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ORIGINAL RESEARCH

Effect of Discretization on Dynamical Behavior in an Epidemiological Model

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Abstract Dynamical behavior of two discrete epidemic models for disease with nonlinear incidence rate is studied. Both discrete models are derived from the continuous case by applying forward and backward Euler methods. The effect of the two different discretizations on the stability behavior of the disease-free equilibrium and endemic equilibrium is discussed. Finally, numerical simulations are presented to illustrate our theoretical results.

Keywords Discrete epidemic model · Forward and backward Euler methods · Stability · Lyapunov functional

Introduction

The discrete-time models or difference equations are more accurate to describe epidemics than the continuous-time models or ordinary differential equations because statistical data on epidemics is collected in discrete time. In addition, the numerical simulations of continuous-time models are obtained by discretizing the models or by using other methods proposed by Khan et al. in [1–3].

In the literature, many discrete models have been developed in order to understand disease transmission dynamics. Zhou et al. [4] formulated a discrete mathematical model to investigate the transmission of severe acute respiratory syndrome (SARS) and their simulation

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results match the statistical data well and indicate that early quarantine and a high quarantine rate are crucial to the control of SARS. Enatsu et al. [5] proposed a discrete SIR epidemic model with a class of nonlinear incidence rate and a distributed latent period by applying a variation of Euler Backward discretization. They proved that the global asymptotic stability of the equilibria is fully determined by the basic reproduction number R_0 when the infection incidence rate has a suitable monotone property. Das et al. [6] proposed a discrete version of SI model with nonlinear incidence rate. They observed that the discrete system converges to a unique equilibrium point for certain effective transmission rate of the disease and beyond which stability of the system is disturbed. In [7], a simple discrete-time West Nile epidemic model is proposed and analysed. The derivation of this model is based on a continuous-time model proposed by Cruz-Pacheco et al. [8]. In [9], Jang and Elaydi formulated a discrete-time epidemic model with immigration of infectives which was obtained from the continuous-time model by using nonstandard discretization technique.

On the other hand, it is well known that the continuous model exhibits a threshold behavior such that the disease always cleared out from the population if the threshold parameter called the basic reproduction number is less than unity and the disease persist in the population if the basic reproduction number exceed unity. However, the corresponding discrete-time epidemic model exhibits more complex dynamical behaviors, such as flip bifurcation, Hopf bifurcation and chaos phenomenon [10,11]. All the studies cited above are based on the discretization of the continuous epidemic models. Therefore, it is important to study the impact of the discretization on the dynamics of continuous models.

We consider the following continuous epidemic model with nonlinear incidence rate:

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - \mu_1 S - \frac{\beta SI}{1+\alpha I}, \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\beta SI}{1+\alpha I} - (\mu_2+\gamma)I, \\ \frac{\mathrm{d}R}{\mathrm{d}t} &= \gamma I - \mu_3 R, \end{aligned} \tag{1}$$

where S, I and R denotes the susceptible, infectious and recovered classes, respectively. Λ is the recruitment rate of new individuals into the susceptible class. μ_1, μ_2 and μ_3 are positive constants representing the death rates in the susceptible, infectious and recovered class, respectively. The average time spent in class I before recovery is $1/\gamma$. β is the contact number and α determines the level at which the force of infection saturates. Since, R does not appear in the first two equations, it is sufficient to analyze the behavior of solutions considering the first two equations of the system (1).

The basic reproduction number of system (1), denoted by R_0 is given by

$$R_0 = \frac{\Lambda\beta}{\mu_1(\mu_2+\gamma)}.$$

System (1) always has a disease-free steady state $E_f(S_0, 0)$ where $S_0 = \Lambda/\mu_1$, and an endemic steady state $E^*(S^*, I^*)$ which exist when $R_0 > 1$, where

$$S^* = \frac{\Lambda\alpha+\mu_2+\gamma}{\beta+\alpha\mu_1}, \quad I^* = \frac{\Lambda\beta-\mu_1(\mu_2+\gamma)}{(\mu_2+\gamma)(\beta+\alpha\mu_1)} = \frac{\mu_1(R_0-1)}{\beta+\alpha\mu_1}.$$

In [12], a detailed analysis of the current model is presented. It is shown that if $R_0 < 1$, then the disease-free equilibrium is globally asymptotically stable. If $R_0 > 1$, then the disease-free equilibrium is unstable, the system is permanent, and there is an endemic equilibrium which

is locally asymptotically stable. In [13], the author McCluskey showed that if $R_0 > 1$, the endemic equilibrium is globally asymptotically stable, without any further conditions on the parameters. The generalization of the results presented in [12] and [13] is given by Hattaf et al. [14] by extending the work of [12] to more general incidence function. Moreover, the case when $R_0 = 1$ is not discussed in [12] and [13], but in [14], Hattaf et al. proved that the disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$.

