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STABILITY THEORY TO A COUPLED SYSTEM OF NONLINEAR FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS

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In the current manuscript, we investigate existence of solutions to a coupled system of fractional hybrid differential equations (FHDEs). With the help of mixed type Lipschitz and Caratheodory conditions, some conditions for the existence of solutions to the considered problem are established. Considering the tools of nonlinear analysis and hybrid fixed points theory, we establish our results. Further some new type results about stability including Ulam-Hyers (UH), generalized Ulam-Hyers (GUH) stability, Ulam-Hyers-Rassias (UHR) and generalized Ulam-Hyers-Rassias (GUHR) stability are developed. A test problem is given to demonstrate the establish results.

Key words : Quadratic perturbation; hybrid FDEs; hybrid fixed point theorem.

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1. Introduction

In last few decades, it has been proved that the area involving fractional differential equations (FDEs) has got considerable attention from many researchers. Because FDEs have many applications in various disciplines of science and technology, we refer few of them in [4, 6, 13, 17, 22, 24]. An attractive area for research in the field of fractional calculus is devoted to find positive solutions to boundary value problems (BVPs) for FDEs, which is widely explored and large numbers of research papers are available in literature, for detail, see some of them as [5, 14, 16, 18-20, 21]. Moreover existence theory of solutions to initial and BVPs for systems of FDEs have given some attentions in last few years, we refer to [1, 3, 23, 25]. However the area involving HDEs is in its initial stage.

Recently the area involving the quadratic perturbation of nonlinear differential equations of first and second kind which is also called HDEs has got considerable attention from researchers. Because the class of HDEs includes the perturbations of original differential equations in various ways. The HDEs have fundamental importance, as they include several dynamic systems as special cases. The study of HDEs is implicit in the works of Krasnoselsikii, Dhage and Lakshmikantham and extensively investigated by many researchers. For the said results, the authors mentioned afore used hybrid fixed point theory. Consequently several papers on HDEs with different perturbations are available, some of them are [8, 11, 12, 15, 26]. In [8], authors studied existence and uniqueness results for the following first order HDE of the form

$$\begin{cases} [w(t) - \phi(t, w(t))]' = \psi(t, w(t)), \quad a.e\; t \in I = [t_0, a + t_0], \\ w(t_0) = w_0 \in \mathrm{R}, \end{cases} \tag{1.1}$$

where $\phi, \psi \in C(I \times \mathrm{R}, \mathrm{R})$. Further they also established some fundamentals inequalities for HDEs which gave initiation to the existence theory of aforesaid equations. In Lu *et al.* [15], discussed a generalization of (1.1) by replacing the classical differentiation by arbitrary order derivative in the Riemann-Liouvile sense as

$$\begin{cases} \mathbf{D}^{\sigma}[w(t) - \phi(t, w(t)] = \psi(t, w(t),\ a.e\; t \in [t_0, a + t_0], \\ w(t_0) = w_0 \in \mathrm{R}, \end{cases}$$

where $0 < \sigma \leq 1,\ f, g \in C(I \times \mathrm{R}, \mathrm{R})$. Hilal *et al.* [11], studied the following BVP for FHDEs involving Caputo's fractional order derivative

$$\begin{cases} \mathbf{D}^{\sigma}[\dfrac{w(t)}{\phi(t, w(t))}] = \psi(t, w(t)), a.e\; t \in \mathfrak{I} = [0, T], \\ \alpha \dfrac{w(0)}{\phi(0, w(0))} + \beta \dfrac{w(T)}{\phi(T, w(T))} = \gamma, \end{cases}$$

where $0 < \sigma < 1$ and $\phi \in C(\mathfrak{I} \times \mathrm{R}, \mathrm{R} \setminus \{0\}),\ \psi \in C(\mathfrak{I} \times \mathrm{R}, \mathrm{R})$. Also α, β, γ are real constants with $\alpha + \beta \neq 0$.

In last few years along with the existence theory of solutions to nonlinear FDEs, another aspect known as stability analysis has been attracted the attentions, see [27, 29, 30, 32, 33]. The stability analysis is very important to establish techniques for numerical solutions as well as in optimization theory of differential equations. Different kinds of stability like exponential, Mittag-Leffler and Lyapunov stability have been studied for the said differential equations. Another kind of stability which greatly attracted the researchers attentions has been recently considered for nonlinear and linear FDEs and fractional partial differential equations, we refer [35-41]. This important form of stability was first pointed out by Ulam in 1940 and was brilliantly explained by Hyers in 1941, see [31]. After that valuable contributions have been done in this regards. In 1997, Rassias extended the aforementioned stability to some other forms known as UHR and GUHR stability. The concerned stability results have been investigated recently for fractional differential equations, ordinary and functional equations, see [28]. The aforementioned stability have been investigated for functional, integral and differential equations very well, see [42-44]. To the best of our knowledge the said stability has not yet considered for FHDEs.

Therefore motivated by the afore mentioned work, in this paper, we study existence of mild solutions to the following coupled systems of FHDEs. The initial conditions are nonhomogeneous and the quadratic perturbation is of second type. The system is given by

$$\begin{cases} {}^c\mathbf{D}^{\sigma}\left(w(t) - \phi_1(t, w(t), z(t))\right) = \psi_1(t, w(t), z(t)),\ a.e\ t \in \mathfrak{I}, \\ {}^c\mathbf{D}^{\varrho}\left(z(t) - \phi_2(t, w(t), z(t))\right) = \psi_2(t, w(t), z(t)),\ a.e\ t \in \mathfrak{I}], \\ w(t)|_{t=0} = w_0,\ z(t)|_{t=0} = z_0, \end{cases} \tag{1.2}$$

where $\sigma, \varrho \in (0, 1],{}^c\mathbf{D}$ is fractional derivative in Caputo's sense and w_0, z_0 are real numbers. Where $\phi_i, \psi_i : I \times \mathrm{R} \times \mathrm{R} \to \mathrm{R},\ i = 1, 2$, are continuous nonlinear functions. In this regard we can claim that the mentioned stability has not investigated for FHDEs yet. So we for the first time study the mentioned stability for the considered problem (1.2). upon using fixed point theorem due to [**?**, **?**, **?**], we form appropriate conditions for existence and uniqueness of at least one solution to the problem under consideration. Further we also investigate different kinds of Ulam stability for the considered system (1.2). In last we give an example to verify our establish results.

