left(t, x, \int_{-h}^{h} v(-\mu, y)\, dy, w(0, 0_{[n]}), q\right) \quad \text{on } \Omega. \tag{7}$$

Then (1) reduces to the differential integral system

$$\partial_t z_k(t, x) = G\left(t, x, \int_{x-h}^{x+h} z(t - \mu, y)\, dy, z(\varphi(t, x)), \partial_x z_k(t, x)\right), \quad k \in \mathbb{N}. \tag{8}$$

It is clear that more complicated examples of differential functional systems can be derived from (1).

2 Integral functional equations

Let $\mathbb{L}([\tau, t], \mathbb{R}^n_+)$, $[\tau, t] \subset \mathbb{R}$, be the class of all $\omega : [\tau, t] \to \mathbb{R}^n_+$ which are integrable on $[\tau, t]$. For $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, we put $\|x\| = |x_1| + \ldots + |x_n|$. We use the symbol "$\circ$" to denote the scalar product in $\mathbb{R}^n$. We denote by $M_{n \times n}$ be the class of all $n \times n$ matrices with real elements. For $A \in M_{n \times n}$ where $A = [a_{ij}]_{i,j=1,\ldots,n}$ we put

$$\|A\|_{n \times n} = \max\left\{\sum_{j=1}^{n} |a_{ij}| : 1 \le i \le n\right\}.$$

Let $M_{\infty \times n}$ be the class of all real matrices $B = [b_{ij}]_{i \in \mathbb{N}, \, 1 \leq j \leq n}$ with the finite norm

$$\|B\|_{\infty \times n} = \sup \left\{ \sum_{j=1}^{n} |b_{ij}| \, : \, i \in \mathbb{N} \right\}.$$

We will use the symbol $M_{\infty \times \infty}$ to denote the set of real matrices $C = [c_{ij}]_{i, j \in \mathbb{N}}$ with the finite norm:

$$\|C\|_{\infty \times \infty} = \sup \left\{ \sum_{j=1}^{\infty} |c_{ij}| \, : \, i \in \mathbb{N} \right\}.$$

For the above $B \in M_{\infty \times n}$ and $C \in M_{\infty \times \infty}$ we write $b_{[i]} = (b_{i1}, \ldots, b_{in})$, and $c_{[i]} = (c_{i1}, c_{i2}, \ldots)$, $i \in \mathbb{N}$.

We will denote by $CL(B, \mathbb{R})$ the class of all linear and continuous operators defined on $C(B, \mathbb{R})$ and taking values in $\mathbb{R}$. The norm in the space $CL(B, \mathbb{R})$ generated by the maximum norm in $C(B, \mathbb{R})$ will be denoted by $\| \cdot \|_\star$. Let $CL(B, M_{\infty \times \infty})$ be the class of all $\Xi = [\Xi_{ij}]_{i, j \in \mathbb{N}}$ such that

$$\|\Xi\|_{\infty \times \infty; \star} = \sup\{\|\Xi_{[i]}\|_{\infty; \star} \, : \, i \in \mathbb{N}\} < +\infty,$$

where

$$\|\Xi_{[i]}\|_{\infty; \star} = \sum_{j=1}^{\infty} \|\Xi_{ij}\|_\star, \quad \Xi_{[i]} = (\Xi_{i1}, \Xi_{i2}, \ldots) \qquad \text{for} \quad i \in \mathbb{N}.$$

Now we define some function spaces. Given $\bar{c} = (c_0, c_1, c_2) \in \mathbb{R}_+^3$, we denote by $\mathbb{X}$ the set of all $\psi = \{\psi_k\}_{k \in \mathbb{N}}$ such that for each $k \in \mathbb{N}$ we have

(i) $\psi_k \in C(E_{0.k}, \mathbb{R})$, the derivatives $\partial_x \psi_k = (\partial_{x_1} \psi_k, \ldots, \partial_{x_n} \psi_k)$ exist on $E_{0.k}$ and $\partial_x \psi_k \in C(E_{0.k}, \mathbb{R}^n)$,
(ii) the estimates

$$|\psi_k(t, x)| \leq c_0, \quad \|\partial_x \psi_k(t, x)\| \leq c_1,$$
$$\|\partial_x \psi_k(t, x) - \partial_x \psi_k(t, \bar{x})\| \leq c_2 \|x - \bar{x}\|$$

are satisfied on $E_{0.k}$.

Let $\psi \in \mathbb{X}$, $\psi = \{\psi_k\}_{k \in \mathbb{N}}$, be given and $\kappa < c \leq a$. We denote by $C_{\psi.c}$ the class of all $z \in C(E_c, l^\infty)$, $z = \{z_k\}_{k \in \mathbb{N}}$, such that $z_k(t, x) = \psi_k(t, x)$ on $E_{0.k}$ for $k \in \mathbb{N}$. For the above ψ and $\kappa < c \leq a$ we denote by $C_{\partial \psi_k.c}$, $k \in \mathbb{N}$, the class of all $\vartheta \in C(E_c, \mathbb{R}^n)$ such that $\vartheta(t, x) = \partial_x \psi_k(t, x)$ on $E_{0.k}$.

Write $\Omega_I = [-b, b] \times C(B, l^\infty) \times C(B, l^\infty) \times \mathbb{R}^n$ and $\Omega_t = S_t \times C(B, l^\infty) \times C(B, l^\infty) \times \mathbb{R}^n$, $t \in [0, a]$.

Assumption $H_0[F]$. The function $F \colon \Omega \to \mathbb{S}$ satisfies the condition (V) and

1. for each $(x, v, w, q) \in \Omega_I$ the function $F \colon (\cdot, x, v, w, q) \colon I[x] \to \mathbb{S}$ is measurable and there is $\alpha \in \mathbb{L}([0, a], \mathbb{R}_+)$ such that

$$\|F(t, x, \theta, \theta, 0_{[n]})\|_\infty \leq \alpha(t) \quad \text{on } E$$

where $\theta \in C(B, l^\infty)$ is given by $\theta(\tau, s) = 0_{l^\infty}$ and 0_{l^∞} is the zero in the space l^∞,

2. for each $P = (t, x, v, w, q) \in \Omega$ there exist the derivatives

$$\partial_x F(P) = [\partial_{x_j} F_i(P)]_{i \in \mathbb{N}, 1 \le j \le n}, \quad \partial_q F(P) = [\partial_{q_j} F_i(P)]_{i \in \mathbb{N}, 1 \le j \le n},$$

and the functions $\partial_x F(\cdot, x, v, w, q), \partial_q F(\cdot, x, v, w, q) \colon I[x] \to M_{\infty \times n}$ are measurable and the functions $\partial_x F(t, \cdot), \partial_q F(t, \cdot) \colon \Omega_t \to M_{\infty \times n}$ are continuous,

3. for each $P = (t, x, v, w, q) \in \Omega$ there exist the Frechét derivatives

$$\partial_v F(P) = [\partial_{v_j} F_i(P)]_{i, j \in \mathbb{N}}, \quad \partial_w F(P) = [\partial_{w_j} F_i(P)]_{i, j \in \mathbb{N}},$$

and for each $\tilde{w} \in C(B, \mathbb{R})$ we have

$$\partial_v F(\cdot, x, v, w, q)\tilde{w}, \partial_q F(\cdot, x, v, w, q)\tilde{w} \colon I[x] \to M_{\infty \times \infty} \text{ are measurable}$$

and

$$\partial_v F(t, \cdot)\tilde{w}, \partial_w F(t, \cdot)\tilde{w} \colon \Omega_I \to M_{\infty \times \infty} \text{ are continuous}$$

where

$$\partial_v F(P)\tilde{w} = [\partial_{v_j} F_i(P)\tilde{w}]_{i, j \in \mathbb{N}}, \quad \partial_w F(P)\tilde{w} = [\partial_{w_j} F_i(P)\tilde{w}]_{i, j \in \mathbb{N}},$$

4. there are $\beta \in \mathbb{L}([0, a], \mathbb{R}_+)$ and $L \in \mathbb{L}([0, a], \mathbb{R}_+^n)$, $L = (L_1, \ldots, L_n)$, such that for $P = (t, x, v, w, q) \in \Omega$ we have

$$\|\partial_x F(P)\|_{\infty \times n} \le \beta(t), \quad \|\partial_v F(P)\|_{\infty \times \infty; \star} \le \beta(t), \quad \|\partial_w F(P)\|_{\infty \times \infty; \star} \le \beta(t),$$

and

$$(|\partial_{q_1} F_k(P)|, \ldots, |\partial_{q_n} F_k(P)|) \le L(t), \quad k \in \mathbb{N},$$

and for $t \in [0, a]$ we have

$$M(t) = \int_0^t L(\xi) \, d\xi.$$

Assumption $H[\varphi]$. The functions $\phi_0 \colon [0, a] \to \mathbb{R}$, $\phi \colon E \to \mathbb{R}^n$, $\phi = (\phi_1, \ldots, \phi_n)$, are continuous and

1. $0 \le \phi_0 \le t$ for $t \in [0, a]$ and $\varphi(t, x) = (\phi_0(t), \phi(t, x)) \in E$,
2. there exist the derivatives

$$\partial_x \phi(t, x) = [\partial_{x_j} \phi_i(t, x)]_{i, j = 1, \ldots, n}$$

and $\partial_x \phi \in C(E, M_{n \times n})$,

3. the constant $Q_0 \in \mathbb{R}_+$ is defined by the relation $\|\partial_x \phi(t, x)\|_{n \times n} \le Q_0$ for $(t, x) \in E$ and there is $Q \in \mathbb{R}_+$ such that

$$\|\partial_x \phi(t, x) - \partial_x \phi(t, \bar{x})\|_{n \times n} \le Q \quad \text{on } E.$$

Suppose that Assumptions $H_0[F]$, $H[\varphi]$ are satisfied and $\psi \in \mathbb{X}$, $z \in C_{\psi.c}$, $u \in C(E_c, M_{\infty \times \infty})$ where $\kappa < c \le a$ and

$$z = \{z_k\}_{k \in \mathbb{N}}, \quad u = [u_{ij}]_{i \in \mathbb{N}, 1 \le j \le n},$$
$$u_{[i]} = (u_{i1}, \ldots, u_{in}), \quad u_{[i]} \in C_{\partial \psi_i.c} \quad \text{for } i \in \mathbb{N}.$$