The purpose of this paper is to study the impact of the discretization on the dynamics of the continuous model (1). First, we study the stability results for the discrete model obtained from (1) by applying forward Euler discretization. By using the linearization method, we establish the criteria on the local stability of the disease-free equilibrium and endemic equilibrium. Moreover, Hopf-bifurcation occurs in our discrete model which is not present in the continuous model (1). Secondly, we formulate another discrete model derived from (1) by the backward Euler discretization. By using the technique of Lyapunov functional we show that this discretization scheme preserves the global stability of both the disease-free and endemic equilibrium for the continuous model (1).

This paper is organized as follows. In the next section, we analyse the discrete model obtained by the forward Euler method. In "Analysis of the discrete model obtained by backward Euler method" section, we propose another discrete model derived from (1) by using the backward Euler method and a detailed stability analysis of the equilibria of this model is discussed. In "Numerical simulation" section, we present the numerical simulations to illustrate our theoretical results. Finally, we give a brief discussion of our results in "Conclusion and discussion" section.

Analysis of the discrete model obtained by forward Euler method

We discretize model (1) by using the forward Euler method, we obtain the following discrete model

$$\begin{aligned} S_{n+1} &= S_n + h\left(\Lambda - \mu_1 S_n - \frac{\beta S_n I_n}{1+\alpha I_n}\right), \\ I_{n+1} &= I_n + h\left(\frac{\beta S_n I_n}{1+\alpha I_n} - (\mu_2+\gamma) I_n\right), \end{aligned} \tag{2}$$

where $h > 0$ is the step size and other parameters are same as in system (1).

The system (2) always has a disease-free fixed point $E_f(\Lambda/\mu_1, 0)$ and one endemic fixed point

$$E^*\left(\frac{\Lambda\alpha + \mu_2 + \gamma}{\beta + \alpha\mu_1}, \frac{\mu_1(R_0 - 1)}{\beta + \alpha\mu_1}\right). \tag{3}$$

Now, we investigate the local behavior of the model around each of the above fixed points. The local stability analysis of the model can be studied by computing the variation matrix corresponding to each fixed point. The Jacobian matrix of the system (2) at an arbitrary point $E(S, I)$ is given by

$$J(E) = \begin{pmatrix} 1 - h\mu_1 - \dfrac{h\beta I}{1+\alpha I} & -\dfrac{h\beta S}{(1+\alpha I)^2} \\ \dfrac{h\beta I}{1+\alpha I} & 1 + \dfrac{h\beta S}{(1+\alpha I)^2} - h(\mu_2+\gamma) \end{pmatrix}.$$

Let λ_1 and λ_2 be two eigenvalues of matrix $J(E)$. We recall some definitions of topological types for a fixed point $E(S, I)$:

(i) $E(S, I)$ is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so the sink is locally asymptotically stable.
(ii) $E(S, I)$ is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so the source is locally unstable.
(iii) $E(S, I)$ is called a saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$).
(iv) $E(S, I)$ is non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Stability of Disease-Free Equilibrium

The Jacobian matrix of model (2) at $E_f(\Lambda/\mu_1, 0)$ is

$$J(E_f) = \begin{pmatrix} 1 - h\mu_1 & -h(\mu_2 + \gamma)R_0 \\ 0 & 1 + h(\mu_2 + \gamma)(R_0 - 1) \end{pmatrix}.$$

The two eigenvalues of $J(E_f)$ are $\lambda_1 = 1 - h\mu_1$ and $\lambda_2 = 1 + h(\mu_2 + \gamma)(R_0 - 1)$. We can immediately obtain the following result.

