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Kadec-Klee properties of Calderón-Lozanovskiĭ sequence spaces

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Abstract We study two Kadec-Klee properties with respect to coordinatewise convergence and with respect to uniform convergence. We shall give full criteria for these properties in Calderón-Lozanovskiĭ sequence spaces. In particular, we obtain the characterizations of Kadec-Klee properties in Orlicz-Lorentz spaces, which have not been known in such generality until now.

Keywords Köthe spaces · Calderón-Lozanovskiĭ spaces · Kadec-Klee properties

Mathematics Subject Classification (2000) 46B20 · 46B42 · 46B45 · 46A45

1 Introduction

The Kadec-Klee properties play important role in the theory of Banach function spaces (see [5,11,17]). The Calderón-Lozanovskiĭ spaces are one of important classes of Banach lattices, especially due to their application in the interpolation theory. Geometry of Calderón-Lozanovskiĭ spaces has been deeply studied recently (see for example [3,7–10,12,14]).

The complete characterization of Kadec-Klee properties with respect to local (global) convergence in measure H_l (H_g) for Orlicz function spaces L_φ has been presented in [6] and later generalized in [13] to the case of Calderón-Lozanovskiĭ function spaces E_φ. Here we consider the respective sequence case. Some partial results concerning Kadec-Klee property with respect to pointwise convergence in generalized Calderón-Lozanovskiĭ and Orlicz-Lorentz sequence spaces have been presented in [7] and [2]. We present full characterizations of

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Kadec-Klee properties with respect to pointwise convergence (with respect to uniform convergence) in Calderón-Lozanovskiĭ sequence spaces. In particular we obtain the respective criteria for these properties in Orlicz-Lorentz sequence spaces.

2 Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ be the sets of real, nonnegative real and positive integer numbers, respectively. As usual $S(X)$ (resp. $B(X)$) stands for the unit sphere (resp. the closed unit ball) of a real Banach space $(X, \|\cdot\|_X)$.

Let $(\mathbb{N}, 2^{\mathbb{N}}, m)$ be the counting measure space and $l^0 = l^0(m)$ be the linear space of all real sequences.

Let $E = (E, \leq, \|\cdot\|_E)$ be a Banach sequence lattice over the measure space $(\mathbb{N}, 2^{\mathbb{N}}, m)$, that is E is a Banach space being a subspace of l_0 endowed with the natural coordinatewise semi-order relation, and E satisfies the conditions:

(i) if $x \in E, y \in l^0, |y| \leq |x|$, i.e. $|y(i)| \leq |x(i)|$ for every $i \in \mathbb{N}$, then $y \in E$ and $\|y\|_E \leq \|x\|_E$,

(ii) there exists a sequence x in E that is positive on the whole $\mathbb{N}$ (see [11] and [17]). Banach sequence lattices are often called *Köthe sequence spaces*.

The symbol $e_i = (0, \ldots, 0, 1, 0, \ldots)$ stands for the ith unit vector. The set $E_+ = \{x \in E : x \geq 0\}$ is called the *positive cone of* E. For any subset $A \subset E$ define $A_+ = A \cap E_+$.

A point $x \in E$ is said to have *order continuous norm* if for any sequence (x_m) in E such that $0 \leq x_m \leq |x|$ and $x_m \to 0$ pointwisely we have $\|x_m\|_E \to 0$. A Köthe sequence space E is called *order continuous* ($E \in (OC)$) if every element of E has an order continuous norm (see [11] and [17]).

Recall that E is said to have Kadec-Klee property ($E \in (KK)$ for short) whenever $\|x_n - x\| \to 0$ for any x and (x_n) in E satisfying $x_n \to x$ in the weak topology $\sigma(E, E^*)$ and $\|x_n\| \to \|x\|$ (see [17]). This property, also called the Radon-Riesz property or property H, has been considered in many classes of Banach spaces (see [1,2,5,7,15]). If we consider E more generally over $\sigma-$finite and complete measure space (T, Σ, μ) and we replace the weak convergence $\sigma(E, E^*)$ by the convergence in measure ($x_n \stackrel{\mu}{\to} x$), by the convergence in measure on every set of finite measure ($x_n \stackrel{\mu}{\to} x$ locally) or by the uniform convergence ($x_n \rightrightarrows x$), then we say that E has the Kadec-Klee property with respect to convergence in measure, local convergence in measure or uniform convergence, respectively (we shall write $E \in (H_g)$, $E \in (H_l)$, $E \in (H_u)$). Clearly, $E \in (H_l) \Rightarrow E \in (H_g) \Rightarrow E \in (H_u)$. Moreover, the converse of any of these implications is not true in general (see [13]). If (T, Σ, μ) is a counting measure space $(\mathbb{N}, 2^{\mathbb{N}}, m)$ then:

(i) $E \in (H_l)$ if and only if $E \in (H_c)$ that means E has the Kadec-Klee property with respect to pointwise convergence.

(ii) $E \in (H_g)$ if and only if $E \in (H_u)$.

In the whole paper φ denotes an *Orlicz function*, i.e. $\varphi : \mathbb{R} \to [0, \infty]$, it is convex, even, vanishing and continuous at zero, left continuous on $(0, \infty)$ and not identically equal to zero. Denote

$$a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\} \quad \text{and} \quad b_\varphi = \sup\{u \geq 0 : \varphi(u) < \infty\}.$$

We write $\varphi > 0$ when $a_\varphi = 0$ and $\varphi < \infty$ if $b_\varphi = \infty$. Let $\varphi_r = \varphi\chi_{G_\varphi}$, where

$$G_\varphi = \begin{cases} [a_\varphi, b_\varphi] & \text{if } \varphi(b_\varphi) < \infty, \\ [a_\varphi, b_\varphi) & \text{otherwise.} \end{cases} \tag{1}$$

Define on L^0 a convex semimodular I_φ by

$$I_\varphi(x) = \begin{cases} \|\varphi \circ x\|_E & \text{if } \varphi \circ x \in E, \\ \infty & \text{otherwise,} \end{cases}$$

where $(\varphi \circ x)(t) = \varphi(x(t))$, $t \in T$. By the Calderón-Lozanovskiĭ space E_φ we mean

$$E_\varphi = \{x \in L^0 : I_\varphi(cx) < \infty \text{ for some } c > 0\}$$

equipped with so called *Luxemburg-Nakano norm* defined by

$$\|x\|_\varphi = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \le 1\}.$$

We generally assume that if $b_\varphi < \infty$, then $a_\varphi < b_\varphi$, because when $0 < a_\varphi = b_\varphi$, then $E_\varphi = L^\infty$ and $\|x\|_\varphi = \frac{1}{b_\varphi}\|x\|_\infty$.

If $E = L^1 (E = l^1)$, then E_φ is the Orlicz function (sequence) space equipped with the Luxemburg norm. If $E = \Lambda_\omega$ (the Lorentz function space) or $E = \lambda_\omega$ (the Lorentz sequence space), then E_φ is the corresponding Orlicz-Lorentz function (sequence) space denoted by $(\Lambda_\omega)_\varphi$ $((\lambda_\omega)_\varphi)$ and equipped with the Luxemburg norm (see [8,12]).

We will assume in the whole paper that E has the *Fatou property*, that is, if $0 \le x_n \uparrow x \in L^0$ with $(x_n)_{n=1}^\infty$ in E and $\sup_n \|x_n\|_E < \infty$, then $x \in E$ and $\|x\|_E = \lim_n \|x_n\|_E$. Since E has the Fatou property, E_φ has also this property, whence E_φ is a Banach space.