Write $S[z, u_{[k]}](\tau, x) = (\tau, x, z_{(\tau,x)}, z_{\varphi(t,x)}, u_{[k]}(\tau, x))$, $k \in \mathbb{N}$. We consider the Cauchy problem

$$\omega'(\tau) = -\partial_q F_k(S[z, u_{[k]}](\tau, \omega(\tau))), \quad \omega(t) = x, \tag{9}$$

where $(t, x) \in E$, $a_k \leq t \leq a$ and $\partial_q F_k = (\partial_{q_1} F_k, \ldots, \partial_{q_n} F_k)$. Let us denote by $g_{[k]}[z, u_{[k]}](\cdot, t, x)$ the solution of (9). The function $g_{[k]}[z, u_{[k]}](\cdot, t, x)$ is the k-th bicharactersitic of (1) corresponding to (z, u). Write

$$u_{(t,x)} = [(u_{ij})_{(t,x)}]_{i \in \mathbb{N}, 1 \leq j \leq n},$$
$$(u_{[k]})_{(t,x)} = ((u_{k1})_{(t,x)}, \ldots, (u_{kn})_{(t,x)}), \quad k \in \mathbb{N},$$

and $P[z, u_{[k]}](\tau, t, x) = S[z, u_{[k]}](\tau, g_{[k]}[z, u_{[k]}](\tau, t, x))$, $k \in \mathbb{N}$. For $P \in \Omega$ and for

$$\tilde{w} \in C(B, l^\infty), \quad \tilde{w} = \{\tilde{w}_k\}_{k \in \mathbb{N}}, \quad \tilde{W} \in C(B, M_{\infty \times n}), \quad \tilde{W} = [\tilde{w}_{ij}]_{i \in \mathbb{N}, 1 \leq j \leq n},$$

we put

$$\partial_v F_k(p) \diamond \tilde{w} = \sum_{j=1}^{\infty} \partial_{v_j} F_k(P) \tilde{w}_j,$$

$$\partial_v F_k(P) \star \tilde{W} = \left[\sum_{j=1}^{\infty} \partial_{v_j} F_k(P) \tilde{w}_{j1}, \ldots, \sum_{j=1}^{\infty} \partial_{v_j} F_k(P) \tilde{w}_{jn} \right],$$

where $k \in \mathbb{N}$. In similar way we define the expressions $\partial_w F_k(P) \diamond \tilde{w}$, $\partial_w F_k(P) \star \tilde{W}$ for $k \in \mathbb{N}$. Let us denote by $\mathbb{F}[z, u] = \{\mathbb{F}_k[z, u]\}_{k \in \mathbb{N}}$ the function defined in the following way:

$$\mathbb{F}_k[z, u](t, x) = \psi_k(t, x) \quad \text{on } E_{0.k}$$

and

$$\mathbb{F}_k[z, u](t, x) = \psi_k(a_k, g_{[k]}[z, u_{[k]}](a_k, t, x)) + \int_{a_k}^{t} F_k(P[z, u_{[k]}](\tau, t, x)) \, d\tau$$

$$- \int_{a_k}^{t} \partial_q F_k(P[z, u_{[k]}](\tau, t, x)) \diamond u_{[k]}(\tau, g_{[k]}[z, u_{[k]}](\tau, t, x)) \, d\tau \quad \text{on } E \cap ([a_k, c] \times \mathbb{R}^n).$$

Moreover we put

$$\mathbb{G}[z, u] = \{\mathbb{G}_{ij}[z, u]\}_{i \in \mathbb{N}, 1 \leq j \leq n},$$
$$\mathbb{G}_{[k]}[z, u] = (\mathbb{G}_{k1}[z, u], \ldots, \mathbb{G}_{kn}[z, u]), \quad k \in \mathbb{N},$$

where

$$\mathbb{G}_{[k]}[z, u](t, x) = \partial_x \psi_k(t, x) \quad \text{on } E_{0.k}$$

and

$$\mathbb{G}_{[k]}[z, u](t, x) = \partial_x \psi_k(a_k, g_{[k]}[z, u_{[k]}](a_k, t, x)) + \int_{a_k}^{t} \partial_x F_k(P[z, u_{[k]}](\tau, t, x)) \, d\tau$$

$$+ \int_{a_k}^{t} \partial_v F_k(P[z, u_{[k]}](\tau, t, x)) \star u_{(\tau, g_{[k]}[z, u_{[k]}](\tau, t, x))} \, d\tau$$

$$+ \int_{a_k}^{t} \partial_w F_k(P[z, u_{[k]}](\tau, t, x)) \star \left[u_{\varphi(\tau, g_{[k]}[z, u_{[k]}](\tau, t, x))} \partial_x \phi(\tau, g_{[k]}[z, u_{[k]}](\tau, t, x)) \right] d\tau$$

on $E \cap ([a_k, c] \times \mathbb{R}^n)$. The functions $u_{\varphi(\tau, y)} \partial_x \phi(\tau, y) : B \to M_{\infty \times n}$, $y = g_{[k]}[z, u_{[k]}](\tau, t, s)$, are defined by

$$u_{\varphi(\tau, y)} \partial_x \phi(\tau, y) = \left[\sum_{\mu=1}^{n} (u_{i\mu})_{\varphi(\tau, y)} \partial_{x_j} \phi_\mu(\tau, y) \right]_{i \in \mathbb{N}, 1 \le j \le n}.$$

We consider the system of functional integral equations

$$z = \mathbb{F}[z, u], \quad u = \mathbb{G}[z, u]. \tag{10}$$

We show that under natural assumptions on given functions there exists a solution $(\bar{z}, \bar{u}) : E_c \to l^\infty \times M_{\infty \times n}$ of (10) and there exist the derivatives $\partial_x \bar{z}_k = (\partial_{x_1} \bar{z}_k, \ldots, \partial_{x_n} \bar{z}_k)$, $k \in \mathbb{N}$, and $\bar{u}_{[k]} = \partial_x \bar{z}_k$ for $k \in \mathbb{N}$.

We first give estimates of solutions to (10). For $z \in C(E_c, l^\infty)$, $\vartheta \in C(E_c, \mathbb{R}^n)$, $u \in C(E_c, M_{\infty \times n})$ we define the seminorms

$$\|z\|_{(t, l^\infty)} = \max\{\|z(\tau, s)\|_\infty : (\tau, s) \in E_t\},$$
$$\|\vartheta\|_{(t, \mathbb{R}^n)} = \max\{\|\vartheta(\tau, s)\|_\infty : (\tau, s) \in E_t\},$$
$$\|v\|_{(t, M_{\infty \times n})} = \max\{\|u(\tau, s)\|_{\infty \times n} : (\tau, s) \in E_t\},$$

where $t \in [0, c]$.

Lemma 2.1 *Suppose that Assumptions $H_0[F]$, $H[\varphi]$ are satisfied and*

1. *$\psi \in \mathbb{X}$ and $\kappa < c \le a$,*
2. *the functions $\bar{z} : E_c \to l^\infty$, $\bar{u} : E_c \to M_{\infty \times n}$ satisfy (10) and $\bar{z} \in C(E_c, l^\infty)$, $\bar{u} \in C(E_c, M_{\infty \times n})$.*

Then

$$\|\bar{z}\|_{(t, l^\infty)} \le \zeta(t), \quad \|\bar{u}\|_{(t, M_{\infty \times n})} \le \chi(t) \quad \text{for } t \in [0, c],$$

where

$$\chi(t) = c_1 \exp\left\{ (1 + Q_0) \int_0^t \beta(\xi) \, d\xi \right\} + \int_0^t \beta(\xi) \exp\left\{ (1 + Q_0) \int_\xi^t \beta(\tau) \, d\tau \right\} d\xi,$$

$$\zeta(t) = c_0 \exp\left\{ 2 \int_0^t \beta(\xi) \, d\xi \right\} + \int_0^t \bar{\gamma}(\xi) \exp\left\{ 2 \int_\xi^t \beta(\tau) \, d\tau \right\} d\xi,$$

$$\bar{\gamma}(t) = \alpha(t) + (\beta(t) + \|L(t)\|) \chi(t).$$

Proof Write $\bar{\zeta}(t) = \|\bar{z}\|_{(t,l^\infty)}$, $\bar{\chi}(t) = \|\bar{u}\|_{(t,M_{\infty \times n})}$, $t \in [0, c]$. It follows from Assumptions $H_0[F]$ and $H[\varphi]$ that the $(\bar{\zeta}, \bar{\chi})$ satisfy the integral inequalities

$$\bar{\zeta}(t) \le c_0 + \int_0^t \alpha(\xi)\, d\xi + 2 \int_0^t \beta(\xi)\bar{\zeta}(\xi)\, d\xi + \int_0^t (\beta(\xi) + \|L(\xi)\|)\bar{\chi}(\xi)\, d\xi,$$

$$\bar{\chi}(t) \le c_1 + \int_0^t \beta(\xi)\, d\xi + (1 + Q_0) \int_0^t \beta(\xi)\bar{\chi}(\xi)\, d\xi, \quad t \in [0, c].$$

The functions (ζ, χ) satisfy the integral equations corresponding to the above inequalities. This proves the lemma.

Suppose that $\psi \in \mathbb{X}$, $\kappa < c \le a$ and $d_0, r_0 \in \mathbb{R}_+$ and $d_0 \ge c_1$, $r_0 \ge c_2$. We denote by $C_{\psi,c}[\zeta, d_0]$ the class of all $z \in C_{\psi,c}$ such that

$$\|z\|_{(t,l^\infty)} \le \zeta(t) \quad \text{for } t \in [0, a]$$

and

$$|z_k(t, x) - z_k(t, \bar{x})| \le d_0 \|x - \bar{x}\| \quad \text{for } (t, x), (t, \bar{x}) \in E \cap ([a_k, a] \times \mathbb{R}^n), \ k \in \mathbb{N}.$$

Let $C_{\partial \psi_k.c}[\chi, r_0]$, $k \in \mathbb{N}$, be the class of all $\vartheta \in C_{\partial \psi_k.c}$ satisfying the conditions:

$$\|\vartheta\|_{(t,\mathbb{R}^n)} \le \chi(t) \quad \text{for } t \in [a_k, c]$$

and

$$\|\vartheta(t, x) - \vartheta(t, \bar{x})\| \le r_0 \|x - \bar{x}\| \quad \text{on } E \cap ([a_k, c] \times \mathbb{R}^n).$$

Write $d = \zeta(a)$, $r = \chi(a)$ and $\Omega[d, r] = E \times K_{C(B,l^\infty)}[d] \times K_{C(B,l^\infty)}[d] \times K_{\mathbb{R}^n}[r]$ where $K_{C(B,l^\infty)}[d] = \{w \in C(B, l^\infty) : \|w\|_B \le d\}$, $K_{\mathbb{R}^n}[r] = \{q \in \mathbb{R}^n : \|q\| \le r\}$.

Assumption $H_\star[F]$. The function $f \colon \Omega \to \mathbb{S}$ satisfies Assumption $H_0[F]$ and there is $\gamma \in \mathbb{L}([0, a], \mathbb{R}_+)$ such that the terms

$$\|\partial_x F(t, x, v, w, q) - \partial_x F(t, \bar{x}, \bar{v}, \bar{w}, \bar{q})\|_{\infty \times n}, \quad \|\partial_q F(t, x, v, w, q)$$
$$- \partial_q F(t, \bar{x}, \bar{v}, \bar{w}, \bar{q})\|_{\infty \times n}$$

and

$$\|\partial_v F(t, x, v, w, q) - \partial_v F(t, \bar{x}, \bar{v}, \bar{w}, \bar{q})\|_{\infty \times \infty; \star}, \quad \|\partial_w F(t, x, v, w, q)$$
$$- \partial_w F(t, \bar{x}, \bar{v}, \bar{w}, \bar{q})\|_{\infty \times \infty; \star}$$

are bounded from above by $\gamma(t)[\|x - \bar{x}\| + \|v - \bar{v}\|_B + \|w - \bar{w}\|_B + \|q - \bar{q}\|]$ on $\Omega[d, r]$.

Remark 2.2 It is important in our considerations that we have assumed the Lipschitz condition for $\partial_x F$, $\partial_v F$, $\partial_w F$, $\partial_q F$ with respect to (x, v, w, q) and the estimates are local with respect to all variables. It is clear that there are differential systems with deviated variables and differential integral systems such that local estimates hold and global inequalities are not satisfied.