Theorem 2.1 (i) *E_f is asymptotically stable if and only if*

$$h < \min\left\{\frac{2}{\mu_1}, \frac{2}{(\mu_2 + \gamma)(1 - R_0)}\right\} \quad \textit{and} \quad R_0 < 1. \tag{4}$$

(ii) *E_f is unstable if and only if*

$$h > \min\left\{\frac{2}{\mu_1}, \frac{2}{(\mu_2 + \gamma)(1 - R_0)}\right\} \quad \textit{and} \quad R_0 < 1 \quad \textit{or} \quad R_0 > 1. \tag{5}$$

(iii) *E_f is non-hyperbolic if and only if*

$$h = \frac{2}{\mu_1} \quad \textit{or} \quad h = \frac{2}{(\mu_2 + \gamma)(1 - R_0)} \quad \textit{and} \quad R_0 < 1 \quad \textit{or} \quad R_0 = 1. \tag{6}$$

Stability of Endemic Equilibrium

In this subsection, we discuss the stability of endemic equilibrium E^*.

The characteristic equation of Jacobian matrix $J(E^*)$ is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0, \tag{7}$$

where

$$\begin{aligned} a_1 &= h(\mu_1 + \mu_2 + \gamma + p - q) - 2, \\ a_2 &= 1 - h(\mu_1 + \mu_2 + \gamma + p - q) + h^2\big((\mu_1 + p)(\mu_2 + \gamma) - q\mu_1\big), \end{aligned}$$

with

$$p = \frac{\beta I^*}{1 + \alpha I^*} = \frac{\beta\mu_1(R_0 - 1)}{\beta + \alpha\mu_1 R_0} \quad \textit{and} \quad q = \frac{\beta S^*}{(1 + \alpha I^*)^2} = \frac{(\mu_2 + \gamma)(\beta + \alpha\mu_1)}{\beta + \alpha\mu_1 R_0}. \tag{8}$$

Theorem 2.2 *Assume that $R_0 > 1$. E^* is asymptotically stable if the following conditions are satisfied:*

(i) $4 + h^2(\mu_2 + \gamma)(\mu_1 + p) + 2hq > 2h(\mu_1 + \mu_2 + \gamma + p) + h^2\mu_1 q$.
(ii) $h < \dfrac{\mu_1 + \mu_2 + \gamma + p - q}{(\mu_2 + \gamma)(\mu_1 + p) - q\mu_1}$.

Proof From the Schur–Cohn criterion [15], the zeros of the characteristic polynomial (7) lie inside the unit disk if and only if

$$P(1) = 1 + a_1 + a_2 > 0, \tag{9}$$
$$P(-1) = 1 - a_1 + a_2 > 0, \tag{10}$$
$$1 - a_2 > 0. \tag{11}$$

We have

$$1 + a_1 + a_2 = h^2(\mu_2 + \gamma)(\mu_1 + p) - h^2\mu_1 q, \tag{12}$$
$$1 - a_1 + a_2 = 4 + h^2(\mu_2 + \gamma)(\mu_1 + p) + 2hq - 2h(\mu_1 + \mu_2 + \gamma + p) - h^2\mu_1 q, \tag{13}$$
$$1 - a_2 = h(\mu_1 + \mu_2 + \gamma + p) + h^2 q\mu_1 - h^2(\mu_2 + \gamma)(\mu_1 + p) - hq. \tag{14}$$

It is easy to verify that $1 + a_1 + a_2 > 0$ when $R_0 > 1$. Hence, E^* is asymptotically stable if the conditions (i) and (ii) are satisfied. □

Analysis of the Discrete Model Obtained by Backward Euler Method

Now, we discretize model (1) by using the backward Euler method. Then, we obtain the following discrete model

$$\begin{aligned} S_{n+1} &= S_n + h\left(\Lambda - \mu_1 S_{n+1} - \frac{\beta S_{n+1} I_n}{1 + \alpha I_n}\right), \\ I_{n+1} &= I_n + h\left(\frac{\beta S_{n+1} I_n}{1 + \alpha I_n} - (\mu_2 + \gamma) I_{n+1}\right), \end{aligned} \tag{15}$$

