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Qualitative Theory of Dynamical Systems

Stability for a New Discrete Ratio-Dependent Predator–Prey System

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Abstract The stability of a new two-species discrete ratio-dependent predator–prey system is considered. By using the linearization method, we obtain some sufficient conditions for the local stability of the positive equilibria. We also obtain a new sufficient condition to ensure that the positive equilibrium is globally asymptotically stable by using an iteration scheme and the comparison principle of difference equations, which generalizes what paper (Chen and Zhou in J Math Anal Appl 27:7358–7366, 2003) has done. The method given in this paper is new and very resultful comparing with articles (Damgaard in J Theor Biol 227:197–203, 2004; Edmunds in Theor Popul Biol 72:379–388, 2007; Fan and Wang in Math Comput Model 35:951–961, 2002; Muroya in J Math Anal Appl 330:24–33, 2007; Huo and Li in Appl Math Comput 153:337–351, 2004; Liao et al. in Appl Math Comput 190:500–509, 2007) and it can also be applied to study other global asymptotic stability for general multiple species discrete population systems. At the end of this paper, we present two open questions.

Keywords Discrete ratio-dependent predator–prey system · Local stability · Variational matrix · Global stability · Iteration scheme method

Mathematics Subject Classification 39A11 · 92D25

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1 Introduction

In recent years, the dynamical behaviors of the discrete-time predator–prey systems have been widely investigated. Many important and interesting results can be found in many articles, such as in [1–27] and the references cited therein. Particularly, the discrete two-species predator–prey systems with ratio-dependent functional responses were studied in [10–17,23,25]. What interested them are the dynamical behaviors, such as, the study for the local and global stability of the equilibria, the persistence, permanence and extinction of species, the existence of positive periodic solutions and positive almost periodic solutions, the bifurcation and chaos phenomenon, etc.. Recently, Chen and Zhou [17] discussed the global stability for a nonautonomous two species discrete competition system. However, the conditions of their results in [17] is strong and complicated. Therefore, as an extension and improvement, we discuss in the present paper the following discrete-time two-species competition system:

$$\begin{cases} x(k+1) = x(k)\exp\left[r_1\left(1 - \frac{x^m(k)}{K_1} - \mu_2 y^n(k)\right)\right], \\ y(k+1) = y(k)\exp\left[r_2\left(1 - \mu_1 x^m(k) - \frac{y^n(k)}{K_2}\right)\right]. \end{cases} \tag{1.1}$$

where $x(k)$ and $y(k)$ represent the sizes or the densities of species x and y at kth generation, respectively. Parameters r_i, K_i and μ_i $(i = 1, 2)$ are positive constants and represent the intrinsic growth rates, the carrying capacities, and the competition coefficients of species x and y, respectively. m and n are arbitrary positive integer.

In this paper, we will introduce a new method to discuss the global asymptotic stability of system (1.1). The main results of this paper is to establish the criteria on the existence and local asymptotic stability of equilibria for system (1.1) by using the linear approximation method, and obtain some new sufficient conditions on the global stability of the positive equilibrium for system (1.1) by using the iterative scheme method and the comparison principle of difference equations.

2 Preliminary Lemmas

Let $(x(k), y(k))$ be any solution of system (1.1) satisfying the initial value $x(0) > 0$ and $y(0) > 0$ considered the biological background of system (1.1). It is clear that any solution $(x(k), y(k))$ of system (1.1) is defined on Z_+ and always remains positive, where Z_+ denotes the set of all nonnegative integers.

What interested us is the positive equilibrium of system (1.1). By a simple computation, we directly obtain the following results.

Lemma 2.1 *If* $1 - \mu_1 K_1 > 0$ *and* $1 - \mu_2 K_2 > 0$, *then system* (1.1) *has a unique positive equilibrium* $E_+(x_0, y_0)$, *where*

$$x_0^m = \frac{K_1(1 - \mu_2 K_2)}{1 - \mu_1 \mu_2 K_1 K_2}, \quad y_0^n = \frac{K_2(1 - \mu_1 K_1)}{1 - \mu_1 \mu_2 K_1 K_2}.$$

Further, we need the following lemma, which can be easily proved by the relations between roots and coefficients of a quadratic equation.