We say an Orlicz function φ satisfies *condition* $\Delta_2(0)$ (resp. $\Delta_2(\infty)$) if there exist $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) > 0$ (resp. $\varphi(u_0) < \infty$) and the inequality $\varphi(2u) \leqslant K\varphi(u)$ holds for all $u \in [0, u_0]$(resp. $u \in [u_0, \infty)$). If there exists $K > 0$ such that $\varphi(2u) \leqslant K\varphi(u)$ for all $u \geqslant 0$, then we say that φ satisfies *condition* $\Delta_2(\mathbb{R}_+)$. We write for short $\varphi \in \Delta_2(0)$, $\varphi \in \Delta_2(\infty)$, $\varphi \in \Delta_2(\mathbb{R}_+)$, respectively.

For a Köthe space E and an Orlicz function φ we say that φ satisfies *condition* Δ_2^E ($\varphi \in \Delta_2^E$ for short) if:

1. $\varphi \in \Delta_2(0)$ whenever $E \hookrightarrow L^\infty$;
2. $\varphi \in \Delta_2(\infty)$ whenever $L^\infty \hookrightarrow E$;
3. $\varphi \in \Delta_2(\mathbb{R}_+)$ whenever neither $L^\infty \hookrightarrow E$ nor $E \hookrightarrow L^\infty$ (see [8]),

where the symbol $E \hookrightarrow F$ stands for the continuous embedding of the space E into the space F.

Relationships between the modular I_φ and the norm $\|\cdot\|_\varphi$ are collected in [12].

3 Results

3.1 Calderón-Lozanovskiĭ sequence spaces

The property H_c.

We will need in the sequel the following refining of Corollary 12 from [14].

Lemma 1 *Suppose that E is a Köthe sequence space. Then $E_\varphi \in (OC)$ if and only if:*

(a) $E \in (OC)$.
(b) $\varphi \in \Delta_2^E$.

Proof *Necessity*. First, we prove that

$$(+)\quad a_\varphi = 0 \quad \text{whenever } E \hookrightarrow l^\infty \quad \text{or} \quad l^\infty \not\hookrightarrow E.$$

Assume that $E \hookrightarrow l^\infty$ and $a_\varphi > 0$. Then $m = \inf_n \|e_n\|_E > 0$. Taking $x = a_\varphi \chi_{\mathbb{N}}$ and $x_n = a_\varphi \chi_{\{e_n\}}$, we get $x_n \to 0$ pointwisely, $I_\varphi(2x_n) = \varphi(2a_\varphi)\,\|e_n\|_E \geq \varphi(2a_\varphi)m > 0$, whence $\|x_n\|_\varphi \nrightarrow 0$. Thus $E_\varphi \notin (OC)$.

Suppose that $l^\infty \not\hookrightarrow E$ and $a_\varphi > 0$. Then $\chi_{\mathbb{N}} \notin E$ and putting $x = a_\varphi \chi_{\mathbb{N}}$ and $x_n = a_\varphi \chi_{A_n}$, where $A_n = \{n, n+1, \ldots\}$, we conclude that $E_\varphi \notin (OC)$.

Next, note that if $E_\varphi \in (OC)$ and $\varphi(b_\varphi) < \infty$, then $E \hookrightarrow c_0 \hookrightarrow l^\infty$ (see the proof of Corollary 12 from [14]). Consequently (a) and (b) in Corollary 12 from [14] means that if $E_\varphi \in (OC)$, then $E \in (OC)$. Thus $E \hookrightarrow c_0\{\|e_n\|\}$. Thus, by $(+)$, we may apply Lemma 2.9 from [12] and Theorem 2.4 from [7] to obtain that $\varphi \in \Delta_2^E$. The *sufficiency* is done by Corollary 12 from [14]. □

Lemma 2 [13, Lemma 8] *If* $\varphi \in \Delta_2(\infty)$, *then for any* $a > 0$ *there are* $\sigma \in (0, 1]$ *and* $u_1 > a_\varphi + a$ *such that*

$$\inf_{v\in(0,a)} \inf_{u\geq u_1} \frac{\varphi(u-v)}{\varphi(u)-\varphi(v)} \geq \sigma.$$

Theorem 3 *The Calderón-Lozanovskiĭ sequence space* $E_\varphi \in (H_c)$ *if and only if:*

(a) *If* $E \hookrightarrow l^\infty$, *then* $\varphi(b_\varphi)\inf_i \|e_i\|_E \geq 1$.
(b) $\varphi \in \Delta_2^E$.
(c) $E \in (H_c)$.

Proof **The sufficiency** has been proved in [7, Theorem 4.4] but under general assumption that $\varphi < \infty$. However, recall that, if $E \hookrightarrow l^\infty$ and $\varphi \in \Delta_2(0)$ then, for any $x \in E_\varphi$, the equivalence $I_\varphi(x) = 1 \Leftrightarrow \|x\|_\varphi = 1$ holds if and only if $\varphi(b_\varphi)\inf_i \|e_i\|_e \geq 1$ (Lemma 1.4(ii) from [12]). Thus the proof can be done as in [7, Theorem 4.4] with small changes when $E \hookrightarrow l^\infty$, $\varphi \in \Delta_2\,(0)$ and $b_\varphi < \infty$. Formally we need also apply 2.9 from [12].

The necessity. Since $H_c \Rightarrow OC$ (see [6]), so, applying Lemma 1, we conclude $\varphi \in \Delta_2^E$ and $E \in (OC)$. Consequently, condition (a) needs to considered only in the case $E \hookrightarrow l^\infty$ and $\varphi \in \Delta_2(0)$. Notice that if $E \in (OC)$ and $E \hookrightarrow l^\infty$ then $l^\infty \not\hookrightarrow E$. Thus condition (a) follows the same way as in the proof of Theorem 9(ii) below.

Suppose for the contrary that $E_\varphi \in (H_c)$ and $E \notin (H_c)$. Recall that property H_c can be equivalently considered only on the positive cone E_+ (Proposition 1 in [9]). Furthermore, the straightforward calculation shows that we may equivalently take x and $\{x_n\}$ in $S(E)$ in the definition of property H_c. Consequently, we find x and $\{x_n\}$ in $(S(E))_+$ with $x_n \to x$ pointwisely and $\|x_n - x\|_E \geq \varepsilon$. Set

$$y_n = \varphi_r^{-1} \circ x_n \quad \text{and} \quad y = \varphi_r^{-1} \circ x.$$

By (a) we conclude that y_n, y are well defined. Furthermore, $I_\varphi(y) = I_\varphi(y_n) = 1$, whence $y, y_n \in S(E_\varphi)$. Moreover, $y_n \to y$ pointwisely. It is enough to prove that there is $\eta > 0$ with

$$\|y_n - y\|_\varphi \geq \eta \tag{2}$$

for infinitely many n. The similar condition has been proved in [12, Thereom 2.12], but under additional assumption that $\varphi > 0$ and $E \in (UM)$. Here the lack of these assumptions requires new techniques in comparison with the respective proof in [12, Thereom 2.12]. We have

$$\|\varphi \circ y_n - \varphi \circ y\|_E \geq \varepsilon. \tag{3}$$

Denote

$$A_n = \{i \in \mathbb{N} : \varphi(y_n(i)) \geq \varphi(y(i))\} \quad \text{and} \quad D_n = \mathbb{N} \backslash A_n. \tag{4}$$

By (3) we get

$$\max\left\{\left\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n}\right\|_E, \left\|(\varphi \circ y_n - \varphi \circ y)\chi_{D_n}\right\|_E\right\} \geq \varepsilon/2 \tag{5}$$

for each n. We assume that $\left\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n}\right\|_E \geq \varepsilon/2$ for infinitely many $n \in \mathbb{N}$, because otherwise the proof is analogous. Notice that two cases:

(i) $l^\infty \hookrightarrow E$ and $\varphi \in \Delta_2(\infty)$,
(ii) $l^\infty \not\hookrightarrow E$, $E \not\hookrightarrow l^\infty$ and $\varphi \in \Delta_2(\mathbb{R}_+)$,

can be done analogously as in the proof of Theorem 8 in [13]. We repeat arguments for readers convenience.

