

Parametric analysis of diffusivity equation in oil reservoirs

Reza Azin¹ · Mohamad Mohamadi-Baghmolaei¹ · Zahra Sakhaei¹

Received: 5 November 2015 / Accepted: 2 April 2016 / Published online: 15 April 2016
© The Author(s) 2016. This article is published with open access at Springerlink.com

Abstract The governing equation of fluid flow in an oil reservoir is generally non-linear PDE which is simplified as linear for engineering proposes. In this work, a comprehensive numerical model is employed to study the role of non-linear term in reservoir engineering problems. The pervasive sensitivity analysis is performed on rock and fluid properties, and it is shown that rock permeability and fluid viscosity are the most significant parameters which influence the pressure profile. Moreover, the critical reservoir radius was determined in four different cases including heavy and light oil along with high and low permeable rocks, in which the pressure difference of linear and non-linear diffusivity equation is significant. Results of this study give an insight into proper simplification of diffusivity equation and provide an engineering-based decision strategy for different reservoir properties having certain rock and fluid properties in oil reservoirs. The results show the significance of depletion time, production rate, and reservoir radius in calculation of pressure drop.

Keywords Diffusivity equation · Non-linear partial differential equation · Numerical solution · Analytical solution

List of symbols

B Formation volume factor, bbl/STB
C Compressibility factor, psi^{-1}
h Formation thickness, ft

K Absolute permeability, mD
p Pressure, psi
Q Flow rate, STB/day
q Constant rate, STB/day
r Radius, ft
t Time, h

Greek symbols

μ Viscosity, cp
 η Diffusivity constant
 ρ Gas density, lb/ft^3
 ϕ Porosity
v Apparent fluid velocity, bbl/day-ft^2

Subscripts

c Critical
e External
f Formation
i Initial
o Oil
pss Pseudo steady-state
sc Standard condition
t Total
w Wellbore

Abbreviations

L Linear
NL Non-linear

Introduction

Understanding the trend of pressure profile in a reservoir during primary depletion is essential in reservoir studies. Especially, for those reservoirs which are susceptible to perform two phases of hydrocarbon due to pressure drop

✉ Reza Azin
reza.azin@pgu.ac.ir

¹ Department of Chemical Engineering, Faculty of Petroleum, Gas and Petrochemical Engineering, Persian Gulf University, Bushehr, Iran

(Danesh 1998). In this regard, any precise approach and model which include less assumption can be a reliable method. The fluid flow in reservoir or in porous medium has been a great interest of physicists, engineers and hydrologists who tried to predict the behaviors of compressible and incompressible fluids. They have designed several experiments so as to validate the implementation of their proposed correlations (Ahmed 2011). The basic equation for predicting pressure distribution in a reservoir is the diffusivity equation. For this equation, the reservoir temperature is supposed to be constant which is a valid assumption in most cases. Several methods have presented to solve the diffusivity equation including numerical and analytical approaches. The diffusivity equation is also solved in dimensionless form (Lee 1996).

Several procedures have been proposed to solve the pressure diffusivity equation. Take Chakrabarty et al. (1993) as an example, who provided a quantitative analysis of the effects of neglecting the quadratic gradient term on solving the diffusion equation governing the transient state (Chakrabarty et al. 1993). It should be noted that among the flow regimes in reservoir, the transient flow is the most significant state upon which such important characteristics as permeability, reservoir capacity, and skin factor can be determined using well test analysis (Van Everdingen 1953; Lee 1992). Regarding this specification, Liang et al. applied the transient pulse decay to measure the permeability of tight rocks and synthetic materials using the pressure diffusion equation, in which due to dependent pressure of gas compressibility the diffusivity equation became non-linear (Liang et al. 1978). They also considered the Klinkenberg effect in their computations. Another common application of diffusivity equation is during pressure transient analyses. Goode offered an analytic solution for three dimensional diffusivity equations during drawdown and buildup of a horizontal model to get some specifications of reservoir. He applied numerical solution for validation of his procedure (Goode 1987). In some cases, the researchers face with heterogeneous reservoirs instead of homogeneous, therefore, they use some synthetic methods to solve the problem (Dake 1983). Loucks et al. solved diffusivity equation using Laplace transformation for finding pressure drop characteristics in a composite system with different permeabilities (Loucks and Guerrero 1961). Also, van Everdingen and Hurst proposed that by use of Laplace transform instead of tedious prior mathematical analysis, the problems encounter with flow equations can be simplified (Van Everdingen and Hurst 1949).

It should be noted that, as the non-linear form of diffusivity equation causes difficulties in solution process (Odeh and Babu 1988). To overcome these difficulties several suggestions and methods have been proposed such

as that of Vik and Jelmert, who planned a solution using pressure logarithm transform for diffusivity equation included pressure squared term (Jelmert and Vik 1996). Xu-Long et al. in their study found the exact solution for radial fluid flow along with pressure squared term. They considered both types of inner boundary conditions, constant flow rate and constant pressure. The generalized Weber transform has been used in their study. They also proposed the analytical solution using Hankel transform for fluid flow in spherical state when the pressure squared term is included. Their researches consisted of analyzing and comparing the linear and non-linear diffusivity equation. Their results showed that, the differences between pressures which had been obtained by two types of equations reaches about 8 percent for long time (Xu-long et al. 2004). Another interesting illustration in this category is the work done by Braeuning et al. who applied modified logarithm transform for solving diffusivity equation. They considered the damage skin and wellbore storage in their solution. They noticed that, the magnitude of error as result of linearization of diffusivity equation depends on the wellbore damage, the pseudo-skin due to partial penetration and non-linear flow parameter (Braeuning et al. 1998). Other example is the usage of Laplace transform in analytical solution of non-linear diffusivity equation which was done by Dusseault and Wang. They claimed that the observed deviation of their approach from existing solutions was occurred due to high pressure gradient, compressibility of the core and the injected fluids (Wang and Dusseault 1991).