Lemma 2.3 *Suppose that Assumptions $H_\star[F]$, $H[\varphi]$ are satisfied and $\kappa < c \le a$ and*

$$\psi, \bar{\psi} \in \mathbb{X}, \quad z \in C_{\psi.c}[\zeta, d_0], \quad \bar{z} \in C_{\bar{\psi}.c}[\zeta, d_0],$$
$$u, \bar{u} \in C(E_c, M_{\infty \times n}), \quad u = [u_{ij}]_{i \in \mathbb{N}, 1 \le j \le n}, \quad \bar{u} = [\bar{u}_{ij}]_{i \in \mathbb{N}, 1 \le j \le n},$$
$$u_{[i]} = (u_{i1}, \ldots, u_{in}), \quad \bar{u}_{[i]} = (\bar{u}_{i1}, \ldots, \bar{u}_{in}), \quad i \in \mathbb{N},$$

and $u_{[i]} \in C_{\partial \psi_i.c}[\chi, r_0]$, $\bar{u}_{[i]} \in C_{\partial \bar{\psi}_i.c}[\chi, r_0]$ for $i \in \mathbb{N}$.

Then for each $k \in \mathbb{N}$ we have:

(i) *the bicharacteristics $g_{[k]}[z, u_{[k]}](\cdot, t, x)$ and $g_{[k]}[\bar{z}, \bar{u}_{[k]}](\cdot, t, x)$, $(t, x) \in E \cap ([a_k, c] \times \mathbb{R}^n)$, exist on intervals $[a_k, \Delta[z, u_{[k]}](t, x)]$ and $[a_k, \Delta[\bar{z}, \bar{u}_{[k]}](t, x)]$ such that for $\tau = \Delta[z, u_{[k]}](t, x)$, $\bar{\tau} = \Delta[\bar{z}, \bar{u}_{[k]}](t, x)$ we have $(\tau, g_{[k]}[z, u_{[k]}](\tau, t, x)) \in \partial E_c$, $(\bar{\tau}, g_{[k]}[\bar{z}, \bar{u}_{[k]}](\bar{\tau}, t, x)) \in \partial E_c$, where ∂E_c is the boundary of E_c,*
(ii) *for each $k \in \mathbb{N}$ the solution of* (9) *is unique and we have the estimates*

$$\|g_{[k]}[z, u_{[k]}](\tau, t, x) - g_{[k]}[z, u_{[k]}](\tau, \bar{t}, \bar{x})\|$$

$$\leq \left[\|x - \bar{x}\| + \left| \int_t^{\bar{t}} \|L(\xi)\| \, d\xi \right| \right] \exp \left\{ \bar{C} \left| \int_\tau^t \gamma(\xi) \, d\xi \right| \right\}, \tag{11}$$

and

$$\|g_{[k]}[z, u_{[k]}](\tau, t, x) - g_{[k]}[\bar{z}, \bar{u}_{[k]}](\tau, t, x)\|$$

$$\leq \left| \int_\tau^t \gamma(\xi) [2\|z - \bar{z}\|_{(\xi, l^\infty)} + \|u_{[k]} - \bar{u}_{[k]}\|_{(\xi, \mathbb{R}^n)}] \, d\xi \right| \exp \left\{ \bar{C} \left| \int_t^\tau \gamma(\xi) \, d\xi \right| \right\}, \tag{12}$$

where $(t, x), (\bar{t}, \bar{x}) \in E \cap ([a_k, c] \times \mathbb{R}^n)$ and $\bar{C} = 1 + d_0(1 + Q_0) + r_0$.

Proof The local existence and uniqueness of the solution to (9) follows from classical theorems on Carethéodory solutions of ordinary differential equations. Suppose that $[t_0, t]$ is the interval on which the bicharacteristic $g_{[k]}[z, u_{[k]}](\cdot, t, x)$ is defined. Then

$$-L(\tau) \leq \frac{d}{d\tau} g_{[k]}[z, u_{[k]}](\tau, t, x) \leq L(\tau), \quad \tau \in [t_0, t],$$

and consequently

$$-b + M(\tau) \leq g_{[k]}[z, u_{[k]}](\tau, t, x) \leq b - M(\tau), \quad \tau \in [t_0, t].$$

We conclude that $(\tau, g_{[k]}[z, u_{[k]}](\tau, t, x)) \in E_c$ for $\tau \in [t_0, t]$ and the bicharacteristic $g_{[k]}[z, u_{[k]}](\cdot, t, x)$ is defined on $[a_k, t]$ and the assertion (i) follows.

Now we prove that for each $k \in \mathbb{N}$ the function $g_{[k]}[z, u_{[k]}](\cdot, t, x) - g_{[k]}[z, u_{[k]}](\cdot, \bar{t}, \bar{x})$ satisfies a linear integral inequality. Note that the functions $z_{(\tau, y)}$ and $z_{(\tau, \bar{y})}$ where $(\tau, y), (\tau, \bar{y}) \in E \cap ([0, c] \times \mathbb{R}^n)$, $y \neq \bar{y}$, have different domains. Hence we need the following construction. Write $B_\star = [-b_0, c] \times [-2b + 2b^-, 2b + 2b^+]$. There is $z_\star \in C(B_\star, l^\infty)$ such that

(i) $z_\star(t, x) = z(t, x)$ on E_c and $\|(z_\star)_{(t,x)}\|_B \leq d$ on $E \cap ([0, c] \times \mathbb{R}^n)$,
(ii) $\|z_\star(t, x) - z_\star(t, \bar{x})\|_\infty \leq d_0\|x - \bar{x}\|$ on $E \cap ([0, c] \times \mathbb{R}^n)$.

Then the functions $(z_\star)_{(\tau, y)}$ and $(z_\star)_{(\tau, \bar{y})}$ where $(\tau, y), (\tau, \bar{y}) \in E \cap ([0, c] \times \mathbb{R}^n)$ are defined on B. It follows form (9) that

$$g_{[k]}[z, u_{[k]}](\tau, t, x) - g_{[k]}[z, u_{[k]}](\tau, \bar{t}, \bar{x}) = x - \bar{x}$$

$$+ \int_\tau^t \partial_q F_k(P[z_\star, u_{[k]}](\xi, t, x)) \, d\xi - \int_\tau^{\bar{t}} \partial_q F_k(P[z_\star, u_{[k]}](\xi, \bar{t}, \bar{x})) \, d\xi$$

and consequently

$$\|g_{[k]}[z, u_{[k]}](\tau, t, x) - g_{[k]}[z, u_{[k]}](\tau, \bar{t}, \bar{x})\|$$

$$\leq \|x - \bar{x}\| + \left| \int_t^{\bar{t}} \|L(\xi)\| \, d\xi \right| + \bar{C} \left| \int_\tau^t \gamma(\xi) \|g_{[k]}[z, u_{[k]}](\xi, t, x) - g_{[k]}[z, u_{[k]}](\xi, \bar{t}, \bar{x})\| d\xi \right|.$$

From the Gronwall inequality we deduce (11). It follows from Assumptions $H[\varphi]$, $H_0[F]$ and from (9) that

$$\|g_{[k]}[z, u_{[k]}](\tau, t, x) - g_{[k]}[\bar{z}, \bar{u}_{[k]}](\tau, t, x)\|$$

$$\leq \left| \int_\tau^t \gamma(\xi)[2\|z - \bar{z}\|_{(\xi, l^\infty)} + \|u_{[k]} - \bar{u}_{[k]}\|_{(\xi, \mathbb{R}^n)}] \, d\xi \right|$$

$$+ \bar{C} \left| \int_\tau^t \gamma(\xi) \|g_{[k]}[z, u_{[k]}](\xi, t, x) - g_{[k]}[\bar{z}, \bar{u}_{[k]}](\xi, t, x)\| \, d\xi \right|.$$

Then we obtain (12) form the Gronwall inequality.

Remark 2.4 Set $\bar{\Omega} = E \times C(B, l^\infty) \times \mathbb{R}^n$ and suppose that $\bar{F} : \bar{\Omega} \to \mathbb{S}$, $\bar{F} = \{\bar{F}_k\}_{k \in \mathbb{N}}$, is a given function of the variables (t, x, v, q). Let us consider the functional differential system

$$\partial_t z_k(t, x) = \bar{F}_k(t, x, z_{(t,x)}, \partial_x z_k(t, x)), \quad k \in \mathbb{N}, \tag{13}$$

which is a particular case of (1).

There are the following motivation for investigations of (1), (2) instead of (13), (2). Differential equations with deviated variables are obtained from (13) in the following way. Suppose that $G : E \times l^\infty \times l^\infty \times \mathbb{R}^n \to \mathbb{S}$, $G = \{G_k\}_{k \in \mathbb{N}}$, is a given function. Write

$$\bar{F}(t, x, v, q) = G(t, x, v(0, 0_{[n]}), v(\varphi(t, x) - (t, x)), q). \tag{14}$$

Then system (13) is equivalent to (4).

Note that Assumption $H_0[F]$ is not satisfied in this case for $\bar{F}$ given by (14). More precisely, the derivatives

$$\partial_x \bar{F}(t, x, v, q) = [\partial_{x_j} \bar{F}_i(t, x, v, q)]_{i \in \mathbb{N}, 1 \leq j \leq n} \tag{15}$$

do not exist on $\bar{\Omega}$. It is clear that under natural assumptions on G the function F given by (3) satisfies Assumption $H_0[F]$.

Let us consider the second example. Suppose that E and E_0 are given by (5) and (6) respectively. For the above G we put

$$\bar{F}(t, x, v, q) = G\left(t, x, \int_{-h}^h v(\mu, y) \, dy, v(\varphi(t, x) - (t, x)), q\right) \quad \text{on } \bar{\Omega}. \tag{16}$$

Then system (13) is equivalent to (8). Note that Assumption $H_0[F]$ is not satisfied for $\bar{F}$ given by (16) because the derivatives (15) do not exist on $\bar{\Omega}$. It is clear that under natural assumptions on G the function F given by (7) satisfies Assumption $H_0[F]$.

With the above motivation we have considered problem (1), (2).

3 Existence of solutions to initial problems

For $\psi \in \mathbb{X}$, $\psi = \{\psi_k\}_{k \in \mathbb{N}}$, we put

$$\|\psi_k\|_{E_{0.k}} = \max\{|\psi_k(t, x)| : (t, x) \in E_{0.k}\},$$
$$\|\partial_x \psi_k\|_{E_{0.k}} = \max\{\|\partial_x \psi_k(t, x)\| : (t, x) \in E_{0.k}\}$$

where $k \in \mathbb{N}$ and

$$\|\psi\|_{\mathbb{X}} = \sup\{\|\psi_k\|_{E_{0.k}} + \|\partial_x \psi_k\|_{E_{0.k}} : k \in \mathbb{N}\}.$$

Write

$$\hat{\Gamma}(t) = c_1 + (1 + d_0 + d_0 Q_0) \int_0^t \beta(\xi)\, d\xi + r\bar{C} \int_0^t \gamma(\xi)\, d\xi + 2r_0 \int_0^t \|L(\xi)\|\, d\xi,$$

$$\Gamma(t) = \hat{\Gamma}(t) \exp\left[\bar{C} \int_0^t \gamma(\xi)\, d\xi\right],$$

$$\Lambda(t) = \left\{c_2 + B_0 \int_0^t \gamma(\xi)\, d\xi + B_1 \int_0^t \beta(\xi)\, d\xi\right\} \exp\left[\bar{C} \int_0^t \gamma(\xi)\, d\xi\right],$$

$$B_0 = \bar{C}(1 + r + r Q_0), \quad B_1 = r_0 + r_0 Q_0^2 + r Q.$$

Assumption $H[F]$. The function $F : \Omega \to \mathbb{S}$ satisfies Assumption $H_\star[F]$ and the constant $c \in (\kappa, a]$ is small enough to satisfy the conditions

$$\Gamma(c) \leq d_0, \quad \Lambda(c) \leq r_0. \tag{17}$$

Remark 3.1 If we assume that $d_0 > c_1$ and $r_0 > c_2$ then there is $c \in (0, a]$ such that condition (17) is satisfied.