(i) Suppose that $l^\infty \hookrightarrow E$ and $\varphi \in \Delta_2(\infty)$. Let $c = \varphi_r^{-1}\left(\frac{\varepsilon}{32\|\chi_{\mathbb{N}}\|_E}\right) > a_\varphi$. Denoting

$$A_n^1 = \{i \in A_n : y_n(i) < c\} \quad \text{and} \quad A_n^2 = \{i \in A_n : y_n(i) \geq c\}$$

we get

$$\varepsilon/2 \leq \left\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n}\right\|_E \leq \left\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n^2}\right\|_E + \varepsilon/4.$$

Thus

$$\left\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n^2}\right\|_E \geq \varepsilon/4. \tag{6}$$

Set

$$c_1 = \frac{c + a_\varphi}{2}, \quad B_n^1 = \{i \in A_n^2 : y(i) < c_1\} \quad \text{and} \quad B_n^2 = \{i \in A_n^2 : y(i) \geq c_1\}.$$

The remaining proof of the case (i) we divide into two parts.

1. Suppose that $\left\|(\varphi \circ y_n - \varphi \circ y)\chi_{B_n^1}\right\|_E \geq \varepsilon/8$. By Lemma 2, for $a = c_1$ there are $\sigma > 0$ and $u_1 > a_\varphi + c_1$ such that

$$\inf_{v \in (0, c_1)} \inf_{u \geq u_1} \frac{\varphi(u - v)}{\varphi(u) - \varphi(v)} \geq \sigma.$$

Set

$$C_n^1 = \{i \in B_n^1 : y_n(i) < u_1\} \quad \text{and} \quad C_n^2 = \{i \in B_n^1 : y_n(i) \geq u_1\}.$$

(a) Assume that $\left\|(\varphi \circ y_n - \varphi \circ y)\chi_{C_n^1}\right\|_E \geq \varepsilon/16$. Taking $0 < \alpha < \frac{c - c_1}{u_1}$ we get

$$\begin{aligned}\left\|\left(\varphi \circ \left(\frac{y_n - y}{\alpha}\right)\right)\chi_{C_n^1}\right\|_E &\geq \left\|\left(\varphi\left(\frac{c - c_1}{\alpha}\right)\right)\chi_{C_n^1}\right\|_E \geq \left\|\varphi(u_1)\chi_{C_n^1}\right\|_E \\ &\geq \left\|(\varphi \circ y_n - \varphi \circ y)\chi_{C_n^1}\right\|_E \geq \varepsilon/16.\end{aligned}$$

Thus $\|y_n - y\|_\varphi \geq \eta_1 = \min\{1, \alpha\varepsilon/16\}$.

(b) Let $\left\| (\varphi \circ y_n - \varphi \circ y)\chi_{C_n^2} \right\|_E \geq \varepsilon/16$. Thus, by Lemma 2,

$$\left\| \varphi \circ (y_n - y)\, \chi_{C_n^2} \right\|_E \geq \sigma \left\| (\varphi \circ y_n - \varphi \circ y)\, \chi_{C_n^2} \right\|_E \geq \frac{\sigma \varepsilon}{16}$$

and consequently $\|y_n - y\|_\varphi \geq \eta_2 = \min\{1, \sigma\varepsilon/16\}$ (cf. [12, Lemma 1.1]).

2. Assume that $\left\| (\varphi \circ y_n - \varphi \circ y)\chi_{B_n^1} \right\|_E < \varepsilon/8$. Then, by (6),

$$\left\| (\varphi \circ y_n - \varphi \circ y)\, \chi_{B_n^2} \right\|_E \geq \varepsilon/8. \tag{7}$$

Since $\varphi \in \Delta_2(\infty)$, for $l = 1 + \varepsilon/32$ and $u_1 = c_1$ there is $a = a(l, u_1) \in (0, 1)$ such that

$$\varphi\left((1 + a)\, u\right) \leq l\varphi\left(u\right) \tag{8}$$

for every $u \geq c_1$ (see [4, Theorem 1.13(4)]). Moreover, we can choose $a > 0$ satisfying

$$\frac{a}{1+a}\varphi\left(c_1\right) \|\chi_T\|_E < \varepsilon/32. \tag{9}$$

Let

$$B_n^{21} = \left\{ i \in B_n^2 : (y_n - y)\,(i) < \frac{ac_1}{1+a} \right\} \text{ and}$$
$$B_n^{22} = \left\{ i \in B_n^2 : (y_n - y)\,(i) \geq \frac{ac_1}{1+a} \right\}. \tag{10}$$

Then, by (8) and (9), using convexity of φ, we get

$$\begin{aligned}
|\varphi \circ y_n - \varphi \circ y|\, \chi_{B_n^{21}} &= (\varphi \circ (y_n - y + y) - \varphi \circ y)\, \chi_{B_n^{21}} \\
&\leq \left(\frac{a}{1+a}\varphi \circ \left(\frac{1+a}{a}\,(y_n - y) \right) + \frac{1}{1+a}\varphi \circ ((1+a)\, y) - \varphi \circ y \right) \chi_{B_n^{21}} \\
&\leq \left(\frac{a}{1+a}\varphi \circ \left(\frac{1+a}{a}\,(y_n - y) \right) + \frac{1}{1+a} l\varphi \circ y - \varphi \circ y \right) \chi_{B_n^{21}} \\
&= \left(\frac{a}{1+a}\varphi \circ \left(\frac{1+a}{a}\,(y_n - y) \right) + \left(\frac{\varepsilon/32 - a}{1+a} \right) \varphi \circ y \right) \chi_{B_n^{21}}.
\end{aligned} \tag{11}$$

Note $f(a) = \frac{\varepsilon/32 - a}{1+a}$ is a decreasing function of $a > 0$. Hence, by (9), $\left\| (\varphi \circ y_n - \varphi \circ y)\chi_{B_n^{21}} \right\|_E < \varepsilon/16$. Then, by (7),

$$\left\| (\varphi \circ y_n - \varphi \circ y)\, \chi_{B_n^{22}} \right\|_E \geq \varepsilon/16.$$