2. Background Materials

For further analysis, some fundamental results and notions are needed throughout this paper which we take from [7-10, 19, 22].

Definition 2.1 — The Riemann-Liouville type integral of order $\sigma \in (0, \infty)$ for a function $z \in L^1([0, T], \mathrm{R})$ is defined as

$$\mathbf{I}^{\sigma} z(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t - \xi)^{\sigma - 1} z(\xi)\, d\xi,$$

provided that integral exists on right side.

Definition 2.2 — The Caputo's derivative of a function z over the interval $[0, T]$ of order $\sigma \in (0, \infty)$ is defined by

$${}^c\mathbf{D}^{\sigma} z(t) = \frac{1}{\Gamma(k - \sigma)} \int_0^t (t - \xi)^{k - \sigma - 1} z^{(k)}(\xi)\, d\xi,$$

where $k=[\sigma]+1$ and $[\sigma]$ is the integer part of σ.

Lemma 2.3 — Corresponding to FDE, we have a result

$$\mathbf{I}^{\sigma}[{}^{c}\mathbf{D}^{\sigma}h(t)] = h(t)+a_0+a_1t+a_2t^2+\cdots+a_{k-1}t^{k-1},$$

for arbitrary $a_i \in \mathrm{R}, \quad i=0,1,2,\ldots,k-1$, where $k=[\sigma]+1$ and $[\sigma]$ is the integer part of σ.

The Banach space of all continuous functions from $\mathfrak{I} \to \mathrm{R}$ is denoted by $\mathbf{X}$ endowed with a norm norm $\|w\| = \max\{|w(t)| : t \in \mathfrak{I}\}$. The space denoted by $\mathbf{E} = \mathbf{X} \times \mathbf{Y}$ is a Banach space endowed with a norm $\|(w,z)\| = \|w\| + \|z\|$.

***Theorem* 2.4** — *(Dhage, et. al. [8]). Let $\mathbf{W} \subset \mathbf{E}$ be the closed and bounded subset and $\mathscr{T} : \mathbf{E} \to \mathbf{E}$, $\mathscr{S} : \mathbf{W} \to \mathbf{E}$ be two operators satisfying*

(A_1) *$\mathscr{T}$ is nonlinear contraction;*

(A_2) *$\mathscr{S}$ is continuous and compact, and*

(A_3) *$(w,z) = \mathscr{T}(w,z) + \mathscr{S}(\bar{\mu},\vec{\nu})$ for all $(\bar{\mu},\bar{\nu}) \in \mathbf{W} \Rightarrow (w,z) \in \mathbf{W}$.*

Then the operator equations $\mathscr{T}(w,z) + \mathscr{S}(w,z) = (w,z)$ has a solution in $\mathbf{W}$.

The following hypothesis are required for our analysis onward in this paper.

(H_1) There exist constants $K,\ L :> 0 \in \mathrm{R}$ such that

$$|\phi_1(t,w,z) - f_1(t,\bar{w},\bar{z})| \le K[|w-\bar{w}| + |z-\bar{z}|], \text{ for all } t \in \mathfrak{I},\ w,\bar{w},z,\bar{z} \in \mathrm{R},$$

and

$$|\phi_2(t,w,z) - \phi_2(t,\bar{w},\bar{z})| \le L[|w-\bar{w}| + |z-\bar{z}|], \text{ for all } t \in \mathfrak{I},\ w,\bar{w},z,\bar{z} \in \mathrm{R};$$

(H_2) There exist continuous functionals $\alpha,\ \beta : [0,1] \to \mathrm{R}$ satisfy

$$|\psi_1(t,w,z)| \le \alpha(t), \text{ and } |\psi_2(t,w,z)| \le \beta(t).$$

(H_3) The following notations are used for easiness

$$\Lambda_1 = \sup_{t\in[0,1]} |\phi_1(t,0,0)|,\ \Lambda_2 = \sup_{t\in[0,1]} |\phi_2(t,0,0)|$$

and

$$\|\alpha\|_{\mathcal{L}} = \int_0^1 |\alpha(s)|ds,\ \|\beta\|_{\mathcal{L}} = \int_0^1 |\beta(s)|ds.$$

Definition 2.5 — We recall that $\mathscr{T} : \mathbf{E} \to \mathbf{E}$ is μ-Lipschitz if there exists constant μ satisfies

$$\|\mathscr{T}(w, z) - \mathscr{T}(\bar{w}, \bar{z})\| \leq \mu \left(\|w - \bar{w}\| + \|z - \bar{z}\|\right),$$

for all $(w, z), (\bar{w}, \bar{z} \in \mathbf{E}$ with $\mu > 0$. Further if $\mu < 1$, then $\mathscr{T}$ is a strict contraction.

For the UH, GUH, UHR and GUHR stability, we give the following definitions and remark, for detail see [45, 46].

Definition 2.6 — The solution $w \in C[0, T]$ of the FDE given by

$$\begin{aligned} &{}^{c}\mathbf{D}^{\sigma}\left[w(t) - \phi_1(t, w(t), z(t))\right] = \psi_1(t, w(t), z(t)),\ t \in \mathfrak{I}, \\ &w(t)|_{t=0} = w_0 \end{aligned} \tag{2.1}$$

is UH stable if we can find a real number $\hat{C}_{\sigma,\varrho,\Delta} > 0$ with the property that for every $\epsilon > 0$ and for every solution $w \in C[0, T]$ of the inequality

$$\left|{}^{c}\mathbf{D}^{\sigma}\left[w(t) - \phi_1(t, w(t), z(t))\right] - \psi_1(t, w(t), z(t))\right| \leq \epsilon,\ t \in [0, T], \tag{2.2}$$

there exists unique solution $x \in C[0, T]$ of the given FDE (2.1) with a constant $\hat{C}_{\sigma,\varrho,\Delta} > 0$ with

$$\|w - x\| \leq \hat{C}_{\sigma,\varrho,\Delta}\epsilon.$$

Definition 2.7 — The solution $w \in C[0, T]$ of the FDE (2.1) is called to be GUH stable , if we can find

$$\theta : (0, \infty) \to \mathrm{R}^{+},\ \theta(0) = 0,$$

such that for each solution $w \in C[0, T]$ of the inequality (2.2), we can find a unique solution $x \in C[0, T]$ of the FDE (2.1) with

$$\|w - x\| \leq \hat{C}_{\sigma,\varrho,\Delta}\theta.$$

Next we recall the definitions of UHR and GUHR stability [28] for our considered problem (2.1) as bellow:

