Review of Philosophy and Psychology (2022) 13:127–149
https://doi.org/10.1007/s13164-021-00524-x

Cognitive Structuralism: Explaining the Regularity of the Natural Numbers Progression

Paula Quinon[1]

Accepted: 11 January 2021 / Published online: 31 March 2021

Abstract

According to one of the most powerful paradigms explaining the meaning of the concept of natural number, natural numbers get a large part of their conceptual content from core cognitive abilities. Carey's bootstrapping provides a model of the role of core cognition in the creation of mature mathematical concepts. In this paper, I conduct conceptual analyses of various theories within this paradigm, concluding that the theories based on the ability to subitize (i.e., to assess an *exact* quantity of the elements in a collection without counting them), or on the ability to approximate quantities (i.e., to assess an *approximate* quantity of the elements in a collection without counting them), or both, fail to provide a conceptual basis for bootstrapping the concept of an exact natural number. In particular, I argue that none of the existing theories explains one of the key characteristics of the natural number structure: the equidistances between successive elements of the natural numbers progression. I suggest that this regularity could be based on another innate cognitive ability, namely sensitivity to the regularity of rhythm. In the final section, I propose a new position within the core cognition paradigm, inspired by structuralist positions in philosophy of mathematics.

Keywords Natural number · Core cognition paradigm · Bootstrapping · Structuralism · Rhythm

1 Introduction

In this paper, I investigate which conceptual content of the concept of natural number can be traced back to innate cognitive abilities, as opposed to being issued from social constructs. In particular, I investigate possible origins of equidistances between the successive elements of the natural numbers progression.

✉ Paula Quinon
paula.quinon@pw.edu.pl

1 International Center for Formal Ontology, Faculty of Administration and Social Sciences, Warsaw University of Technology, Warsaw, Poland

Many researchers claim today that there exist cognitive systems that process quantitative information and that generate non-verbal representations used in inferential processes. Those systems are observable already in infancy, and their core character is confirmed by the fact that they are also detected in many non-human animals. Spelke (2000) speaks of "core cognitive systems". I call the "core cognition paradigm" any framework that commits to the view that the conceptual content of the natural number concept is scaffolded on core cognition.

Despite the fact that the role of core cognition in the area of concept acquisition has been since long extensively explored, there is no full agreement which core cognitive systems play a role in acquisition of the number concept. Some researchers believe that the Parallel Individuation System (PIS), which is connected to the ability to subitize, is the primary cognitive system in this development (e.g., Carey 2009; Izard et al. 2008; Sarnecka and Carey 2008; Carey et al. 2017). Others assign the primary role to the Approximate Number System (ANS) that allows for rough estimation and comparison of quantities (e.g., Dehaene 2011; Halberda and Feigenson 2008; Dehaene and Brannon 2011). Yet others hold that both systems need to be in place for the natural number concept to appear (e.g., Spelke 2000; Leibovich et al. 2017; Van Marle et al. 2018).

In this paper, I strongly rely on the research of cognitive scientists, but I want to shift attention of the inquiry into a more analytical direction. Instead of modeling the process of concept acquisition from an ontogenetic perspective, I focus on the conceptual content and the development of this content—especially content that can be traced back to core cognition—in both ontogenesis and phylogeny. My objective is to investigate the semantics[1] of numerical expressions and formulate sound arguments disclosing which conceptual content is most likely inherited from core cognitive representations, and which can be considered as coming from cultural experience. The two perspectives, the concept acquisition and the concept development, are entwined and their conclusions frequently overlap, but they differ in methodological terms. The concept acquisition is studied experimentally in the context of personal development, whereas the concept content and the content development, require analytic tools applied in both the individual growth and the history of idea perspective.

I argue in favor of an inclusive version of the core cognition paradigm, which could be called "the pluralist core cognition paradigm". In this version, I claim that the conceptual content of natural numbers is composed of partial conceptual contents of concepts such as "successor", "one-to-one correspondence", "infinite progression", etc. This is my general attitude, because I do not choose any particular combination of these partial contents or any particular model of cognition. Such choices should be supported by experimental work and serious modeling endeavour that are not mine.

To begin with, I am considering two innate systems that generate representations related to numerosity that are most often taken into account, namely the PIS and the ANS. I take the following content as 'standard' interpretations of the conceptual content correlated with these two systems: the PIS correlates numerical unity with the concept of a discrete individual entity; the ANS accounts for the ascending

[1] Including semantical properties of syntax.

sequence of growing magnitudes. Further, I claim that the conceptual content that can be issued from the two systems does not fully cover the conceptual content of the concept of natural number. I claim that neither the PIS nor the ANS can express the idea that there are equidistances between each couple of elements forming the natural numbers progression. In this paper, I suggest that one of the possible explanation of this regularity could be beat induction, that is the sensitivity to syncopation (the ability to detect when the rhythm becomes irregular).

The pluralist approach that I propose is in line with the theoretical structure of the concept of natural number. This concept is composed of various partial concepts such as the concept of natural number, concepts of individual natural numbers[2] or the concept of the sequence of natural numbers.[3] The advantage of the pluralistic approach is that it also explains various mathematical representations of natural numbers. It explains why some want to speak of "cardinal meaning" of natural numbers and some others of "structuralist meaning". In order to include all those partial meanings simultaneously, I use the expression "numerosity", which I consider highly neutral.

Finally, I conclude by suggesting that, as in the philosophy of mathematics, those positions within the core cognition paradigm that make significant use of the conceptual content responsible for the structural aspect of the natural numbers progression, should be thought of in structuralist terms. Structuralism in philosophy of mathematics is summarised by the famous slogan "mathematics is the science of structure". I suggest to call those positions which both refer in an important way to structural properties and make appeal to core cognitive resources, "cognitive structuralism".

In Section 2 "Core Cognition, Enculturation, and Bootstrapping of the Conceptual Content of Numerical Expressions", I first present core cognitive and cultural systems that are most often studied in the context of natural number concept acquisition and development. Then, I present the theory which explains the mechanisms of emergence of new concepts from partial, cognitive or cultural input, called "bootstrapping". In Section 3 "Analytical Method and Structural Properties of the Natural Numbers Progression", I justify my analytical method as complementary to other methods used in the research on numerical cognition, in particular in the context of studies of the conceptual content development in both ontogenesis and phylogeny. I show how applying the analytical method leads to the conclusion that structural properties of the natural numbers progression are as important for fully accounting for the concept of natural number, as properties of individual numbers. Then, I claim that there exists one aspect of the natural numbers progression, crucial for the concept of natural number, that is not accounted for by any of the well-studied core cognitive systems: natural numbers form a regular recursive ω-progression,[4] where by regular I understand the fact that the distances between the successive elements of this progression are equal. Thus, in Section 4 "Sensitivity to Rhythm", I present the main argument of this paper: while this regularity cannot be accounted for by the Parallel Individuation System nor the Approximate Number System, it can be explained

[2] It corresponds to Fregean and neofregean approach.

[3] As in structuralism.

[4] An ω-progression is a discrete progression of elements that has a starting point, does not form loops and develops in infinity.

by appealing to the sensitivity to rhythm, which is also an ability possessed already by human infants. I finish the paper by postulating, in the Section 5 "Cognitive Structuralism", that a lot of progress in philosophical, and maybe also in cognitive sciences, could be made by considering the idea of "cognitive structuralism", which—as any other structuralist position in philosophy of mathematics—highlights the importance of structural properties of the concept of natural numbers, and—as any standpoint belonging to the core cognition paradigm—traces back at least part of the explication of the natural number concept acquisition and development to core cognition.

2 Core Cognition, Enculturation, and Bootstrapping of the Conceptual Content of Numerical Expressions

In this section, I review cognitive and cultural resources possibly involved in processing quantitative information and I present *bootstrapping*, a mechanism that enables the emergence of new concepts from various resources.