Lemma 2.2 *Consider the function* $F(\lambda) = \lambda^2 + p\lambda + q$, *here, both* p *and* q *are constants. Suppose* $F(1) > 0$ *and* λ_1, λ_2 *are two roots of the quadratic equation* $F(\lambda) = 0$. *Then we can easily prove that*

1. $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $q < 1$;
2. $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) < 0$;
3. $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $q > 1$;
4. $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $F(-1) = 0$ *and* $p \neq 0, 2$;
5. λ_1 *and* λ_2 *is a pair of conjugate complex root and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $p^2 - 4q < 0$ *and* $q = 1$.

Here, with λ_1 *and* λ_2 *be the two roots of the characteristic equation* $F(\lambda) = \lambda^2 + p\lambda + q = 0$ *of* $J(x, y)$, *we have the following definitions.*

1. *If* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$, *then* $J(x, y)$ *is called a sink and is locally asymptotic stable;*
2. *If* $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$, *then* $J(x, y)$ *is called a source and is unstable;*
3. *If* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*(or* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$*)* , *then* $J(x, y)$ *is called a saddle and is unstable;*
4. *If* $|\lambda_1| = 1$ *or* $|\lambda_2| = 1$, *then* $J(x, y)$ *is called non-hyperbolic.*

Lemma 2.3 *Let* $f(u) = u\exp(\alpha - \beta u^n)$, *where,* α *and* β *are both positive constants,* n *is any a positive integer, then* $f(u)$ *is nondecreasing on* $u \in \left(0, \sqrt[n]{\frac{1}{n\beta}}\right]$.

Lemma 2.4 *If the sequence* $\{u(k)\}$ *satisfies*

$$u(k+1) = u(k)\exp\left(\alpha - \beta u^n(k)\right), \quad k = 1, 2, \ldots,$$

here, α *and* β *are both positive constants,* n *is any a positive integer and* $u(0) > 0$. *Then*

1. *If* $\alpha < \frac{2}{n}$, *then* $\lim_{k\to\infty} u(k) = \sqrt[n]{\frac{\alpha}{\beta}}$.
2. *If* $\alpha \leq \frac{1}{n}$, *then* $u(k) \leq \sqrt[n]{\frac{1}{\beta n}}$ *for all* $k = 2, 3, \ldots$.

Proof Conclusion (1) can be proved using Theorem 2.8 in [4], so we omit it.

Note that the function $x\exp(\alpha - \beta x^n)$ has a unique maximum in $x = \sqrt[n]{\frac{1}{\beta n}}$, then

$$\begin{aligned} u(k+1) &= u(k)\exp\left(\alpha - \beta u^n(k)\right) \\ &\leq \sqrt[n]{\frac{1}{\beta n}}\exp\left(\alpha - \frac{1}{n}\right) \leq \sqrt[n]{\frac{1}{\beta n}}, \quad n = 1, 2, \ldots, \end{aligned}$$

then conclusion (2) is proved. This ends the proof. □

Lemma 2.5 (see [23]) *Assume that functions* $f, g : Z_+ \times [0, \infty) \to [0, \infty)$ *satisfy* $f(n, x) \le g(n, x)(f(n, x) \ge g(n, x))$ *for* $n \in Z_+$ *and* $x \in [0, \infty)$, $g(n, x)$ *is nondecreasing for* $x > 0$. *Let sequences* $\{x(n)\}$ *and* $\{u(n)\}$ *be the nonnegative solutions of the following difference equations*

$$x(n+1) = f(n, x(n)), \quad u(n+1) = g(n, u(n)), \quad n = 0, 1, 2, \dots,$$

respectively, with $x(0) \le u(0)(x(0) \ge u(0))$, *then we have for all* $n \ge 0$

$$x(n) \le u(n)(x(n) \ge u(n)).$$

3 Local Stability

In this section, we use the eigenvalues of the variational matrix of system (1.1) at the equilibria $E_+(x_0, y_0)$ to study its local stability.

Let $J(E_+)$ be the variational matrix of system (1.1) at equilibrium $E_+(x_0, y_0)$, then

$$J(E_+) = \begin{pmatrix} 1 - \frac{mr_1 x_0^m}{K_1} & -nr_1\mu_2 x_0 y_0^{n-1} \\ -mr_2\mu_1 x_0^{m-1} y_0 & 1 - \frac{nr_2 y_0^n}{K_2} \end{pmatrix}.$$

The corresponding characteristic equation of $J(E_+)$ can be written as

$$F(\lambda) = \lambda^2 + p\lambda + q = 0, \tag{3.1}$$

where

$$p = -\left(2 - \frac{mr_1 x_0^m}{K_1} - \frac{nr_2 y_0^n}{K_2}\right),$$
$$q = \left(1 - \frac{mr_1 x_0^m}{K_1}\right)\left(1 - \frac{nr_2 y_0^n}{K_2}\right) - mnr_1r_2\mu_1\mu_2 x_0^m y_0^n.$$

Then we have the following result.