Since $\varphi \in \Delta_2(\infty)$, for $u_3 = c_1$ and $\beta = \frac{1+a}{a}$ there is $k_2 > 0$ such that $\varphi(\beta u) \leq k_2\varphi(u)$ for each $u \geq u_3$ (see [4]). Taking $0 < \gamma < \frac{a}{1+a}$ and applying (8), we get

$$\begin{aligned}
|\varphi \circ y_n - \varphi \circ y|\, \chi_{B_n^{22}} &= (\varphi \circ (y_n - y + y) - \varphi \circ y)\, \chi_{B_n^{22}} \leq \left(\varphi \circ \left(\frac{y_n - y}{\gamma} + y \right) - \varphi \circ y \right) \chi_{B_n^{22}} \\
&\leq \left(\frac{a}{1+a}\varphi \circ \left(\frac{1+a}{a} \left(\frac{y_n - y}{\gamma} \right) \right) + \frac{1}{1+a}\varphi \circ ((1+a)\, y) - \varphi \circ y \right) \chi_{B_n^{22}} \\
&\leq \left(\frac{a}{1+a} k_2\varphi \circ \left(\frac{y_n - y}{\gamma} \right) + \frac{1}{1+a} l\varphi \circ y - \varphi \circ y \right) \chi_{B_n^{22}} \\
&= \left(\frac{a}{1+a} k_2\varphi \circ \left(\frac{y_n - y}{\gamma} \right) + \left(\frac{\varepsilon/32 - a}{1+a} \right) \varphi \circ y \right) \chi_{B_n^{22}}.
\end{aligned} \tag{12}$$

Then $\left\|\varphi \circ (\frac{y_n-y}{\gamma})\right\|_E \geq \frac{\varepsilon(1+a)}{32ak_2}$ and consequently $\|y_n - y\|_\varphi \geq \eta_3 = \min\left\{1, \frac{\gamma\varepsilon(1+a)}{32ak_2}\right\}$ (see Lemma 1.1 in [12]). Combining cases 1 and 2 we get (2) with $\eta = \min\{\eta_1, \eta_2, \eta_3\}$.

(ii) Suppose that $l^\infty \not\hookrightarrow E$ and $E \not\hookrightarrow l^\infty$. Since $\varphi \in \Delta_2(\mathbb{R}_+)$, so for every $l > 1$ there is $a = a(l) \in (0, 1)$ such that $\varphi((1+a)u) \leq l\varphi(u)$ for every $u \geq 0$ (see [4, Theorem 1.13(4)]). Moreover for every $\beta > 0$ there is $k > 0$ such that $\varphi(\beta u) \leq k\varphi(u)$ for each $u \geq 0$. Then the proof is analogous as in case (i) (it is simpler and shorter).

Note that the necessity when $E \hookrightarrow l^\infty$ has not been discussed in the function case in [13], because if E is a Köthe function space with $E \hookrightarrow L^\infty$, then $E_a = \{0\}$.

(iii) Suppose that $E \hookrightarrow l^\infty$ and $\varphi \in \Delta_2(0)$. We divide the proof into three parts.

A. Assume that $\varphi < \infty$. By $E \hookrightarrow l^\infty$ we get $\|\varphi \circ y_n\|_{l^\infty} \leq M \|\varphi \circ y_n\|_E \leq M$. Hence $y_n \leq \varphi^{-1}(M)$. Moreover, $y_n \chi_{A_n} \geq y\chi_{A_n}$, by $\varphi \circ y_n \chi_{A_n} \geq \varphi \circ y\chi_{A_n}$ and $y_n, y \geq 0$. From $\varphi \in \Delta_2(0)$ and $\varphi < \infty$ we conclude that:

(a) for each $l > 1$ and $u_0 > 0$ there is $a \in (0, 1)$ such that $\varphi((1 + a)u) \leq l\varphi(u)$ for each $0 \leq u \leq u_0$.

(b) for each $\beta > 0$ and $u_0 > 0$ there is $k > 0$ such that $\varphi(\beta u) \leq k\varphi(u)$ for each $0 \leq u \leq u_0$.

Applying (a) take $a \in (0, 1)$ for $l = 1 + \varepsilon/32$ and $u_0 = \varphi^{-1}(M)$. Let $k > 0$ be from (b) for $\beta = \frac{1+a}{a}$ and $u_0 = \varphi^{-1}(M)$. Consequently we obtain

$$\begin{aligned} |\varphi \circ y_n - \varphi \circ y|\, \chi_{A_n} &= (\varphi \circ (y_n - y + y) - \varphi \circ y)\, \chi_{A_n} \\ &\leq \left(\frac{a}{1+a}\varphi \circ \left(\frac{1+a}{a}\,(y_n - y)\right) + \frac{1}{1+a}\varphi \circ ((1+a)\, y) - \varphi \circ y\right) \chi_{A_n} \\ &\leq \left(\frac{a}{1+a}\varphi \circ (\beta\,(y_n - y)) + \frac{1}{1+a} l\varphi \circ y - \varphi \circ y\right) \chi_{A_n} \\ &\leq \left(\frac{ak}{1+a}\varphi \circ (y_n - y) + \left(\frac{\varepsilon/32 - a}{1+a}\right)\varphi \circ y\right) \chi_{A_n}. \end{aligned} \tag{13}$$

Note $f(a) = \frac{\varepsilon/32-a}{1+a}$ is a decreasing function of $a > 0$. Therefore $\|\varphi \circ (y_n - y)\|_E \geq \frac{\varepsilon(1+a)}{4ak}$ and we are done.

B. Suppose that $\varphi(b_\varphi) < \infty$. From $\varphi \in \Delta_2(0)$ we conclude that:

(a) for each $l > 1$ there is $a \in (0, 1)$ such that $\varphi((1 + a)u) \leq l\varphi(u)$ for each $0 \leq u \leq b_\varphi/(1+a)$.

(b) for each $\beta > 1$ there is $k_0 > 0$ such that $\varphi(\beta u) \leq k_0\varphi(u)$ for each $0 \leq u \leq b_\varphi/\beta$.

Applying (a) take a number $0 < a < 1$ for $l = 1 + \varepsilon/32$. Take k_0 from (b) for $\beta = \frac{1+a}{a}$. Set

$$A_n^1 = \left\{i \in A_n : y\,(i) \geq \frac{b_\varphi}{1+a}\right\} \quad \text{and} \quad A_n^2 = \left\{i \in A_n : y\,(i) < \frac{b_\varphi}{1+a}\right\}.$$

Since $E_\varphi \in (H_c)$, so $E_\varphi \in (OC)$ (see [6]) and consequently $y \in (E_\varphi)_a$, whence, by Theorem 11 from [14], $|y(i)| \to 0$ as $i \to \infty$. Then $m(A) < \infty$, where $A = \bigcup A_n^1$. On the other hand $\varphi \circ y_n \to \varphi \circ y$ pointwisely, so $\left\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n^1}\right\|_E \to 0$. Thus $\left\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n^2}\right\|_E \geq \varepsilon/2$ for almost all n. Put

$$A_n^{21} = \left\{i \in A_n^2 : (1+a)\,(y_n - y)\,(i) \leq ab_\varphi\right\} \quad \text{and} \quad A_n^{22} = A_n^2 \backslash A_n^{21},$$

We have $\sigma_0 = \inf_i \|e_i\|_E > 0$, because $E \hookrightarrow l^\infty$. If $\left\|(\varphi \circ y_n - \varphi \circ y)\chi_{A_n^{22}}\right\|_E \geq \varepsilon/4$ for infinitely many n, then

$$\left\|\varphi \circ (y_n - y)\, \chi_{A_n^{22}}\right\|_E \geq \varphi\left(\frac{ab_\varphi}{1+a}\right)\left\|\chi_{A_n^{22}}\right\|_E \geq \sigma_0\varphi\left(\frac{ab_\varphi}{1+a}\right) > 0,$$

whence $\|y_n - y\|_\varphi \geq \eta_1 = \min\left\{1, \sigma_0\varphi\left(\frac{ab_\varphi}{1+a}\right)\right\}$. This is again the contradiction with the fact that $E_\varphi \in (H_c)$.