2.1 Systems of Core Cognition and Symbolic Representations

In 1967, Mehler and Bever (1967) modified Piaget's number conservation task, and shown that even very young children pass the test of "where is more" when the arrays are made of candy. They concluded that the choice is dictated by some innate cognitive process not engaging conscious counting. Since then, many researchers (e.g., Spelke 2000, 2003) argued that the ability to process quantitative information in a non-symbolic manner, but still in such a way that representations generated in the process can be involved in inferential processes, stem from core cognitive systems.

According to what I call "core cognition paradigm"—which underpins my perspective in this paper—the core cognitive systems appear early in ontogeny, long before the onset of systems of symbolic representations, e.g., processing quantities can be observed in human infants (Wynn 1992; Izard et al. 2008), and function throughout individual's life span (Spelke 2000). Moreover, the similarity between data on humans and non-human animals suggest that these systems have a long and shared evolutionary history (Brannon and Merritt 2011). In the paradigm, non-symbolic representations generated by core cognitive systems serve as building blocks, or support, for new conceptual content to arise. Two core cognitive systems are studied in the context of quantities processing (Carey 2009; Dehaene 2011). These systems have different functions and they are located in different regions of the brain (Feigenson and et al. 2004).

The first of those systems is called the Parallel Individuation System (PIS). This core system enables subitizing (Carey 2009),[5] i.e., determining the quantity of objects

[5]The early history of research on subitizing is presented in Mandler and Shebo (1982). In that paper, for instance, the authors show that the explanation for subitizing could also be not numerical in nature (not based on recognizing a set of individual items), but based on shape-recognition

(up to three or four) in one's field of vision without counting. Thus, a widely accepted theory explaining subitizing states that discrete quantities are recognized by forming distinct mental representations for each of the apprehended objects. For instance, three boxes might be represented in an implicit manner by "box", "box", "box", instead of there being a representation for "three" (Carey 2009; Sarnecka and Carey 2008). However, rather than being a general system for discrete quantities, subitizing only works for small quantities due to constraints such as the available space in working memory (Beck 2017).

The second system is called the Approximate Number System (ANS). The ANS allows forming approximate representations of cardinal quantities without counting, and thus is used for estimating the number of observed objects, or for apprehending the difference in quantity between collections of objects (Dehaene 2011). Unlike the PIS, which processes quantities only implicitly and is used for other cognitive tasks, the ANS is numerosity-specific. Unlike subitizing, the ANS is not limited to small quantities, its drawback is that it is intrinsically inexact (Carey 2009).

Since neither the PIS (non-quantity specific, forming only implicit representations), nor the ANS (increasingly inaccurate as the quantities become bigger) adequately represent the natural numbers progression, which is formed from discrete entities and is potentially infinite,[6] it is often claimed that in addition to the two core cognitive systems, the natural numbers progression gets its conceptual content from yet another system: the Exact Number System (ENS). The ENS processes available quantitative information and generates meaningful external symbolic representations forming a specific symbol progression (e.g., Izard et al. 2008; Castronovo and Göbel 2012; Sarnecka 2015; a similar idea can be found in Butterworth 1999). It is not clear if the ENS shall count as core cognition (Butterworth goes in this direction) or is acquired through cultural and social interactions (e.g., Carey 2009; Sarnecka 2015); this idea is also very tightly related to the idea of enculturation of Menary (2015).

Independently of the character of the ENS (core cognitive or result of enculturation), the system of numerals (i.e., symbolic representations of natural numbers) is crucial for the concept of natural number (its conceptual structure and its acquisition). As claimed by Everett (2019), numerals are the invention that enables structuring the way humans acquire and apply quantitative information about the world. Without them humans can have a rough or limited sense of quantity, but it is the appearance of numerals that makes it possible to conceptualize quantities in an exact and discrete manner for the entire natural numbers progression. The importance of the system of symbolic representations in the number concept acquisition is highlighted by other researchers (e.g., Dehaene 2011; Hauser et al. 2002; Sarnecka 2015).

According to the mainstream research, children learn the first numerals by heart in a purely syntactic way, without any strong connection to any cardinal or ordinal meaning. Those meanings, and many other, are only further assigned to them (Fuson

[6] In everyday life, we obviously use just an initial segment of this progression. There exist philosophical discussions about the relation between the concept of natural number, and the concept used in finite arithmetical structures, or in sequences of finite models (Potter 2002). In this paper, I will not elaborate on these discussions.

et al. 1982; Fuson 1988; Wynn 1990; Griffin and Case 1996). At some point, as highlighted by Gelman and Gallistel (1978), the natural numbers progression becomes stabilised ("stable order principle") and the numerals are always recited in the same order. It has been established by Fuson (1988) that the progression of syntactic numerals converges towards a specific structure. Sarnecka and Carey (2008) refer to it as a "placeholder structure" for numerals.

2.2 Conceptual Content and Bootstrapping

Many questions about numerical cognition relate to children's ability to understand symbolic representations on the basis of conceptual content issued from cognitive and cultural resources. The question is, which resources exactly and in which order provide meaning. As presented above, in the core cognition paradigm two cognitive systems are considered the most frequently. In this paper, I propose to extend the usual picture dominated with studies of the conceptual content that can be assigned to individual natural numbers, with investigations into the conceptual content contained by symbolic structure of natural numbers itself. I claim that the semantical features of the symbolic structure are as important for the meaning of the number concept, as are cardinal meanings of individual numerals, and as such should be accounted for. This fact is usually not sufficiently highlighted by researchers in numerical cognition.

At first, children do not know how to match the placeholder list of numerals neither with the cardinalities of collections nor with ordinals indicating places of numerals in the progression.

The supporters of the PIS claim that the process of the number concept acquisition starts with children learning the meanings of the number words "one", "two", "three" and "four". Children initially do so one numeral at a time, slowly understanding which conceptual content is quantity-specific (Sarnecka 2015). Children appear to connect these words to the existing PIS-based mental models of one, two, three and four objects (Carey 2009). Thus, these number words become children's first symbols for exact cardinalities, and it is plausible that the PIS provides at least a partial reference for them. The process of meaning formation and concept acquisition possibly continues with the learning agent understanding that the last number on the counting list refers to the cardinality of the collection that is being enumerated, that is with her grasping the cardinality principle (Carey 2009; Sarnecka 2015; *see also* Davidson et al. 2012; Gärdenfors and Quinon 2020). Since the PIS-based first four quantity representations are separated by the addition of one individual, children become not only the cardinality principle knowers, but also might get a grasp of the successor (Sarnecka and Carey 2008).[7]

The main alternative account in the literature states that number concepts are primarily based on the Approximate Number System and that the conceptual content

[7] It is also possible that early learning (up to number 4) is not supported by the PIS (Wagner and Johnson 2011). In their recent work, Carey and Barner suggest that ANS might not be a part of the number concept at all (Carey and Barner 2019).

of exact numerals emerges from, or supervenes on, the experience of approximating quantities with the ANS. The process of approximation is a subject of two effects. The *magnitude effect* makes the assessment easier for smaller magnitudes (it is easier to compare ten with twenty elements than 200 with 210). According to the *distance effect* the larger the ratio, the easier to assess the difference (it is easier to compare ten with twenty elements than ten with eleven). According to Dehaene, through the constant experience with local estimations and comparisons there emerges a sense of progression of quantities or magnitudes (Halberda and Feigenson 2008), what he calls the "number sense" (Dehaene 2011).[8]

Other authors claim that both systems play a role in the process of the number concept formation and acquisition. Spelke (2011) has proposed a theory in which the sense of progression of natural numbers can be drawn from the ANS whereas the conception of discrete quantities forming a progression comes from the PIS. The way these two functionally distinct core systems are combined, according to Spelke, is by acquiring the verbal counting sequence. When a child learns number words, she associates them with both PIS- and ANS-based quantity representations. Van Marle et al. (2018) proposed an enhancement of the two-system account of Spelke. In their "merge model" the PIS performance only predicts cardinal knowledge during its early stage (i.e., the beginning of preschool), whereas ANS performance continues to predict cardinal knowledge also in later stage. Thus the merge model of van Marle et al. differs from Spelke's model in how enduring the influence of the two core systems is in the development of numerical knowledge.[9]

As I see it, from my analytical perspective, the main difference between PIS-generated and ANS-generated representations on the one side, and ENS-generated representations on the other, is that the latter can potentially get conceptual content from both, its own syntactic structure and internal cognitive resources (and also the external cultural ones). In consequence, the PIS and the ANS get a double representational roles: both systems continue to generate non-symbolic representations in response to quantitative information throughout our lives, and both contribute to generate the conceptual content of symbolic representations issued from the ENS.