Theorem 3.1 *Assume that* $1 - \mu_1 K_1 > 0$ *and* $1 - \mu_2 K_2 > 0$, *then we have*

1. $E_+(x_0, y_0)$ *is a sink if one of the following conditions holds:*
   (a) $r_1 < t_2, r_2 < t_1, r_2 \le \frac{1}{n(1-\mu_1 K_1)}$, *where*

$$t_1 = \frac{2(1 - \mu_1\mu_2 K_1 K_2)}{n(1 - \mu_1 K_1)}, \quad t_2 = \frac{2[2(1 - \mu_1\mu_2 K_1 K_2) - nr_2(1 - \mu_1 K_1)]}{m(1 - \mu_2 K_2)[2 - nr_2(1 - \mu_1 K_1)]}.$$

   (b) $t_1 > r_2 > \frac{1}{n(1-\mu_1 K_1)}$ *and* $r_1 < \min\{t_2, t_3\}$, *where*

$$t_3 = \frac{nr_2(1 - \mu_1 K_1)}{m(1 - \mu_2 K_2)(nr_2(1 - \mu_1 K_1) - 1)}.$$

(c) $r_2 > t_4$ *and* $t_3 > r_1 > t_2$,*where* $t_4 = \frac{2}{n(1-\mu_1 K_1)}$.

2. $E_+(x_0, y_0)$ *is a source if one of the following conditions holds:*
   (a) $\frac{1}{1-\mu_1 K_1} \le r_2 < t_1$ *and* $t_3 < r_1 < t_2$*;*
   (b) $r_2 > t_4$ *and* $r_1 > \max\{t_2, t_3\}$.
3. $E_+(x_0, y_0)$ *is non-hyperbolic if one of the following conditions holds:*
   (a) $r_1 = t_2$ *and* $r_2 = t_1$*;*
   (b) $r_1 = t_2$ *and* $r_2 > t_4$*.;*
4. $E_+(x_0, y_0)$ *is a saddle if one of the following conditions holds:*
   (a) $r_2 < t_1$ *and* $r_1 > t_2$*;*
   (b) $t_1 \le r_2 \le t_4$*;*
   (c) $r_2 > t_4$ *and* $r_1 < t_2$.

Proof Here, we only prove conclusion (1) of Theorem 3.1. The others can also be proved by the same way.

From (3.1), we have

$$F(1) = 1 + p + q = mnr_1r_2x_0^m y_0^n \frac{1-\mu_1\mu_2 K_1 K_2}{K_1 K_2} > 0,$$

$$\begin{aligned} F(-1) = 1 - p + q &= 4 - 2\left(\frac{mr_1x_0^m}{K_1} + \frac{nr_2y_0^n}{K_2}\right) + mnr_1r_2x_0^m y_0^n \frac{1-\mu_1\mu_2K_1K_2}{K_1K_2} \\ &= \frac{4(1-\mu_1\mu_2K_1K_2) - 2nr_2(1-\mu_1K_1)}{1-\mu_1\mu_2K_1K_2} \\ &\quad - \frac{mr_1(1-\mu_2K_2)[2-nr_2(1-\mu_1K_1)]}{1-\mu_1\mu_2K_1K_2}, \end{aligned}$$

and

$$q - 1 = \frac{mr_1(1-\mu_2K_2)[nr_2(1-\mu_1K_1)-1] - nr_2(1-\mu_1K_1)}{1-\mu_1\mu_2K_1K_2}.$$

If $2(1-\mu_1\mu_2K_1K_2) - nr_2(1-\mu_1K_1) > 0$, then we have $r_2 < t_1$ and $2 - nr_2(1-\mu_1K_1) > 0$. Hence, $F(-1) > 0$ if

$$r_1 < \frac{2[2(1-\mu_1\mu_2K_1K_2) - nr_2(1-\mu_1K_1)]}{m(1-\mu_2K_2)[2-nr_2(1-\mu_1K_1)]} \triangleq t_2.$$

