Quasinormal Modes of Black Holes*

Small Perturbations of Black Holes

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One typically disturbs a system from equilibrium and studies its behavior to understand its stability. This procedure gives insight into the dynamics at play. It is one of the few techniques used in all areas of physics, be it Newtonian mechanics, quantum mechanics, or general relativity. We have continued this effort by disturbing black holes. We will see that a black hole chimes like a bell when it is disturbed. We will study this chiming mathematically in Schwarzschild black holes.

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Introduction

Perturbations play a significant role in the stability of a system. It is easy to reason that a car parked on a peak will go downhill if it is disturbed. But such a deduction is not trivial when we are dealing with complex mathematical objects. One such object is produced by the gravitational collapse of huge stars—first proposed by Oppenheimer and Snyder in 1939—and today known as the black holes.

Everyone wanted to know if these exotic objects existed in reality or were just a mathematical anomaly. Hence, they asked the question, what happens if black holes are disturbed from their initial state. Do they fizz out into oblivion, or do they come back to a stable state?

The question was answered was provided 30 years later by C V Vishveshwara, who proved that perturbed black holes returned to a stable state by emitting waves in some characteristic frequencies. And hence, these exotic objects were stable. Little did he

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know that his insight into perturbations of black holes would become the backbone for the detection of gravitational waves.

When one hits a bell with a hammer, the bell starts ringing, and this clamor gradually fades away. This analogy can be extended to black holes when they are impinged upon by some gravitational disturbance—say two black holes are inspiraling towards each other, and they ultimately merge into a single black hole. The resulting black hole starts chiming in some characteristic frequencies, and they become fainter and fainter until the black hole settles to a stable state.

Ringdown is the last stage of a black hole merger preceded by inspiral and merger.

Huge gravitational wave detectors observe these dying notes in the ringdown phase of the black hole merger. We have a special name for these notes, called the 'quasinormal modes'. The qualifier *quasi-* is used to indicate that these modes are similar to, but not exactly equal to normal modes. We will see that quasinormal mode frequencies also have an imaginary part which acts as a damping term. This article aims to find these quasinormal modes. We start with the simplest of these complex mathematical objects, a non-rotating, non-charged and spherical black hole, also known as 'Schwarzschild black hole'.

We start with a formalism of gravity, the Einstein gravity in 4 dimensions (3 space + 1 time). The equations which relate spacetime curvature to matter in the space are the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \qquad \mu, \nu = 0, 1, 2, 3 \tag{1}$$

where $T_{\mu\nu}$ is the measure of energy in matter and radiation, called the stress-energy tensor. The spacetime is governed by $g_{\mu\nu}$, which encodes the information about distance between two spacetime points, and it is called the metric tensor. $R_{\mu\nu}$ is the measure of curvature of spacetime which depends on $g_{\mu\nu}$, called the Ricci tensor. The trace of Ricci tensor is known as the Ricci scalar, $R = g^{\mu\nu}R_{\mu\nu}$[1]. (1) can be written succinctly as

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2}$$

[1] We have used Einstein sum convention. If there is a repeated lower and upper index, it implies sum over that index. For example, $x_\mu y^\mu = \sum_{\mu=0}^{3} x_\mu y^\mu$.

with

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

$G_{\mu\nu}$ is known as the Einstein tensor. An important point to note is that $G_{\mu\nu}$ and $T_{\mu\nu}$ are 4×4 symmetric matrices (symmetric under the exchange of indices, μ and ν) and hence, the Einstein equations are a set of 10 equations[2]. One, in general, is interested in finding the spacetime (the metric tensor) given a particular arrangement of matter (the stress-energy tensor). We then perturb these solutions away from the background spacetime ($g^0_{\mu\nu}$) by adding a small perturbation, $h_{\mu\nu}$ so that $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$ still satisfies the Einstein equations (1) up to linear order in $h_{\mu\nu}$.

[2]In an n-dimensional spacetime, there are $\frac{n(n+1)}{2}$ independent equations.