These kinds of approximations and considerations may cause some changes in pressure drop profile along the radius of reservoir which may be significant or not. On this subject in the current study first, the comparison between analytical and numerical solution for both linear and non-linear diffusivity equation will be performed at wellbore radius, and second the comprehensive sensitivity analysis will be done to find the significant physical properties of reservoir which influence the pressure drop. Next, a comprehensive analysis of involved parameters such as depletion time, reservoir radius and production rate will be performed to find where the pressure differences of linear and non-linear diffusivity equations are significant.

Mathematical considerations

For slightly compressible, unsteady-state fluid flow through a radial reservoir, the diffusivity equation is written as (Ahmed 2011). For more details, refer to “Appendix 1”:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + c \left(\frac{\partial p}{\partial r} \right)^2 = \frac{1}{\eta} \frac{\partial p}{\partial t} \quad (1)$$

Many authors neglect the $\left(\frac{\partial p}{\partial r}\right)^2$ term assuming small magnitude of compressibility factor and make Eq. 1 linear. This assumption may be valid for slightly compressible fluids. However, when either compressibility factor or pressure gradient terms are large, the non-linear term cannot be removed from diffusivity equation and makes the PDE non-linear. In this paper, both numerical and analytical solutions will be applied to understand the difference between both linear and non-linear equation in unsteady-state condition.

Boundary conditions and analytical solution

As a rule for second order differential equations, two boundary conditions are required. The inner and outer boundary conditions are set based on reservoir conditions. A flush for determination of reservoir state is pseudo steady-state time, t_{pss} , defined by Eq. (2) (Ahmed 2011):

$$t_{\text{pss}} = \frac{12000\phi\mu c_i r_e^2}{K} \tag{2}$$

Before this time the transient state is governed and after this the pseudo steady-state is started. As said, solving the partial differential equation of diffusivity equation is possible using proper inner and outer boundary conditions. The inner boundary condition for both states can be one either constant rate or constant pressure, however, the second is most common (Ahmed 2011). The constant rate boundary condition is:

$$\lim_{r \rightarrow r_w} \left(\frac{2\pi r h K}{\mu} \frac{dp}{dr} \right) = q \tag{3}$$

Nevertheless, outer boundary condition is different for transient and pseudo steady-state. The infinite acting boundary is used for transient state and no flow outer boundary commonly used for pseudo steady-state. Both boundary conditions are illustrated below, respectively (Ahmed 2011):

$$\lim_{r \rightarrow \infty} p(r, t) = p_i \tag{4}$$

$$\lim_{r \rightarrow \infty} \left(\frac{2\pi r h K}{\mu} \frac{dp}{dr} \right) = 0 \tag{5}$$

Both numerical and analytical solutions of diffusivity equation have attracted the considerations of many researchers (Eymard and Sonier 1994). Some useful approaches have been applied to solve the mentioned equation such as Laplace transform, Boltzmann transform, dimensionless form and *Ei* function (Loucks and Guerrero 1961; Odeh and Babu 1988; Marshall 2009). Also, Cole–Hopf transform is used to simplify pressure diffusivity equation included pressure dependence permeability, porosity and

density which cause the flow equation non-linear (Marshall 2009). The *Ei* function approach is functional for the current case of study of analytical solution. The pressure variation in reservoir for transient state using *Ei* function is presented (Ahmed 2011; Craft et al. 1959):

$$p(r, t) = p_i + \left[\frac{70.6 Q_o \mu B_o}{Kh} \right] Ei[-x] \tag{6}$$

Non-linear diffusivity equation needs an extra step for linearization to be solved analytically. This makes the non-linear diffusivity equation unfavorable and tedious in comparison with linear equations. Odeh and Babu proposed an analytical approach for solving non-linear diffusivity equation. Regarding “Appendix 2”, the final pressure function for constant flow at inner boundary condition and infinite reservoir is described as Eq. (7) (Odeh and Babu 1988):

$$\Delta p = \left(\frac{-1}{c} \right) \left(\ln \left(1 + \lambda \left(\ln \left(\frac{T}{r_w^2} \right) + 0.2319 \right) \right) + \frac{0.5772\lambda}{1 + \lambda \left(\frac{\ln T}{r_w^2} + 0.2319 \right)} \right) \tag{7}$$

Results and discussion

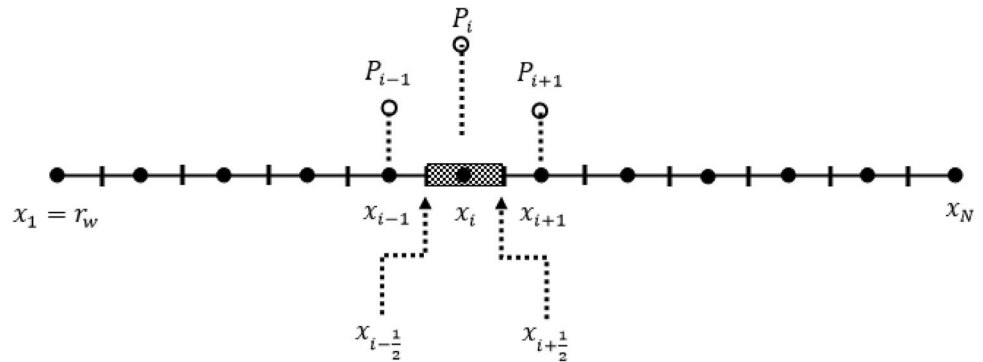
The models described in previous section are applied for six cases reported in Table 1. The comprehensive analytical numerical analysis will be accomplished in three sections. In the first section, the comparison between analytical and numerical solutions for both linear and non-linear diffusivity equations is performed by employing reported data of case 1. The validity of linear and non-linear solutions is checked using CMG reservoir simulator (version 2013). The second section is allocated to the sensitivity analysis of physical properties of rock and fluid using reported physical properties of case 2. The critical radius of reservoir, defined as the maximum radius of a reservoir with certain rock and fluid properties in which the pressure differences resulted from linear and non-linear solutions is considerable, was used to analyze the results. The last section is attributed to real cases analysis to find the critical radius of reservoir. Cases 3–6 are belonged to the last section. The rock and fluid properties of each case of study are reported in Table 1.