The system (15) always has a disease-free fixed point $E_f(\Lambda/\mu_1, 0)$ and one endemic fixed point

$$E^*\left(\frac{\Lambda\alpha + \mu_2 + \gamma}{\beta + \alpha\mu_1}, \frac{\mu_1(R_0 - 1)}{\beta + \alpha\mu_1}\right). \tag{16}$$

Stability of Disease-Free Equilibrium

For an arbitrary equilibrium $E(S, I)$, the characteristic equation is given by

$$\begin{vmatrix} \frac{1 + \alpha I}{(1+\alpha I)(1+h\mu_1)+h\beta I} - \lambda & -\frac{h\beta(S+h\Lambda)}{\left((1 + \alpha I)(1+h\mu_1)+h\beta I\right)^2} \\ \frac{h\beta I}{\left((1 + h(\mu_2 + \gamma)\right)\left((1 + \alpha I)(1+h\mu_1)+h\beta I\right)} & \frac{1}{1+h(\mu_2 + \gamma)}\left[1 + \frac{h\beta(1+h\mu_1)(S+h\Lambda)}{\left((1 + \alpha I)(1+h\mu_1)+h\beta I\right)^2}\right] - \lambda \end{vmatrix} = 0. \tag{17}$$

The characterization of the local stability of the disease-free equilibrium is given by the following theorem.

Theorem 3.1

- *If $R_0 < 1$, then E_f is locally asymptotically stable.*
- *If $R_0 > 1$, then E_f is unstable.*

Proof At E_f, (17) reduces to

$$\left(\frac{1}{1+h\mu_1}-\lambda\right)\left(\frac{1+hR_0(\mu_2+\gamma)}{1+h(\mu_2+\gamma)}-\lambda\right)=0. \tag{18}$$

Clearly,

$$\lambda_1=\frac{1}{1+h\mu_1} \quad \text{and} \quad \lambda_2=\frac{1+hR_0(\mu_2+\gamma)}{1+h(\mu_2+\gamma)} \tag{19}$$

are the roots of this equation. We have, $0<\lambda_1<1$ and $0<\lambda_2<1$ if $R_0<1$. Then E_f is locally asymptotically stable when $R_0<1$.

On the other hand, it easy to show that $\lambda_2>1$ when $R_0>1$. Then E_f is unstable. This proves the theorem. □

Theorem 3.1 only establishes local stability of E_f. However, the following theorem establishes the global asymptotic stability of the disease-free equilibrium.

Theorem 3.2 *If $R_0\le 1$, the disease-free equilibrium E_f is globally asymptotically stable.*

Proof Consider the following sequence $\{V(n)\}_{n=0}^{+\infty}$ defined by

$$V(n)=S_0 g\left(\frac{S_n}{S_0}\right)+I_n+\frac{h\beta S_0 I_n}{1+\alpha I_n},$$

where $S_0=\Lambda/\mu_1$ and $g(x)=x-1-\ln x$, $x\in\mathbb{R}^+$. Obviously, $g:\mathbb{R}^+\to\mathbb{R}^+$ attains its strict global minimum at $x=1$ and $g(1)=0$.

$$V(n+1)-V(n)=S_{n+1}-S_n+S_0\ln\left(\frac{S_n}{S_{n+1}}\right)+I_{n+1}-I_n+\frac{h\beta S_0 I_{n+1}}{1+\alpha I_{n+1}}-\frac{h\beta S_0 I_n}{1+\alpha I_n}$$

From $\ln x\le x-1$, we have

$$\begin{aligned}
V(n+1)-V(n)&\le\left(1-\frac{S_0}{S_{n+1}}\right)(S_{n+1}-S_n)+I_{n+1}-I_n+\frac{h\beta S_0 I_{n+1}}{1+\alpha I_{n+1}}-\frac{h\beta S_0 I_n}{1+\alpha I_n}\\
&=\left(1-\frac{S_0}{S_{n+1}}\right)h\left(\Lambda-\mu_1 S_{n+1}-\frac{\beta S_{n+1}I_n}{1+\alpha I_n}\right)\\
&\quad+h\left(\frac{\beta S_{n+1}I_n}{1+\alpha I_n}-(\mu_2+\gamma)I_{n+1}\right)+\frac{h\beta S_0 I_{n+1}}{1+\alpha I_{n+1}}-\frac{h\beta S_0 I_n}{1+\alpha I_n}\\
&=-h\mu_1 S_{n+1}\left(1-\frac{S_0}{S_{n+1}}\right)^2+h(\mu_2+\gamma)I_{n+1}\left(\frac{R_0}{1+\alpha I_{n+1}}-1\right)\\
&\le-h\mu_1 S_{n+1}\left(1-\frac{S_0}{S_{n+1}}\right)^2+h(\mu_2+\gamma)I_{n+1}(R_0-1).
\end{aligned}$$