If $nr_2(1-\mu_1K_1) - 1 \le 0$, then $q < 1$. If $nr_2(1-\mu_1K_1) - 1 > 0$, then $q < 1$ is equivalent to the following inequality

$$r_1 < \frac{nr_2(1-\mu_1K_1)}{m(1-\mu_2K_2)(nr_2(1-\mu_1K_1)-1)} \triangleq t_3.$$

Hence, if condition (a) or (b) of conclusion (1) of Theorem 3.1 holds, then we have $F(-1) > 0$ and $q < 1$. From Lemma 2.2, we can obtain $E_+(x_0, y_0)$ in system (1.1) is a sink.

On the other hand, if $r_2 > \frac{2}{n(1-\mu_1K_1)} \triangleq t_4$, then we have $2(1-\mu_1\mu_2K_1K_2) - nr_2(1-\mu_1K_1) < 0$. Hence, $F(-1) > 0$ if $r_1 < t_3$. Since $r_2 > t_4$, a similar argument as in above we have $q < 1$ if $r_1 < t_3$. Hence, if condition (c) of conclusion (1) of Theorem 3.1 holds, then we have $F(-1) > 0$ and $q < 1$. From Lemma 2.2, we obtain $E_+(x_0, y_0)$ in system (1.1) is also a sink. This completes the proof. □

4 Global Stability

In this section, we will use the method of iteration scheme and the comparison principle of difference equations to study the global stability of the positive equilibrium of system (1.1).

Theorem 4.1 *Assume that* $1 - \mu_1 K_1 > 0$ *and* $1 - \mu_2 K_2 > 0$. *If* $r_1 \le \frac{1}{m}$ *and* $r_2 \le \frac{1}{n}$, *then equilibrium* $E_+(x_0, y_0)$ *of system* (1.1) *is globally asymptotically stable.*

Proof Assume that $(x(k), y(k))$ is any a solution of system (1.1) with initial value $x(0) > 0$ and $y(0) > 0$. Let

$$U_1 = \limsup_{k\to\infty} x(k) \qquad V_1 = \liminf_{k\to\infty} x(k),$$
$$U_2 = \limsup_{k\to\infty} y(k) \qquad V_2 = \liminf_{k\to\infty} y(k).$$

In the following, we will prove that $U_1 = V_1 = x_0$ and $U_2 = V_2 = y_0$.

From the first equation of system (1.1) we obtain

$$x(k+1) \le x(k) \exp\left(r_1 - \frac{r_1}{K_1} x^m(k)\right), \quad k = 0, 1, 2, \dots.$$

Consider the auxiliary equation

$$u(k+1) = u(k) \exp\left(r_1 - \frac{r_1}{K_1} u^m(k)\right). \tag{4.1}$$

Let $u(k)$ be any a solution of Eq. (4.1) with initial value $u(0) > 0$. For $0 < r_1 \le \frac{1}{m}$, by conclusion (2) of Lemma 2.4, we have that $u(k) \le \sqrt[m]{\frac{K_1}{mr_1}}$ for all $n \ge 2$. From Lemma 2.3, we have $f(u) = u \exp(r_1 - \frac{r_1}{K_1} u^m)$ is nondecreasing for $u \in \left(0, \sqrt[m]{\frac{K_1}{mr_1}}\right]$.

Hence, from Lemma 2.5, we have $x(k) \le u(k)$ for all $k \ge 2$, where $u(k)$ is the solution of Eq. (4.1) with $u(2) = x(2)$. By conclusion (1) of Lemma 2.4, we further obtain

$$U_1 = \limsup_{k\to\infty} x(k) \le \lim_{k\to\infty} u(k) = \sqrt[m]{K_1} \triangleq M_1^x.$$

Hence, for any $\varepsilon > 0$ small enough, there exists a $N_1 > 2$ such that if $n \ge N_1$, then $x(k) \le M_1^x + \varepsilon$.

From the second equation of system (1.1) we have

$$y(k+1) \le y(k)\exp\left(r_2 - \frac{r_2}{K_2}y^n(k)\right), \quad k \ge N_1.$$

By the same way, we can obtain

$$U_2 = \limsup_{k\to\infty} y(k) \le \sqrt[n]{K_2} \triangleq M_1^y.$$

Hence, for any $\varepsilon > 0$ small enough, there exists a $N_2 > N_1$ such that if $k \ge N_2$, then $y(k) \le M_1^y + \varepsilon$.