Pressure changes for oil reservoir considering numerical and analytical solutions

Solution of pressure diffusivity equation is a major concern of many researchers. Analytical and numerical solutions are the conventional ways of solving pressure diffusivity

Table 1 Rock and fluid properties for each case

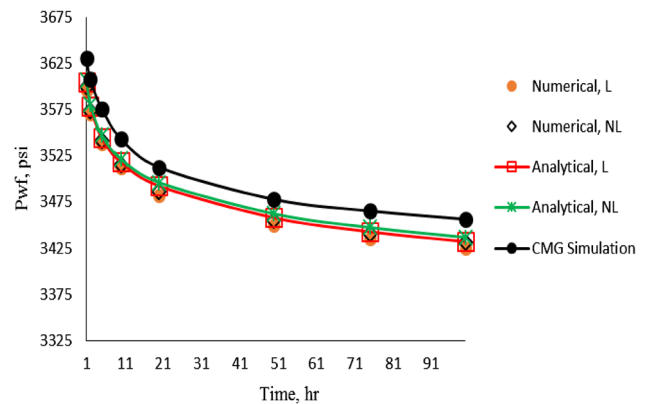
Index	B_o (bbl/STB)	μ_o (cp)	C_i (psi^{-1})	K_o (md)	h (ft)	ϕ (%)	r_w (ft)	p_i (psi)	Q (STB/D)	References
Case 1	1.20	1.5	$30 (10^{-6})$	85.1	20	25	0.333	4000	500	(Craft et al. 1959)
Case 2	1.25	1.5	$30 (10^{-6})$	60	15	15	0.250	4000	300	(Ahmed 2011)
Case 3	1.26	0.65 (L)	$21 (10^{-6})$	7.23 (L)	82	12	0.280	3770	419	(Ahmed 2011)
Case 4	1.20	1.5 (L)	$25 (10^{-6})$	20 (L)	30	15	0.250	4500	100	(Ahmed 2011)
Case 5	1.30	25.33 (L)	$20 (10^{-6})$	782 (H)	15	28.8	0.25	4000	300	(Torabi et al. 2016)
Case 6	1.50	400.2 (H)	$18 (10^{-6})$	2820 (H)	15	25.4	0.25	4000	200	(Xu et al. 2016)

Fig. 1 Discretization of the domain

equation. COMSOL Multiphasic 5.0 software was employed for numerical solution of both linear and non-linear diffusivity equations. The governing partial differential equations of current problem were solved by finite element method with prevailing assumptions and boundary conditions presented in “Mathematical considerations”. Application of four general types of equations in partial differential module of COMSOL software including coefficient form, general form, weak form and wave form, facilitate the solution of complicated problems. For the current problem, the coefficient form of differential equation along with one dimensional radial geometry was employed. The discretization of current problem is displayed in Fig. 1 where, 1788 elements were generated. In addition, 1 h time step was considered for the sake of time discretization regarding 0.001 relative tolerances. It is worth mentioning that, numerical solution of diffusivity equation is started from initial production time and is finished at pseudo steady-state time (t_{pss}).

To compare analytical and numerical solutions of linear and non-linear diffusivity equations, the solid-symbolic lines of each approach are depicted in Fig. 2. The abbreviations NL and L refer to non-linear and linear solutions correspondingly.

The results of analytical and numerical solutions of linear and non-linear pressure diffusivity equation validate using reservoir simulator (Commercial CMG 2013). For this reason, the single layer one dimensional radial reservoir along with one production well located in center of the reservoir was built up. The flow was considered to be liquid

**Fig. 2** The comparison between analytical and numerical approaches

single phase. Other required parameters for simulation are the same as data reported in case 1 of Table 1.

As cleared from Fig. 2, the results obtained by numerical and analytical solutions are in great agreement with those of reservoir simulator which justify the solution approach. Thus, the numerical approach is used for further analysis. Both numerical and analytical solutions have resulted in a semi-similar pressure distribution trend.

Sensitivity analysis of physical properties of rock and fluid

In this part, sensitivity analysis of rock and fluid physical properties is performed using reported data of case 2. Numerical solution is applied to recognize which involving