Definition 2.8 — FDE (2.1) is said to be UHR stable with respect to $\hbar \in C([0, T], \mathrm{R})$ if there exists a non zero positive real constant $\hat{C}_{\hbar,\Delta} > 0$ for each $\epsilon > 0$ such that for every solution $w \in C[0, T]$ of the inequality

$$\left|{}^{c}\mathbf{D}^{\sigma}\left[w(t) - \phi_1(t, w(t), z(t))\right] - \psi_1(t, w(t), z(t))\right| \leq \epsilon\hbar(t),\ t \in [0, T], \tag{2.3}$$

there exists a solution $x \in C[0,T]$ of the equation (2.1), such that

$$|w(t) - x(t)| \leq \hat{C}_{\hbar,\Delta}\hbar(t),\ t \in [0,T].$$

Definition 2.9 — The equation (2.1) is said to be GUHR stable with respect to $\hbar \in C[0,T]$, if there exists a real number $\hat{C}_{\hbar,\Delta} > 0$ such that for each solution $w \in C[0,T]$ of the inequality

$$\left| {}^{c}\mathbf{D}^{\sigma}\left[w(t) - \phi_1(t, w(t), z(t))\right] - \psi_1(t, w(t), z(t)) \right| \leq \hbar(t),\ t \in [0,T], \tag{2.4}$$

there exists a solution $w \in C[0,T]$ of the equation (2.1) such that $|w(t) - x(t)| \leq \hat{C}_{\hbar,\Delta}\hbar(t), t \in [0,T]$.

Remark 2.10 : A function $w \in C[0,T]$ is said to be the solution of inequality given in (2.2) if and only if, we can find a function $\hbar \in C[0,1]$ depends on w only such that

(i) $|\hbar(t)| \leq \epsilon$, for all $t \in [0,T]$;

(ii) ${}^{c}\mathbf{D}^{\sigma}\left[\mathbf{w(t)} - \phi_{\mathbf{1}}(\mathbf{t}, \mathbf{w(t)}, \mathbf{z(t)})\right] = \varphi(\mathbf{t}) + \hbar(\mathbf{t}),$ for all $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$.

3. Main Results

This portion is devoted to investigate the conditions for mild solutions to the proposed system of FHDEs (1.2).

***Theorem* 3.1** — *Let $\varphi : \Im \to \mathrm{R}$, then the solution of the FHDE*

$$\begin{aligned} &{}^{c}\mathbf{D}^{\sigma}\left[w(t) - \phi_1(t, w(t), z(t))\right] = \varphi(t),\ \sigma \in (0,1],\ t \in \Im, \\ &w(t)|_{t=0} = w_0 \end{aligned} \tag{3.1}$$

is provided by

$$w(t) = w_0 - \phi_1(0, w_0, z_0) + \phi_1(t, w(t), z(t)) + I^{\sigma}\psi_1(t, w(t), z(t)) \tag{3.2}$$

PROOF : In view of application of $\mathbf{I}^{\sigma}$ on ${}^{c}\mathbf{D}^{\sigma}\mathbf{w(t)} = \mathbf{h(t)}$ and using Lemma 2.3, (3.2) yields

$$w(t) - \phi_1(t, w(t), z(t)) = \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\varphi(\xi)d\xi + a_0 \tag{3.3}$$

applying boundary conditions $w(t)|_{t=0} = w_0$ on (3.3) and after calculating the values of a_0, we obtain

$$w(t) = w_0 - \phi_1(0, w_0, z_0) + \phi_1(t, w(t), z(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)} \psi_1(\xi, w(\xi), z(\xi)) d\xi. \tag{3.4}$$

Similarly repeating the above process with second part of (1.2), we get the solution as

$$z(t) = z_0 - \phi_2(0, w_0, z_0) + \phi_2(t, w(t), z(t)) + \int_0^t \frac{(t-\xi)^{\varrho-1}}{\Gamma(\varrho)} \psi_2(\xi, w(\xi), z(\xi)) d\xi. \tag{3.5}$$

To obtain the conditions for at least one solution to (1.2), we give the following theorem.

***Theorem* 3.2** — *Assume that the assumptions* $(H_1) - (H_3)$ *and there exist bounded and closed ball* $\mathbf{W} = \{(w, z) \in \mathbf{E} : \|(w, z)\| \leq \mathcal{R}\}$ *where*

$$\mathcal{R} \geq |\sqsupseteq_\prime - \phi_\infty(\prime, \sqsupseteq_\prime, \ddagger_\prime)| + |\ddagger_\prime - \phi_\in(\prime, \sqsupseteq_\prime, \ddagger_\prime)| + \mathcal{K} + \mathcal{L} + *_\infty + *_\in + \frac{\mathcal{T}^\sigma \|\alpha\|_\mathcal{L}}{-(\sigma + \infty)} + \frac{\mathcal{T}^\varrho \|\beta\|_\mathcal{L}}{-(\varrho + \infty)}. \tag{3.6}$$

Then coupled system of FHDEs (1.2) *has a solution defined in* $\mathbf{W}$.

PROOF : Let us define a closed ball $\mathbf{W} = \{(w, z) \in \mathbf{E} : \|(w, z)\| \leq \mathcal{R}\} \subset \mathbf{E}$. obviously $\mathbf{W}$ is bounded closed subset of $\mathbf{E}$. Further we define the operators $\mathscr{T} = (\mathscr{T}_1, \mathscr{T}_2) : \mathbf{E} \to \mathbf{E}$ and $\mathscr{S} = (\mathscr{S}_1, \mathscr{S}_2) : \mathbf{W} \to \mathbf{E}$ by $\mathscr{T}(w, z) = (\mathscr{T}_1(w, z), \mathscr{T}_2(w, z))$ and $\mathscr{S}(w, z) = (\mathscr{S}_1(w, z), \mathscr{S}_2(w, z))$

$$\mathscr{T}_1(w(t), z(t)) = \phi_1(t, w(t), z(t)),\ \mathscr{T}_2(w(t), z(t)) = \phi_2(t, w(t), z(t)),\ t \in \mathfrak{J}, \tag{3.7}$$

$$\mathscr{S}_1(w(t), z(t)) = w_0 - \phi_1(0, w_0, z_0) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)} \psi_1(\xi, w(\xi), z(\xi)) d\xi,\ t \in \mathfrak{J}, \tag{3.8}$$

and

$$\mathscr{S}_2(w(t), z(t)) = z_0 - \phi_2(0, w_0, z_0) + \int_0^t \frac{(t-\xi)^{\varrho-1}}{\Gamma(\varrho)} \psi_2(\xi, w(\xi), z(\xi)) d\xi,\ t \in \mathfrak{J}. \tag{3.9}$$