Theorem 3.2 *If Assumption $H[\varphi]$, $H[F]$ are satisfied and $\psi \in \mathbb{X}$ then there exists a solution $\hat{z} : E_c \to l^\infty$ to (1), (2) and*

$$\|\hat{z}\|_{(t, l^\infty)} \leq d, \quad \|\partial_x \hat{z}\|_{(t, M_{\infty \times n})} \leq r \quad for\ t \in [0, c], \tag{18}$$

and

$$\|\partial_x \hat{z}(t, x) - \partial_x \hat{z}(t, \bar{x})\|_{\infty \times n} \leq r_0 \|x - \bar{x}\| \quad on\ E_c. \tag{19}$$

If $\tilde{\psi} \in \mathbb{X}$, $\tilde{\psi} = \{\tilde{\psi}_k\}_{k \in \mathbb{N}}$, and $\tilde{z} : E_c \to l^\infty$ is a solution to (1) with the initial conditions

$$z_k(t, x) = \tilde{\psi}_k(t, x) \quad on\ E_{0.k} \quad for\ k \in \mathbb{N}, \tag{20}$$

then

$$\|\hat{z} - \tilde{z}\|_{(t, l^\infty)} + \|\partial_x \hat{z} - \partial_x \tilde{z}\|_{(t, M_{\infty \times n})} \leq \|\psi - \tilde{\psi}\|_{\mathbb{X}} \exp\left[\int_0^t \Gamma_\star(\tau)\, d\tau\right], \quad t \in [0, c], \tag{21}$$

where

$$\Gamma_\star(\tau) = B_\star \gamma(\tau) + 2\beta(\tau) + 2\|L(\tau)\|, \quad B_\star = 2[\Gamma(c) + \Lambda(c) + 1 + r + r Q_0].$$

Proof We have divided the proof into a sequence of steps. We use a method of successive approximations.

I. We consider the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ where

$$z^{(m)} : E_c \to l^\infty, \quad u^{(m)} : E_c \to M_{\infty \times n},$$

$$z^{(m)} = \{z_k^{(m)}\}_{k \in \mathbb{N}}, \quad u^{(m)} = \{u_{[k]}^{(m)}\}_{k \in \mathbb{N}}, \quad u_{[k]}^{(m)} = (u_{k1}^{(m)}, \dots, u_{kn}^{(m)}), \quad k \in \mathbb{N}.$$

We put first

$$z_k^{(0)}(t, x) = \psi_k(t, x) \text{ on } E_{0.k} \text{ and } z_k^{(0)} = \psi_k(a_k, x) \text{ on } E \cap ([a_k, c] \times \mathbb{R}^n) \quad \text{for } k \in \mathbb{N},$$

and

$$u_k^{(0)}(t, x) = \partial_x \psi_k(a_k, x) \text{ on } E \cap ([a_k, c] \times \mathbb{R}^n) \quad \text{for } k \in \mathbb{N}.$$

If $z^{(m)} : E_c \to l^\infty$ and $u^{(m)} : E_c \to M_{\infty \times n}$ are known functions then for each $k \in \mathbb{N}$ the function $u_{[k]}^{(m+1)}$ is a solution of the equation

$$\vartheta(t, x) = \mathbb{G}_k^{(m)}[\vartheta](t, x) \tag{22}$$

where $\vartheta = (\vartheta_1, \dots, \vartheta_n)$ and $\mathbb{G}_k^{(m)}[\vartheta] = (\mathbb{G}_{k1}^{(m)}[\vartheta], \dots, \mathbb{G}_{kn}^{(m)}[\vartheta])$ and $\mathbb{G}_k^{(m)}[\vartheta]$ is defined by

$$\mathbb{G}_k^{(m)}[\vartheta](t, x) = \partial_x \psi_k(t, x) \quad \text{on } E_{0.k}$$

and

$$\mathbb{G}_k^{(m)}[\vartheta](t, x) = \partial_x \psi_k(a_k, g_{[k]}[z^{(m)}, \vartheta](a_k, t, x))$$

$$+ \int_{a_k}^{t} \partial_x F_k(P[z^{(m)}, \vartheta](\tau, t, x)) \, d\tau + \int_{a_k}^{t} \partial_v F_k(P[z^{(m)}, \vartheta](\tau, t, x)) \star u_{(\tau, g_{[k]}[z^{(m)}, \vartheta](\tau, t, x))}^{(m)} \, d\tau$$

$$+ \int_{a_k}^{t} \partial_w F_k(P[z^{(m)}, \vartheta](\tau, t, x)) \star [u_{\varphi(\tau, g_{[k]}[z^{(m)}, \vartheta](\tau, t, x))}^{(m)} \partial_x \phi(\tau, g_{[k]}[z^{(m)}, \vartheta](\tau, t, x))] \, d\tau$$

$$\tag{23}$$

on $E \cap ([a_k, c] \times \mathbb{R}^n)$. The function $z^{(m+1)}$ is given by

$$z^{(m+1)}(t, x) = \mathbb{F}[z^{(m)}, u^{(m+1)}](t, x) \quad \text{on } E_c. \tag{24}$$

II. We prove that

(A_m) the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are defined on E_c and for $m \geq 0$ we have

$$z^{(m)} \in C_{\psi.c}[\zeta, d_0], \quad u_k^{(m)} \in C_{\partial \psi_k.c}[\chi, r_0] \quad \text{for } k \in \mathbb{N},$$

(B_m) there are $\lambda, \lambda_0 \in \mathbb{L}([0, c], \mathbb{R}_+)$ such that for any $m \geq 0$ we have

$$\|z^{(m)}(t, x) - z^{(m)}(\tilde{t}, x)\|_\infty \leq \left| \int_{t}^{\tilde{t}} \lambda_0(\tau) \, d\tau \right|, \quad \|u^{(m)}(t, x) - u^{(m)}(\tilde{t}, x)\|_{\infty \times n} \leq \left| \int_{t}^{\tilde{t}} \lambda(\tau) \, d\tau \right|,$$

where $(t, x), (\tilde{t}, x) \in E_c, 0 \leq t, \tilde{t} \leq c$,

(C_m) there exists the sequence $\{\partial_x z^{(m)}\}$ and for $m \geq 0$ we have: $\partial_x z^{(m)}(t, x) = u^{(m)}(t, x)$ on E_c.

We prove $(A_m) - (C_m)$ by induction. It is clear that conditions $(A_0) - (C_0)$ are satisfied. Supposed now that $(A_m) - (C_m)$ hold for a given $m \geq 0$, we will prove that there exists $u^{(m+1)} : E_c \to M_{\infty \times n}$ and $u_{[k]}^{(m+1)} \in C_{\partial \psi_k . c}[\chi, r_0]$ for $k \in \mathbb{N}$. We first prove that

$$\mathbb{G}_k^{(m)} : C_{\partial \psi_k . c}[\chi, r_0] \to C_{\partial \psi_k . c}[\chi, r_0], \quad k \in \mathbb{N}. \tag{25}$$

It follows from Assumptions $H[\varphi]$ and $H[F]$ that for $\vartheta \in C_{\partial \psi_k . c}[\chi, r_0], \ k \in \mathbb{N}$, we have

$$\|\mathbb{G}_k^{(m)}[\vartheta](t, x)\| \leq c_1 + (1 + Q_0) \int_0^t \beta(\xi) \chi(\xi) \, d\xi + \int_0^t \beta(\xi) \, d\xi = \chi(t)$$

and

$$\|\mathbb{G}_k^{(m)}[\vartheta](t, x) - \mathbb{G}_k^{(m)}[\vartheta](t, \tilde{x})\| \leq \Lambda(t)\|x - \tilde{x}\| \leq r_0 \|x - \tilde{x}\| \quad \text{on } E_c \cap ([a_k, c] \times \mathbb{R}^n).$$

From the above estimates we deduce (25). It follows easily that for $\vartheta, \ \tilde{\vartheta} \in C_{\partial \psi_k . c}[\chi, r_0]$ we have

$$\|\mathbb{G}_k^{(m)}[\vartheta](t, x) - \mathbb{G}_k^{(m)}[\tilde{\vartheta}](t, x)\| \leq \Lambda(c) \int_{a_k}^t \gamma(\xi)\|\vartheta - \tilde{\vartheta}\|_{(\xi, \mathbb{R}^n)} \, d\xi \quad \text{on } E \cap ([a_k, c] \times \mathbb{R}^n).$$

For the above $\vartheta, \ \tilde{\vartheta}$ we put

$$[|\vartheta - \tilde{\vartheta}|] = \max \left\{ \|\vartheta - \tilde{\vartheta}\|_{(t, \mathbb{R}^n)} \exp \left[-2\Lambda(c) \int_{a_k}^t \gamma(\xi) \, d\xi \right] : t \in [a_k, c] \right\}.$$

We deduce from Assumption $H[F]$ that

$$[|\mathbb{G}_k^{(m)}[\vartheta] - \mathbb{G}_k^{(m)}[\tilde{\vartheta}]|] \leq \frac{1}{2}[|\vartheta - \tilde{\vartheta}|].$$

It follows from the Banach fixed point theorem that for each $k \in \mathbb{N}$ there exists exactly one solution of Eq. (22). Then there exists $u^{(m+1)} : E_c \to M_{\infty \times n}$ and $u_k^{(m+1)} \in C_{\partial \psi_k . c}[\chi, r_0]$ for $k \in \mathbb{N}$.

We deduce from Assumption $H[\varphi]$, $H[F]$ and from (24) that

$$\|z^{(m+1)}\|_{(t, \infty)} \leq c_0 + \int_0^t \alpha(\tau) \, d\tau + 2 \int_0^t \beta(\tau)\zeta(\tau) \, d\tau + 2 \int_0^t \|L(\tau)\|\chi(\tau) \, d\tau = \zeta(t)$$

where $t \in [0, c]$ and

$$|z_k^{(m+1)}(t, x) - z_k^{(m+1)}(t, \tilde{x})| \leq \Gamma(t)\|x - \tilde{x}\| \quad \text{on } E \cap ([a_k, c] \times \mathbb{R}^n).$$

The above relations and Assumption $H[c]$ show that $z^{(m+1)} \in C_{\psi . c}[\zeta, d_0]$.