Fig. 3 Wellbore pressure variation by increasing 40 % of involving parameters

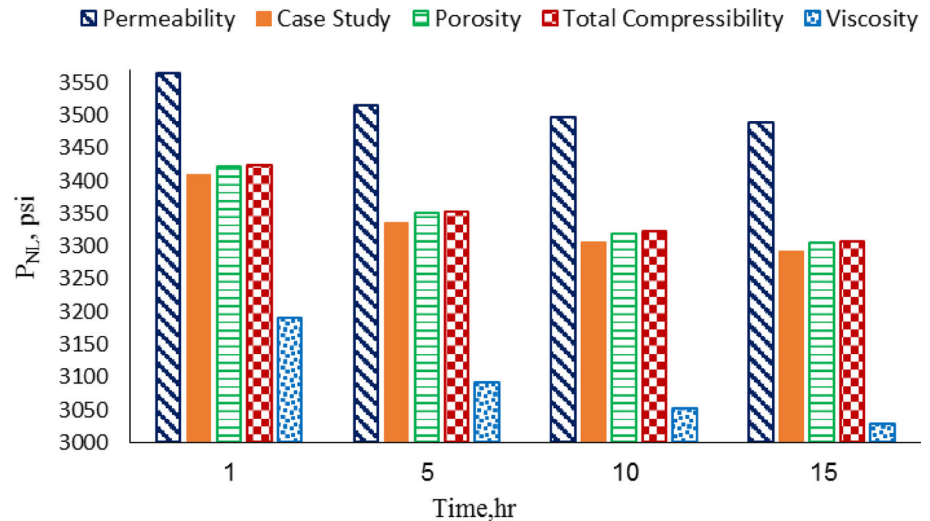
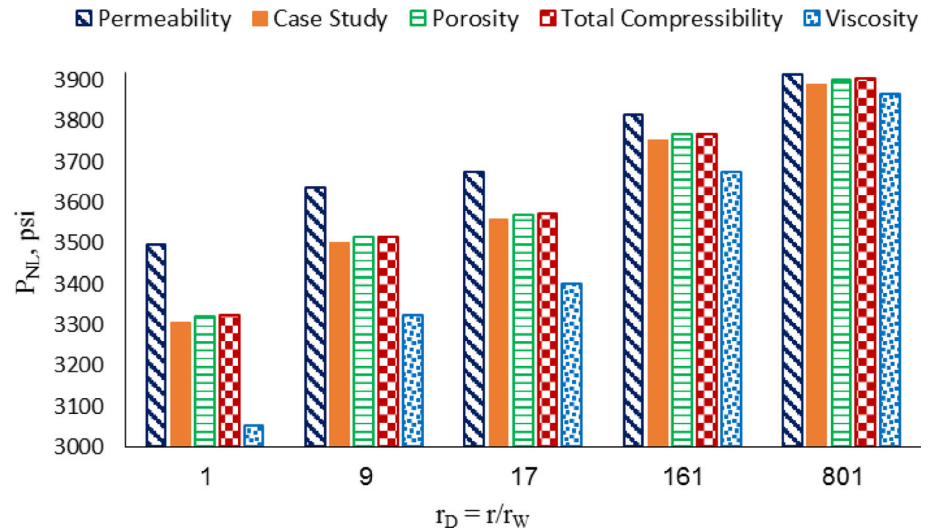


Fig. 4 The pressure variation of reservoir in 10th h in the case that involving parameters increase 40 %



physical properties in oil reservoir have the most influence on pressure profile. Permeability, porosity, total compressibility and viscosity are changed 40 % more and less. The presented results in Fig. 3 obtained in wellbore radius and various times. It is obvious that the inner boundary condition may change by variation of the mentioned parameters and time at wellbore radius. Also, the analysis was performed in 10th hour and various radius as displayed in Fig. 4. The Y-axis of the bar charts denotes the reservoir pressure in different times and different radius, respectively. As observed from Figs. 3 and 4, the effect of changes in porosity and total compressibility is equal. This can be interpreted by considering the inner boundary condition and diffusivity equation when both parameters have the same role in diffusivity equation and their absence at inner boundary condition. These figures show the most effective parameters on pressure profile which are

permeability and viscosity. This can be recognized from the highest and shortest bars related to permeability and viscosity, respectively.

This scenario is repeated for 40 % reduction of permeability, porosity, total compressibility and viscosity. The results are shown in Figs. 5 and 6 as well. The results are absolutely the same as the last state. The increase in absolute permeability causes the reduction in pressure gradient (considering Darcy law). In contrary, increasing the fluid viscosity, results the rise in pressure gradient.

Real cases analysis

As understood from previous section, the two major important parameters are viscosity and permeability. Thus, four different real cases are chosen from open literature (cases 3–6). It was tried to select wide range of viscosity

Fig. 5 Wellbore pressure variation by reducing 40 % of involving parameters

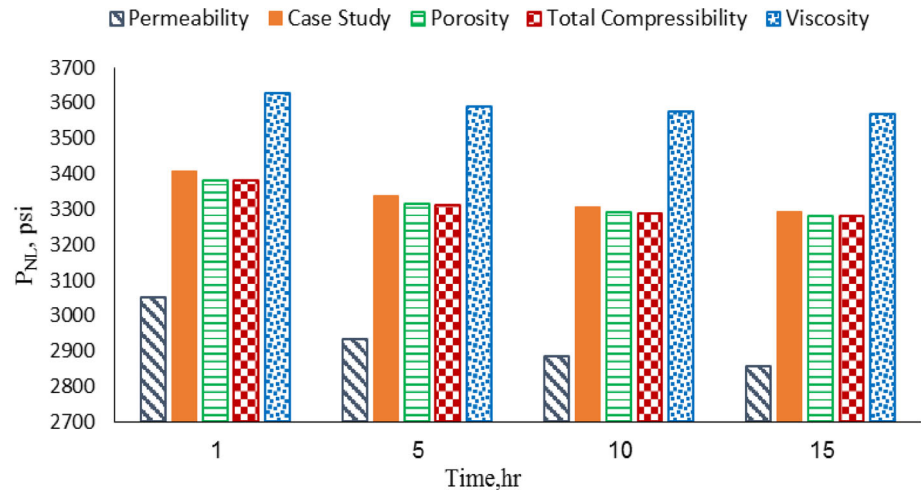
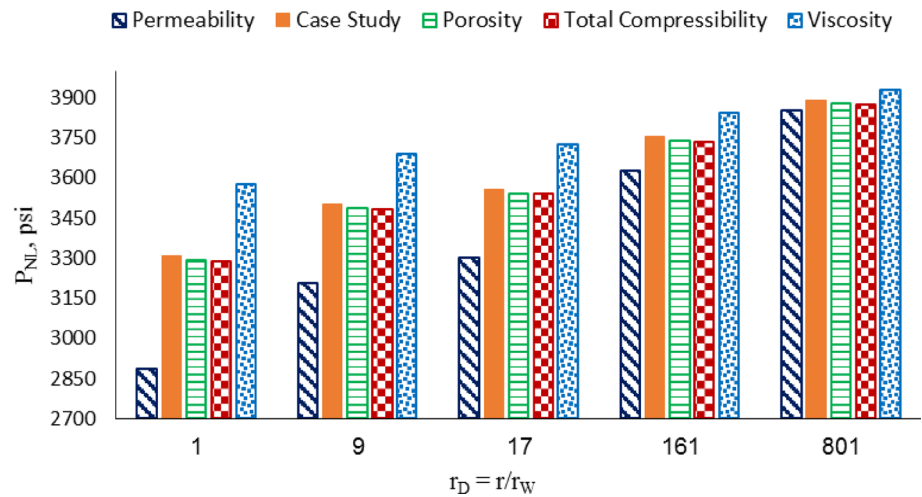