Supposing that $\left\|(\varphi \circ y_n - \varphi \circ y)\, \chi_{A_n^{21}}\right\|_E \geq \varepsilon/4$ we follow the same way as in the proof of inequality (13).

We need to discuss additionally the case $\|(\varphi \circ y_n - \varphi \circ y)\chi_{D_n}\|_E \geq \varepsilon/2$ for infinitely many $n \in \mathbb{N}$, where $D_n = \{i \in \mathbb{N} : \varphi(y(i)) > \varphi(y_n(i))\}$ is defined in (4). We decompose set D_n analogously

$$D_n^1 = \left\{i \in D_n : y(i) \geq \frac{b_\varphi}{1+a}\right\}, \quad D_n^2 = \left\{i \in D_n : y(i) < \frac{b_\varphi}{1+a}\right\},$$
$$D_n^{21} = \left\{i \in D_n^2 : (1+a)(y - y_n)(i) \leq ab_\varphi\right\} \quad \text{and} \quad D_n^{22} = D_n^2 \backslash D_n^{21}.$$

Notice that $(1+a)y_n(i) \leq (1+a)y(i) \leq b_\varphi$ for each $i \in D_n^2$. Then we step analogously as above but replacing roles of elements y_n and y.

C. Suppose that $b_\varphi < \infty$ and $\varphi(b_\varphi) = \infty$. Then, from $\varphi \in \Delta_2(0)$, we get:

(i) for each $l > 1$ and $u_0 < b_\varphi$ there is $a \in (0,1)$ such that $\varphi((1+a)u) \leq l\varphi(u)$ for each $0 \leq u \leq u_0/(1+a)$.

(ii) for each $\beta > 1$ and $u_0 < b_\varphi$ there is $k_0 > 0$ such that $\varphi(\beta u) \leq k_0\varphi(u)$ for each $0 \leq u \leq u_0/\beta$. Thus we step as in case B with $u_0 = \varphi_r^{-1}(M)$, where M is chosen as in case A. Note also that we have to take u_0 instead of b_φ in the respective definitions of sets A_n^1, A_n^2, A_n^{21} and A_n^{22}. □

The property H_u.

We will need in the sequel the following

Lemma 4 [13, Lemma 9]

(i) *Suppose $x_n, x \in l^0$. If $x_n \rightrightarrows x$, $\varphi \circ x_n$, $\varphi \circ x$ are finitely valued and $\varphi \circ x \in l^\infty$, then $\varphi \circ x_n \rightrightarrows \varphi \circ x$.*

(ii) *If $x_n \rightrightarrows x$, $x_n, x \in E_+$ and $\varphi_r^{-1} \circ x_n$, $\varphi_r^{-1} \circ x$ are well defined functions, then $\varphi_r^{-1} \circ x_n \rightrightarrows \varphi_r^{-1} \circ x$.*

The proof of (i) has been done under the assumption that E is symmetric Köthe space. However, if we replace this assumption by $\varphi \circ x \in l^\infty$, the proof is the same.

Lemma 5 [13, Lemma 5] *Suppose that E is a Köthe sequence space. Then $E \in (H_u)$ if and only if $E \in (H_u)_+$.*

Definition 6 Assume that $\sup_{i \in \mathbb{N}} \|e_i\|_E = \infty$. Then by $\mathbb{B}$ we denote a subset of $\mathbb{N}$ such that for any infinite subset B of $\mathbb{B}$ we have $\sup_{i \in B} \|e_i\|_E = \infty$.

Lemma 7 *If $E \in (H_u)$ then $E|_{\mathbb{B}} \in (OC)$.*

Proof Let $x \in E|_{\mathbb{B}}$, $x \geq 0$, $x \notin E_a$ and $\|x\|_E = 1$. Then $\mathbb{B}$ is infinite and there are a number $\delta > 0$ and a sequence (B_n) pairwise disjoint subsets of $\mathbb{B}$ with $\|x\chi_{B_n}\|_E \geq \delta$ for each n. Setting $B_n = \left\{i_1^{(n)}, i_2^{(n)}, \ldots, i_{k(n)}^{(n)}, \ldots\right\}$ we conclude that for each k_0 there is N_0 such that $i_1^{(n)} \geq k_0$ for any $n \geq N_0$. Notice also that $x \in c_0$. Thus $x\chi_{B_n} \to 0$ uniformly. Moreover, $x\chi_{B_n} \to 0$ weakly (see the proof of Proposition 2.1 from [6]. Put

$$y = x \quad \text{and} \quad y_n = x - x\chi_{B_n}.$$

Since $y_n \to y$ weakly and $\|y_n\|_E \leq \|y\|_E$, so $\|y_n\|_E \to \|y\|_E$ (because the norm is lower semicontinuous with respect to the weak topology). Finally, $y_n \to y$ uniformly and $\|y_n - y\|_E \geq \delta$. □

Remark 8 The same proof shows that if $E \in (H_u)$ then $E \cap c_0 \in (OC)$. Notice also that $x \in c_0$ need not imply that $x \in E_a$. The required example may be constructed in particular Marcinkiewicz sequence space.

We set $c_0\{\|e_i\|_E\} = \left\{x \in l^0 : |x(i)|\,\|e_i\|_E \to 0\right\}$. Clearly, if $E \in (OC)$, then $E \hookrightarrow c_0\{\|e_i\|_E\}$ and the converse is not true (see [7]).

Theorem 9 (i) *If $l^\infty \hookrightarrow E$ or $a_\varphi > 0$, then $E_\varphi \in (H_u)$.*

(ii) *Let $l^\infty \not\hookrightarrow E$, $E \hookrightarrow l^\infty$ and $a_\varphi = 0$. Then the Calderón-Lozanovskiĭ sequence space $E_\varphi \in (H_u)$ if and only if:*

(a) $\varphi(b_\varphi)\inf_i \|e_i\|_E \geq 1$.

(b) $\varphi \in \Delta_2(0)$ *and* $E \in (H_u)$.

(iii) *Suppose that $l^\infty \not\hookrightarrow E$, $E \not\hookrightarrow l^\infty$, $E \hookrightarrow c_0\{\|e_i\|_E\}$ and $a_\varphi = 0$. Then the Calderón-Lozanovskiĭ sequence space $E_\varphi \in (H_u)$ if and only if $\varphi \in \Delta_2(\mathbb{R}_+)$ and $E \in (H_u)$.*

Proof (i) Take $x, x_n \in E_\varphi, n \in \mathbb{N}$, $\|x_n\|_\varphi \to \|x\|_\varphi$ and $x_n \to x$ uniformly. Let $\lambda, \varepsilon > 0$. If $l^\infty \hookrightarrow E$, then there is $\sigma > 0$ such that $\varphi(\lambda\sigma)\,\|\chi_{\mathbb{N}}\|_E < \varepsilon$. Moreover, there is a number N_0 such that $|x_n(i) - x(i)| < \sigma$ for each $n \geq N_0$ and $i \in \mathbb{N}$. Consequently

$$\|\varphi \circ (\lambda\,|x_n - x|)\|_E \leq \varphi(\lambda\sigma)\,\|\chi_{\mathbb{N}}\|_E < \varepsilon$$

for $n \geq N_0$. This means that $\|x_n - x\|_\varphi \to 0$. If $a_\varphi > 0$, then, taking N_0 such that $\lambda\,|x_n(i) - x(i)| < a_\varphi$ for $n \geq N_0$ and each i, we get $\|\varphi \circ (\lambda\,|x_n - x|)\|_E = 0 < \varepsilon$.