Since $R_0\le 1$, we have $V(n+1)-V(n)\le 0$ for any $n\ge 0$. Then, $V(n)$ is monotone decreasing sequence. Since $V(n)\ge 0$, there is a limit $\lim_{n\to+\infty}V(n)\ge 0$. Hence $\lim_{n\to+\infty}\big(V(n+1)-V(n)\big)=0$, from which we get $\lim_{n\to+\infty}S_{n+1}=S_0$ and $\lim_{n\to+\infty}\big(I_{n+1}(R_0-1)\big)=0$. We will discuss the following two cases:

- If $R_0<1$, then $\lim_{n\to+\infty}I_{n+1}=0$.
- If $R_0=1$. From $\lim_{n\to+\infty}S_n=S_0$ and the first equation of (15), we have

$$\lim_{n\to+\infty}I_n=0.$$

By the above discussion, we conclude that E_f is globally asymptotically stable. □

Stability of Endemic Equilibrium

Note that the disease-free equilibrium E_f is unstable when $R_0 > 1$. Now, we establish the global stability of the endemic equilibrium E^*.

Theorem 3.3 *Assume $R_0 > 1$. Then the endemic equilibrium E^* is globally asymptotically stable.*

Proof By using a Lyapunov functional for the continuous-time model (1), McCluskey [13] proved that the endemic equilibrium of (1) is globally asymptotically stable if $R_0 > 1$. In order to prove the global stability of the endemic equilibrium for the discrete-time model (15), we consider the following sequence $\{W(n)\}_{n=0}^{+\infty}$ defined by

$$W(n) = \frac{1}{h\beta f(I^*)} g\left(\frac{S_n}{S^*}\right) + \frac{I^*}{h\beta S^* f(I^*)} g\left(\frac{I_n}{I^*}\right) + g\left(\frac{f(I_n)}{f(I^*)}\right),$$

where $f(x) = x/(1+\alpha x)$. We have

$$\begin{aligned}
W(n+1) - W(n) &= \frac{1}{h\beta f(I^*)}\left(\frac{S_{n+1}-S_n}{S^*} + \ln\left(\frac{S_n}{S_{n+1}}\right)\right) \\
&\quad + \frac{I^*}{h\beta S^* f(I^*)}\left(\frac{I_{n+1}-I_n}{I^*} + \ln\left(\frac{I_n}{I_{n+1}}\right)\right) \\
&\quad + g\left(\frac{f(I_{n+1})}{f(I^*)}\right) - g\left(\frac{f(I_n)}{f(I^*)}\right) \\
&\le \frac{1}{h\beta S^* f(I^*)}\left(1 - \frac{S^*}{S_{n+1}}\right)(S_{n+1} - S_n) \\
&\quad + \frac{1}{h\beta S^* f(I^*)}\left(1 - \frac{I^*}{I_{n+1}}\right)(I_{n+1} - I_n) + g\left(\frac{f(I_{n+1})}{f(I^*)}\right) - g\left(\frac{f(I_n)}{f(I^*)}\right) \\
&= \frac{1}{\beta S^* f(I^*)}\left(1 - \frac{S^*}{S_{n+1}}\right)\left(\Lambda - \mu_1 S_{n+1} - \frac{\beta S_{n+1} I_n}{1+\alpha I_n}\right) \\
&\quad + \frac{1}{\beta S^* f(I^*)}\left(1 - \frac{I^*}{I_{n+1}}\right)\left(\frac{\beta S_{n+1} I_n}{1+\alpha I_n} - (\mu_2+\gamma) I_{n+1}\right) \\
&\quad + g\left(\frac{f(I_{n+1})}{f(I^*)}\right) - g\left(\frac{f(I_n)}{f(I^*)}\right)
\end{aligned}$$

Note that $\Lambda = \mu_1 S^* + \beta S^* f(I^*)$ and $(\mu_2+\gamma) I^* = \beta S^* f(I^*)$.