Fig. 6 The pressure variation of reservoir in 10th hr in the case of 40 % parameters decrease



and permeability for the chosen cases. Therefore, both light and heavy oil along with low and high permeable rock used in analysis are described in Table 1. *L* and *H* denote light and heavy for oil and low and high for permeability, respectively (Ahmed 2011; Craft et al. 1959; Torabi et al. 2016; Xu et al. 2016).

To evaluate other important parameters which potentially may affect the pressure differences of linear and non-linear diffusivity equation, the changes of production rate, depletion time and radius are explored for the aforementioned cases. Figures 7 and 8 show the pressure differences considering the variation of time, production rate and reservoir radius. As seen, each case has different behavior with respect to time, flow rate and radius changes. The severest changes belong to case 6 (heavy oil and high permeability), whereas the least important changes are those of case 3 (light oil and low permeability). In addition, the pressure differences increase by growing the depletion time, production rate and decreasing reservoir radius.

Simultaneous changes of radius, depletion time and production rate are considered to find the significant pressure differences for each of 4 cases. In the present paper, critical pressure (P_c) difference is arbitrarily set to 5 psi, and pressure differences higher than 5 psi are considered to be significant, as defined by Eq. (8).

$$\Delta p_c = |p_L - p_{NL}| = 5psi \tag{8}$$

The reservoir radiuses in which the pressure differences are thought to be critical are reported in Tables 2, 3, 4 and 5. These critical radiuses vary by changing depletion time and production rate as observed in these tables. Beyond that radius, pressure difference between linear and non-linear becomes less than ΔP_c , and both linear and non-linear solutions apply with negligible error.

It should be taken into account that, by analyzing the obtained results it was understood that, for the production rates less than 500, 200, 230, 50 STB/day for each case respectively, the pressure differences were not significant

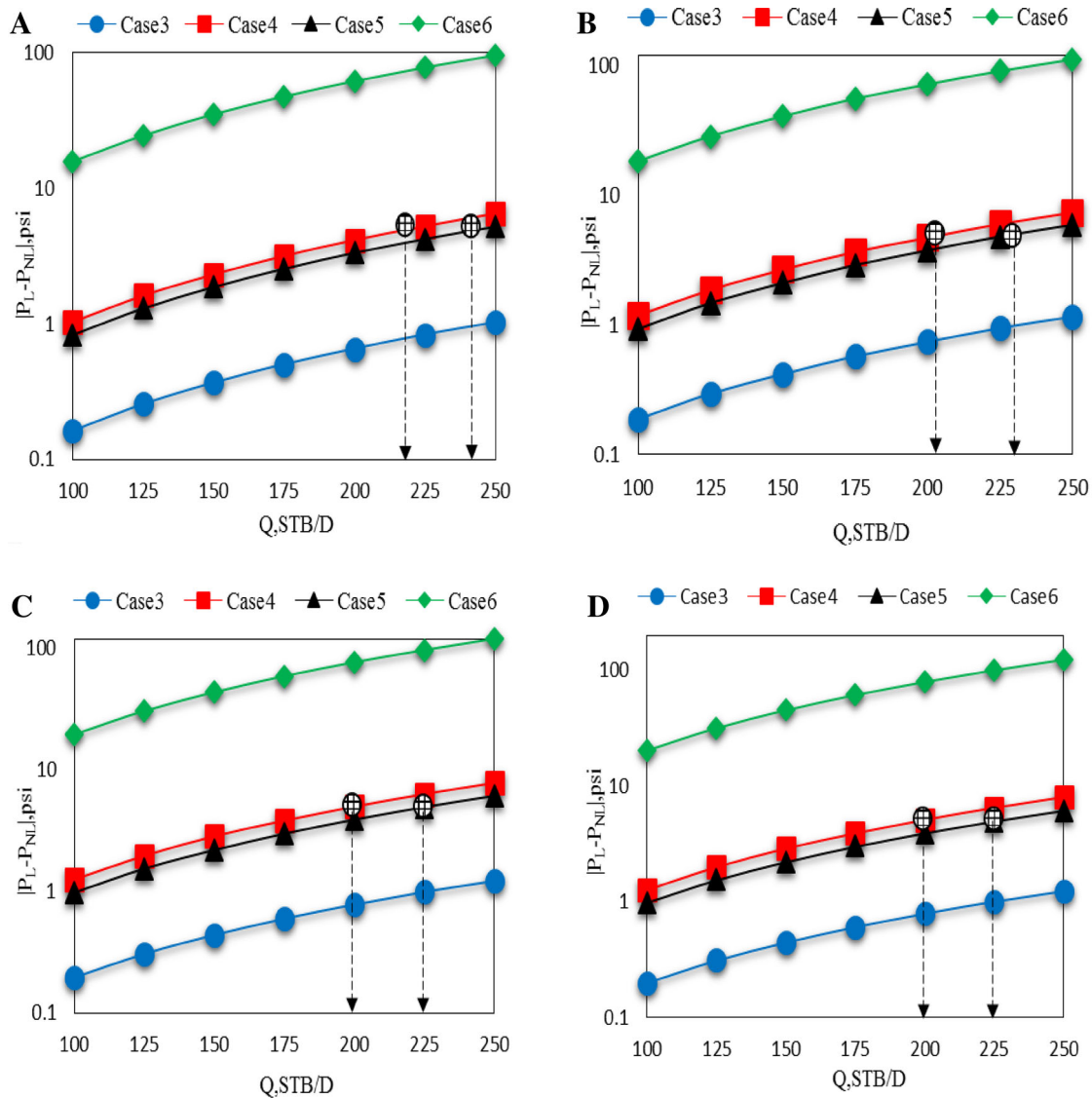


Fig. 7 Comparison between different flow rates and depletion times considering in wellbore radius. **a** $t = 10$ h; **b** $t = 30$ h; **c** $t = 50$ h; **d** $t = 70$ h