An easy computation shows that condition (B_{m+1}) is satisfied with

$$\lambda_0(\tau) = (\Gamma(c) + 2r)\|L(\tau)\| + 2d\beta(\tau) + \alpha(\tau),$$
$$\lambda(\tau) = \Lambda(c)\|L(\tau)\| + (1 + r + rQ_0)\beta(\tau), \quad \tau \in [0, c].$$

Now we prove (C_{m+1}). Write

$$\mathbb{D}_k(t, x, y) = z_k^{(m+1)}(t, y) - z_k^{(m+1)}(t, x) - u_{[k]}^{(m+1)}(t, x) \circ (y - x), \quad k \in \mathbb{N},$$

where (t, x), $(t, y) \in E \cap ([a_k, c] \times \mathbb{R}^n)$. We prove that there is $C_\star \in \mathbb{R}_+$ such that

$$|\mathbb{D}_k(t, x, y)| \le C_\star \|x - y\|^2, \quad k \in \mathbb{N}. \tag{26}$$

Set $g_{[k]}^{(m)}(\tau, t, x) = g_{[k]}[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)$. Then we have

$$\mathbb{D}_k(t, x, y) = \mathbb{F}_k[z^{(m)}, u^{(m+1)}](t, y) - \mathbb{F}[z^{(m)}, u^{(m+1)}](t, x) - \mathbb{G}_{[k]}[u_{[k]}^{(m+1)}](t, x) \circ (y - x)$$

$$= \psi_k(a_k, g_{[k]}^{(m)}(a_k, t, y)) - \psi_k(a_k, g_{[k]}^{(m)}(a_k, t, x))$$

$$+ \int_{a_k}^t [F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, y)) - F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x))] \, d\tau$$

$$- \int_{a_k}^t \partial_q F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, y)) \circ u_{[k]}^{(m+1)}(\tau, g_{[k]}^{(m)}(\tau, t, y)) \, d\tau$$

$$+ \int_{a_k}^t \partial_q F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)) \circ u_{[k]}^{(m+1)}(\tau, g_{[k]}^{(m)}(\tau, t, x)) \, d\tau$$

$$- \mathbb{G}_{[k]}[u_{[k]}^{(m+1)}](t, x) \circ (y - x), \quad k \in \mathbb{N}.$$

We transform the above expressions in the following way. We apply the Hadamard mean value theorem to the differences

$$F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, y)) - F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)), \quad k \in \mathbb{N},$$

and we denote by

$$Q^{(m)}(\xi, \tau, t, x, y) = \xi P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, y)$$

$$+ (1 - \xi) P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x), \quad \xi \in [0, 1], \ k \in \mathbb{N},$$

suitable intermediate points. Let us denote by $\mathbb{D}_{k.0}(t, x, y)$, $\mathbb{D}_{k.1}(t, x, y)$, $\mathbb{D}_{k.2}(t, x, y)$, $\mathbb{D}_{k.3}(t, x, y)$ the expressions defined by

$$\mathbb{D}_{k.0}(t, x, y) = \psi_k(a_k, g_{[k]}^{(m)}(a_k, t, y)) - \psi_k(a_k, g_{[k]}^{(m)}(a_k, t, x))$$

$$- \partial_x \psi_k(a_k, g_{[k]}^{(m)}(a_k, t, x)) \circ [g_{[k]}^{(m)}(a_k, t, y) - g_{[k]}^{(m)}(a_k, t, x)],$$

$$\mathbb{D}_{k.1}(t, x, y) = \int_{a_k}^t \int_0^1 \Big[\partial_x F_k(Q^{(m)}(\xi, \tau, t, x, y))$$

$$- \partial_x F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)) \Big] \circ [g_{[k]}^{(m)}(\tau, t, y) - g_{[k]}^{(m)}(\tau, t, x)] \, d\xi \, d\tau$$

$$+ \int_{a_k}^t \int_0^1 [\partial_v F_k(Q^{(m)}(\xi, \tau, t, x, y)) - \partial_v F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x))]$$

$$\star \left[z_{(\tau, g_{[k]}^{(m)}(\tau, t, y))}^{(m)} - z_{(\tau, g_{[k]}^{(m)}(\tau, t, x))}^{(m)} \right] d\xi \, d\tau$$

$$+ \int_{a_k}^t \int_0^1 [\partial_w F_k(Q^{(m)}(\xi, \tau, t, x, y)) - \partial_w F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x))]$$

$$\star \left[z^{(m)}_{\varphi(\tau, g^{(m)}_{[k]}(\tau, t, y))} - z^{(m)}_{\varphi(\tau, g^{(m)}_{[k]}(\tau, t, x))} \right] d\xi \, d\tau$$

$$+ \int_{a_k}^{t} \int_{0}^{1} [\partial_q F_k(Q^{(m)}(\xi, \tau, t, x, y)) - \partial_q F_k(P[z^{(m)}, u^{(m+1)}_{[k]}](\tau, t, x))]$$

$$\circ [u^{(m+1)}_{[k]}(\tau, g^{(m)}_{[k]}(\tau, t, y)) - u^{(m+1)}_{[k]}(\tau, g^{(m)}_{[k]}(\tau, t, x))] \, d\xi \, d\tau$$

and

$$\mathbb{D}_{k.2}(t, x, y) = \int_{a_k}^{t} \partial_v F_k(P[z^{(m)}, u^{(m+1)}_{[k]}](\tau, t, x)) \star \left[z^{(m)}_{(\tau, g^{(m)}_{[k]}(\tau, t, y))} \right.$$

$$\left. - z^{(m)}_{(\tau, g^{(m)}_{[k]}(\tau, t, x))} - (u^{(m)})_{(\tau, g^{(m)}_{[k]}(\tau, t, x))} (g^{(m)}_{[k]}(\tau, t, y) - g^{(m)}_{[k]}(\tau, t, x))^T \right] d\tau$$

$$+ \int_{a_k}^{t} \partial_w F_k(P[z^{(m)}, u^{(m+1)}_{[k]}](\tau, t, x)) \star \left[z^{(m)}_{\varphi(\tau, g^{(m)}_{[k]}(\tau, t, y))} - z^{(m)}_{\varphi(\tau, g^{(m)}_{[k]}(\tau, t, x))} \right.$$

$$\left. - (u^{(m)})_{\varphi(\tau, g^{(m)}_{[k]}(\tau, t, x))} \partial_x \phi(\tau, g^{(m)}_{[k]}(\tau, t, x))(g^{(m)}_{[k]}(\tau, t, y) - g^{(m)}_{[k]}(\tau, t, x))^T \right] d\tau.$$

Moreover we set

$$\mathbb{D}_{k.3}(t, x, y) = \partial_x \psi_k(a_k, g^{(m)}_{[k]}(a_k, t, x)) \circ [[g^{(m)}_{[k]}(a_k, t, y) - g^{(m)}_{[k]}(a_k, t, x)] - (y - x)]$$

$$+ \int_{a_k}^{t} [\partial_q F_k(P[z^{(m)}, u^{(m+1)}_{[k]}](\tau, t, x)) - \partial_q F_k(P[z^{(m)}, u^{(m+1)}_{[k]}](\tau, t, y))] \circ u^{(m+1)}_{[k]}$$

$$\times (\tau, g^{(m)}_{[k]}(\tau, t, y)) \, d\tau$$

$$+ \int_{a_k}^{t} \partial_x F_k(P[z^{(m)}, u^{(m+1)}_{[k]}](\tau, t, x)) \circ [[g^{(m)}_{[k]}(\tau, t, y) - g^{(m)}_{[k]}(\tau, t, x)] - (y - x)] \, d\tau$$

$$+ \int_{a_k}^{t} \partial_v F_k(P[z^{(m)}, u^{(m+1)}_{[k]}](\tau, t, x)) \star (u^{(m)})_{(\tau, g^{(m)}_{[k]}(\tau, t, x))}] \circ [[g^{(m)}_{[k]}(\tau, t, y)$$

$$- g^{(m)}_{[k]}(\tau, t, x)] - (y - x)] \, d\tau$$

$$+ \int_{a_k}^{t} \{\partial_w F_k(P[z^{(m)}, u^{(m+1)}_{[k]}](\tau, t, x)) \star [(u^{(m)})_{\varphi(\tau, g^{(m)}_{[k]}(\tau, t, x))} \partial_x \phi(\tau, g^{(m)}_{[k]}(\tau, t, x))]\}$$

$$\circ [[g^{(m)}_{[k]}(\tau, t, y) - g^{(m)}_{[k]}(\tau, t, x)] - (y - x)] \, d\tau.$$

We put $k \in \mathbb{N}$ in the above definitions. Then we have

$$\mathbb{D}_k(t, x, y) = \mathbb{D}_{k.0}(t, x, y) + \mathbb{D}_{k.1}(t, x, y) + \mathbb{D}_{k.2}(t, x, y) + \mathbb{D}_{k.3}(t, x, y), \quad k \in \mathbb{N}. \quad (27)$$

Since $\psi \in \mathbb{X}$, there is $C_0 \in \mathbb{R}_+$ such that

$$|\mathbb{D}_{k.0}(t, x, y)| \le C_0 \|x - y\|^2, \quad k \in \mathbb{N}, \quad (28)$$

where (t, x), $(t, y) \in E \cap ([a_k, c] \times \mathbb{R}^n)$. It is easily seen that

$$|\mathbb{D}_{k.1}(t, x, y)| \le C_1 \|x - y\|^2, \quad k \in \mathbb{N}, \tag{29}$$

where

$$C_1 = (\tilde{c}\bar{C})^2 \int_0^c \gamma(\xi) \, d\xi, \quad \tilde{c} = \exp\left[\bar{C} \int_0^c \gamma(\xi) \, d\xi\right].$$

It follows from (C_m) that there is $c_\star \in \mathbb{R}_+$ such that

$$|z_k^{(m)}(t, y) - z_k^{(m)}(t, x) - u_{[k]}^{(m)}(t, x) \circ (y - x)| \le c_\star \|x - y\|^2, \tag{30}$$

where (t, x), $(t, y) \in E \cap ([a_k, c] \times \mathbb{R}^n)$. We conclude from Assumptions $H[F]$, $H[\varphi]$ and from (11), (30) that there is $C_2 \in \mathbb{R}_+$ such that

$$|\mathbb{D}_{k.2}(t, x, y)| \le C_2 \|x - y\|^2, \; k \in \mathbb{N}, \tag{31}$$

where (t, x), $(t, y) \in E \cap ([a_k, c] \times \mathbb{R}^n)$.

We transform the expressions $\mathbb{D}_{k.3}(t, x, y)$, $k \in \mathbb{N}$, in the following way. Write

$$
\begin{aligned}
\mathbb{V}_k(\xi, \tau, t, x, y) &= \left\{ \partial_x F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)) \right. \\
&\quad + \partial_v F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)) \star (u^{(m)})_{(\tau, g_{[k]}^{(m)}(\tau, t, x))} \\
&\quad + \left. \partial_w F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)) \star [(u^{(m)})_{(\tau, g_{[k]}^{(m)}(\tau, t, x))} \partial_x \phi(\tau, g_{[k]}^{(m)}(\tau, t, x))] \right\} \\
&\quad \circ [\partial_q F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\xi, t, y)) - \partial_q F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\xi, t, x))]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{W}_k(\xi, t, x) &= \partial_x \psi_k(a_k, g_{[k]}^{(m)}(a_k, t, x)) \\
&\quad + \int_{a_k}^{\xi} [\partial_x F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)) + \partial_v F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)) \\
&\quad \star (u^{(m)})_{(\tau, g_{[k]}^{(m)}(\tau, t, x))}] \, d\tau \\
&\quad + \int_{a_k}^{\xi} \partial_w F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)) \star [(u^{(m)})_{(\tau, g_{[k]}^{(m)}(\tau, t, x))} \partial_x \phi(\tau, g_{[k]}^{(m)}(\tau, t, x))] \, d\tau, \quad k \in \mathbb{N},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{D}_{k.4}(t, x, y) &= \int_{a_k}^{t} \int_{\tau}^{t} \mathbb{V}_k(\xi, \tau, t, x, y) \, d\xi \, d\tau \\
&\quad + \partial_x \psi_k(a_k, g_{[k]}^{(m)}(a_k, t, x)) \circ \int_{a_k}^{t} [\partial_q F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\xi, t, y)) \\
&\quad - \partial_q F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\xi, t, x))] \, d\xi
\end{aligned}
\tag{32}
$$

where $k \in \mathbb{N}$. Then we have

$$\int\limits_{a_k}^{t} \int\limits_{\tau}^{t} \mathbb{V}_k(\xi, \tau, t, x, y) \, d\xi \, d\tau = \int\limits_{a_k}^{t} \int\limits_{a_k}^{\xi} \mathbb{V}_k(\xi, \tau, t, x, y) \, d\tau \, d\xi$$

and consequently

$$\mathbb{D}_{k.4}(t, x, y)$$
$$= \int\limits_{a_k}^{t} \mathbb{W}_k(\xi, t, x) \circ [\partial_q F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\xi, t, y)) - \partial_q F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\xi, t, x))] \, d\xi.$$