Then the coupled system of hybrid integral equations (3.4) and (3.5) is transformed to the system of operators equation given below

$$\begin{aligned}
&\mathscr{T}(w, z) + \mathscr{S}(w, z) = (w, z),\ t \in \mathfrak{J} \\
&(\mathscr{T}_1(w, z), \mathscr{T}_2(w, z)) + (\mathscr{S}_1(w, z), \mathscr{S}_2(w, z)) = (w, z),\ t \in \mathfrak{J} \\
&\mathscr{T}_1(w, z) + \mathscr{S}_1(w, z) = w,\ \mathscr{T}_2(w, z) + \mathscr{S}_2(w, z) = z,\ t \in \mathfrak{J}.
\end{aligned}$$

Onward we will show that the operators $\mathscr{T}, \mathscr{S}$ satisfy all the assumptions of Theorem 2.4. In this regard, we will prove that $\mathscr{T} = (\mathscr{T}_1, \mathscr{T}_2)$ is Lipschitz operator on $\mathbf{E}$ with constants $K + L$ and $\mathscr{S}$ is completely continuous operator from $\mathbf{W}$ to $\mathbf{E}$. Let $(w, z) \in \mathbf{E}$. Then by using (H_1) we have

$$|\mathscr{T}_1(w,z)(t) - \mathscr{T}_1(\bar{w},\bar{z})(t)| = |\phi_1(t, w(t), z(t)) - \phi_1(t, \bar{w}(t), \bar{z}(t))|$$
$$\leq K[|w(t) - \bar{w}(t)| + |z(t) - \bar{z}(t)|] \leq K[\|w - \bar{w}\| + \|z - \bar{z}\|], \text{ for all } t \in \mathfrak{I},$$

taking supremum over t, we get

$$\|\mathscr{T}_1(w,z) - \mathscr{T}_1(\bar{w},\bar{z})\| \leq K[\|w - \bar{w}\| + \|z - \bar{z}\|], \text{ for all } (w,z), (\bar{w},\bar{z}) \in \mathbf{E}.$$

Following the same fashion, one can easily show that $\mathscr{T}_2$ is Lipschitz with constant L as

$$\|\mathscr{T}_2(w,z) - \mathscr{T}_2(\bar{w},\bar{z})\| \leq L[\|w - \bar{w}\| + \|z - \bar{z}\|], \text{ for every} (w,z), (\bar{w},\bar{z}) \in \mathbf{E}.$$

Hence $\mathscr{T}$ is Lipschitz with constant $K + L$ as

$$\|\mathscr{T}(w,z) - \mathscr{T}(\bar{w},\bar{z})\| \leq (K + L)[\|w - \bar{w}\| + \|z - \bar{z}\|], \text{ for every } (w,z), (\bar{w},\bar{z}) \in \mathbf{E}.$$

To show that $\mathscr{S} = (\mathscr{S}_1, \mathscr{S}_2)$ is continuous and compact operator from $\mathbf{W}$ to $\mathbf{E}$. For continuity of $\mathscr{S}$, let (w_n, z_n) be a sequence in $\mathbf{W}$ converging to a point $(w, z) \in \mathbf{W}$. Due to Lebesgue dominated convergence theorem , one has

$$\lim_{n\to\infty} \mathscr{S}_1(w_n, z_n)(t) = \lim_{n\to\infty} \left[w_0 - \phi_1(0, w_0, z_0) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)} \psi_1(\xi, w_n(\xi), z_n(\xi)) d\xi \right]$$
$$= w_0 - \phi_1(0, w_0, z_0) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)} \lim_{n\to\infty} \psi_1(\xi, w_n(\xi), z_n(\xi)) d\xi$$
$$= w_0 - \phi_1(0, w_0, z_0) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)} \psi_1(\xi, w(\xi), z(\xi)) d\xi = \mathscr{S}_1(w,z)(t), \text{ for all } t \in \mathfrak{I},$$

similarly

$$\lim_{n\to\infty} \mathscr{S}_2(w_n, z_n)(t) = \mathscr{S}_2(w,z)(t), \text{ for all } t \in \mathfrak{I}.$$

Hence $\mathscr{S}(w_n, z_n) = (S_1(w_n, z_n),\ \mathscr{S}_2(w_n, z_n))$ is converges to $\mathscr{S}(w, z)$ point wise on $\mathfrak{I}$. Further, we will show that $\{S(w_n, z_n)\}$ is equi-continuous sequence of functions in $\mathbf{E}$. We have to prove that $\mathscr{S}(\mathbf{W})$ is uniformly bounded and equi-continuous set in $\mathbf{E}$. Then for any $(w, z) \in \mathbf{W}$ and using

(H_2) one consider

$$\begin{aligned}|\mathscr{S}_1(w,z)(t)| &= \left|w_0 - \phi_1(0,w_0,z_0) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right| \\ &\leq |w_0 - \phi_1(0,w_0,z_0)| + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}|\psi_1(\xi,w(\xi),z(\xi))|d\xi \\ &\leq |w_0 - \phi_1(0,w_0,z_0)| + \sup_{t\in\mathfrak{I}}\int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}|\alpha(\xi)|d\xi \end{aligned} \tag{3.10}$$

which implies that $\|\mathscr{S}_1(w,z)\| \leq |w_0 - \phi_1(0,w_0,z_0)| + \dfrac{T^\sigma\|\alpha\|_{\mathcal{L}}}{\Gamma(\sigma+1)}$.