From the first equations of system (1.1) again, we further have

$$x(k+1) \ge x(k)\exp\left[r_1\left(1 - \frac{1}{K_1}x^m(k) - \mu_2\left(M_1^y + \varepsilon\right)^n\right)\right], \quad k \ge N_2.$$

Consider the auxiliary equation

$$u(k+1) = u(k)\exp\left[r_1\left(1 - \frac{1}{K_1}u^m(k) - \mu_2\left(M_1^y + \varepsilon\right)^n\right)\right]. \tag{4.2}$$

From the arbitrariness of ε, we can let $\varepsilon < \frac{1-\sqrt[n]{\mu_2}M_1^y}{\sqrt[n]{\mu_2}}$. From $1 - \mu_2 K_2 > 0$, we have $0 < r_1(1 - \mu_2(M_1^y + \varepsilon)^n) < \frac{1}{m}$. By conclusion (2) of Lemma 2.4, we conclude that $u(k) \le \sqrt[m]{\frac{K_1}{mr_1}}$ for all $k \ge N_2$, where $u(k)$ is any solution of Eq. (4.2) with initial value $u(0) > 0$. From Lemma 2.3, we have that $f(u) = u\exp(r_1 - r_1\mu_2(M_1^y + \varepsilon)^n - \frac{r_1}{K_1}u^m)$ is nondecreasing for $u \in \left(0, \sqrt[m]{\frac{K_1}{mr_1}}\right]$. Hence from Lemma 2.5 we have that $x(k) \ge u(k)$ for all $k \ge N_2$, where $u(k)$ is the solution of Eq. (4.2) with $u(N_2) = x(N_2)$. From conclusion (1) of Lemma 2.4 again, we have

$$V_1 = \liminf_{n\to\infty} x(k) \ge \lim_{k\to\infty} u(k) = \sqrt[m]{K_1\left(1 - \mu_2\left(M_1^y + \varepsilon\right)^n\right)}.$$

From the arbitrariness of $\varepsilon > 0$, we have $V_1 \ge N_1^x$, where

$$N_1^x = \sqrt[m]{K_1\left(1 - \mu_2(M_1^y)^n\right)}.$$

Hence, for $\varepsilon > 0$ small enough, there exists a $N_3 > N_2$ such that if $k \ge N_3$, then $x(k) \ge N_1^x - \varepsilon$.

From the second equations of system (1.1) we further have

$$y(k+1) \ge y(k)\exp\left[r_2\left(1 - \frac{1}{K_2}y^n(k) - \mu_1\left(M_1^x + \varepsilon\right)^m\right)\right], \quad k \ge N_3.$$

By the same way, we can obtain

$$V_2 = \liminf_{k\to\infty} y(k) \geq \sqrt[n]{K_2\left(1-\mu_1\left(M_1^x+\varepsilon\right)^m\right)}.$$

From the arbitrariness of $\varepsilon > 0$, we get $V_2 \geq N_1^y$, where

$$N_1^y = \sqrt[n]{K_2\left(1-\mu_1\left(M_1^x\right)^m\right)} < \sqrt[n]{K_2}.$$

Hence, for $\varepsilon > 0$ small enough, there exists a $N_4 \geq N_3$ such that if $k \geq N_4$, then $y(k) \geq N_1^y - \varepsilon > 0$.

Further, from the first equations of system (1.1) we have

$$x(k+1) \leq x(k)\exp\left[r_1\left(1-\mu_2\left(N_1^y-\varepsilon\right)^n-\frac{x^m(k)}{K_1}\right)\right], \quad k \geq N_4.$$

Using the similar argument as in above, we can get

$$U_1 = \limsup_{k\to\infty} x(k) \leq \sqrt[m]{K_1\left(1-\mu_2\left(N_1^y-\varepsilon\right)^n\right)}.$$

From the arbitrariness of $\varepsilon > 0$, we claim that $U_1 \leq M_2^x$, where

$$M_2^x = \sqrt[m]{K_1\left(1-\mu_2\left(N_1^y\right)^n\right)} < \sqrt[m]{K_1}.$$

Hence, for any $\varepsilon > 0$ small enough, there exists a $N_5 \geq N_4$ such that if $k \geq N_5$, then $x(k) \leq M_2^x + \varepsilon$.