at all. This shows the significance of production rate at which for every depletion time and reservoir radius the pressure difference was insignificant. To put it another way, while the flow rates of each cases are under the boundary values, the pressure differences are less than critical and unimportant. To clarify the presented issue, the critical flow rates with variation of time and reservoir radius are determined by nodes in Figs. 7 and 8. As seen in Fig. 7, the pressure differences are highly significant for presented case 6 where the flow rates are higher than 50 STB/day. On the other side, there is no critical flow rate for case 3 regarding the smaller flow in compare with 500 STB/day index. Also, the indicated nodes on plotted

curve of cases 4 and 5 show the range of flow rates at wellbore radius at which pressure differences are critical or not. The similar scenario was repeated in Fig. 8, in which the pressure differences were determined versus flow rates for 10th and various radiuses. As observed from Fig. 8, the designated nodes on curves of cases 4 and 5 belonging to state A (wellbore radius) determine the border of critical pressure. Whereas, the pressure differences of cases 3 and 6 are unimportant and highly critical, respectively. As shown in Fig. 8, the pressure differences become unimportant for further distances. Hence, the pressure differences of other states (B–D) for each flow rates are normal and unimportant as radius is extended.

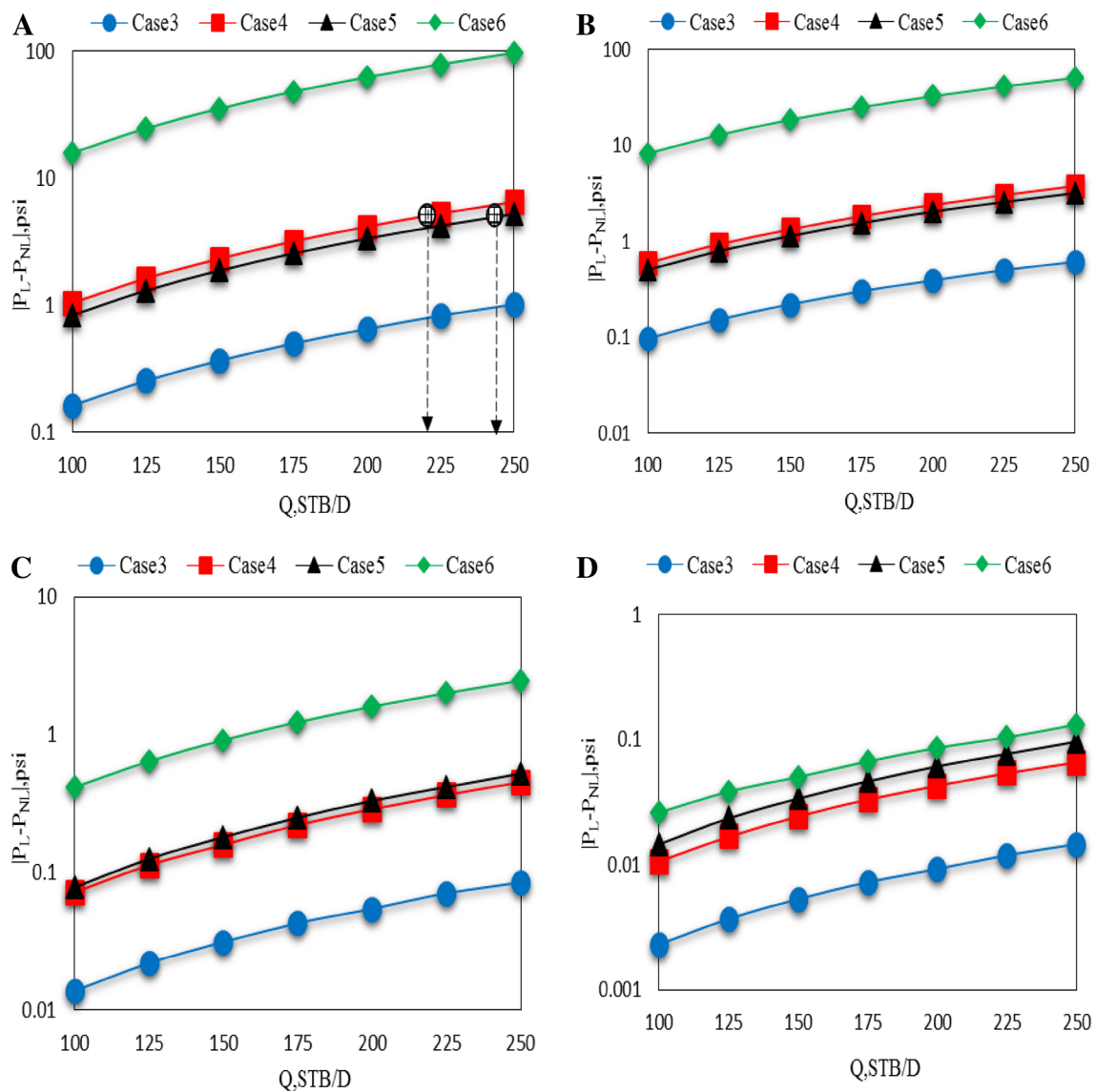


Fig. 8 Comparison between different flow rates and radiuses at $t = 10$ h. **a** $r = r_w$; **b** $r = 30.2$ Ft; **c** $r = 430.2$ Ft; **d** $r = 800.1$ Ft