The process of development and acquisition of the number concept has a dynamic character, best described by the bootstrapping metaphor. The theory of bootstrapping provides an informal model of how this process goes (e.g., Sarnecka and Carey 2008; Carey 2009; Spelke 2011). The idea of bootstrapping originally came from Quine (1960), who describes the conceptual structure of the language as a holistic interconnected web containing core cognitive concepts alongside with historical and cultural constructions. A conceptual change, or a shift in meaning, of one of the concepts within the web induces conceptual changes in the neighboring concepts and in this

[8]The claim that ANS predicts performance with symbolic numbers is not always supported (e.g., Holloway and Ansari 2009; Sasanguie et al. 2014).

[9]Noël and Rouselle (2011) argue that research on developmental dyscalculia suggests that ANS if it is a part of the process at all, it gets involved in the concept acquisition late in ontogenesis.

way sometimes triggers a supervenience of a new concept. Bootstrapping explains changes in concepts in the phylogeny due to cultural interactions, but it also applies to the processing and learning of concepts in the ontogeny by individuals.

While "Quinean bootstrapping" is a term generally used in discussions on concept formation and acquisition, in the context of numerical cognition, "bootstrapping" refers most often specifically to Carey's bootstrapping theory based on the Parallel Individuation System and the cardinality principle grasping (Carey 2009). However, in her book, Carey also discusses another path for bootstrapping number concepts based on core cognitive abilities, namely on the basis of the Approximate Number System. Unlike in the case of Carey's version, the proponents of the ANS version do not use the term "bootstrapping". Carey also seems to be open to possibilities of other concepts configurations that could lead to bootstrapping the concept of natural number.

To sum up, according to all the hypotheses above, from various initially unrelated conceptual inputs—either innate or learned—the child builds the idea that numerals form a progression of individual discrete entities representing increasingly growing magnitudes. However, and this is what I argue for in this paper, neither Carey's bootstrapping account, nor any of the competing theories secure sufficient conceptual resources to explain the equidistances between subsequent elements of the natural numbers progression. This paper's central argument is that equidistances have to be accounted for, and that one of the possibilities is rhythm induction.

3 Analytical Method and Structural Properties of the Natural Numbers Progression

In this section, I justify my analytical method as complementary to other methods used in the research on numerical cognition, and I show how applying this method leads to the observation that structural properties of the natural numbers progression are as important as properties of individual numbers.

The analytical perspective used in this paper, is in line with those investigations that ask what conceptual content young children attribute to numerals and numerical expressions,[10] and at which number-knower level are they able to distinguish one meaning from another (Sarnecka and Gelman 2004; Slusser et al. 2013; Slusser and Sarnecka 2011). The analytical method, in addition to considering number concept *acquisition* in ontogenesis, takes into account number concept *development* in both ontogenesis and philogeny.[11] Therefore, it begins with a careful study of the meanings of numerical expressions at different stages of development of the concept's phylogeny, with the objective of detecting which partial conceptual content is necessary to account for those meanings. In parallel, a philosopher needs to evaluate

[10] Numerals, quantifiers, etc.

[11] In the spirit of the "open texture"– that is, the idea that concepts evolve from partial meanings and eventually reach maturity (Makovec and Shapiro 2019)—I believe that concepts develop over time.

which conceptual content can be conveyed by the cognitive (or cultural) resources.[12] For instance, can the PIS convey the partial meaning of the natural number concept, such that the natural numbers progression starts with a single element, or that the progression continues to the infinity? The analytical approach does aim at proposing any full fledged model of number concept acquisition, or the number concept development, instead it aims at critically assess conceptual content that could come out of the cognitive representations.

In philosophy of arithmetics, which studies properties of the concept of natural number, two competing positions play a distinguished role, and often come back-to-back as points of reference: *Neofregeanism*, which aims to define what is the cardinal meaning of individual natural numbers (MacBride 2003); and *Structuralism*, which focuses on properties of natural numbers progression (Shapiro 1991).[13] Similarly, within the core cognitive paradigm, there exist studies of conceptual content of individual natural numbers (e.g., what is the approximated or exact quantity of elements corresponding to the given natural number), and of the "progression" that those individual natural numbers form (e.g., whether it is linear or logarithmic). Thus, in both, the perspective of concepts in mathematics at various stages of conceptual development and the perspective of the conceptual structure of everyday concepts in the core cognitive paradigm, the two aspects of numerosity are taken into account: i.e., the meaning of the individual elements forming the progression and the meaning inherited from the properties of the progression. However, the latter is much less explored.

As I already explained earlier, my objective is to foster the idea—not sufficiently explored in the research on numerical cognition—that in addition to studying meanings of individual numerical expressions, it is crucial—for the overall natural number concept understanding—to consider that numerals form a specific structure and that structural properties influence the meaning of the number concept. Focusing on the structural properties of the natural numbers progression is also justified from other than theoretical reasons. Contemporary numeral systems, such as the one used in English, are the product of a long line of cultural evolution and have some important characteristics that are not general to all numeral systems (for instance, they do

[12] I do not want to delve too much into methodological investigations, but I consider it important to mention that the analytical method which is mine, could be called "reverse conceptual engineering". This method is a back-and-forth approach consisting in both: (i) studying the contexts of use of the concept of natural number, and - after analysing the conceptual structure in those contexts - detecting where the partial meanings originate; (ii) studying which conceptual content of symbols can be traced back to the representations generated by core cognitive systems. "Conceptual engineering" is a method used to conform a given intuitive term to serve in a specific scientific, philosophical, or social context (Cappelen 2018); Tanswell writes specifically about the use of conceptual engineering in mathematics (Tanswell 2017). Just as "conceptual engineering" builds upon engineering design where the process of the final production is preceded by a successful proof of concept, "reverse conceptual engineering" builds upon reverse engineering where the object is deconstructed to reveal its designs and architecture.

[13] The idea of relying directly on analytical work of philosophers can be found in Rips et al. (2008) for structuralism and the Peano-Dedekind's axioms, and in Buijsman (2019) for axioms of Frege's arithmetic.

not characterize numeral systems based on body parts). Before it could develop in the way we know it today, there has been a great multitude of ways in which human beings have used expressions and notations for quantities, beginning from various types of tallying (e.g., Butterworth 1999; Dehaene 2011; Everett 2019).

Putting the acquisition of natural number concepts in its proper cultural context, it becomes obvious that the role of numerals and of the sequence of numerals, is crucial for number concept development and acquisition. In particular, because symbolic representations serve as placeholders that can and are filled with conceptual content and because the sequence of numerals is by itself an important carrier of meaning. In our application of the Hindu-Arabic number system, the regularity of the natural numbers progression (i.e., the equality of distances between subsequent elements of the progression) is fundamentally present in the method of constructing numerals. In the English language, the first twelve numerals do not show a regular structure: they could just as well be twelve random words. After that, however, the numerals start showing regularity. For example, the sequence "twenty-five, twenty-six, twenty-seven" differs from the sequence "eighty-five, eighty-six, eighty-seven" only in the "tens" part of the numeral, thus suggesting that the steps to the next numeral are similar in both sequences.