From the second equations of system (1.1) we further obtain

$$y(k+1) \leq y(k)\exp\left[r_2\left(1-\mu_1\left(N_1^x-\varepsilon\right)^m-\frac{y^n(k)}{K_2}\right)\right], \quad k \geq N_5.$$

Similarly to the above argument, we can obtain

$$U_2 = \limsup_{k\to\infty} y(k) \leq \sqrt[n]{K_2\left(1-\mu_1\left(N_1^x-\varepsilon\right)^m\right)}.$$

From the arbitrariness of $\varepsilon > 0$, we obtain $U_2 \leq M_2^y$, where

$$M_2^y = \sqrt[n]{K_2\left(1-\mu_1\left(N_1^x\right)^m\right)} < \sqrt[n]{K_2}.$$

Hence, for $\varepsilon > 0$ small enough, there exists a $N_6 > N_5$ such that if $k \geq N_6$, $y(k) \leq M_2^y + \varepsilon$.

Further, from the first equations of system (1.1) we obtain

$$x(k+1) \geq x(k) \exp\left[r_1\left(1-\mu_2\left(M_2^y+\varepsilon\right)^n-\frac{x^m(k)}{K_1}\right)\right], \quad k \geq N_6.$$

Using a similar argument, we again can obtain

$$V_1 = \liminf_{k\to\infty} x(k) \geq \sqrt[m]{K_1\left(1-\mu_2\left(M_2^y+\varepsilon\right)^n\right)}.$$

From the arbitrariness of $\varepsilon > 0$, we get that $V_1 \geq N_2^x$, where

$$N_2^x = \sqrt[m]{K_1\left(1-\mu_2\left(M_2^y\right)^n\right)} > \sqrt[m]{K_1\left(1-\mu_2\left(M_1^y\right)^n\right)} = N_1^x.$$

Hence, for any $\varepsilon > 0$ small enough, there exists a $N_7 > N_6$ such that if $k \geq N_7$, $x(k) \geq N_2^x - \varepsilon > 0$.

From the second equations of system (1.1) we further have

$$y(k+1) \geq y(k) \exp\left[r_2\left(1-\mu_1\left(M_2^x+\varepsilon\right)^m-\frac{y^n(k)}{K_2}\right)\right], \quad k \geq N_7.$$

Using a similar discussion, we again can obtain

$$V_2 = \liminf_{k\to\infty} y(k) \geq \sqrt[n]{K_2\left(1-\mu_1\left(M_2^x+\varepsilon\right)^m\right)}.$$

From the arbitrariness of $\varepsilon > 0$, we claim that $V_2 \geq N_2^y$, where

$$N_2^y = \sqrt[n]{K_2\left(1-\mu_1\left(M_2^x\right)^m\right)} > \sqrt[n]{K_2\left(1-\mu_1\left(M_1^x\right)^m\right)} = N_1^y.$$

Repeating the above process, we can finally obtain four sequences $\{M_k^x\}$, $\{N_k^x\}$, $\{M_k^y\}$ and $\{N_k^y\}$ such that

$$M_k^x = \sqrt[m]{K_1\left(1-\mu_2\left(N_{k-1}^y\right)^n\right)} \qquad M_k^y = \sqrt[n]{K_2\left(1-\mu_1\left(N_{k-1}^x\right)^m\right)}, \tag{4.3}$$

and

$$N_k^x = \sqrt[m]{K_1\left(1-\mu_2\left(M_k^y\right)^n\right)} \qquad N_k^y = \sqrt[n]{K_2\left(1-\mu_1\left(M_k^x\right)^m\right)}. \tag{4.4}$$

Clearly, we have for any integer $k > 0$

$$N_k^x \leq V_1 \leq U_1 \leq M_k^x \qquad N_k^y \leq V_2 \leq U_2 \leq M_k^y. \tag{4.5}$$

In the following, we will prove that $\{M_k^x\}$ and $\{M_k^y\}$ are monotonically decreasing, $\{N_k^x\}$ and $\{N_k^y\}$ are monotonically increasing, by means of inductive method.