Hence,

$$\begin{aligned}
W(n+1) - W(n) &\le -\frac{\mu_1 (S_{n+1} - S^*)^2}{\beta S^* f(I^*) S_{n+1}} + \left(1 - \frac{S^*}{S_{n+1}}\right)\left(1 - \frac{S_{n+1}}{S^*}\frac{f(I_n)}{f(I^*)}\right) \\
&\quad + \left(1 - \frac{I^*}{I_{n+1}}\right)\left(\frac{S_{n+1}}{S^*}\frac{f(I_n)}{f(I^*)} - \frac{I_{n+1}}{I^*}\right) + g\left(\frac{f(I_{n+1})}{f(I^*)}\right) - g\left(\frac{f(I_n)}{f(I^*)}\right) \\
&= -\frac{\mu_1 (S_{n+1} - S^*)^2}{\beta S^* f(I^*) S_{n+1}} + 3 - \frac{S^*}{S_{n+1}} - \frac{I_{n+1}}{I^*} - \frac{I^*}{I_{n+1}}\frac{S_{n+1}}{S^*}\frac{f(I_n)}{f(I^*)} \\
&\quad + \ln\left(\frac{f(I_n)}{f(I^*)}\right) + g\left(\frac{f(I_{n+1})}{f(I^*)}\right)
\end{aligned}$$

$$= -\frac{\mu_1(S_{n+1}-S^*)^2}{\beta S^* f(I^*) S_{n+1}} - g\left(\frac{S^*}{S_{n+1}}\right) - g\left(\frac{I^*}{I_{n+1}}\frac{S_{n+1}}{S^*}\frac{f(I_n)}{f(I^*)}\right)$$
$$+ \frac{f(I_{n+1})}{f(I^*)} - \frac{I_{n+1}}{I^*} + \ln\left(\frac{I_{n+1}}{I^*}\frac{f(I^*)}{f(I_{n+1})}\right)$$

We have

$$\frac{f(I_{n+1})}{f(I^*)} - \frac{I_{n+1}}{I^*} + \ln\left(\frac{I_{n+1}}{I^*}\frac{f(I^*)}{f(I_{n+1})}\right) \le \frac{f(I_{n+1})}{f(I^*)} - \frac{I_{n+1}}{I^*} + \frac{I_{n+1}}{I^*}\frac{f(I^*)}{f(I_{n+1})} - 1$$
$$= \left(1 - \frac{f(I_{n+1})}{f(I^*)}\right)\left(\frac{I_{n+1}}{I^*}\frac{f(I^*)}{f(I_{n+1})} - 1\right)$$
$$= -\frac{\alpha(I_{n+1}-I^*)^2}{I^*(1+\alpha I^*)(1+\alpha I_{n+1})}.$$

Hence

$$W(n+1) - W(n) \le -\frac{\mu_1(S_{n+1}-S^*)^2}{\beta S^* f(I^*) S_{n+1}} - \frac{\alpha(I_{n+1}-I^*)^2}{I^*(1+\alpha I^*)(1+\alpha I_{n+1})} - g\left(\frac{S^*}{S_{n+1}}\right)$$
$$-g\left(\frac{I^*}{I_{n+1}}\frac{S_{n+1}}{S^*}\frac{f(I_n)}{f(I^*)}\right).$$

Consequently, $W(n+1) - W(n) \le 0$ for any $n \ge 0$. Then, $W(n)$ is monotone decreasing sequence. Since $W(n) \ge 0$, there is a limit $\lim_{n\to+\infty} W(n) \ge 0$. Hence $\lim_{n\to+\infty}\big(W(n+1) - W(n)\big) = 0$, from which we get $\lim_{n\to+\infty} S_{n+1} = S^*$ and $\lim_{n\to+\infty} I_{n+1} = I^*$. This completes the proof of the theorem. □