(33)

It is clear that the bicharacteristics satisfy the relations

$$g_{[k]}^{(m)}(\tau, \xi, g_{[k]}^{(m)}(\xi, t, x)) = g_{[k]}^{(m)}(\tau, t, x), \quad k \in \mathbb{N},$$

where $(t, x) \in E \cap ([a_k, c] \times \mathbb{R}^n)$. This gives

$$u_{[k]}^{(m+1)}(\xi, g_{[k]}^{(m)}(\xi, t, x)) = \mathbb{W}_k(\xi, t, x), \quad k \in \mathbb{N}.$$

We conclude from (32), (33) that

$$\mathbb{D}_{k.3}(t, x, y) = \int\limits_{a_k}^{t} \left[\partial_q F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, x)) \right.$$

$$\left. - \partial_q F_k(P[z^{(m)}, u_{[k]}^{(m+1)}](\tau, t, y)) \right] \circ [u_{[k]}^{(m+1)}(\tau, g_{[k]}^{(m)}(\tau, t, y))$$

$$- u_{[k]}^{(m+1)}(\tau, g_{[k]}^{(m)}(\tau, t, x))] \, d\tau, \quad k \in \mathbb{N}.$$

Hence, there is $C_3 \in \mathbb{R}_+$ such that

$$|\mathbb{D}_{k.3}(t, x, y)| \le C_3 \|x - y\|^2, \quad k \in \mathbb{N},$$

(34)

where $(t, x), \ (t, y) \in E \cap ([a_k, c] \times \mathbb{R}^n)$.

It follows from (27)–(29), (31), (34) that estimates (26) are satisfied with $C^\star = C_0 + C_1 + C_2 + C_3$. Hence, for each $k \in \mathbb{N}$ there exists $\partial_x z^{(m+1)}(t, x)$ for $(t, x) \in E \cap ([a_k, c] \times \mathbb{R}^n)$ and $\partial_x z^{(m+1)} = u_{[k]}^{(m+1)}$. This proves (C_{m+1}). Thus $(A_m) - (C_m)$ follow by induction.

III. Now we prove that the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are uniformly convergent on E_c. It follows from Assumptions $H[F]$, $H[\varphi]$ and from (23) that there are $\Upsilon_0, \ \Upsilon_1 \in \mathbb{L}([0, c], \mathbb{R}_+)$ such that

$$\|u_{[k]}^{(m+1)} - u_{[k]}^{(m)}\|_{(t, \mathbb{R}^n)} \le \int\limits_{0}^{t} \Upsilon_0(\tau) \|u_{[k]}^{(m+1)} - u_{[k]}^{(m)}\|_{(\tau, \mathbb{R}^n)} \, d\tau$$

$$\int\limits_{0}^{t} \Upsilon_1(\tau) [\|z^{(m)} - z^{(m-1)}\|_{(\tau, \mathbb{R}^n)} + \|u^{(m)} - u^{(m-1)}\|_{(\tau, M_{\infty \times n})}] \, d\tau, \quad k \in \mathbb{N}.$$

By using the Gronwall inequality we get

$$\|u^{(m+1)}_{[k]} - u^{(m)}_{[k]}\|_{(t,\mathbb{R}^n)}$$
$$\leq \exp\left[\int_0^t \Upsilon_0(\tau)\, d\tau\right] \int_0^t \Upsilon_1(\tau)[\|z^{(m)} - z^{(m-1)}\|_{(\tau,\mathbb{R}^n)} + \|u^{(m)} - u^{(m-1)}\|_{(\tau,M_{\infty\times n})}]\, d\tau,$$

(35)

where $k \in \mathbb{N}$. We conclude from (10) and from Assumption $H[F]$, $H[\varphi]$ that there is $\Upsilon_2 \in \mathbb{L}([0,c],\mathbb{R}_+)$ such that

$$\|z^{(m+1)} - z^{(m)}\|_{(t,l^\infty)}$$
$$\leq \int_0^t \Upsilon_2(\tau)[\|z^{(m)} - z^{(m-1)}\|_{(\tau,l^\infty)} + \|u^{(m+1)} - u^{(m)}\|_{(\tau,M_{\infty\times n})}]\, d\tau.$$

(36)

Write

$$K^{(m)}(t) = \|z^{(m+1)} - z^{(m)}\|_{(t,l^\infty)} + \|u^{(m+1)}_{[k]} - u^{(m)}_{[k]}\|_{(t,\mathbb{R}^n)}, \quad t \in [0,c].$$

We deduce from (35), (36) that there is $\Upsilon \in \mathbb{L}([0,c],\mathbb{R}_+)$ such that

$$K^{(m)}(t) \leq \int_0^t \Upsilon(\tau)K^{(m-1)}(\tau)\, d\tau, \quad m \geq 1.$$

Set

$$[|K^{(m)}|] = \max\left\{ K^{(m)}(t)\exp\left[-2\int_0^t \Upsilon(\tau)\, d\tau\right] : t \in [0,c]\right\}.$$

Then we have

$$K^{(m)}(t) \leq [|K^{(m-1)}|]\int_0^t \Upsilon(\tau)\exp\left[2\int_0^\tau \Upsilon(\xi)\, d\xi\right] d\tau$$
$$\leq [|K^{(m-1)}|]\exp\left[2\int_0^t \Upsilon(\tau)\, d\tau\right]$$

and consequently

$$[|K^{(m)}|] \leq \frac{1}{2}[|K^{(m-1)}|] \quad \text{for } m \geq 1.$$

Then $\lim_{m\to\infty}[|K^{(m)}|] = 0$ and consequently there are the limits

$$\tilde{z}(t,x) = \lim_{m\to\infty} z^{(m)}(t,x), \quad \tilde{u}(t,x) = \lim_{m\to\infty} u^{(m)}(t,x) \quad \text{uniformly on } E_c,$$

where $\tilde{z} = \{\tilde{z}_k\}_{k\in\mathbb{N}}$, $\tilde{u} = [\tilde{u}_{ij}]_{i\in\mathbb{N},\, 1\leq j\leq n}$ and $\tilde{u}_{[k]} = (\tilde{u}_{k1},\ldots,\tilde{u}_{kn})$ for $k \in \mathbb{N}$.

It follows from (C_m) that there exist the derivatives $\partial_x\tilde{z}, k \in \mathbb{N}$ and $\partial_x\tilde{z}_k = \tilde{u}_{[k]}$ for $k \in \mathbb{N}$.

IV. Now we prove that $\tilde{z} : E_c \to \mathbb{S}$ is a solution to (1), (2). Write $\tilde{g}_{[k]}(\cdot, t, x) = g_{[k]}[\tilde{z}, \partial_x \tilde{z}](\cdot, t, x)$, $(t, x) \in E \cap ([a_k, c] \times \mathbb{R}^n)$, $k \in \mathbb{N}$. It follows from (**III**) that $\tilde{z}_k(t, x) = \psi_k(t, x)$ on $E_{0.k}$ for $k \in \mathbb{N}$ and

$$\partial_t \tilde{z}_k(t, x) = \psi_k(a_k, \tilde{g}_{[k]}(a_k, t, x)) + \int\limits_{a_k}^{t} F_k(P[\tilde{z}, \partial_x \tilde{z}_k](\tau, t, x)) \, d\tau$$

$$- \int\limits_{a_k}^{t} \partial_q F_k(P[\tilde{z}, \partial_x \tilde{z}_k](\tau, t, x))) \circ \partial_x \tilde{z}_k(\tau, \tilde{g}_{[k]}(\tau, t, x)) \, d\tau, \qquad (37)$$

where $(t, x) \in E \cap ([a_k, c] \times \mathbb{R}^n)$, $k \in \mathbb{N}$. Suppose that $k \in \mathbb{N}$ is fixed. For given $(t, x) \in E \cap ([a_k, c] \times \mathbb{R}^n)$, let us put $y = \tilde{g}_{[k]}(a_k, t, x)$. It follows that the relations $x = \tilde{g}_{[k]}(t, a_k, y)$ and $y = \tilde{g}_{[k]}(a_k, t, x)$ are equivalent. We conclude from (37) that

$$\tilde{z}_k(t, \tilde{g}_{[k]}(t, a_k, x)) = \psi_k(a_k, y)$$

$$+ \int\limits_{a_k}^{t} F_k(\tilde{T}_k(\tau, y)) \, d\tau - \int\limits_{a_k}^{t} \partial_q F_k(\tilde{T}_k(\tau, y)) \circ \partial_x \tilde{z}_k(\tau, \tilde{g}_{[k]}(\tau, a_k, y)) \, d\tau \qquad (38)$$

where

$$\tilde{T}_k(\tau, y) = (\tau, \tilde{g}_{[k]}(\tau, a_k, y), \tilde{z}_{(\tau, \tilde{g}_{[k]}(\tau, a_k, y))}, \tilde{z}_{\varphi(\tau, \tilde{g}_{[k]}(\tau, a_k, y))}, \partial_x \tilde{z}_k(\tau, \tilde{g}_{[k]}(\tau, a_k, y))).$$

By differentiating (38) with respect to t and by putting again $x = \tilde{g}_{[k]}(t, a_k, y)$, we obtain that $\tilde{z}$ is a weak solution of (1), (2).

V. It follows form $(A_m) - (C_m)$ that the sequences $\{z^{(m)}\}$ and $\{\partial_x z^{(m)}\}$ satisfy the conditions

$$\|z^{(m)}\|_{(t, l^\infty)} \le d, \quad \|\partial_x z^{(m)}\|_{(t, M_{\infty \times n})} \le r$$

and

$$\|\partial_x z^{(m)}(t, x) - \partial_x z^{(m)}(t, \bar{x})\|_{(t, M_{\infty \times n})} \le r_0 \|x - \bar{x}\|$$

where $m \in \mathbb{N}$, (t, x), $(t, \bar{x}) \in E_c$. From the above inequalities we obtain in the limit, letting m tend to ∞, estimates (18), (19).

VI. Now we prove (21). It follows from Assumption $H[F]$ that

$$\|\hat{z} - \tilde{z}\|_{(t, l^\infty)} + \|\partial_x \hat{z}^{(m)} - \partial_x \tilde{z}^{(m)}\|_{(t, M_{\infty \times n})}$$

$$\le \|\psi - \tilde{\psi}\|_{\mathbb{X}} + \int\limits_{0}^{t} \Gamma_\star(\tau)[\|\hat{z} - \tilde{z}\|_{(\tau, l^\infty)} + \|\partial_x \hat{z}^{(m)} - \partial_x \tilde{z}^{(m)}\|_{(\tau, M_{\infty \times n})}] \, d\tau, \quad t \in [0, c].$$

Then we obtain (21) from the Gronwall inequality. This completes the proof of the theorem.