Actually, as depicted by Borges in "Funes the Memorious", only those progressions of elements whose regular structure is encoded in their symbolic representations, are suitable for representing natural numbers as we know them today. Borges story goes as follows, Funes the Memorious, a fictional character from one of Borges' novels (Borges 2000, pages 91–99), happened to have an infinite working memory due to a severe brain injury. At the same time, he was critically lacking any ability to systematize his knowledge. Beyond other extraordinary aspects of Funes' condition that Borges describes, Funes invented his own numerical system:

> He told me that in 1886 he had invented a numbering system original with himself, and that within a very few days he had passed the twenty-four thousand mark. (...) Instead of seven thousand thirteen (7013), he would say, for instance, "Máximo Pérez"; instead of seven thousand fourteen (7014), "the railroad"; other numbers were "Luis Melián Lafinur," "Olimar," "sulfur," "clubs," "the whale," "gas," "a stewpot," "Napoleon," "Augustín de Vedia." Instead of five hundred (500), he said "nine." Every word had a particular figure attached to it, a sort of marker; the later ones were extremely complicated...

For the sake of our thought experiment, imagine a child who learns numbers only from Funes. According to the model we work in, as an infant, this child will learn an arbitrary sequence of Funes numerals the way other infants do with standard numerals. It might go something like: "Luis Melián Lafinur", "Olimar", "sulfur", "clubs", "the whale", "gas"... Through the PIS and the ability to subitize, she will learn to associate the meaning of "Luis Melián Lafinur" (that might be Funes' one) to collections of one element, "Olimar" Funes' two) to collections of two elements, "sulphur" (Funes' three) to collections of three elements and also "the reins" (Funes' four) to collections of four elements. Thanks to the training in associating number names with representations given by the Parallel Individuation System and/or the Approximate Number System, as well as the cultural context and conceptual interconnections, she will eventually understand that numerals refer to finite cardinalities and/or that they form a natural numbers progression. Thanks to the growing magnitudes of cardinalities, perhaps due to a sense of "more" arising from the functioning of the ANS, she will apprehend the idea that the quantities form a progression of growing magnitudes. Finally, thanks to the cardinality principle, she will eventually learn that adding an element to the set will allow her to progress to the next name on her number line. Funes' child will learn all that from the Funes' number line. But—I argue—this does not guarantee that the Funes' child conceptual content of representations coincide with the expected conceptual content of representations generated by the Exact Number System.

This is precisely what Borges himself objects to in Funes' numerals:

> I tried to explain to Funes that his rhapsody of unconnected words was exactly the opposite of a number system. I told him that when one said "365" one said "three hundreds, six tens, and five ones," a breakdown impossible with the "numbers" [...] Timoteo or a ponchoful of meat. Funes either could not or would not understand me.

One of the commonly accepted characteristics of natural numbers, as used on a daily basis, is that each successive element of the natural numbers progression is generated in a predictable and regular manner and that the distances between these elements are equal. While number systems in languages such as English are generated on a regular basis, Funes' system is not.

There are two ways in which the equidistances between subsequent elements of the natural numbers progression are explained in the research on numerical cognition. From the one hand side, Butterworth (1999) claims that humans have an innate cognitive module providing the concept of unity and the concept of a regular successor, from the other, such researchers as Carey (2009), Sarnecka (2015), and Everett (2019) or Menary (2015) claim that the regularity of the exact natural numbers progression comes from cultural impact.

When it comes to the first way, whether or not regularity of progression is or is not an innate or somehow basic property is still a matter of debate. Pica et al. (2004) report that members of the Munduruku tribe living in the Amazon, tend to place quantities on a number logarithmic progression, presumably due to having exclusively ANS-based numerosity representations (Dehaene et al. 2008). Similarly, reports about people experiencing synesthesia show preferences to a logarithmic representation (Galton 1881). However, this natural tendency seems to be replaced in the learning process by a progression of natural numbers with regular intervals. Moeller et al. (2009) found that even though young children tend to perceive the number line in a logarithmic manner early in life, they gradually switch to a linear one as they grow. Furthermore, Sarnecka and Lee (2009) argue that linear progression can be equally natural when exercised in favorable circumstances. Furthermore, Sarnecka and Lee (2009) argue that linear progression can be equally natural when exercised in favorable circumstances. Their experiment suggests that children spontaneously generate a linear and regular progression of quantities when they are not spatially constrained, i.e., when instead of a segment, as in the experiments by Dehaene et al. (2008) and Moeller et al. (2009), they are given a potentially infinite line. On the basis of those results, it is not possible to draw the conclusion regarding an innate or cultural provenience of the regularity.

Independently of its provenance, the starting point of this study is—similarly to e.g., Fuson (1988); Butterworth (1999); Everett (2019)—that the numeral system humans learn is crucial for grasping natural number concepts. Additionally, I claim that the specific sequence of numerals that children are taught, influences the conceptual content of natural numbers as we know them today. Numerals from Funes' number line, for example, do not embrace the idea of equal distances between the elements. The Funes numerals are separate entities that show no internal syntactical structure. In this way, for the Funes' number line, the ANS and the PIS could indeed be sufficient in order to bootstrap the structure of natural numbers. The representations of the Exact Number System, however, show syntactical structure which regularity accounts for the idea that the distance, say, from twenty-four to twenty-five is the same as that from four to five.

In this paper, I explore the hypothesis that this regularity of the natural numbers progression represented in the symbolic progression has cognitive basis, and comes from a cognitive faculty different from the PIS and/or the ANS. In the next section, I suggest that the disposition toward regularity could be connected to another well-established phenomenon in which humans experience regularity and equidistances between the subsequent elements of a discrete progression: sensitivity to rhythm.

4 Sensitivity to Rhythm

In this section, I investigate the possibility of tracing back the conceptual content corresponding to equidistances between subsequent elements of the natural numbers progression used in everyday life, to innate cognitive resources.

4.1 Regularity of the Natural Numbers Progression

As I argued earlier, the concept of natural number is closely related to the concept of natural number progression, both in philosophy of mathematics (various structuralist positions) and in numerical cognition (core cognition paradigm). However, what exactly is meant by progression can importantly differ depending of the context.

The natural numbers that are transferred as a cultural inheritance—that are used daily for assessing the number of elements of finite sets and for enumerating, or ordering, elements of a discrete set—form a discrete endless loop-free endless loop-free progression that starts with a single point, and is formed from entities separated by *equidistant* spaces. The ENS-generated symbolic representations that are usually identified with natural numbers, are—in the Western culture—symbolically represented by the sequence "1, 2, 3, ...".

In formal arithmetic, the natural numbers progression is identified with an ω-progression. An ω-progression is a discrete progression, which has an initial element and then progresses into an infinity without loops, however, interestingly, the *distances* between its elements *do not have to be equal*. The Fibonacci sequence, a logarithmic progression or a non-computable ω-sequence forming a weird permutation, are still accepted models of arithmetic.[14]

In the search for properties of the everyday mature concept of natural numbers that—I believe—should be bootstrapped from the partial cognitive input, one should refer to the middle ground between uninformed and expert knowledge. The requirement that the distances between successive elements of the progression are equal seems to belong to this middle ground.

According to the mainstream version of the core cognitive paradigm the Parallel Individuation System and the Approximate Number System (one or both of them) provide an important part of the cognitive basis for the bootstrapping process. I claim in this paper that they are not enough to account for the idea of the regular and equal intervals between elements of the natural number structure. The PIS has a limited scope with numerosity represented only implicitly. Carey (2009), indeed, explains regular intervals between the subsequent elements of the number structure by generalisation upon the result of the cognitive access to small discrete quantities. The weak point of this explanation is that one can subitize objects that are not identical, and it can be objected—for instance—that the "distance" between a red toy-car and a blue-dress doll is different from the "distance" between this toy-car and a tennis ball. Also, subitizing as such represents numerosity only in an implicit manner and it does not imply the order of the successive subitized quantities.