(ii) **The necessity**. (a) Suppose that $\varphi(b_\varphi)\inf_i \|e_i\|_E < 1$. Let

$$x = b_\varphi e_{i_0} \quad \text{and} \quad x_n = x + a_n\chi_{A_n},$$

where $\varphi(b_\varphi)\left\|e_{i_0}\right\|_E < 1$, (A_n) is an increasing sequence of subsets of $\mathbb{N}\backslash\{i_0\}$ with $\left\|\chi_{A_n}\right\|_E \to \infty$ (that can be achieved by $E \in (FP)$) and for each n a number a_n is such that $\varphi(b_\varphi)\left\|e_{i_0}\right\|_E + \varphi(a_n)\left\|\chi_{A_n}\right\|_E = 1$. Then $x, x_n \in S(E_\varphi)$ and $x_n \to x$ uniformly. Finally,

$$\|x_n - x\|_\varphi \geq I_\varphi(x_n - x) = \varphi(a_n)\left\|\chi_{A_n}\right\|_E = 1 - \varphi\left(b_\varphi\right)\left\|e_{i_0}\right\|_E.$$

(b) The proof of necessity that $\varphi \in \Delta_2(0)$ we divide into two parts.

I. Assume that $\sup_{i\in\mathbb{N}} \|e_i\|_E = \infty$. Then $\mathbb{B}$ is infinite. By Lemma 7 applied for $E = E_\varphi$ we conclude that $E_\varphi|_{\mathbb{B}} \in (OC)$. Thus, by Lemma 1, we get $\varphi \in \Delta_2^E = \Delta_2(0)$.

II. Suppose $M = \sup_{i\in\mathbb{N}} \|e_i\|_E < \infty$ and $\varphi \notin \Delta_2(0)$. Then there is a sequence $\{u_n\}$ in $\mathbb{R}_+$ with $u_n \to 0$ and

$$\varphi(2u_n) > 2^n\varphi(u_n)$$

for any $n \in \mathbb{N}$. Without loss of generality, passing to a subsequence if necessary, we can assume that $\varphi(u_n) \le 2^{-n}$. Really, since $\varphi(u_n) \to 0$, there is an increasing sequence (n_k) of positive integers such that $\varphi(u_{n_k}) \le 1 \big/ 2^k$ for any $k \in \mathbb{N}$. Noticing that $n_k \ge k$ for any $k \in \mathbb{N}$, we have

$$\varphi\left(2u_{n_k}\right) > 2^{n_k}\varphi\left(u_{n_k}\right) \ge 2^k\varphi\left(u_{n_k}\right).$$

To get the desired subsequence it is enough to put $v_k = u_{n_k}$ for any $k \in \mathbb{N}$.

By (a), if $\varphi(b_\varphi) < \infty$, there is $a \in (0, b_\varphi]$ with $\varphi(a)\,\|e_1\|_E = 1$. If $\varphi(b_\varphi) = \infty$, the existence of such a is obvious. Set

$$x = a\chi_{\{e_1\}} \quad \text{and} \quad A = \{i_2, i_3, \ldots\}.$$

Since $\chi_A \notin E$, for each $n \in \mathbb{N}$ we denote by $k = k(n) \in \mathbb{N}$ the smallest number satisfying

$$\varphi\left(u_n\right)\left\|\chi_{B_n}\right\|_E > 2^{-n}, \quad \text{where } B_n = \left\{i_2, i_3, \ldots, i_{k(n)}\right\} \subset A.$$

Then

$$\varphi\left(u_n\right)\left\|\chi_{B_n}\right\|_E \le \varphi\left(u_n\right)\left\|\chi_{\{i_2,i_3,\ldots,i_{k(n)-1}\}}\right\|_E + \varphi\left(u_n\right)\left\|\chi_{\{i_{k(n)}\}}\right\|_E \le 2^{-n} + 2^{-n}M.$$

Set

$$x_n = x + \frac{u_n}{2}\chi_{B_n}.$$

We have $\|x_n\|_\varphi \ge \|x\|_\varphi = 1$. Moreover,

$$I_\varphi\left(x_n\right) \le 1 + \frac{1}{2}\varphi\left(u_n\right)\left\|\chi_{B_n}\right\|_E \le 1 + \frac{1}{2}2^{-n}\left(1+M\right) \to 1,$$

whence $1 \le \|x_n\|_\varphi \le I_\varphi(x_n) \to 1$. Moreover, $x_n \to x$ uniformly. Finally,

$$I_\varphi\left(4\left(x_n - x\right)\right) \ge \varphi\left(2u_n\right)\left\|\chi_{B_n}\right\|_E \ge 2^n\varphi\left(u_n\right)\left\|\chi_{B_n}\right\|_E > 1.$$

It means that $\|x_n - x\|_\varphi \ge \frac{1}{4}$.

Finally, suppose that $\varphi \in \Delta_2(0)$ and $E \notin (H_u)$. Then, by Lemma 5, we find $x \in S(E)_+$ and $\{x_n\}$ in E_+ with $\|x_n\|_E \to \|x\|_E$ with $x_n \to x$ uniformly and $\|x_n - x\|_E \ge \varepsilon$. Set

$$y_n = \varphi_r^{-1} \circ x_n \quad \text{and} \quad y = \varphi_r^{-1} \circ x.$$

By (a) we conclude that y_n, y are well defined. Then $I_\varphi(y) = 1$ and $I_\varphi(y_n) \to 1$, whence $y \in S(E_\varphi)$ and $\|y_n\|_\varphi \to 1$. Moreover, $y_n \to y$ uniformly, by Lemma 4(ii). It is enough to prove that there is $\eta > 0$ with

$$\|y_n - y\|_\varphi \ge \eta \tag{14}$$

for infinitely many n. To prove inequality (14) we follow analogously as in the proof of Theorem 3 cases (iii) A, B, C. In the respective case B denote

$$A = \left\{i \in \mathbb{N} : y\left(i\right) \ge \frac{b_\varphi}{1+a}\right\} \quad \text{and} \quad B = \left\{i \in \mathbb{N} : y\left(i\right) < \frac{b_\varphi}{1+a}\right\}.$$

Then $\chi_A \in E_\varphi$, whence $\chi_A \in E$. Since $\varphi \circ y_n \to \varphi \circ y$ uniformly so $\|(\varphi \circ y_n - \varphi \circ y)\chi_A\|_E \to 0$. Thus $\|(\varphi \circ y_n - \varphi \circ y)\chi_B\|_E \ge \varepsilon/2$ for almost all n. The rest of the proof is the same as in case B.