As we will see in the later sections, when perturbations ($h_{\mu\nu}$) are applied to Einstein's equations, we get a Schrödinger-like differential equation for the perturbation,

$$\frac{d^2\Psi}{dr_*^2} + \left(\omega^2 - V(r_*)\right)\Psi = 0, \tag{3}$$

where $V(r_*)$ is the analogue of potential in Schrödinger equation, ω is the frequency of oscillations, and r_* is known as the tortoise coordinate in the literature. The exact expressions for $V(r_*)$ and r_* depend on the background spacetime $g^0_{\mu\nu}$.

Compare equation (3) with the time-independent Schrödinger equation in 1 dimension,

$$\frac{d^2\Psi}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))\Psi = 0,$$

where $\hbar$ is the Planck constant. E, and m are the energy and mass of the particle moving in a potential $V(x)$. In quantum mechanics, the particle is described by a complex valued function called the wavefunction Ψ.

These quasinormal frequencies are independent of the processes which give rise to oscillations. By analyzing the ringdown behavior, we can probe black holes to determine their mass, charge, and angular momentum, in addition to confirming their existence. Quasinormal frequencies serve as a unique fingerprint for oscillating black holes.

In this article, we have used natural units, that is, $c = 1 = G$, where c is the speed of light, and G is Newton's gravitational constant. In the following sections, we will discuss the boundary conditions and will ultimately find the differential equations for perturbations of Schwarzschild (non-rotating and uncharged) black holes.

1. Quasinormal Modes

We wish to find the quasinormal mode (QNM) spectrum, that is, the frequencies[3] of oscillation which are subject to certain boundary conditions. We choose boundary conditions in a manner such that the solutions are physically valid. We are also interested in enumerating the number of such ω's. Finally, we want to find a way to calculate them.

[3] The frequency ω is, in general, a complex number.

To review boundary conditions, we take Schwarzschild black holes as an example. The Schwarzschild line element[4] for a spherical object of mass M is given by

[4] A line element can be thought of as the distance between two points which are infinitesimally close to each other.

$$\begin{aligned} ds^2 &= g^0_{\mu\nu} dx^\mu dx^\nu, \\ &= \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin\theta^2 d\varphi^2). \end{aligned} \tag{4}$$

The above solution for metric tensor is found for no matter or radiation outside the spherical body, that is, $T_{\mu\nu} = 0$. The Einstein's equations read as, $R_{\mu\nu} = 0$ since R can be proved to be zero.

A particular surface of interest is the event horizon where the metric (4) becomes singular. Anything which crosses the event horizon, cannot counter the gravitational pull of the black hole, and therefore, cannot escape the black hole. For a Schwarzschild black hole, the event horizon is at $r = 2M$ where the coefficient of dr^2 in (4) blows up![5]

[5] However, by a suitable coordinate transformation, the singularity can be removed. Hence, the singularity is a manifestation of the coordinate system we used, but there is no physical singularity.

Another useful trick is to change the radial coordinate (r) to tortoise coordinate (r_*), which are related by,

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right). \tag{5}$$

Observe that, as $r \to 2M$, $r_* \to -\infty$ and as $r \to \infty$, $r_* \to \infty$. As we approach the event horizon, r changes slowly with r_*, that is, for a large change in r_*, the change in r is small. That is why r_* is called the tortoise coordinate.

If nothing can escape the event horizon, the gravitational waves should be no exception. So, QNMs should be purely ingoing at the event horizon of black holes ($r_* \to -\infty$). Asymptotically as $r_* \to -\infty$, Ψ should be an ingoing plane wave,

$$\Psi \sim e^{i\omega(t+r_*)}. \tag{6}$$

At spatial infinity, none of the QNMs should reflect back. There is no physical significance of wave coming in from spatial infinity. Hence, they are purely outgoing. As $r_* \to \infty$,

$$\Psi \sim e^{i\omega(t-r_*)}, \tag{7}$$

which is a purely outgoing plane wave.

To summarize, QNMs are purely ingoing at the event horizon and purely outgoing at the spatial infinity. Also, no incident radiation is given to the system.