Firstly, it is clear that

$$M_2^x \leq M_1^x, \quad M_2^y \leq M_1^y, \quad N_2^x \geq N_1^x, \quad N_2^y \geq N_1^y.$$

For $k(k \geq 2)$,we assume that $M_k^x \leq M_{k-1}^x$ and $N_k^x \geq N_{k-1}^x$, then we further have

$$M_k^y = \sqrt[n]{K_2\left(1-\mu_1\left(N_{k-1}^x\right)^m\right)} \leq \sqrt[n]{K_2\left(1-\mu_1\left(N_k^x\right)^m\right)} = M_{k-1}^y, \tag{4.6}$$

and

$$N_k^y = \sqrt[n]{K_2\left(1-\mu_1\left(M_k^x\right)^m\right)} \geq \sqrt[n]{K_2\left(1-\mu_1\left(M_{k-1}^x\right)^m\right)} = N_{k-1}^y. \tag{4.7}$$

From (4.6) and (4.7) we have

$$\begin{aligned}
\left[M_{k+1}^x\right]^m - \left[M_k^x\right]^m &= K_1\left(1-\mu_2\left(N_k^y\right)^n\right) - K_1\left(1-\mu_2\left(N_{k-1}^y\right)^n\right) \\
&= -K_1\mu_2\left[N_k^y)^n - N_{k-1}^y)^n\right] \\
&\leq 0. \qquad (4.8) \\
\left[M_{k+1}^y\right]^n - \left[M_k^y\right]^n &= K_2\left(1-\mu_1\left(N_k^x\right)^m\right) - K_2\left(1-\mu_1\left(N_{k-1}^x\right)^m\right) \\
&= -K_2\mu_1\left[N_k^x)^m - N_{k-1}^x)^m\right] \\
&\leq 0. \qquad (4.9)
\end{aligned}$$

Note that $a^n - b^n$ and $a-b$ have the same sign, when both a and b are positive constants. Therefore, from (4.8) and (4.9), we have $M_{k+1}^x \leq M_k^x$ and $M_{k+1}^y \leq M_k^y$.

From (4.8) and (4.9) we further have

$$\begin{aligned}
\left[N_{k+1}^x\right]^m - \left[N_k^x\right]^m &= K_1\left(1-\mu_2\left(M_{k+1}^y\right)^n\right) - K_1\left(1-\mu_2\left(M_k^y\right)^n\right) \\
&= -K_1\mu_2\left[\left(M_{k+1}^y\right)^n - \left(M_k^y\right)^n\right] \\
&\geq 0.
\end{aligned}$$

and

$$\begin{aligned}
\left[N_{k+1}^y\right]^n - \left[N_k^y\right]^n &= K_2\left(1-\mu_1\left(M_{k+1}^x\right)^m\right) - K_2\left(1-\mu_1\left(M_k^x\right)^m\right) \\
&= -K_2\mu_1\left[\left(M_{k+1}^x\right)^m - \left(M_k^x\right)^m\right] \\
&\geq 0.
\end{aligned}$$

This means that $\{M_k^x\}$ and $\{M_k^y\}$ are monotonically decreasing, $\{N_k^x\}$ and $\{N_k^y\}$ are monotonically increasing. Therefore, by the criterion of monotone bounded, we have proved that every one of this four sequences has a limit.

From (4.3) and (4.4), we can obtain

$$\left(M_k^x\right)^m = K_1\left[1-\mu_2\left(N_{k-1}^y\right)^n\right] = K_1\left[1-\mu_2K_2\left(1-\mu_1\left(M_{k-1}^x\right)^m\right)\right]$$

and

$$\left(M_k^y\right)^n = K_2\left[1-\mu_1\left(N_{k-1}^x\right)^m\right] = K_2\left[1-\mu_1K_1\left(1-\mu_2\left(M_{k-1}^y\right)^n\right)\right].$$

Taking $k \to \infty$ in both sides of the above two equations, respectively, then we have

$$\lim_{k\to\infty} M_k^x = x_0, \qquad \lim_{k\to\infty} M_k^y = y_0.$$

By the same way, we also can obtain

$$\lim_{k\to\infty} N_k^x = x_0, \qquad \lim_{k\to\infty} N_k^y = y_0.$$

It follows from (4.5) that

$$U_1 = V_1 = x_0, \qquad U_2 = V_2 = y_0.$$

Therefore, we finally have

$$\lim_{k\to\infty} x(k) = x_0, \qquad \lim_{k\to\infty} y(k) = y_0.$$

This shows that equilibrium $E_+(x_0, y_0)$ of system (1.1) is globally attractive.