Numerical Simulation

In this section, we present the numerical simulations to illustrate our theoretical results. We use the following data set: $\Lambda = 4.45$, $\mu_1 = 0.5$, $\mu_2 = 0.2$, $\beta = 0.5$, $\gamma = 0.5$ and $\alpha = 0.01$. The equilibria of two discrete models (2) and (15) are: $E_f(8.9, 0)$ and $E^*(1.4075, 5.3518)$. By calculation, we have $R_0 = 6.3571$. Hence the disease-free equilibrium E_f is unstable. Now we choose $h = 0.9091$ which represent the step size of discretization. It is easy to verify that the first condition of Theorem 2.2 is not satisfied. In this case, periodic solutions of model (2) arise due to Hopf bifurcation which is shown in Fig. 1. However, all solutions of model (15) converge to E^* which is globally asymptotically stable (see Fig. 2).

In Fig. 3, we show that there exists a $h^* > 0$ such as if $h \in (0, h^*)$, then endemic equilibrium E^* of model (2) is locally asymptotically stable, and when $h > h^*$, we obtain that E^* loses stability and there appears Hopf bifurcation and chaos phenomenon in model (2). Whereas, the endemic equilibrium E^* of model (15) is globally asymptotically stable independent of the time step size h (see Fig. 4).

Conclusion and Discussion

In this work, we have proposed and analyzed two discrete epidemic models with nonlinear incidence rate which are derived from the continuous case by applying forward and backward Euler methods. We have proved that the equilibria of the continuous-time and discrete-time models are the same. Moreover, we have established the local stability of these

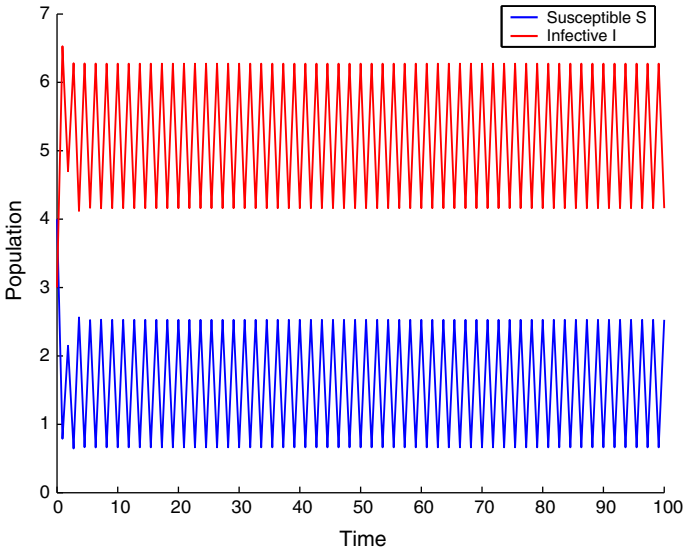

Fig. 1 This figure illustrates trajectories of system (2) for $\Lambda = 4.45,\ \mu_1 = 0.5,\ \mu_2 = 0.2,\ \beta = 0.5,\ \gamma = 0.5,\ \alpha = 0.01$

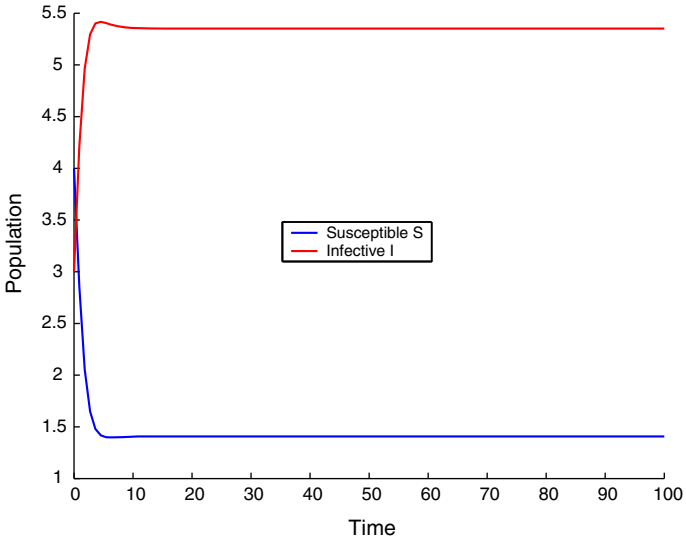

Fig. 2 This figure illustrates trajectories of system (15) for $\Lambda = 4.45,\ \mu_1 = 0.5,\ \mu_2 = 0.2,\ \beta = 0.5,\ \gamma = 0.5,\ \alpha = 0.01$

equilibria if we choose forward Euler discretization. This local stability of the equilibria is only obtained for a small time step size and under certain restrictions on the parameter values. The numerical simulations given in "Numerical simulation" section show that along with time step size h increase the stability properties of the endemic equilibrium will lose and Hopf-bifurcation occurs in our discrete model which is not present in the continuous case.