Similarly one can show that

$$\|\mathscr{S}_2(w,z)\| \leq |z_0 - \phi_2(0,w_0,z_0)| + \frac{T^\varrho\|\beta\|_{\mathcal{L}}}{\Gamma(\varrho+1)}. \tag{3.11}$$

Hence from (4.6) and (4.7) we have

$$\|\mathscr{S}(w,z)\| \leq |w_0 - \phi_1(0,w_0,z_0)| + |z_0 - \phi_2(0,w_0,z_0)| + \frac{T^\sigma\|\alpha\|_{\mathcal{L}}}{\Gamma(\sigma+1)} + \frac{T^\varrho\|\beta\|_{\mathcal{L}}}{\Gamma(\varrho+1)}. \tag{3.12}$$

Thus $\mathscr{S}$ is uniformly bounded on $\mathbf{W}$. Now let $t,\tau \in \mathfrak{I}$ with $t<\tau$, then for any $(w,z)\in\mathbf{W}$, we have

$$\begin{aligned}&|\mathscr{S}_1(w,z)(t) - \mathscr{S}_1(w,z)(\tau)| \\ &\leq \left|\int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi - \int_0^\tau \frac{(\tau-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right| \\ &+ \left|\int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi - \int_0^t \frac{(\tau-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right| \\ &\leq \frac{\|T^\sigma\alpha\|_{\mathcal{L}}}{\Gamma(\sigma+1)}[2(t^\sigma - \tau^\sigma) + (\tau-t)^\sigma]\end{aligned}$$

which implies that

$$|\mathscr{S}_1(w,z)(t) - \mathscr{S}_1(w,z)(\tau)| \leq \frac{2T^\sigma\|\alpha\|_{\mathcal{L}}}{\Gamma(\sigma+1)}[(t^\sigma - \tau^\sigma) + (\tau-t)^\sigma], \tag{3.13}$$

similarly we can prove that

$$|\mathscr{S}_2(w,z)(t) - \mathscr{S}_2(w,z)(\tau)| \leq \frac{2T^\varrho\|\beta\|_{\mathcal{L}}}{\Gamma(\varrho+1)}[(t^\varrho - \tau^\varrho) + (\tau-t)^\varrho]. \tag{3.14}$$

As $t\to\tau$ then right hand sides of (3.13) and (3.14) tend to zero. Thus $\mathscr{S}$ is equi-continuous for all $t\in\mathfrak{I}$ and for all $(w,z)\in\mathbf{W}$. Hence $S(\mathbf{W})$ is equi-continuous set in $\mathbf{E}$. By using Arzel$\acute{a}$ Ascoli theorem $\mathscr{S}$ is compact operator and consequently completely continuous.

Now to prove condition (A_3) of Theorem 2.4, let $(w,z)\in\mathbf{W}$ and using (H_1) we have

$$
\begin{aligned}
&|\mathscr{T}(w,z)(t)+S(w,z)(t)|\leq|\mathscr{T}(w,z)|+|\mathscr{S}(w,z)|\\
&\leq|\mathscr{T}_1(w,z)(t)|+|\mathscr{T}_2(w,z)(t)|+|\mathscr{S}_1(w,z)(t)|+|\mathscr{S}_2(w,z)(t)|\\
&\leq|w_0-\phi_1(0,w_0,z_0)|+|z_0-\phi_2(0,w_0,z_0)|+|\phi_1(t,w(t),z(t))|+|\phi_2(t,w(t),z(t))|\\
&+\left|\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right|+\left|\int_0^t\frac{(t-\xi)^{\varrho-1}}{\Gamma(\varrho)}\psi_2(\xi,w(\xi),z(\xi))d\xi\right|\\
&\leq|w_0-\phi_1(0,w_0,z_0)|+|z_0-\phi_2(0,w_0,z_0)|+|\phi_1(t,w(t),z(t))-\phi_1(t,0,0)|+|\phi_1(t,0,0)|\\
&+|\phi_2(t,w(t),z(t))-\phi_2(t,0,0)|+|\phi_2(t,0,0)|+\left|\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right|\\
&+\left|\int_0^t\frac{(t-\xi)^{\varrho-1}}{\Gamma(\varrho)}\psi_2(\xi,w(\xi),z(\xi))d\xi\right|\\
&\leq|w_0-\phi_1(0,w_0,z_0)|+|z_0-\phi_2(0,w_0,z_0)|+K+L+\sup_{t\in\Im}|\phi_1(t,0,0)|+\sup_{t\in\Im}|\phi_2(t,0,0)|\\
&+\sup_{t\in\Im}\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}|\alpha(\xi)|d\xi+\sup_{t\in\Im}\int_0^t\frac{(t-\xi)^{\varrho-1}}{\Gamma(\varrho)}|\beta(\xi)|d\xi\\
&\leq|w_0-\phi_1(0,w_0,z_0)|+|z_0-\phi_2(0,w_0,z_0)|+K+L+\Lambda_1+\Lambda_2+\frac{T^\sigma\|\alpha\|_{\mathcal{L}}}{\Gamma(\sigma+1)}+\frac{T^\varrho\|\beta\|_{\mathcal{L}}}{\Gamma(\varrho+1)}\\
&\leq\mathcal{R}.
\end{aligned}
$$

Therefore the criteria of Theorem 2.4 is fulfilled. Hence the proposed problem (1.2) of FHDEs has a mild solution in $\mathbf{W}$. □

4. Ulam Stability Analysis of the Solutions of Coupled System of FHDEs (1.2)

In this section, we prove necessary and sufficient conditions for the UH and GUH stability as well as UHR and GUHR stability of the solutions to considered coupled system (1.2) of nonlinear FHDEs. To come across the required result, we give the following Remarks first:

Remark 4.1 : The pair of functions $(w,z)\in\mathbf{E}$ is said to be solution of inequality given by

$$
\begin{cases}\left|{}^c\mathbf{D}^\sigma\left(w(t)-\phi_1(t,w(t),z(t))\right)-\psi_1(t,w(t),z(t))\right|<\epsilon_1,\ a.e\ t\in\Im,\\ \left|{}^c\mathbf{D}^\varrho\left(z(t)-\phi_2(t,w(t),z(t))\right)-\psi_2(t,w(t),z(t))\right|<\epsilon_2,\ a.e\ t\in\Im],\end{cases}\tag{4.1}
$$

for $\epsilon_1,\epsilon_2.0$ if and only if, we can find functions $\hbar_1,\hbar_2\in C[0,T]$ depend only on w,z respectively such that

(i) $|\hbar_1(t)|\leq\epsilon_1,\ |\hbar_2(t)|\leq\epsilon_2$for all $t\in[0,T]$

(ii) and

$$\begin{cases} {}^{c}\mathbf{D}^{\sigma}\left(w(t)-\phi_1(t,w(t),z(t))\right)=\psi_1(t,w(t),z(t))+\hbar_1(t),\ a.e\ t\in\Im, \\ {}^{c}\mathbf{D}^{\varrho}\left(z(t)-\phi_2(t,w(t),z(t))\right)=\psi_2(t,w(t),z(t))+\hbar_2(t),\ a.e\ t\in\Im]. \end{cases} \tag{4.2}$$