Table 2 The critical reservoir radius with respect to depletion time and production rate for case 3

Case 3					
$Q = 1200$ STB/D		$Q = 1000$ STB/D		$Q = 800$ STB/D	
$r_D < 791.85$	$t = 10$ h	$r_D < 504.4$	$t = 10$ h	$r_D = 215.2$	$t = 10$ h
$r_D < 1546.9$	$t = 50$ h	$r_D < 1081.1$	$t = 50$ h	$r_D = 516.9$	$t = 50$ h
$r_D < 1654.1$	$t = 100$ h	$r_D < 1163.2$	$t = 100$ h	$r_D = 574.1$	$t = 100$ h

Conclusion

In this work, a parametric analysis was studied on pressure diffusivity equation. Results of numerical and analytical solutions for both linear and non-linear pressure diffusivity equations was presented and discussed. The critical radius

of reservoir, defined as the maximum radius of a reservoir with certain rock and fluid properties in which the pressure differences resulted from linear and non-linear solutions is considerable, was used to analyze the results. Moreover, four different cases including light and heavy oil along with low and high permeable rock used in the analysis to

Table 3 The critical reservoir radius with respect to depletion time and production rate for case 4

Case 4					
$Q = 1000$ STB/D		$Q = 800$ STB/D		$Q = 500$ STB/D	
$r_D < 2012.5$	$t = 10$ h	$r_D < 1644.6$	$t = 10$ h	$r_D < 860.8$	
$r_D < 3268.2$	$t = 50$ h	$r_D < 2900.3$	$t = 50$ h	$r_D < 1816.6$	
$r_D < 3360.2$	$t = 100$ h	$r_D < 3048.2$	$t = 100$ h	$r_D < 1998.5$	

Table 4 The critical reservoir radius with respect to depletion time and production rate for case 5

Case 5					
$Q = 800$ STB/D		$Q = 600$ STB/D		$Q = 400$ STB/D	
$r_D < 1756.6$	$t = 10$ h	$r_D < 1188.7$	$t = 10$ h	$r_D < 420.9$	$t = 10$ h
$r_D < 2756.3$	$t = 50$ h	$r_D < 2060.5$	$t = 50$ h	$r_D < 874.8$	$t = 50$ h
$r_D < 2820.3$	$t = 100$ h	$r_D < 2138.5$	$t = 100$ h	$r_D < 924.8$	$t = 100$ h

Table 5 The critical reservoir radius with respect to depletion time and production rate for case 6

Case 6					
$Q = 200$ STB/D		$Q = 150$ STB/D		$Q = 100$ STB/D	
$r_D < 1088.7$	$t = 10$ h	$r_D < 770.8$	$t = 10$ h	$r_D < 332.9$	$t = 10$ h
$r_D < 2456.4$	$t = 50$ h	$r_D < 1828.5$	$t = 50$ h	$r_D < 888.8$	$t = 50$ h
$r_D < 2948.3$	$t = 100$ h	$r_D < 2354.4$	$t = 100$ h	$r_D < 1244.7$	$t = 100$ h

investigate the critical radius of reservoir at which the pressure differences of linear and non-linear diffusivity equations are significant. It was found that both numerical and analytical solution techniques concluded accurate results. Results show the significance of depletion time, production rate and reservoir radius in calculation of pressure drop and determining critical radius. Sensitivity analysis of results shows that permeability and viscosity are the most effective parameters on pressure profile. The implemented approach for finding critical radius is strongly useful for petroleum engineers in natural depletion.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Appendix 1: Derivation and solution of the diffusivity equation

By writing mass balance for a definite element of porous medium, the continuity equation is determined (Ahmed 2011).

$$\frac{1}{r} \frac{\partial}{\partial r} [r(v\rho)] = \frac{\partial}{\partial t} (\phi\rho) \tag{9}$$

Liquid velocity is determined by getting help from Darcy’s law:

$$v = (0.0002637) \frac{K}{\mu} \frac{\partial p}{\partial r} \tag{10}$$

Substituting Eq. 9 in Eq. 10 resulted:

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r \left(0.0002637 \frac{K\rho}{\mu} \frac{\partial p}{\partial r} \right) \right] = \frac{\partial}{\partial t} (\phi\rho) \tag{11}$$

Using formation compressibility factor c_f , the general partial differential equation is:

$$c_f = \frac{1}{\phi} \frac{\partial \phi}{\partial p} \tag{12}$$

$$\frac{0.0002637}{r} \frac{\partial}{\partial r} \left[\left(\frac{K}{\mu} (\rho r) \frac{\partial p}{\partial r} \right) \right] = \phi \rho c_f \frac{\partial p}{\partial t} + \phi \frac{\partial \rho}{\partial t} \tag{13}$$

For constant viscosity, porosity and permeability the diffusivity equation for slightly compressible fluids:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + c \left(\frac{\partial p}{\partial r} \right)^2 = \frac{1}{\eta} \frac{\partial p}{\partial t} \tag{14}$$

$$c = \frac{1}{\rho} \frac{\partial \rho}{\partial p} \quad (15)$$

$$\eta = \frac{0.0002637K}{\phi \mu C_t} \quad (16)$$

where c is the compressibility factor of slightly compressible fluid.

Appendix 2: Analytical solution for linear and non-linear diffusivity equation and errors

The analytical solution of linear pressure diffusivity equation concluded to Ei function (Ahmed 2011):

$$P(r, t) = P_i + \left[\frac{70.6Q_0 \mu B_0}{Kh} \right] Ei[-x] \quad (17)$$

$$x = \frac{948 \phi \mu C_t r^2}{Kt} \quad (18)$$

where B_0 is the formation volume factor.