The ANS can be seen as providing the general idea that quantities form an indefinitely growing progression of magnitudes. However, the ANS does not conform to the idea that the intervals between successive elements of the structure are equal or regular. What we are looking for in the concept of natural number is that in any segment of the structure the distances between its elements are equal, e.g., the progression 4-5-6-7 has intervals of the same length as, say, the progression 4555-4556-4557-4558. This is not the case with representations of approximate quantities because the ability to distinguish between the elements of the progression of approximate numbers weakens as they grow. In addition, the borders between subsequent representations become blurred, so to speak, which does not support the expected discreteness of the natural numbers progression (Pantsar and Quinon 2018).

Another way is to suggest that the regularity comes from the language. Sarnecka (2015) points at counting routine as a necessary aspect of the natural number concept learning process. Explanations based on language resources get additional support in the fact that some notation systems display an obvious regularity. The notation that the most explicitly endorses regularity is tallying. In addition to using body parts for

[14] Recursivity of ω-progression has been studied in the context of models of weak-arithmetic in the school of Andrzej Mostowski. Computational structuralism is a position in philosophy of mathematics according to which the concept of natural number is best characterised as a model of the first-order Peano Arithmetic, where the arithmetical functions, including the successor function, are recursive (e.g., Halbach and Horsten 2005; Quinon and Zdanowski 2007; Button and Smith 2012; Dean 2014; Quinon 2014; see also Benacerraf 1965, 1996). So far, there is no philosophical study of the regularity of the natural numbers progression used in everyday life.

counting, the earliest forms of keeping track of quantities are various types of tallying, in which adding one stroke corresponds to the successor principle (Dehaene 2011). Importantly for the matter at hand, tallying systems are usually also regular as all strokes are *a priori* the same. The stroke notation continues to be used in parallel to the linguistic recursive number system, and it is plausible that tallying was developed to have a way of communicating the idea of the regular progression of natural numbers.[15] It might be the case that the regularity of tallying can be today seen in the regularity of modern systems of notation, however it is not clear if tallying is the result of cultural development (Everett 2019), or it is rather innate and belonging to the structure of human cognition (Butterworth 1999; Hauser et al. 2002).

All the above proposals on where the regularity of progression comes from are worth further research. Perhaps some of them could be refined enough to provide a valuable explanation. In this paper, however, I would like to suggest another possibility, the idea that regularity could be based on sensitivity to rhythm.

4.2 Rhythm Induction

If in the bootstrapping process we could identify the module responsible for regularity, the bootstrapped concept of natural number would generate a regular ω-progression. What if this ability was the sensitivity to rhythm that empirical studies have found to be well established in young children?

The sensitivity to rhythm is thought to be related to the process of beat induction, i.e., the process of inferring an isochronous pulse from a sequence of sounds (Patel 2008). As of yet, the sensitivity to rhythm has not been applied in explanations of the cognitive basis of natural numbers. This is surprising, because there is an obvious connection between sensitivity to rhythm and a general sense of regularity (e.g., repetitive moves like a steady walk), which has promising explanatory power in accounting for the regularity of the natural numbers progression. Since the regularity of the natural numbers progression remains unexplained, the tendency to expect regularity in rhythm deserves our attention.

Already at two days old, humans manifest sensitivity to regular rhythm and syncopation (e.g., Zenter and Eerola 2010; Winkler 2009; Honing 2012). This sensitivity appears to play a key role in the development of linguistic ability. The study of Dellatolas et al. (2009), for instance, shows that an ability for rhythm reproduction already in kindergarten age is predictive of later reading performance. Przybylski et al. (2013) also show a connection between recognizing rhythmic patterns and grasping the structure of language.

The rhythmic patterns that children are sensitive to in the experiments are regular, and as such they can be characterized in a recursive manner.[16] Just as the

[15] A close proximity between the natural numbers progression and the system of tallies is confirmed by a formal result. Quine (1946) shows that $PA1$ is equivalent to the arithmetic of concatenation.

[16] As I explained above, not every successor function is recursive, and not every recursive function is regular (i.e., grants equidistances between successive elements). However, what is important for this argument, every regular function is recursive.

successor function of natural numbers is the most often in everyday life characterized recursively as "plus one" (or equivalent), rhythmic patterns can be characterized recursively as repeating the previous patterns. Aside from music and mathematics, such repeatable, recursive processes are much discussed in the literature on linguistic development. According to Chomsky (2002), the deep structure of human languages is determined by the recursivity of the syntax. Indeed, Hauser et al. (2002) have famously argued that recursion, and the lack of upper bounds in the length of syntactical constructions, could plausibly be the only uniquely human part of our languages. If sensitivity to rhythm was a module that plays the role in bootstrapping the concept of natural number, that could also mean that it is a cognitive background for recursivity.

There is little doubt about the existence of the sensitivity to rhythm already in very young children. It is also widely accepted that languages are structured in a recursive and often regular manner. The correlation between the two also seems highly plausible. Languages have characteristic rhythms and there are numerous ways in which the natural connection between language and rhythm is established, from the rhythmic chanting of many tribes to the common use of songs in teaching language to children. Since the connection between rhythm and language is so ubiquitous, our next question is how rhythm and the learning of number concepts could be connected.

4.3 Conceptual Content Conveyed by Rhythm Induction

While the connections between rhythm, regularity and the progression of numerals have not been explicitly studied, they are related to a well-known discussion about the recursivity of the natural number structure. Famously, Hauser et al. (2002) explained the recursivity as being a byproduct of the recursive structure of the language. Indeed, the way we can add more words or syllables to phrases in general seems to closely correlate with creating and learning new numerals. Numeral systems in many languages are syntactically built to follow the idea that numerals form a progression that can be continued indefinitely.[17]

The relation between the recursive structure of language and the recursivity of numerals is much more difficult. It is not possible to know, for example, whether numerals developed their recursive character because the languages are recursively structured, or whether this development happened the other way around: one can hypothesize that the pre-verbal conception of natural numbers included the idea of regular progression, and numerals were developed to express this idea. It could also be the case that the sensitivity to rhythm and regularity influenced both the linguistic and numerical abilities, but the two processes happened (at least partially) independently. As important as these questions are for my present purposes, the possible answers make little difference. I only want to suggest that the innate sensitivity to rhythm plays a role in bootstrapping the mature concept of natural numbers, which fulfills the general intuition that subsequent elements of the number line do not only

[17] As such, it facilitates teaching of this recursivity of numbers to new generations, thus being a key feature in what Menary (2015) calls "enculturated" arithmetical cognition.

form a progression of discrete elements, but also that those elements are equally spaced.

Thus, I claim that the sensitivity to rhythm has the potential to be the missing piece in the bootstrapping process that generates a regular ω-progression. The grasping of the concept of natural number could then look as follows[18]

1. the PIS gives us the concept of a individual entities;
2. the ANS gives the idea that numerals form a progression;
3. finally, the sensitivity to rhythm ensures that the progression is regular.

The idea that the sensitivity to rhythm plays a role, would explain many of the problems the current accounts have in explaining bootstrapping. Firstly, it would explain why, while bootstrapping the concept of natural number, we bootstrap the linear system instead of a logarithmic one: we can have a pre-linguistic conception of regularity that corresponds with the regularity of the symbolic representations of natural numbers. Secondly, it could help explain why the grammatical structure of language and the natural numbers progression both have a recursive structure. Both could be determined, at least partially, by the sensitivity to rhythm and regular progressions. Thirdly, it could help explain the occurrences of regularity when it comes to symbolic representations of quantities that do not belong to natural languages, such as stroke or binary notations.

In the above argument, I do not want to claim that sensitivity to rhythm and regularity is a core cognitive ability comparable to the Parallel Individuation System or the Approximate Number System. Indeed, while there is already abundant data showing how prevalent the sensitivity to rhythm is, there does not seem to be any consensus on what innate cognitive ability is responsible for it and how it relates to our cognitive capacities. I am also aware of Piaget's (1952) proposal that sensitivity to regularities and symmetries is obtained through the process of empirical experience. Furthermore, I do not want to claim that incorporating sensitivity to rhythm into the research paradigm will immediately solve all the problems in numerical cognition.