(H_4) Let for continuous functions $\alpha,\beta\in([0,1],\mathrm{R})$ we have

$$|\psi_1(t,w(t),z(t))-\psi_1(t,x(t),y(t))|\leq\alpha(t)[|w-x|+|z-y|]$$

and

$$|\psi_2(t,w(t),z(t))-\psi_2(t,x(t),y(t))|\leq\beta(t)[|w-x|+|z-y|].$$

Lemma 4.2 — Under the assumption given as $(H_1)-(H_2)$ and Remark 4.1, the solution $(w,z)\in$ $\mathbf{E}$ of the system of FHDEs given by

$$\begin{cases} {}^{c}\mathbf{D}^{\sigma}\left(w(t)-\phi_1(t,w(t),z(t))\right)=\psi_1(t,w(t),z(t))+\hbar_1(t),\ a.e\ t\in\Im, \\ {}^{c}\mathbf{D}^{\varrho}\left(z(t)-\phi_2(t,w(t),z(t))\right)=\psi_2(t,w(t),z(t))+\hbar_2(t),\ a.e\ t\in\Im], \\ w(t)|_{t=0}=w_0,\ z(t)|_{t=0}=z_0. \end{cases} \tag{4.3}$$

satisfies the relation given by

$$\begin{aligned} &\left|w(t)-\left(-w_0+\phi_1(0,w_0,z_0)\phi_1(t,w(t),z(t))-\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right)\right| \\ &\leq\frac{\epsilon_1}{\Gamma(\sigma+1)},\ t\in\Im, \\ &\left|z(t)-\left(-z_0+\phi_2(0,w_0,z_0)-\phi_2(t,w(t),z(t))-\int_0^t\frac{(t-\xi)^{\varrho-1}}{\Gamma(\varrho)}\psi_2(\xi,w(\xi),z(\xi))d\xi\right)\right| \\ &\leq\frac{\epsilon_2}{\Gamma(\varrho+1)},\ t\in\Im. \end{aligned} \tag{4.4}$$

PROOF : In view of Theorem 3.1, we get the solution of system (4.13) as given by

$$\begin{cases} w(t)=w_0-\phi_1(0,w_0,z_0)\phi_1(t,w(t),z(t))+\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi \\ \qquad +\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\hbar_1(\xi)d\xi,\ t\in\Im, \\ z(t)=z_0-\phi_2(0,w_0,z_0)+\phi_2(t,w(t),z(t))+\int_0^t\frac{(t-\xi)^{\varrho-1}}{\Gamma(\varrho)}\psi_2(\xi,w(\xi),z(\xi))d\xi \\ \qquad +\int_0^t\frac{(t-\xi)^{\varrho-1}}{\Gamma(\varrho)}\hbar_2(\xi)d\xi,\ t\in\Im. \end{cases} \tag{4.5}$$

Now it is easy to obtain (4.14) from the solution given as (4.5). □

***Theorem* 4.3** — *Under the assumptions $(H_1),(H_4)$ and Lemma 4.2 together with the condition that $1>(\mathbf{P}+\mathbf{Q})$, where*

$$\mathbf{P}=K+\frac{T^{\sigma}\|\alpha\|_L}{(\sigma+1)},\ \mathbf{Q}=L+\frac{T^{\varrho}\|\beta\|_L}{(\varrho+1)},$$

then the solutions of considered problem (1.2) are UH and consequently GUH stable.

PROOF : Let $(w,z)\in\mathbf{E}$ is any solution of coupled system (1.2) of FHDEs and $(x,y)\in\mathbf{E}$ be the unique solution of the proposed system (1.2), then consider

$$\begin{aligned}
|x(t)-w(t)| &= \left|x(t)-\left(w_0-\phi_1(0,w_0,z_0)+\phi_1(t,w(t),z(t))+\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right)\right|\\
&\le \left|x(t)-\left(x_0-\phi_1(0,x_0,x_0)+\phi_1(t,x(t),y(t))+\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,x(\xi),y(\xi))d\xi\right)\right|\\
&+ \left|\left(x_0-\phi_1(0,x_0,x_0)+\phi_1(t,x(t),y(t))+\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right)\right.\\
&- \left.\left(w_0-\phi_1(0,w_0,z_0)+\phi_1(t,w(t),z(t))+\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right)\right|\\
&\le \frac{\epsilon_1}{\Gamma(\sigma+1)}+|\phi_1(t,w(t),z(t)-\phi_1(t,x(t),y(t))|\\
&+ \int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}|\psi_1(\xi,w(\xi),z(\xi))-\psi_1(\xi,x(\xi),y(\xi))|d\xi\\
&\le \frac{\epsilon_1}{\Gamma(\sigma+1)}+K[|w-x|+|z-y|]+\int_0^t\frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}|\alpha(\xi)|[\|w-x\|+\|z-y\|]d\xi
\end{aligned}$$

and hence

$$\|x-w\|\le\frac{\epsilon_1}{\Gamma(\sigma+1)}+K[\|w-x\|+\|z-y\|]+[\|w-x\|+\|z-y\|]\frac{T^{\sigma}\|\alpha\|_L}{\Gamma(\sigma+1)}. \tag{4.6}$$

Similarly for second equation of the system (1.2) we have

$$\|y-z\|\le\frac{\epsilon_2}{\Gamma(\varrho+1)}+L[\|w-x\|+\|z-y\|]+[\|w-x\|+\|z-y\|]\frac{T^{\varrho}\|\beta\|_L}{\Gamma(\varrho+1)}. \tag{4.7}$$

From (4.6) and (4.7), we have

$$\begin{aligned}
\left[1-\left(K+\frac{T^{\sigma}\|\alpha\|_L}{(\sigma+1)}\right)\right]\|x-w\|-\left(K+\frac{T^{\sigma}\|\alpha\|_L}{\Gamma(\sigma+1)}\right)\|z-y\| &\le \frac{\epsilon_1}{\Gamma(\sigma+1)}\\
-\left(L+\frac{T^{\varrho}\|\beta\|_L}{(\varrho+1)}\right)\|x-w\|+\left[1-\left(L+\frac{T^{\varrho}\|\beta\|_L}{\Gamma(\varrho+1)}\right)\right]\|y-z\| &\le \frac{\epsilon_2}{\Gamma(\varrho+1)}
\end{aligned}$$

which further can be written as

$$\begin{bmatrix} 1-\left(K+\frac{T^{\sigma}\|\alpha\|_L}{(\sigma+1)}\right) & -\left(K+\frac{T^{\sigma}\|\alpha\|_L}{\Gamma(\sigma+1)}\right)\\ -\left(L+\frac{T^{\varrho}\|\beta\|_L}{(\varrho+1)}\right) & 1-\left(L+\frac{T^{\varrho}\|\beta\|_L}{\Gamma(\varrho+1)}\right)\end{bmatrix}\begin{bmatrix}\|x-w\|\\ \|y-z\|\end{bmatrix}\le\begin{bmatrix}\frac{\epsilon_1}{\Gamma(\sigma+1)}\\ \frac{\epsilon_2}{\Gamma(\varrho+1)}\end{bmatrix}. \tag{4.8}$$