The mathematical function of Ei is called exponential integral (Ahmed 2011):

$$Ei[-x] = - \int_x^\infty \frac{e^{-u} du}{u} = \left[\ln x - \frac{x}{1!} + \frac{x^2}{2(2!)} - \frac{x^3}{3(3!)} \right] \quad (19)$$

For the values of x less than 0.01, the below approximation is applicable:

$$Ei[-x] = \ln(1.781x) \quad (20)$$

And for range of x between 0.01 and 3:

$$Ei[-x] = a_1 + a_2 \ln(x) + a_3 [\ln(x)]^2 + a_4 [\ln(x)]^3 + a_5 x + a_6 x^2 + a_7 x^3 + \frac{a_8}{x} \quad (21)$$

$$\begin{aligned} a_1 &= -0.33153973 & a_2 &= -0.81512322 \\ a_3 &= 5.22123384 \times 10^{-2} & a_4 &= 5.9849819 \times 10^{-3} \\ a_5 &= 0.662318450 & a_6 &= -0.12333524 \\ a_7 &= 1.0832566 \times 10^{-2} & a_8 &= 8.6709776 \times 10^{-4} \end{aligned}$$

For most applicable ranges, there are some useful figures in which the value of Ei can be determined knowing x (Ahmed 2011).

The analytical solution of non-linear diffusivity equation needs changing variables to perform linear equation to be solved analytically (Odeh and Babu 1988).

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + c \left(\frac{\partial p}{\partial r} \right)^2 = \frac{\phi \mu c \partial p}{kt \partial t} \quad (22)$$

$$\Delta p = p - p_i, T = \frac{Kt}{\phi \mu c} \text{ and } \Delta p = \frac{1}{c} \ln p^*$$

The linearized form is presented as:

$$\frac{\partial^2 p^*}{\partial r^2} + \frac{1}{r} \frac{\partial p^*}{\partial r} = \frac{\partial p^*}{\partial T} \quad (23)$$

The linear equation can be solved using Laplace transform and concludes:

$$\Delta p = \left(\frac{-1}{c} \right) \left(\ln \left(1 + \lambda \left(\ln \left(\frac{T}{r_w^2} \right) + 0.2319 \right) \right) + \frac{0.5772\lambda}{1 + \lambda \left(\frac{\ln T}{r_w^2} + 0.2319 \right)} \right) \quad (24)$$

$$\lambda = \frac{q \mu c}{4\pi K h}$$

where h is the formation thickness.

References

- Ahmed T, McKinney P (2011) Advanced reservoir engineering. Gulf Professional Publishing, Oxford
- Braeuning S, Jelmert TA, Vik SA (1998) The effect of the quadratic gradient term on variable-rate well-tests. J Petrol Sci Eng 21(3):203–222
- Chakrabarty C, Ali S, Tortike W (1993) Analytical solutions for radial pressure distribution including the effects of the quadratic-gradient term. Water Resour Res 29(4):1171–1177
- Craft BC, Hawkins MF, Terry RE (1959) Applied petroleum reservoir engineering, vol 199. Prentice-Hall, Englewood Cliffs
- Dake LP (1983) Fundamentals of reservoir engineering. Elsevier, Amsterdam
- Danesh A (1998) PVT and phase behaviour of petroleum reservoir fluids, vol 47. Elsevier, Amsterdam
- Eymard R, Sonier F (1994) Mathematical and numerical properties of control-volume finite-element scheme for reservoir simulation. SPE Reserv Eng 9(04):283–289
- Goode P (1987) Pressure drawdown and buildup analysis of horizontal wells in anisotropic media. SPE Form Eval 2(04):683–697
- Jelmert T, Vik S (1996) Analytic solution to the non-linear diffusion equation for fluids of constant compressibility. J Petrol Sci Eng 14(3):231–233
- Lee WJ (1992) Pressure transient testing: part 9. production engineering methods. In: Development Geology Reference Manual. AAPG Methods in Exploration No. 10, Tulsa, OK, pp 477–481
- Lee WJ, Wattenbarger RA (1996) Gas reservoir engineering, SPE Textbook Series, Vol. 5. Society of Petroleum Engineers, p 349
- Liang Y et al (2001) Nonlinear pressure diffusion in a porous medium: approximate solutions with applications to permeability measurements using transient pulse decay method. J Geophys Res Solid Earth 106(B1):529–535
- Loucks T, Guerrero E (1961) Pressure drop in a composite reservoir. Soc Petrol Eng J 1(03):170–176
- Marshall SL (2009) Nonlinear pressure diffusion in flow of compressible liquids through porous media. Transp Porous

- Media 77(3):431–446
- Odeh A, Babu D (1988) Comparison of solutions of the nonlinear and linearized diffusion equations. *SPE Reserv Eng* 3(04):1202–1206
- Torabi F, Mosavat N, Zarivnyy O (2016) Predicting heavy oil/water relative permeability using modified Corey-based correlations. *Fuel* 163:196–204
- Van Everdingen A (1953) The skin effect and its influence on the productive capacity of a well. *J Petrol Technol* 5(06):171–176
- Van Everdingen A, Hurst W (1949) The application of the Laplace transformation to flow problems in reservoirs. *J Petrol Technol* 1(12):305–324
- Wang Y, Dusseault MB (1991) The effect of quadratic gradient terms on the borehole solution in poroelastic media. *Water Resour Res* 27(12):3215–3223
- Xu J et al (2016) Study on relative permeability characteristics affected by displacement pressure gradient: experimental study and numerical simulation. *Fuel* 163:314–323
- Xu-long C, Deng-ke T, Rui-he W (2004) Exact solutions for nonlinear transient flow model including a quadratic gradient term. *Appl Math Mech* 25(1):102–109