From Theorem 3.1, we can obtain that equilibrium $E_+(x_0, y_0)$ of system (1.1) is locally asymptotically stable. Therefore, we finally obtain that $E_+(x_0, y_0)$ is globally asymptotically stable. This completes the proof. □

Remark 1 The main results obtained in the present paper is for any positive integer m and n, which generalizes what paper [7] has obtained. The method given in this paper is new and very resultful comparing with articles [6,9,10,14,16,19,22] on the study of global stability for discrete predator–prey systems. Note that our conditions is more better than the conditions of theorem 3 in paper [7]. For example, the conditions of theorem 3 in paper [7] has been obtained as follows:

(H_1) $1 - \mu_1 x^* > 0$ and $1 - \mu_2 y^* > 0$, where

$$x^* = \frac{K_1}{r_1}\exp(r_1 - 1), \qquad y^* = \frac{K_2}{r_2}\exp(r_2 - 1).$$

(H_2)

$$\lambda_1 = \max\left\{\left|1 - \frac{r_1}{K_1}x^*\right|, \left|1 - \frac{r_1}{K_1}x_*\right|\right\} + \mu_2 r_1 y^* < 1$$

and

$$\lambda_2 = \max\left\{\left|1 - \frac{r_2}{K_2} y^*\right|, \left|1 - \frac{r_2}{K_2} y_*\right|\right\} + \mu_1 r_2 x^* < 1,$$

where

$$x_* = K_1(1 - \mu_2 y^*) \exp\left[r_1\left(1 - \mu_2 y^* - \frac{x^*}{K_1}\right)\right]$$

and

$$y_* = K_2(1 - \mu_1 x^*) \exp\left[r_2\left(1 - \mu_1 x^* - \frac{y^*}{K_2}\right)\right].$$

Note that $\frac{\exp(r-1)}{r} > 1$ for $r > 0$, therefore, it is easy to see that condition (H_1) is stronger than $1 - \mu_1 K_1 > 0$ and $1 - \mu_2 K_2 > 0$.

We can also see that condition (H_2) is complicated comparing with our conditions $r_1 \leq \frac{1}{m}$ and $r_2 \leq \frac{1}{n}$, and not easy to verify. Furthermore, if taking $r_1 = r_2 = 1$, then we have $x^* = K_1$, $y^* = K_2$, $x_* = K_1(1 - \mu_2 K_2)\exp(-\mu_2 K_2)$ and $y_* = K_2(1 - \mu_1 K_1)\exp(-\mu_1 K_1)$. Then

$$\lambda_1 = 1 + \mu_2 K_2 \exp(-\mu_2 K_2) - \exp(-\mu_2 K_2) + \mu_2 K_2.$$

It is clear to see that $\lambda_1 > 1$ for $\mu_2 K_2 > \frac{1}{2}$. This shows that (H_2) is stronger than $r_1 = r_2 = 1$, here $m = n = 1$.

Remark 2 According to Theorem 4.1 of this paper, we have known that the equilibrium $E_+(x_0, y_0)$ of system (1.1) is globally asymptotically stable for $r_1 \leq \frac{1}{m}$, $r_2 \leq \frac{1}{n}$, and is locally asymptotically stable for $r_1 < t_2$, $r_2 < t_1$ and $r_2 \leq \frac{1}{n(1-\mu_1 K_1)}$ (Theorem 3.1). However, whether the equilibrium $E_+(x_0, y_0)$ is also globally asymptotically stable for $\frac{1}{m} < r_1 < t_2$, $\frac{1}{n} < r_2 < t_1$ and $r_2 \leq \frac{1}{1-\mu_1 K_1}$, it is still open.

Remark 3 Another important and interesting open question is whether we can also obtain the same inequality (4.5) but do not apply the comparison principle. If it is possible, then the conditions on the global stability of positive equilibrium of system (1.1) may be extended.

Remark 4 The condition in Theorem 3.1 is to guarantee the existence of positive equilibrium $E_+(x_0, y_0)$ of system (1.1), and the possibility of how the two species can coexist. If the conditions in conclusion (1) of Theorem 3.1 do not hold, then the positive equilibrium of system (1.1) will be unstable.

Remark 5 The approach can also be devoted to studying the global asymptotic stability of positive equilibrium for the other general multiple species discrete population systems. We would like to do some valuable research about the subject.

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