Instead of making such claims, in the spirit of narrowing the gap between philosophers and empirical researchers of mathematical cognition, the purpose of this paper is to propose a hypothesis that opens up new possibilities for future studies, both empirical and philosophical. However, even with the sizable gaps in our current knowledge of the topic, the plausibility of the hypothesis is firmly grounded in empirical studies. In addition to drawing from the empirical data on numerical cognition, the data on sensitivity to rhythm clearly points to its importance for cognitive processes.

While lacking in direct empirical evidence on numerical cognition, the hypothesis about sensitivity to rhythm receives support from a neighboring field: language cognition (e.g., Dellatolas et al. 2009; Przybylski et al. 2013). Given the already widely

[18] I do not want to take any radical stand regarding the choice of the natural number acquisition process, I just point out one plausible possibility.

accepted connection between numerical cognition and linguistic cognition, one can expect the sensitivity to rhythm to receive more attention in the future, also when it comes to the foundations of arithmetic. It will obviously take a great deal of empirical work before the connection between rhythm and bootstrapping of the exact natural number concept can be established and explained. But I hope to have shown that on a theoretical level that connection carries a lot of potential for future research.

5 Cognitive Structuralism

In this section, I develop a position which I call "cognitive structuralism" and in which (1) the importance of structural properties of the natural numbers progression is strongly highlighted (as in a structuralist position); (2) the core cognitive paradigm provides basis for tracing back structural properties of natural numbers progression.

5.1 Structuralism

There is a strong correlation between my current proposal and the positions studied in philosophy of mathematics. In the most general lines, structuralism in philosophy of mathematics is best described by the slogan "mathematics is the science of structure". Shapiro (1997) in his book devoted to structuralism explains in the most general lines what follows:

> The subject matter of arithmetic is the natural-number structure, the pattern common to any system of objects that has a distinguished initial object and a successor relation that satisfies the induction principle. Roughly speaking, the essence of a natural number is the relations it has with other natural numbers. There is no more to being the natural number 2 than being the successor of the successor of 0, the predecessor of 3, the first prime, and so on. The natural-number structure is exemplified by the von Neumann finite ordinals, the Zermelo numerals, the arabic numerals, a sequence of distinct moments of time, and so forth. The structure is common to all of the reductions of arithmetic. Similarly, Euclidean geometry is about Euclidean-space structure, topology about topological structures, and so on. As articulated here, structuralism is a variety of realism.
>
> A natural number, then, is a place in the natural-number structure. (Shapiro 1997, pages 5–6)

As such, structuralism is based on Dedekindian conception of natural number.

> If in the consideration of a simply infinite system [...] set in order by a transformation [...] we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation [...], then are these elements called natural numbers or ordinal numbers or simply numbers. (Dedekind 1888/2012, Section 73)

5.2 Core Cognition Paradigm and Structural Concepts

As explained above, there is no agreement as to which concepts and in which order provide the basis for the bootstrapping of the mature concept of the exact natural number. According to some versions of the core cognitive paradigm the first conceptual content that is correlated with what will later become a natural number concept—which is in our Western society often symbolised by an Arabic numeral—refers to the cardinality of a small collection; according to some other versions, the first conceptual content is correlated to structural features.

Therefore, a core cognition paradigm where the mature number concept is bootstrapped from the ANS with the highlight on "number sense", amounts to a cognitive structuralism, whereas the position based on the PIS and cardinality principle does not. Similarly, the version of the core cognitive paradigm which requires that the successor of the progression be such a recursive function that always add an identical element, thus guaranteeing the regularity of this progression, amounts to a cognitive structuralism. This is because its main focus is on structural properties of the progression and not on the individual elements.[19]

5.3 Philosophy of Mathematics and Core Cognitive Paradigm

Traditionally, structuralism is opposed to Fregean account, which is focused on the individual natural numbers, and on its cardinality meanings. According to the neofregean account, natural numbres are best defined by the so called Hume's Principle. Hume's Principle establishes a co-relation between a possibility to put the number falling under some concept (like the number "four" falls under the concepts "horse's legs", "car's wheels" etc., the number "five" falls under "my right hand fingers" and under "my left hand fingers", etc.) and the equality between two numbers.

> The number of F's is equal to the number of G's if and only if there is a one-to-one correspondence between the F's and the G's.

Structuralism is related to Peano's Arithmetic. Most classical structuralisms refer to the Second-Order Peano's Arithmetic ($PA2$), however, there also exist positions that refer rather to the First-Order Peano's Arithmetic ($PA1$) such as "computational structuralism".[20]

Many debates in philosophy of mathematics relate the superiority of Fregean vs. structuralist account. The so-called Frege-Hilbert controversy most possibly preceded all further development, motivating researchers in philosophy of mathematics to take a stand (Shapiro 2005; Blanchette 2018; Reck and Schiemer 2020). Already

[19] In philosophy, various positions are called "structuralism". According to *ante rem* structuralism of Shapiro (1997) is a realist position where structures exist independently of human apprehension; the *computational structuralism* rely on the idea that arithmetical functions defined on the elements of the structure are computable (e.g., Halbach and Horsten 2005; Quinon and Zdanowski 2007).

[20] The discussions as to the reasons of choice between the second-order and the first-order arithmetic, are very complex. One of the arguments states that $PA1$ has a more "dynamic","practical" or "functional" character, rather than a "descriptive" one as $PA2$.

Carnap famously argued that Fregean's account is an explication of the concept of natural number, whereas $PA2$ is not.[21] He based his claim on the idea that Frege's Arithmetic (*FA*) relates to the intuitive concept of natural number used on everyday bases, where numbers are used to assess quantities (Carnap 1950).

Similarly, within the core cognitive paradigm there are two ways of relating the learning of natural numbers with systems of axioms. Buijsman (2019) proposes such an interpretation of the results in numerical cognition in which children learn the first natural numbers on the basis of *FA*, in lines with neofregean accounts. He also claims, still in line with neofregeanism, that the knowledge of the structure of natural number is acqiured on the basis of the Peano-Dedekind Arithmetic, which can be deduced from the Frege Arithmetic.[22] On the other hand, Rips et al. (2008) suggest that a coherent way of arriving at the concept of natural number is through acquiring and putting to work Peano-Dedekind's axioms.

6 Conclusions and Future Research

In this paper, I have argued that various conceptual aspects generated by core cognitive systems can work in combination and I analysed how they can amount to the conceptual structure of the (exact) natural number concept. In this way, I defend a pluralist version of the core cognitive paradigm. Rather than insisting that the paradigm should focus on one single core cognitive system, I believe it should be open to conceptual contribution from multiple systems. Moreover, I do not presuppose a model in which the acquisition of conceptual content needs to be temporally or functionally ordered in any specific manner. I have therefore not presented any feature of natural number concepts as primary, in the sense that this feature is required as a foundation for the other conceptual content.

I have proposed a way to identify the conceptual content using the example of three systems that influence numerical cognition. The Parallel Individuation System and the Approximate Number System are both widely studied core cognitive systems and their involvement in the natural number concept acquisition is recognized. In addition to the PIS and the ANS, I suggest to consider sensitivity to rhythm as a cognitively influential capacity which has never been studied in this particular context. This hypothesis is suggested as a general proposition to take sensitivity to rhythm into consideration in future research both theoretical as experimental.

Consequently, the account of the number concept development I have presented in this paper retains a high level of flexibility. The natural number concept acquisition can be a complex and nonlinear process and it can be realised in different ways in different individuals. This pertains also to the influence of the cognitive core systems. It could be that, based on their learning context, different individuals use the systems in different ways in the process. Subitizing, for example, is often associated with visual ability. People blind from birth could thus employ the PIS in a different

[21] In Quinon (2019), I claim that it depends on the clarification of the concept.