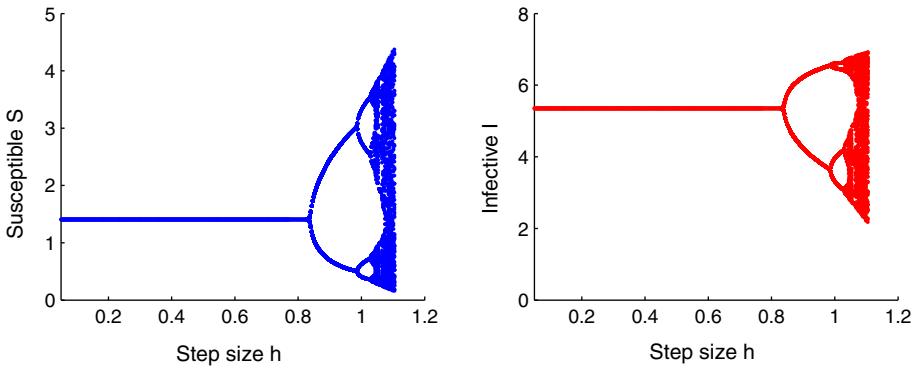

Fig. 3 The dynamical behaviors of $S - h$ and $I - h$ for model (2) with $\Lambda = 4.45,\ \mu_1 = 0.5,\ \mu_2 = 0.2,\ \beta = 0.5,\ \gamma = 0.5,\ \alpha = 0.01$

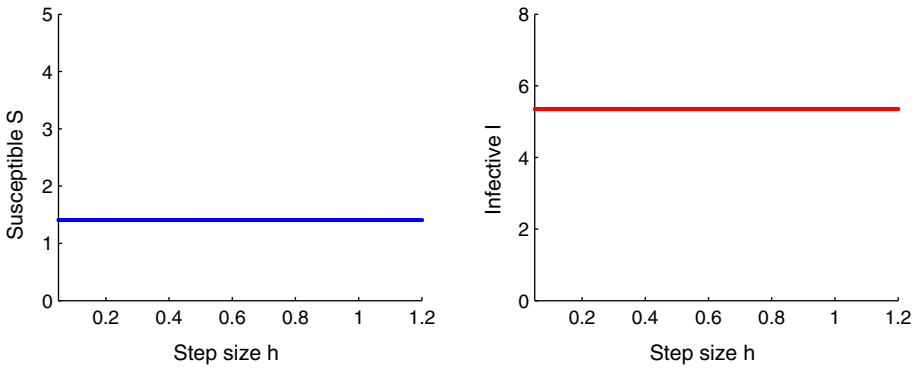

Fig. 4 The dynamical behaviors of $S - h$ and $I - h$ for model (15) with $\Lambda = 4.45,\ \mu_1 = 0.5,\ \mu_2 = 0.2,\ \beta = 0.5,\ \gamma = 0.5,\ \alpha = 0.01$

For the discrete epidemic model obtained by backward Euler discretization, we have established the global asymptotic stability of the disease-free equilibrium and the endemic equilibrium by using suitable Lyapunov functionals. We have shown that the disease-free equilibrium, E_f, is globally asymptotically stable if the basic reproduction number satisfies $R_0 \leq 1$. In this case, all positive solutions converge to E_f and the disease is unable to maintain the infection and will go extinct. When $R_0 > 1$, E_f becomes unstable and there appears an endemic equilibrium E^* which is globally asymptotically stable independent of the time step size h. In this case, all positive solutions converge to E^* and the disease persist in the population. Then, the solutions are not periodic.

From our theoretical and numerical results, we conclude that the discretization can cause periodic oscillations if we consider the forward Euler method. Whereas, the backward Euler method preserves the global asymptotic stability of equilibria.

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