The sufficiency. We follow as in the proof of sufficiency in case (iii) below. To prove the respective condition

$$\|\varphi \circ x_n - \varphi \circ x\|_E \to 0$$

notice that $\varphi \circ x \in E \hookrightarrow l^\infty$ whence $A = \varnothing$ and $B = \mathbb{N}$. Then we show

$$\|\varphi \circ (x_n - x)\|_E \to 0 \tag{15}$$

(see the proof of case (iii) below). Take $\lambda > 1$. We need to show that

$$\|\varphi \circ (\lambda (x_n - x))\|_E \to 0. \tag{16}$$

Since $\varphi \in \Delta_2(0)$ there is $u_0 > 0$ ($u_0 < b_\varphi/\lambda$ when $b_\varphi < \infty$) and $K > 0$ and $K > 0$ such that $\varphi(\lambda u) \leq K\varphi(u)$ for all $u \leq u_0$. Take N_0 big enough to satisfy $|(x_n - x)(i)| \leq u_0$ for each $n \geq N_0$ and $i \in \mathbb{N}$. Then $\varphi \circ (\lambda(x_n - x)) \leq K\varphi \circ (x_n - x)$ for $n \geq N_0$. Thus, by (15), the condition (16) is proved.

(iii) **The necessity**. First we discuss the necessity of condition $\varphi \in \Delta_2(\mathbb{R}_+)$. Note that if $E_\varphi \in (H_u)$, then the implication

$$\|u\|_\varphi = 1 \Rightarrow I_\varphi(u) = 1 \tag{17}$$

is true for any $u \in E_\varphi$. Really, otherwise we find $u \in (E_\varphi)_+$ satisfying $\|u\|_\varphi = 1$ and $I_\varphi(u) < 1$. We divide the proof into two parts.

a. If $\varphi(b_\varphi) < \infty$ and $u(i_0) = b_\varphi$ for some i_0 then taking

$$y = u\chi_{\{i_0\}}$$

we get $\|y\|_\varphi = 1$ and $I_\varphi(y) < 1$. Take an increasing sequence (A_n) in $\mathbb{N}\setminus\{i_0\}$ with $\left\|\chi_{A_n}\right\|_E \to \infty$ and a sequence (a_n) of positive real numbers satisfying $\varphi(a_n) \to 0$ and $\varphi(a_n)\left\|\chi_{A_n}\right\|_E = 1 - I_\varphi(y)$. Setting

$$y_n = y + a_n\chi_{A_n}$$

we get $y_n \to y$ uniformly. Moreover, $y_n \in S(E_\varphi)$ because $I_\varphi(y_n) \leq 1$ and $y \leq y_n$. Finally, $\|y_n - y\|_\varphi \geq I_\varphi(y_n - y) = 1 - I_\varphi(y) > 0$, whence $E_\varphi \notin (H_u)$.

b. Suppose that $u(i) < b_\varphi$ for each i. Take an increasing sequence of finite sets (A_n) in $\mathbb{N}$ with $\left\|\chi_{A_n}\right\|_E \to \infty$ and a sequence (a_n) of positive real numbers satisfying $\varphi(a_n) \to 0$ and $\varphi(a_n)\left\|\chi_{A_n}\right\|_E = 1 - I_\varphi(u)$. Set

$$y = u \quad \text{and} \quad y_n = y - a_n\chi_{A_n}.$$

We will prove that $y_n \in S(E_\varphi)$. First notice that, by superadditivity of φ on $\mathbb{R}_+$ we get

$$I_\varphi(y_n) \leq \left\|\varphi \circ y - \varphi(a_n)\chi_{A_n}\right\|_E \leq 1.$$

Note that the function $f(\lambda) = I_\varphi(\lambda y)$ is convex function of λ. Thus if f is finite valued in the interval $[0, \lambda_0]$ then f is continuous in the interval $[0, \lambda_0]$. Consequently from facts $\|y\|_\varphi = 1$ and $I_\varphi(y) < 1$ we conclude that $I_\varphi(y/\lambda) = \infty$ for each $\lambda < 1$. Moreover, for each n there is $\lambda_n < 1$ with $I_\varphi(\frac{y}{\lambda_n}\chi_{A_n}) < \infty$. Then $I_\varphi(\frac{y}{p}\chi_{A_n}) < \infty$ for each $\lambda_n < p < 1$, whence $I_\varphi(\frac{y}{p}\chi_{\mathbb{N}\setminus A_n}) = \infty$. Therefore

$$I_\varphi\left(\frac{y_n}{p}\right) = \left\|\varphi \circ \left(\frac{y - a_n}{p}\right)\chi_{A_n} + \varphi \circ \left(\frac{y}{p}\right)\chi_{\mathbb{N}\setminus A_n}\right\|_E \geq \left\|\varphi \circ \left(\frac{y}{p}\right)\chi_{\mathbb{N}\setminus A_n}\right\|_E = \infty.$$

Thus $\|y_n\|_\varphi > p$. Finally $\|y_n\|_\varphi = 1$ because $p < 1$ may be taken arbitrary close to 1. The rest of the proof is the same as in case a.

Applying condition (17), Lemma 2.9 from [12] and the proof of Lemma 2.4 [7] we conclude that $\varphi \in \Delta_2^E = \Delta_2(\mathbb{R}_+)$.

Finally, suppose that $\varphi \in \Delta_2(\mathbb{R}_+)$ and $E \notin (H_u)$. Then, by Lemma 5, we find $x \in S(E)_+$ and $\{x_n\}$ in E_+ with $\|x_n\|_E \to \|x\|_E$, $x_n \to x$ uniformly and $\|x_n - x\|_E \geq \varepsilon$. Set

$$y_n = \varphi_r^{-1} \circ x_n \quad \text{and} \quad y = \varphi_r^{-1} \circ x.$$

Then $I_\varphi(y) = 1$, $I_\varphi(y_n) \to 1$, whence $y \in S(E_\varphi)$ and $\|y_n\|_\varphi \to 1$. Moreover, $y_n \to y$ uniformly, by Lemma 4(ii). It is enough to prove that there is $\eta > 0$ with

$$\|y_n - y\|_\varphi \geq \eta \tag{18}$$

for infinitely many n. Then, to prove inequality (18) we follow analogously as in the proof of inequality (13) (the respective inequalities $\varphi((1+a)u) \leq l\varphi(u)$ and $\varphi\left(\frac{1+a}{a}u\right) \leq k\varphi(u)$ hold for all u).