[22] Frege's Theorem states that PA can be derived from FA.

way from people without problems in vision. Rather than suggesting a strictly specified developmental account, the pluralistic account I defend is meant to be used for clarifying a variety of aspects of the number concept in order to help identify their cognitive origin. Because of its inherent plurality, this framework can be potentially used to reconcile various positions trying to explain the concept of natural number.

The method that I apply comes from analytical philosophy, and it is a version of conceptual analysis.[23] It consists in going back-and-forth between the study of the contexts of use of the concept of natural number, in order to disclose its conceptual structure proper to those contexts and to detect where the partial meanings originate; and—on the other hand—the study of the possibilities of conceptual content that can be traced back to the representations generated by core cognitive systems.

The analytic perspective is worth considering as a philosophical contribution to various classical approaches in the research on numerical cognition, such as the research of developmental psychologists who conduct experiments in order to disclose how children learn the number concept, cognitive scientists who are interested in model building or anthropologists who study cultural differences in the use of number concepts. Furthermore, the philosophical work that I undertake, differs from work of my fellow philosophers: those who follow cognitive scientists in developing models of the number concept acquisition with methods of empirically informed philosophy (e.g., Beck 2017; Pantsar 2014), and those who are more metaphysically minded and who—for instance—try to understand if it is possible to conduct empirically informed philosophy in the area of the number concept acquisition if one does not stand on a platonist position (e.g., Panza 2014; Benis-Sinaceur 2014; De Cruz 2016).

My work is not aiming at competing with experimental work of researchers in cognitive sciences. My purpose is only to philosophically motivate the need for experiments studying the connection between sensitivity to rhythm and numerical cognition.

Acknowledgements I want to thank Jan Duszyński for many wonderful conversations about music, the idea that rhythm can play an important role in bootstrapping the concept of natural number came from these conversations that go back to 2013, it just took me a long time to understand exactly what that role is. I am grateful to Peter Gärdenfors for his continued support of my work, for the endless discussions we regularly have about cognition, learning, concepts, and numbers, and for his patience in reading successive versions of this article. Last but not least, I am grateful to Markus Pantsar, with whom we collaborated on bootstrapping of the concept of natural number, and even though we have drifted apart, the shared interests have certainly remained. Markus presents his side of the story in (Pantsar 2021). Finally, thanks to the reviewers and to Christophe Heintz, the editor, for fruitful comments that enabled me to make this piece much better.

[23] I call it "reverse conceptual engineering".

References

Beck, J. 2017. Can bootstrapping explain concept learning? *Cognition* 158: 110–121.
Benacerraf, P. 1965. What numbers could not be. *The Philosophical Review* 74(1): 47–73.
Benacerraf, P. 1996. Recantation or any old ω-sequence would do after all. *Philosophia Mathematica* 3(1): 184–189.
Benis-Sinaceur, H. 2014. Neuropsychology of numbers Lecture at the Mathematics: Objectivity by representation, ANR-DFG-Workshop MCMP (Watch the Lecture @ LMUcast).
Blanchette, P. 2018. The Frege-Hilbert controversy. In *The Stanford encyclopedia of philosophy (Fall 2018 Edition)*, ed. Edward N. Zalta. https://plato.stanford.edu/archives/fall2018/entries/frege-hilbert/.
Borges, J.L. 2000. Funes, his memory. In *Fictions*, 91–99. Penguin Books.
Brannon, E.M., and D.J. Merritt. 2011. Evolutionary foundations of the approximate number system. In *Space, time and number in the brain: Searching for the foundations of mathematical thought*, eds. S. Dehaene, and E. Brannon, 207–224. Elsevier Academic Press.
Buijsman, S. 2019. Learning the natural numbers as a child. *Noûs* 53(1): 3–22.
Butterworth, B. 1999. *What counts: How every brain is hardwired for math*. New York: Free Press.
Button, T., and P. Smith. 2012. The philosophical significance of Tennenbaum's theorem. *Philosophia Mathematica* 20(1): 114–121.
Cantrell, L., and L.B. Smith. 2013. Open questions and a proposal: A critical review of the evidence on infant numerical abilities. *Cognition* 128(3): 331–352.
Cappelen, H. 2018. *Fixing language: An essay on conceptual engineering*. Oxford: Oxford University Press.
Carey, S. 2009. *The origin of concepts*. New York: Oxford University Press.
Carey, S., and D. Barner. 2019. Ontogenetic origins of human integer representations. *Trends in Cognitive Sciences* 23(10): 823–835.
Carey, S., A. Shusterman, P. Haward, and R. Distefano. 2017. Do analog number representations underlie the meanings of young children's verbal numerals? *Cognition* 168: 243–255.
Carnap, R. 1950. *Logical foundations of probability*. London: Routledge and Kegan Paul.
Castronovo, J., and S.M. Göbel. 2012. Impact of high mathematics education on the number sense. *PloS One* 7(4): e33832.
Chomsky, N. 2002. *Syntactic structures*. Walter de Gruyter.
Davidson, K., K. Eng, and D. Barner. 2012. Does learning to count involve a semantic induction? *Cognition* 123: 162–173.
De Cruz, H. 2016. Numerical cognition and mathematical realism. *Philosophers' Imprint* 16: 1–13.
Dean, W. 2014. Models and computability. *Philosophia Mathematica* 22(2): 143–166.
Dedekind, J.W.R. 1888/2012. *Was sind und was sollen die Zahlen? (reprinted 2012)*. Cambridge: Cambridge University Press.
Dehaene, S. 2011. *The number sense: How the mind creates mathematics, 2nd edn*. Oxford: Oxford University Press.
Dehaene, S., and E. M. Brannon (Eds.) 2011. *Space, time and number in the brain: Searching for evolutionary foundations of mathematical thought*: Elsevier. Oxford.
Dehaene, S., V. Izard, E. Spelke, and P. Pica. 2008. Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures. *Science* 320(5880): 1217–1220.
Dellatolas, G., L. Watier, M.T. Le Normand, T. Lubart, and C. Chevrie-Muller. 2009. Rhythm reproduction in kindergarten, reading performance at second grade, and developmental dyslexia theories. *Archives of Clinical Neuropsychology* 24(6): 555–563.
Everett, C. 2019. *Numbers and the making of us: Counting and the course of human cultures*. Cambridge: Harvard University Press.
Feigenson, et al. 2004. Core systems of number. *Trends in Cognitive Sciences* 8(7): 307–314.
Fuson, K.C. 1988. *Children's counting and concepts of number*. Berlin: Springer.

Fuson, K.C., J. Richards, and D.J. Briars. 1982. The acquisition and elaboration of the number word sequence. In *Children's logical and mathematical cognition*, ed. C.J. Brainerd, 33–92. Berlin: Springer.

Galton, F. 1881. Visualised numerals. *The Journal of the Royal Anthropological Institute* 10: 85–102.

Gärdenfors, P., and P. Quinon. 2020. Situated counting. *Review of Philosophy and Psychology*, 1–24. Published online 9 October. https://doi.org/10.1007/s13164-020-00508-3.

Gelman, R., and C.R. Gallistel. 1978. *The child's understanding of number*. Cambridge: Harvard University Press.

Griffin, S., and R. Case. 1996. Evaluating the breadth and depth of training effects, when central conceptual structures are taught. *Monographs of the Society for Research in Child Development* 61(1/2, Serial No. 246): 83–102.

Halbach, Volker, and Leon Horsen. 2005. Computational struturalism. *Philosophia Mathematica* 13(2): 174–186.

Halberda, J., and L. Feigenson. 2008. Set representations required for acquisition of the natural number concept. *Behavioral and Brain Sciences* 31(6): 655–656.

Hauser, M.D., N. Chomsky, and W.T. Fitch. 2002. The faculty of language: What is it, who has it, and how did it evolve? *Science* 298(5598): 1569–1579.

Holloway, I.D., and D. Ansari. 2009. Mapping numerical magnitudes onto symbols: The numerical distance effect and individual differences in children's mathematics achievement. *Journal of Experimental Child Psychology* 103: 17–29.