4 The main result

Suppose that Assumptions $H[\varphi]$, $H[F]$ are satisfied and $\psi \in \mathbb{X}$. Let us denote by $\Xi[\psi]$ the solution to (1), (2). It follows from Theorem 3.2 that $\Xi[\psi]$ exists on E_c and it is unique. Then we have: $\Xi : \mathbb{X} \to C(E_c, l^\infty)$. We will denote by $\mathbb{Y}$ the class of all $\psi \in \mathbb{X}$ satisfying the conditions:

$$\sup\{\|\psi_k\|_{E_{0.k}} : k \in \mathbb{N}\} < c_0, \quad \sup\{\|\partial_x \psi_k\|_{E_{0.k}} : k \in \mathbb{N}\} < c_1,$$

$$\sup\left\{\frac{1}{\|x - \bar{x}\|}\|\partial_x \psi_k(t, x) - \partial_x \psi_k(t, \bar{x})\| : (t, x), (t, \bar{x}) \in E_{0.k}, \ x \neq \bar{x}, \ k \in \mathbb{N}\right\} < c_1.$$

We prove that for each $\psi \in \mathbb{Y}$ there exists the Fréchet derivative $\partial\Xi[\psi]$ of Ξ at the point ψ. Moreover, if $\psi \in \mathbb{Y}$, $\pi \in \mathbb{X}$ and $\bar{\mathbf{z}} = \partial\Xi[\psi]\pi$ then $\bar{\mathbf{z}}$ is a solution of a linear system of integral functional equations generated by (1), (2).

Suppose that Assumptions $H[\varphi]$, $H[F]$ are satisfied. Write

$$\hat{\zeta}(t) = c_0 \, \exp\left[2\int_0^t \beta(\tau)\, d\tau\right],$$

$$\hat{\vartheta}(t) = \hat{d}\left\{\exp\left[\hat{d}\int_0^t \beta(\tau)\, d\tau\right] + \int_0^t \gamma(\mu) \exp\left[\hat{d}\int_\mu^t \beta(\tau)\, d\tau\right] d\mu\right\},$$

where $t \in [0, c]$ and

$$\hat{C} = \hat{\zeta}(c), \quad \hat{d} = \exp\left[\bar{C}\int_0^c \gamma(\tau)\, d\tau\right] \max\{c_1, \ 2\bar{c}\hat{C}, \ 1 + Q_0\}. \tag{39}$$

Suppose that $\pi \in \mathbb{X}$, $\pi = \{\pi_k\}_{k\in\mathbb{N}}$. Let us denote by $C_{\pi.c}[\hat{\zeta}, \hat{\vartheta}]$ the class of all $z : E_c \to l^\infty$, $z = \{z_k\}_{k\in\mathbb{N}}$, satisfying the conditions:

(i) $z \in C(E_c, l^\infty)$ and $z_k(t, x) = \pi_k(t, x)$ for $(t, x) \in E_{0.k}$ and $k \in \mathbb{N}$,
(ii) $\|z\|_{(t,l^\infty)} \leq \hat{\zeta}(t)$ for $t \in [0, c]$ and

$$\sup\left\{\frac{1}{\|x - \bar{x}\|}\|z(\tau, x) - z(\tau, \bar{x})\|_\infty : (\tau, x), (\tau, \bar{x}) \in E_c, \ \tau \leq t\right\} \leq \hat{\vartheta}(t) \quad \text{for } t \in [0, c].$$

In this section we denote by $z(\cdot \, ; \psi) = \{z_k(\cdot \, ; \psi)\}_{k\in\mathbb{N}}$ the solution to (1), (2). Let us consider the Cauchy problem

$$\omega'(\tau) = -\partial_q F_k\big(\tau, \omega(\tau), (z(\cdot \, ; \psi))_{(\tau, \omega(\tau))}, \ (z(\cdot \, ; \psi))_{\varphi(\tau, \omega(\tau))}, \ \partial_x z(\tau, \omega(\tau); \psi)\big), \quad \omega(t) = x, \tag{40}$$

where $(t, x) \in E \cap \big([a_k, c] \times \mathbb{R}^n\big)$ and $k \in \mathbb{N}$. The solution to (40) will be denote by $g_{[k]}[\psi](\cdot, t, x)$. If Assumptions $H[\varphi]$, $H[F]$ are satisfied and $\psi \in \mathbb{X}$ then for each $k \in \mathbb{N}$ the solution $g_{[k]}[\psi](\cdot, t, x)$ is defined on $[a_k, t]$. For $k \in \mathbb{N}$ we put

$$T_k[\psi](\tau, t, x)$$
$$= \big(\tau, g_{[k]}[\psi](\tau, t, x), (z(\cdot \, ; \psi))_{(\tau, g_{[k]}[\psi](\tau, t, x))}, (z(\cdot \, ; \psi))_{\varphi(\tau, g_{[k]}[\psi](\tau, t, x))},$$
$$\times \partial_x z(\tau, g_{[k]}[\psi](\tau, t, x); \psi)\big).$$

Suppose that $\pi \in \mathbb{X}$, $\pi = \{\pi_k\}_{k\in\mathbb{N}}$, and $z \in C_{\pi.c}[\hat{\zeta}, \hat{\vartheta}]$, $z = \{z_k\}_{k\in\mathbb{N}}$. Let us denote by $W[z] = \{W_k[z]\}_{k\in\mathbb{N}}$ the function defined by

$$W_k[z](t, x) = \pi_k(t, x) \quad \text{for } (t, x) \in E_{0.k} \tag{41}$$

and

$$W_k[z](t, x) = \pi_k(a_k, g_{[k]}[\psi](a_k, t, x)) + \int_{a_k}^{t} \partial_v F_k\big(T_k[\psi](\tau, t, x)\big) \diamond z_{(\tau, g_{[k]}[\psi](\tau, t, x))} \, d\tau$$

$$+ \int_{a_k}^{t} \partial_w F_k\big(T_k[\psi](\tau, t, x)\big) \diamond z_{\varphi(\tau, g_{[k]}[\psi](\tau, t, x))} \, d\tau \quad \text{for } (t, x) \in E \cap \big([a_k, c] \times \mathbb{R}^n\big), \quad (42)$$

where $k \in \mathbb{N}$. We consider the linear system of integral functional equations

$$z = W[z]. \tag{43}$$

Lemma 4.1 *Suppose that Assumptions $H[\varphi]$, $H[F]$ are satisfied and $\pi \in \mathbb{X}$. Then there exists exactly one solution $\bar{\mathbf{z}} : E_c \to l^\infty$ of* (43) *and*

$$\|\bar{\mathbf{z}}\|_\infty \leq \hat{C} \quad on \ E_c, \tag{44}$$

$$\|\bar{\mathbf{z}}(t, x) - \bar{\mathbf{z}}(t, \bar{x})\|_\infty \leq \hat{L}\|x - \bar{x}\| \quad on \ E_c \tag{45}$$

where $\hat{C}$ is given by (39) *and $\hat{L} = \hat{\vartheta}(c)$.*

Proof We prove that

$$W : C_{\pi.c}[\hat{\zeta}, \hat{\vartheta}] \to C_{\pi.c}[\hat{\zeta}, \hat{\vartheta}]. \tag{46}$$

It follows from Assumptions $H[\varphi]$, $H[F]$ that for $z \in C_{\pi.c}[\hat{\zeta}, \hat{\vartheta}]$, $(t, x) \in E_c$, $\tau \leq t$, we have

$$|W_k[z](\tau, x)| \leq c_0 + 2 \int_0^t \beta(\mu) \, \hat{\zeta}(\mu) \, d\mu = \hat{\zeta}(t)$$

and consequently

$$\|W[z]\|_{(t, l^\infty)} \leq \hat{\zeta}(t) \quad \text{for } t \in [0, c]. \tag{47}$$

For the above z and for $(\tau, x), (\tau, \bar{x}) \in E_c$, $\tau \leq t$, we have

$$|W_k[z](\tau, x) - W_k[z](\tau, \bar{x})| \leq \hat{d} \left[1 + \int_0^t \gamma(\mu) \, d\mu + \int_0^t \beta(\mu) \, \hat{\vartheta}(\mu) \, d\mu \right] \|x - \bar{x}\|$$

$$= \hat{\vartheta}(t)\|x - \bar{x}\|, \quad k \in \mathbb{N}.$$

This gives

$$\|W[z](\tau, x) - W[z](\tau, \bar{x})\|_\infty \leq \hat{\vartheta}(t)\|x - \bar{x}\|, \tag{48}$$

where $(\tau, x), (\tau, \bar{x}) \in E_c$, $\tau \leq t$. From (47), (48) we deduce (46).

For $z, z_\star \in C_{\pi.c}[\hat{\zeta}, \hat{\vartheta}]$ and for $k \in \mathbb{N}$ we have

$$W_k[z](t, x) - W_k[z_\star](t, x) = 0 \quad \text{on } E_{0.k}, \tag{49}$$

and

$$|W_k[z](t, x) - W_k[z_\star](t, x)| \leq 2 \int_0^t \beta(\tau) \, \|z - z_\star\|_{(\tau, l^\infty)} \, d\tau \quad \text{on } E \cap \big([a_k, c] \times \mathbb{R}^n\big). \tag{50}$$

Write

$$[|z - z_\star|] = \max \left\{ \|z - z_\star\|_{(\tau, l^\infty)} \exp \left[-4 \int_0^t \beta(\tau)\, d\tau \right] : t \in [0, c] \right\}.$$

It follows from (49), (50) that for $z, z_\star \in C_{\pi.c}[\hat{\zeta}, \hat{\vartheta}]$ we have

$$[|W[z] - W[z_\star]|] \leq \frac{1}{2}[|z - z_\star|].$$

From the Banach fixed point theorem we deduce that there is exactly one solution $\bar{\mathbf{z}} \in C_{\pi.c}[\hat{\zeta}, \hat{\vartheta}]$ of (43). We conclude from (46) that conditions (44), (45) are satisfied. This completes the proof.

Theorem 4.2 *Suppose that Assumptions $H[\varphi]$, $H[F]$ are satisfied. Then*

1. *for each $\psi \in \mathbb{Y}$ there exists the Fréchet derivative $\partial \Xi[\psi]$,*
2. *if $\psi \in \mathbb{Y}$, $\pi \in \mathbb{X}$ and $\bar{\mathbf{z}} = \partial \Xi[\psi]\pi$, $\bar{\mathbf{z}} = \{\bar{\mathbf{z}}_k\}_{k \in \mathbb{N}}$, then $\bar{\mathbf{z}}$ is a solution to (43) with W given by (41), (42).*

Proof The proof will be divided into three steps.
I. Let $\psi \in \mathbb{Y}$ and $\pi \in \mathbb{X}$ be fixed. There is $\varepsilon_0 > 0$ such that for $\xi \in I_0 = (-\varepsilon_0, \varepsilon_0)$ we have $\psi + \xi\pi \in \mathbb{X}$. Let $\Delta_\xi : E_c \to l^\infty, \xi \in I_0, \xi \neq 0$, be defined by

$$\Delta_\xi = \left\{\Delta_{\xi.k}\right\}_{k \in \mathbb{N}}, \quad \Delta_{\xi.k}(t, x) = \frac{1}{\xi}\left[z_k(t, x; \psi + \xi\pi) - z_k(t, x; \psi)\right], \quad k \in \mathbb{N}.$$

It follows from Lemma 4.1 that there is exactly one solution $\bar{\mathbf{z}} = \{\bar{\mathbf{z}}_k\}_{k \in \mathbb{N}}$ to (43), $\bar{\mathbf{z}} \in C(E_c, l^\infty)$ and $\bar{\mathbf{z}}$ satisfies (44), (45). We prove that

$$\lim_{\xi \to 0} \|\Delta_\xi(t, x) - \bar{\mathbf{z}}(t, x)\|_\infty = 0 \quad \text{uniformly on } E_c. \tag{51}$$