The sufficiency. We apply Lemma 5. Take $x, x_n \in (E_\varphi)_+, n \in \mathbb{N}$, $\|x_n\|_\varphi \to \|x\|_\varphi = 1$ and $x_n \to x$ uniformly. By $\varphi \in \Delta_2^E = \Delta_2(\mathbb{R}_+)$ we get $\|\varphi \circ x\|_E = 1$ and $\|\varphi \circ x_n\|_E \to 1$. Set

$$A = \{i \in \mathbb{N} : \varphi(x(i)) \geq 1\} \quad \text{and} \quad B = \{i \in \mathbb{N} : \varphi(x(i)) < 1\}. \tag{19}$$

in the case $\varphi \circ x \notin l^\infty$ and $A = \varnothing$, $B = \mathbb{N}$ if $\varphi \circ x \in l^\infty$. Since $\chi_A \in E$, so

$$\|\varphi \circ (x_n - x)\, \chi_A\|_E \to 0. \tag{20}$$

Applying (20) and $\varphi \in \Delta_2(\mathbb{R}_+)$ one can obtain $\|(\varphi \circ x_n - \varphi \circ x)\chi_A\|_E \to 0$ (this can be done using similar arguments as in (13)). Let

$$z_n = \varphi \circ x \chi_A + \varphi \circ x_n \chi_B.$$

Consequently

$$\begin{aligned} \|z_n\|_E &= \|\varphi \circ x_n \chi_B + (\varphi \circ x - \varphi \circ x_n)\, \chi_A + \varphi \circ x_n \chi_A\|_E \\ &\leq \|\varphi \circ x_n\|_E + \|(\varphi \circ x - \varphi \circ x_n)\, \chi_A\|_E \to 1. \end{aligned}$$

Furthermore, setting

$$A_1 = \{i \in A : \varphi(x(i)) \geq \varphi(x_n(i))\} \text{ and } A_2 = \{i \in A : \varphi(x(i)) < \varphi(x_n(i))\}$$

we get

$$\begin{aligned} &\|z_n\|_E = \\ &= \left\|\varphi \circ x_n \chi_B + (\varphi \circ x - \varphi \circ x_n)\, \chi_{A_1} + \varphi \circ x_n \chi_{A_1} + (\varphi \circ x_n - \varphi \circ x)\, \chi_{A_2} - \varphi \circ x_n \chi_{A_2}\right\|_E \\ &\geq \left\|\varphi \circ x_n \chi_B + \varphi \circ x_n \chi_{A_1} + \varphi \circ x_n \chi_{A_2} - (\varphi \circ x_n - \varphi \circ x)\, \chi_{A_2}\right\|_E \\ &\geq \left|\|\varphi \circ x_n\|_E - \left\|(\varphi \circ x_n - \varphi \circ x)\, \chi_{A_2}\right\|_E\right| \to 1. \end{aligned}$$

Thus $\|z_n\|_E \to 1$. Moreover, $\varphi \circ x_n \chi_B \to \varphi \circ x \chi_B$ uniformly by Lemma 4(i). Thus $z_n \to \varphi \circ x$ uniformly and, by $E \in (H_u)$, we conclude $\|z_n - \varphi \circ x\|_E \to 0$. Thus

$$\|(\varphi \circ x_n - \varphi \circ x)\, \chi_B\|_E \to 0.$$

By superadditivity of φ on $\mathbb{R}_+$ we get

$$\|\varphi \circ (x_n - x)\, \chi_B\|_E \leq \|(\varphi \circ x_n - \varphi \circ x)\, \chi_B\|_E \to 0,$$

which together with (20) yields $\|\varphi \circ (x_n - x)\|_E \to 0$. Applying $\varphi \in \Delta_2(\mathbb{R}_+)$ we get $\|x_n - x\|_\varphi \to 0$. □

Remark 10 Discussing assumptions of Theorem 9(iii) notice that conditions $l^\infty \not\hookrightarrow E$, $E \not\hookrightarrow l^\infty$ need not imply that $E \hookrightarrow c_0\{\|e_i\|_E\}$ in general.

Proof Denote $\mathbb{N}_1 = \{i \in \mathbb{N} : i \text{ is odd}\}$ and $\mathbb{N}_2 = \{i \in \mathbb{N} : i \text{ is even}\}$. Take

$$E = \left\{ x \in l^0 : \|x\| = \sum_{i \in \mathbb{N}_1}^{\infty} \left[|x(i)| \frac{1}{i^2} \right] + \sup_{i \in \mathbb{N}_2} \{|x(i)|\, i\} < \infty \right\}.$$

Then $l^\infty \not\hookrightarrow E$, because $x = (0, 1, 0, 1, \ldots) \notin E$. Next, $E \not\hookrightarrow l^\infty$ since $x = (1, 0, \sqrt{3}, 0, \sqrt{5}, 0, \ldots) \in E$. Finally, we conclude that $E \not\hookrightarrow c_0\{\|e_i\|_E\}$ by taking $x = (0, 1/2, 0, 1/4, 0, 1/6, \ldots)$. □

Remark 11 Note that the necessity of condition $\varphi \in \Delta_2(0)$ in Theorem 9(ii) can be deduced analogously as the necessity of $\varphi \in \Delta_2(\mathbb{R}_+)$ in Theorem 9(iii). However, in (iii) we have additionally to assume that $E \hookrightarrow c_0\{\|e_i\|_E\}$ in order to apply results from [7]. In order to show that conditions $l^\infty \not\hookrightarrow E$, $E \hookrightarrow l^\infty$ need not imply that $E \hookrightarrow c_0\{\|e_i\|_E\}$ in general it is enough to apply some modification of above example from Remark 10. Consequently, using the direct proof of necessity of condition $\varphi \in \Delta_2(0)$ in Theorem 9(ii) we obtain result concerning the larger class of Köthe sequence spaces than applying the proof of Lemma 2.4 from [7] which requires the assumption that $E \hookrightarrow c_0\{\|e_i\|_E\}$.

3.2 Orlicz-Lorentz sequence spaces

Recall that Lorentz sequence space λ_ω consists of all sequences $x = (x(i))$ such that $\|x\|_{\lambda_\omega} = \sum_{i=1}^{\infty} x^*(i)\omega(i) < \infty$, where $\omega = (\omega(i))$ is a *weight sequence*, that is ω is a nonincreasing sequence of nonnegative real numbers, and x^* is the nonincreasing rearrangement of x (see [16]).

Lemma 12 (i) *$\lambda_\omega \in (H_c)$ if and only if $\sum_{i=1}^{\infty} \omega(i) = \infty$.*
(ii) *$\lambda_\omega \hookrightarrow c_0$ if and only if $\sum_{i=1}^{\infty} \omega(i) = \infty$. The inclusion $l^\infty \hookrightarrow \lambda_\omega$ holds if and only if $\sum_{i=1}^{\infty} \omega(i) < \infty$.*

Proof (i) Since $H_c \Rightarrow OC$ (see [6]), the necessity follows from Lemma 3.2 from [12]. For the sufficiency it is enough to apply Theorem 7 from [2]. (ii) It is obvious. □

Taking $E = \lambda_\omega$ in Theorems 3, 9 and applying Lemma 12 we get immediately the following new characterization

Corollary 13 *Let $(\lambda_\omega)_\varphi$ be the Orlicz-Lorentz sequence space.*

(a) *$(\Lambda_\omega)_\varphi \in (H_c)$ if and only if $\sum_{i=1}^{\infty} \omega(i) = \infty$, $\varphi \in \Delta_2(0)$ and $\varphi(b_\varphi)\omega(1) \geq 1$.*
(b) (i) *If $\sum_{i=1}^{\infty} \omega(i) < \infty$ or $a_\varphi > 0$, then $(\lambda_\omega)_\varphi \in (H_u)$.*
  (ii) *Suppose that $\sum_{i=1}^{\infty} \omega(i) = \infty$ and $a_\varphi = 0$. Then $(\lambda_\omega)_\varphi \in (H_u)$ if and only if:*
    1. *$\varphi(b_\varphi)\omega(1) \geq 1$.*
    2. *$\varphi \in \Delta_2(0)$.*

Obviously, if $\omega(i) = 1$ for each i, then $(\lambda_\omega)_\varphi = l_\varphi$, the Orlicz sequence space. Thus, applying the previous corollary in this case it is easy to get the respective characterizations for l_φ.

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