Honing, H. 2012. Without it no music: Beat induction as a fundamental musical trait. *Annals of the New York Academy of Sciences* 1252: 85–91.

Izard, V., P. Pica, E.S. Spelke, and S. Dehaene. 2008. Exact equality and successor function: Two key concepts on the path towards understanding exact numbers. *Philosophical Psychology* 21(4): 491–505.

Leibovich, T., N. Katzin, M. Harel, and A. Henik. 2017. From "sense of number" to "sense of magnitude": The role of continuous magnitudes in numerical cognition. *Behavioral and Brain Sciences* 40, E164. doi:10.1017/S0140525X16000960.

MacBride, F. 2003. Speaking with shadows: A study of neo-logicism. *The British Journal for the Philosophy of Science* 54(1): 103–163.

Makovec, D., and S. Shapiro (Eds.) 2019. *Friedrich Waismann. The Open Texture of Analytic Philosophy*: Springer. Berlin.

Mandler, G., and B.J. Shebo. 1982. Subitizing: An analysis of its component processes. *Journal of Experimental Psychology: General* 111(1): 1–22.

Mehler, J., and T.G. Bever. 1967. Cognitive capacity of very young children. *Science* 158(3797): 141–142.

Menary, R. 2015. Mathematical cognition: A case of enculturation. In *Open MIND*.

Moeller, K., S. Pixner, L. Kaufmann, and H.C. Nuerk. 2009. Children's early mental number line: Logarithmic or decomposed linear? *Journal of Experimental Child Psychology* 103(4): 503–515.

Noël, M.-P., and L. Rouselle. 2011. Developmental changes in the profiles of dyscalculia: And explanation based on a double exact-and-approximate number representation model. *Frontiers in Human Neuroscience* 5: 1–4.

Pantsar, M. 2014. An empirically feasible approach to the epistemology of arithmetic. *Synthese* 191(17): 4201–4229.

Pantsar M. 2021. Bootstrapping of integer concepts: the stronger deviant-interpretation challenge (and how to solve it) Synthese. Pages 1-24. Published online 3 February 2021.

Pantsar, M., and P. Quinon. 2018. Critical extended review of "Numbers and the making of us" by Caleb Everett. Harvard University Press 2017, Journal of Numerical Cognition. 4, 2, p. 494–504.

Panza, M. 2014. What are the challenges of Benacerrafs Dilemma? A Reinterpretation Lecture at the Mathematics: Objectivity by Representation, ANR-DFG-Workshop MCMP (Watch the Lecture @ LMUcast.

Patel, A. 2008. Beat-based rhythm processing as a key research area. In *Music, language and the brain*, 402–415. Oxford: Oxford University Press.

Piaget, J. 1952. The child's conception of number. Basic Books (original in French, 1941).

Pica, P., C. Lemer, V. Izard, and S. Dehaene. 2004. Exact and approximate arithmetic in an Amazonian indigene group. *Science* 306(5695): 499–503.

Potter, M. 2002. *Reason's nearest kin: Philosophies of arithmetic from Kant to Carnap*. Oxford: Oxford University Press.

Przybylski, L., S. Krifi-Papoz, D. Roch, S.A. Kotz, N. Bedoin, V. Herbillon, L. Léculier, and B. Tillmann. 2013. Rhythmic auditory stimulation influences syntactic processing in children with developmental language disorders. *Neuropsychology* 27(1): 121.

Quine, W.V.O. 1946. Concatenation as a basis for arithmetic. *J. Symbolic Logic* 11(4): 105–114.

Quine, W.V.O. 1960. *Word and object*. Cambridge: MIT Press.

Quinon, P. 2014. From computability over strings of characters to natural numbers. In *Church's, thesis, logic, mind and nature*, eds. Olszewski, and et al. Copernicus Center Press.

Quinon, P. 2019. Can Church's Thesis be Viewed as a Carnapian Explication? Synthese 128. Published online 18, In Intentionality in Mathematics, Synthese Special Issue, eds. M. Antonutti, and P. Quinon. https://doi.org/10.1007/s11229-019-02286-7.

Quinon, P., and K. Zdanowski. 2007. Intended model of arithmetic. Argument from Tennenbaum's theorem. In *Computation and logic in the real world, CiE*, eds. S.B. Cooper, T.F. Kent, B. Löwe, and A. Sorbi, 313–317.

Reck, E.H., and G. Schiemer. 2020. *The prehistory of mathematical structuralism*. Oxford: Oxford University Press.

Rips, L.J., A. Bloomfield, and J. Asmuth. 2008. From numerical concepts to concepts of number. *Behavioral and Brain Sciences* 31(6): 623–642.

Sarnecka, B.W. 2015. Learning to represent exact numbers. Synthese: 1–18. Published online 27 August 2015. In *Intentionality in Mathematics, Synthese Special Issue*, eds. M. Antonutti, and P. Quinon.

Sarnecka, B.W., and S. Carey. 2008. How counting represents number: What children must learn and when they learn it. *Cognition* 108(3): 662–674.

Sarnecka, B.W., and S.A. Gelman. 2004. Six does not just mean a lot: Preschoolers see number words as specific. *Cognition* 92(3): 329–352.

Sarnecka, B.W., and M.D. Lee. 2009. Levels of number knowledge during early childhood. *Journal of Experimental Child Psychology* 103(3): 325–337.

Sasanguie, D., E. Defever, B. Maertens, and B. Reynvoet. 2014. The approximate number system is not predictive for symbolic number processing in kindergarteners. *The Quarterly Journal of Experimental Psychology* 67(2): 217–280.

Shapiro, S. 1991. *Foundations without foundationalism*. Oxford: Oxford University Press.

Shapiro, S. 1997. *Philosophy of mathematics. Structure and ontology*. Oxford: Oxford University Press.

Shapiro, S. 2005. Categories, structures, and the Frege-Hilbert controversy: The status of meta-mathematics. *Philosophia Mathematica* 13(1): 61–77.

Slusser, E.B., and B.W. Sarnecka. 2011. Find the picture of eight turtles: A link between children's counting and their knowledge of number word semantics. *Journal of Experimental Child Psychology* 110(1): 38–51.

Slusser, E., A. Ditta, and B. Sarnecka. 2013. Connecting numbers to discrete quantification: A step in the child's construction of integer concepts. *Cognition* 129(1): 31–41.

Spelke, E. 2000. Core knowledge. *American Psychologist* 55: 1233–1243.

Spelke, E. 2003. What makes us smart? Core knowledge and natural language. In *Language in mind: Advances in the investigation of language and thought*, eds. D. Gentner, and S. Goldin-Meadow. Cambridge: MIT Press.

Spelke, E. 2011. Natural number and natural geometry. In *Space, time and number in the brain*, ed. Brannon Dehaene, 287–318. London: Academic Press.

Tanswell, F.S. 2017. Conceptual engineering for mathematical concepts. *Inquiry* 61(8): 881–913.

Van Marle, K., F.W. Chu, Y. Mou, J.H. Seok, J. Rouder, and D.C. Geary. 2018. Attaching meaning to the number words: Contributions of the object tracking and approximate number systems. *Developmental Science* 21(1) https://doi.org/10.1111/desc.12495.

Wagner, J.B., and S.C. Johnson. 2011. An association between understanding cardinality and analog magnitude representations in preschoolers. *Cognition* 119: 10–22.

Winkler, I., G. Haden, O. Ladinig, et al. 2009. Newborn infants detect the beat in music. *Proceedings of the National Academy of Sciences USA* 106: 2468–2471.

Wynn, K. 1990. Children's understanding of counting. *Cognition* 36(2): 155–193.

Wynn, K. 1992. Children's acquisition of the number words and the counting system. *Cognitive Psychology* 24(2): 220–251.

Zentner, M., and T. Eerola. 2010. Rhythmic engagement with music in infancy. *PNAS* 107(13): 5768–5773.