II. It follows from Theorem 3.2 that for $k \in \mathbb{N}, \xi \in I_0, \xi \neq 0$ we have

$$\partial_t \Delta_{\xi.k}(t, x)$$
$$= \frac{1}{\xi}\Big\{ F_k\big(t, x, (z(\cdot\,; \psi + \xi\pi))_{(t,x)}, (z(\cdot\,; \psi + \xi\pi))_{\varphi(t,x)}, \partial_x z_k(t, x; \psi + \xi\pi)\big)$$
$$- F_k\big(t, x, (z(\cdot\,; \psi))_{(t,x)}, (z(\cdot\,; \psi))_{\varphi(t,x)}, \partial_x z_k(t, x; \psi)\big) \quad \text{on } E \cap \big([a_k, c] \times \mathbb{R}^n\big), \tag{52}$$

and

$$\Delta_{\xi.k}(t, x) = \pi_k(t, x) \quad \text{on } E_{0.k}. \tag{53}$$

Set

$$\overline{Q}_k(t, x; \psi) = \big(t, x, (z(\cdot\,; \psi))_{(t,x)}, (z(\cdot\,; \psi))_{\varphi(t,x)}, \partial_x z_k(t, x; \psi)\big), \quad k \in \mathbb{N},$$

and

$$\Theta_k[\psi, \pi](t, x; \xi, \eta) = (1 - \eta)\overline{Q}_k(t, x; \psi) + \eta\overline{Q}_k(t, x; \psi + \xi\pi), \quad k \in \mathbb{N},$$

where $\eta \in [0, 1]$. Then we obtain from (52) that

$$\partial_t \Delta_{\xi.k}(t, x) = \int_0^1 \partial_v F_k(\Theta_k[\psi, \pi](t, x; \xi, \eta)) \diamond (\Delta_\xi)_{(t,x)} \, d\eta$$

$$+ \int_0^1 \partial_w F_k(\Theta_k[\psi, \pi](t, x; \xi, \eta)) \diamond (\Delta_\xi)_{\varphi(t,x)} \, d\eta$$

$$+ \int_0^1 \partial_q F_k(\Theta_k[\psi, \pi](t, x; \xi, \eta)) \circ \partial_x \Delta_{\xi.k}(t, x) \, d\eta, \quad (t, x) \in E \cap ([a_k, c] \times \mathbb{R}^n), \quad (54)$$

where $k \in \mathbb{N}$. Let us denote by $g_{[k]}[\psi, \pi; \xi](\cdot, t, x)$ the solution of the Cauchy problem

$$\omega'(\tau) = - \int_0^1 \partial_q F_k(\Theta_k[\psi, \pi](\tau, \omega(\tau); \xi, \eta)) \, d\eta, \quad \omega(t) = x,$$

where $(t, x) \in E \cap ([a_k, c] \times \mathbb{R}^n)$. Write

$$Q_k[\psi, \pi; \xi, \eta](\tau, t, x) = \Theta_k[\psi, \pi](\tau, g_{[k]}[\psi, \pi; \xi](\tau, t, x); \xi, \eta), \quad k \in \mathbb{N}.$$

It follows from (54) that Δ_ξ satisfies relations

$$\Delta_{\xi.k}(t, x) = \pi_k(a_k, g_{[k]}[\psi, \pi; \xi](a_k, t, x))$$

$$+ \int_{a_k}^t \int_0^1 \partial_v F_k(Q_k[\psi, \pi; \xi, \eta](\tau, t, x)) \diamond (\Delta_\xi)_{(\tau, g_{[k]}[\psi, \pi; \xi](\tau, t, x))} \, d\eta \, d\tau$$

$$+ \int_{a_k}^t \int_0^1 \partial_w F_k(Q_k[\psi, \pi; \xi, \eta](\tau, t, x)) \diamond (\Delta_\xi)_{\varphi(\tau, g_{[k]}[\psi, \pi; \xi](\tau, t, x))} \, d\eta \, d\tau \quad (55)$$

where $(t, x) \in E \cap ([a_k, c] \times \mathbb{R}^n)$ and $k \in \mathbb{N}$.

III. We construct an integral functional inequality for $\Delta_\xi - \bar{\mathbf{z}}$. Write

$$A_k(t, x) = \pi_k(a_k, g_{[k]}[\psi, \pi; \xi](a_k, t, x)) - \pi_k(a_k, g_{[k]}[\psi](a_k, t, x)),$$

$$B_k(t, x) = \int_0^t \int_0^1 \partial_v F_k(Q_k[\psi, \pi; \xi, \eta](\tau, t, x)) \diamond (\Delta_\xi - \bar{\mathbf{z}})_{(\tau, g_{[k]}[\psi, \pi; \xi](\tau, t, x))} \, d\eta \, d\tau$$

$$+ \int_0^t \int_0^1 \partial_w F_k(Q_k[\psi, \pi; \xi, \eta](\tau, t, x)) \diamond (\Delta_\xi - \bar{\mathbf{z}})_{\varphi(\tau, g_{[k]}[\psi, \pi; \xi](\tau, t, x))} \, d\eta \, d\tau$$

and

$$
C_k(t, x) = \int_{a_k}^{t} \int_0^1 \partial_v F_k \big(Q_k[\psi, \pi; \xi, \eta](\tau, t, x) \big) \diamond [\bar{\mathbf{z}}_{(\tau, g_{[k]}[\psi, \pi; \xi](\tau, t, x))}
$$

$$
\times - \bar{\mathbf{z}}_{(\tau, g_{[k]}[\psi](\tau, t, x))}] \, d\eta \, d\tau
$$

$$
+ \int_{a_k}^{t} \int_0^1 \partial_w F_k \big(Q_k[\psi, \pi; \xi, \eta](\tau, t, x) \big) \diamond [\bar{\mathbf{z}}_{\varphi(\tau, g_{[k]}[\psi, \pi; \xi](\tau, t, x))} - \bar{\mathbf{z}}_{\varphi(\tau, g_{[k]}[\psi](\tau, t, x))}] \, d\eta \, d\tau,
$$

$$
D_k(t, x) = \int_{a_k}^{t} \int_0^1 \big[\partial_v F_k \big(Q_k[\psi, \pi; \xi, \eta](\tau, t, x) \big) - \partial_v F_k \big(T_k[\psi](\tau, t, x) \big) \big]
$$

$$
\times \diamond \bar{\mathbf{z}}_{(\tau, g_{[k]}[\psi](\tau, t, x))} \, d\eta \, d\tau
$$

$$
+ \int_{a_k}^{t} \int_0^1 \big[\partial_w F_k \big(Q_k[\psi, \pi; \xi, \eta](\tau, t, x) \big) - \partial_w F_k \big(T_k[\psi](\tau, t, x) \big) \big] \diamond \bar{\mathbf{z}}_{\varphi(\tau, g_{[k]}[\psi](\tau, t, x))} \, d\eta \, d\tau
$$

where $k \in \mathbb{N}$. Then we have

$$
\Delta_{\xi.k}(t, x) - \bar{\mathbf{z}}_k(t, x) = A_k(t, x) + B_k(t, x) + C_k(t, x) + D_k(t, x), \quad k \in \mathbb{N}. \tag{56}
$$

It follows from Assumption $H[F]$ and from (21) that

$$
\| g_{[k]}[\psi, \pi; \xi](\tau, t, x) - g_{[k]}[\psi](\tau, t, x) \|
$$

$$
\leq \bar{C} \left| \int_\tau^t \gamma(\mu) \| g_{[k]}[\psi, \pi; \xi](\mu, t, x) - g_{[k]}[\psi](\mu, t, x) \| \, d\mu \right| + |\xi| A_\star, \quad k \in \mathbb{N},
$$

where

$$
A_\star = 3 \|\pi\|_\mathbb{X} \exp \left[\int_0^c \Gamma_\star(\tau) \, d\tau \right] \int_0^c \gamma(\tau) \, d\tau.
$$

This gives

$$
\| g_{[k]}[\psi, \pi; \xi](\tau, t, x) - g_{[k]}[\psi](\tau, t, x) \| \leq |\xi| \hat{A}, \quad k \in \mathbb{N}, \tag{57}
$$

where

$$
\hat{A} = A_\star \exp \left[\bar{C} \int_0^c \gamma(\tau) \, d\tau \right],
$$

and consequently

$$
\big| A_k(t, x) \big| \leq c_1 \hat{A} |\xi|, \quad k \in \mathbb{N}. \tag{58}
$$

It is clear that

$$
\big| B_k(t, x) \big| \leq 2 \int_0^t \beta(\tau) \| \Delta_\xi - \bar{\mathbf{z}} \|_{(\tau, l^\infty)} \, d\tau, \quad k \in \mathbb{N}. \tag{59}
$$

We conclude from Assumption $H[F]$ and from (45) that

$$\left| C_k(t, x) \right| \leq \hat{L}(1 + Q_0) \int_{a_k}^{t} \beta(\tau) \| g_{[k]}[\psi, \pi; \xi](\tau, t, x) - g_{[k]}[\psi](\tau, t, x) \| \, d\tau$$

$$\leq |\xi| \hat{L}(1 + Q_0) \int_0^c \beta(\tau) \, d\tau \, \hat{A}, \quad k \in \mathbb{N}. \tag{60}$$

It follows from Assumption $H[F]$ and from (21) that the expressions

$$\int_{a_k}^{t} \int_0^1 \left\| \partial_v F_k \big(Q_k[\psi, \pi; \xi, \eta](\tau, t, x) \big) - \partial_v F_k \big(T_k[\psi](\tau, t, x) \big) \right\|_{\infty; \star} d\eta \, d\tau,$$

$$\int_{a_k}^{t} \int_0^1 \left\| \partial_w F_k \big(Q_k[\psi, \pi; \xi, \eta](\tau, t, x) \big) - \partial_w F_k \big(T_k[\psi](\tau, t, x) \big) \right\|_{\infty; \star} d\eta \, d\tau, \quad k \in \mathbb{N},$$

can be estimated from above by

$$\bar{C} \int_{a_k}^{t} \gamma(\tau) \| g_{[k]}[\psi, \pi; \xi](\tau, t, x) - g_{[k]}[\psi](\tau, t, x) \| \, d\tau + |\xi| \hat{B}, \quad k \in \mathbb{N},$$

where

$$\hat{B} = 3 \int_0^c \gamma(\tau) \, d\tau \, \| \pi \|_{\mathbb{X}} \, \exp \left[\int_0^c \Gamma_\star(\tau) \, d\tau \right].$$

We conclude from the above relations and from (44), (57) that

$$\left| D_k(t, x) \right| \leq |\xi| \hat{D}, \quad k \in \mathbb{N}, \tag{61}$$

where

$$\hat{D} = 2\hat{C} \left[\bar{C} \hat{A} \int_0^c \gamma(\tau) \, d\tau + \hat{B} \right].$$

It follows from (56) and (58)–(61) that there is $\hat{Q} \in \mathbb{R}_+$ such that

$$\left\| \Delta_\xi - \bar{\mathbf{z}} \right\|_{(t, l^\infty)} \leq |\xi| \hat{Q} + 2 \int_0^c \beta(\tau) \left\| \Delta_\xi - \bar{\mathbf{z}} \right\|_{(\tau, l^\infty)} d\tau, \quad t \in [0, c].$$

By the Gronwall inequality we obtain

$$\left\| \Delta_\xi - \bar{\mathbf{z}} \right\|_{(t, l^\infty)} \leq |\xi| \hat{Q} \, \exp \left[2 \int_0^c \beta(\tau) \, d\tau \right], \quad t \in [0, c].$$

This completes the proof of (51).

The assertion of the theorem follows from (51).

Remark 4.3 It is easy to see that Theorems 3.2 and 4.2 can be applied to problems (4), (2) and (8), (2).

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