Let use $\mathbf{P} = K + \frac{T^{\sigma}\|\alpha\|_L}{(\sigma+1)}$, $\mathbf{Q} = L + \frac{T^{\varrho}\|\beta\|_L}{(\varrho+1)}$, then the above system (4.8) can be written as

$$\begin{bmatrix} \|x-w\| \\ \|y-z\| \end{bmatrix} \leq \begin{bmatrix} 1-\mathbf{P} & -\mathbf{P} \\ -\mathbf{Q} & 1-\mathbf{Q} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\epsilon_1}{\Gamma(\sigma+1)} \\ \frac{\epsilon_2}{\Gamma(\varrho+1)} \end{bmatrix}. \tag{4.9}$$

Upon simplification and using $\Delta = 1 - (\mathbf{P} + \mathbf{Q})$ (4.9) gives

$$\|x-w\| \leq \frac{1-\mathbf{Q}}{\Delta}\frac{\epsilon_1}{\Gamma(\sigma+1)} + \frac{\mathbf{P}}{\Delta}\frac{\epsilon_2}{\Gamma(\varrho+1)} \tag{4.10}$$

$$\|y-z\| \leq \frac{1-\mathbf{P}}{\Delta}\frac{\epsilon_2}{\Gamma(\varrho+1)} + \frac{\mathbf{Q}}{\Delta}\frac{\epsilon_1}{\Gamma(\sigma+1)}. \tag{4.11}$$

Now from (4.10), and using $\max\{\epsilon_1, \epsilon_2\} = \epsilon$, we have

$$\|(x,y)-(w,z)\| \leq \hat{C}_{\sigma,\varrho,\Delta}\epsilon, \text{ where } \hat{C}_{\sigma,\varrho,\Delta} = \frac{1}{\Delta}\left[\frac{1}{\Gamma(\sigma+1)} + \frac{1}{\Gamma(\varrho+1)}\right]. \tag{4.12}$$

Hence the solution of the considered system (1.2) is UH stable. Further let us set $\theta(\epsilon) = \hat{C}_{\sigma,\varrho,\Delta}\epsilon$ which gives on $\theta(0) = 0$. Then the solution of the proposed coupled system (1.2) are GUH stable. □

(H_5) Let for the functions $\hbar_1, \hbar_2$ the inequalities given by

$$I^{\sigma}\hbar_1(t) \leq \lambda_{\hbar_1}\hbar_1(t),\ I^{\varrho}\hbar_2(t) \leq \lambda_{\hbar_2}\hbar_2(t)$$

holds.

Lemma 4.4 — Under the assumption (H_5), the solution $(w, z) \in \mathbf{E}$ of the system of FHDEs given by

$$\begin{cases} {}^c\mathbf{D}^{\sigma}\left(w(t) - \phi_1(t, w(t), z(t))\right) = \psi_1(t, w(t), z(t)) + \hbar_1(t),\ a.e\ t \in \mathfrak{J}, \\ {}^c\mathbf{D}^{\varrho}\left(z(t) - \phi_2(t, w(t), z(t))\right) = \psi_2(t, w(t), z(t)) + \hbar_2(t),\ a.e\ t \in \mathfrak{J}], \\ w(t)|_{t=0} = w_0,\ z(t)|_{t=0} = z_0. \end{cases} \tag{4.13}$$

satisfies the relation given by

$$\begin{aligned} &\left| w(t) - \left(-w_0 + \phi_1(0, w_0, z_0)\phi_1(t, w(t), z(t)) - \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi, w(\xi), z(\xi))d\xi \right) \right| \\ &\leq \epsilon_1\lambda_{\hbar_1}\hbar_1(t),\ t \in \mathfrak{J}, \\ &\left| z(t) - \left(-z_0 + \phi_2(0, w_0, z_0) - \phi_2(t, w(t), z(t)) - \int_0^t \frac{(t-\xi)^{\varrho-1}}{\Gamma(\varrho)}\psi_2(\xi, w(\xi), z(\xi))d\xi \right) \right| \\ &\leq \epsilon_2\lambda_{\hbar_2}\hbar_2(t),\ t \in \mathfrak{J}. \end{aligned} \tag{4.14}$$

PROOF : The proof is similarly can be obtained as in Lemma 4.2. □

***Theorem* 4.5** — *Under the assumptions $(H_1), (H_4), (H_5)$ and Lemma 4.4 together with the condition that $1 > (\mathbf{P}+\mathbf{Q})$, where*

$$\mathbf{P} = K + \frac{T^{\sigma}\|\alpha\|_L}{(\sigma+1)},\ \mathbf{Q} = L + \frac{T^{\varrho}\|\beta\|_L}{(\varrho+1)},$$

then the solutions of considered problem (1.2) are UHR and consequently GUHR stable.

PROOF : Let $(w,z) \in \mathbf{E}$ is any solution of coupled system (1.2) of FHDEs and $(x,y) \in \mathbf{E}$ be the unique solution of the proposed system (1.2), then consider

$$\begin{aligned}
|w(t)-x(t)| &= \left|x(t) - \left(w_0 - \phi_1(0,w_0,z_0) + \phi_1(t,w(t),z(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right)\right| \\
&\le \left|x(t) - \left(x_0 - \phi_1(0,x_0,x_0) + \phi_1(t,x(t),y(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,x(\xi),y(\xi))d\xi\right)\right| \\
&+ \left|\left(x_0 - \phi_1(0,x_0,x_0) + \phi_1(t,x(t),y(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right)\right. \\
&- \left.\left(w_0 - \phi_1(0,w_0,z_0) + \phi_1(t,w(t),z(t)) + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}\psi_1(\xi,w(\xi),z(\xi))d\xi\right)\right| \\
&\le \epsilon_1\lambda_{\hbar_1}\hbar_1(t) + |\phi_1(t,w(t),z(t) - \phi_1(t,x(t),y(t))| \\
&+ \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}|\psi_1(\xi,w(\xi),z(\xi)) - \psi_1(\xi,x(\xi),y(\xi))|d\xi \\
&\le \epsilon_1\lambda_{\hbar_1}\hbar_1(t) + K[|w-x|+|z-y|] + \int_0^t \frac{(t-\xi)^{\sigma-1}}{\Gamma(\sigma)}|\alpha(\xi)|[\|w-x\|+\|z-y\|]d\xi.
\end{aligned}$$

Hence we have

$$\|w-x\| \le \epsilon_1\lambda_{\hbar_1}\hbar_1(t) + K[\|w-x\|+\|z-y\|] + K[\|w-x\|+\|z-y\|]\frac{T^{\sigma}\|\alpha\|_L}{\Gamma(\sigma+1)}. \tag{4.15}$$

Similarly for second equation we have

$$\|z-y\| \le \epsilon_2\lambda_{\hbar_2}\hbar_2(t) + L[\|w-x\|+\|z-y\|] + L[\|w-x\|+\|z-y\|]\frac{T^{\varrho}\|\beta\|_L}{\Gamma(\varrho+1)}. \tag{4.16}$$

Now like Theorem 4.3, in matrix form we can write (4.15) and (4.16) as

$$\begin{bmatrix} 1-\mathbf{P} & -\mathbf{Q} \\ -\mathbf{P} & 1-\mathbf{Q} \end{bmatrix} \begin{bmatrix} \|w-x\| \\ \|z-y\| \end{bmatrix} \le \begin{bmatrix} \epsilon_1\lambda_{\hbar_1}\hbar_1(t) \\ \epsilon_2\lambda_{\hbar_2}\hbar_2(t) \end{bmatrix}. \tag{4.17}$$

Let $\max\{\lambda_{\hbar_1}\hbar_1(t), \lambda_{\hbar_2}\hbar_2(t)\} = \lambda_{\hbar}\hbar(t), \max\{\epsilon_1,\epsilon_2\} = \epsilon$ and solving system (4.17), we get

$$\|w-x\| + \|z-y\| \le \hat{C}_{\hbar,\Delta}\epsilon\hbar(t),\ \text{where } \hat{C}_{\hbar,\Delta} = \frac{2\lambda_{\hbar}}{\Delta}. \tag{4.18}$$

Hence the solutions of the considered coupled system of FHDEs are UHR stable with respect to $\hbar$. Obviously one can prove that the solutions of the considered coupled system of FHDEs are GUHR stable with respect to $\hbar$. □

5. EXAMPLE

To demonstrate our above theatrical results, consider the following test problem.

Example 5.1 : Taking the given system of FHDEs

$$\begin{cases} {}^c\mathbf{D}^{\frac{1}{2}}\left[w(t) - \frac{1}{4}\left(\sin(t) + w(t) + z(t)\right)\right] = \frac{1}{3}[t + w(t) + z(t)],\ t \in \Im, \\ {}^c\mathbf{D}^{\frac{3}{5}}\left[z(t) - \frac{1}{8}\left(\cos(t) + w(t) + z(t)\right)\right] = \frac{1}{4}[t + w(t) + z(t)],\ t \in \Im, \\ w(t)|_{t=0} = 1, \quad z(t)|_{t=0} = 1. \end{cases} \tag{5.1}$$

From (5.1) we get

$$\phi_1(t, w(t), z(t)) = \frac{1}{4}[\sin(t) + w(t) + z(t)],\ \phi_2(t, w(t), z(t)) = \frac{1}{8}[\cos(t) + w(t) + z(t)],$$
$$\psi_1(t, w(t), z(t)) = \frac{1}{3}[t + w(t) + z(t)],\ \psi_2(t, w(t), z(t)) = \frac{1}{4}[t + w(t) + z(t)].$$

Now it is easy to calculate $K = \frac{1}{4},\ L = \frac{1}{8},\ |\psi_1(t, w(t), z(t))| \leq \frac{t}{3},\ |\psi_2(t, w(t), z(t))| \leq \frac{t}{4}$, and $\Lambda_1 = 0,\ \Lambda_2 = \frac{1}{8}, T = 1$. Hence we conclude that

$$|w_0 - \phi_1(0, w_0, z_0)| + |z_0 - \phi_2(0, w_0, z_0)| + K + L + \Lambda_1 + \Lambda_2 + \frac{\|\alpha\|_{\mathcal{L}}}{\Gamma(\sigma+1)} + \frac{\|\beta\|_{\mathcal{L}}}{\Gamma(\varrho+1)}$$
$$= 1 + \frac{7}{8} + \frac{1}{4} + \frac{1}{8} + 0 + \frac{\frac{1}{3}}{\Gamma(1.5)} + \frac{\frac{1}{4}}{\Gamma(1.6)} \leq 5.$$

Hence by Theorem 3.2 coupled system of FHDEs (5.1) has a mild solution in $\mathbf{W} = \{(w, z) \in \mathbf{E} : \|(w, z)\| \leq 5\}$. Further $\mathbf{P} = K + K\frac{\|\alpha\|_{\mathcal{L}} T^{\sigma}}{\Gamma(\sigma+1)} = 0.3440, \mathbf{Q} = L + L\frac{\|\beta\|_{\mathcal{L}} T^{\varrho}}{\Gamma(\varrho+1)} = 0.15997$. From which we have $\mathbf{P} + \mathbf{Q} < 1$ holds and hence the solution of (5.1) is UH stable and similarly it can be easily shown that the concerned solution is also GUH stable. Further taking $\hbar_1(t) = \hbar_2(t) = t$, then the conditions of UHR stability and GUHR stability as discussed in Theorem 4.5 can be easily derived.

6. CONCLUSION

With the help of hybrid fixed point theorem due to Dhage, we successfully established some adequate conditions for at least one solutions to a system of FHDEs. Under the the application of the afore said fixed point theorem, a coupled system with initial conditions of nonlinear FHDEs has been investigated. The newly formed conditions revealed the existence of at least one solutions to such type interesting system of FHDEs. The idea can be further extended to more complicated problems of dynamics involving hybrid differential equations. Some interesting and new results about different kinds of Ulam's stability have been developed by using the techniques of nonlinear functional analysis.

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