Compare the boundary conditions on QNMs with propagating waves which can be written as $\cos(\omega t - kx + \phi_0)$ or $\cos(\omega t + kx + \phi_0)$, if they are moving in $+x$ or $-x$ direction respectively.

With these boundary conditions in mind, we need to solve for the angular frequencies, ω from differential equation (3) viewed as an eigenvalue problem:

$$\mathscr{L}\Psi = -\omega^2\Psi, \tag{8}$$

where

$$\mathscr{L} = \frac{\mathrm{d}^2}{\mathrm{d}r_*^2} - V. \tag{9}$$

Now, the problem boils down to finding the eigenvalues of the $\mathscr{L}$ operator. The problem is simple to state but a nightmare to solve. Later, we will discuss the Wentzel–Kramers–Brillouin (WKB) approximation with a slight modification used to find QNMs.

2. Schwarzschild Black Hole

Karl Schwarzschild solved Einstein equations (1) in 1916 for the gravitational field outside a spherical object. It was the first solution found for Einstein equations, which can be used for approximating the spacetime outside stars and planets.

Schwarzschild line element (4) can be put into a compact form:

$$ds^2 = g^0_{\mu\nu} dx^\mu dx^\nu, \tag{10}$$

where $\mu = 0, 1, 2, 3$ corresponds to t, r, θ, ϕ respectively, and $g^0_{\mu\nu}$ is the metric tensor given by,

$$g^0_{\mu\nu} = \begin{bmatrix} \left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin\theta^2 \end{bmatrix}. \tag{11}$$

As noted previously, we add a perturbation to the above metric,

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}. \tag{12}$$

By imposing the symmetry condition on the metric, the number of independent components of $h_{\mu\nu}$ are reduced from 16 to 10. Further, there is a neat analogy with the transverse nature of electromagnetic waves prompting us to set three components of $h_{\mu\nu}$ to be zero. To capture the remaining variables, we introduce a metric which is non-stationary, axisymmetric[6], and also satisfies the Einstein equation in the vacuum[7]:

[6]For simplicity and clarity, we focus only on QNMs arising from axisymmetric perturbations.

$$ds^2 = e^{2\nu} dt^2 - e^{2\zeta}(d\varphi - \Omega dt - q_1 dr - q_2 d\theta)^2 - e^{2\mu_1}(dr)^2 - e^{2\mu_2}(d\theta)^2. \tag{13}$$

[7]This metric can be proved to be the most general solution of the non-stationary, axisymmetric spacetimes [1].

The Schwarzschild metric is a specific case of this generalization with no rotation, that is, $\Omega = q_1 = q_2 = 0$ and

$$e^{2\nu} = e^{-2\mu_1} = \kappa/r^2 \equiv 1 - 2M/r, \tag{14}$$

$$\kappa = r^2 - 2Mr, \tag{15}$$

$$e^{\mu_2} = r, \tag{16}$$

$$e^{\zeta} = r \sin\theta. \tag{17}$$

When the metric is slightly perturbed, Ω, q_1, q_2 may become non-zero, and ν, ζ, μ_1, μ_2 may experience small increments. This is an indication that these two types of perturbations must be dealt with independently. We will consider the case of non-zero Ω, q_1, q_2—collectively known as axial perturbations. These impart rotation to the black hole. The perturbations of remaining parameters are known as polar perturbations[8].

[8]In literature, axial perturbations are also known as Regge–Wheeler or odd perturbations or vector-type perturbations. The polar perturbations are also known as Zerilli or even perturbations or scalar-type perturbations. The names, odd and even, are inspired from their parity which are $(-1)^{l+1}$ and $(-1)^l$ respectively.

Chandrasekhar showed that QNMs obtained from these two types of perturbations are the same, that is, they are *isospectral*. In particular, he showed that the form of potential in equation (3) obtained from axial perturbations could be transformed to the one obtained from polar perturbations.

It is a common saying that hindsight is 20/20. And with the above insight in mind, we will only deal with axial perturbations in this article. For a complete treatment, Chandrasekhar's book is an excellent resource, and apart from some difference in notation, we have followed Chandrasekhar's book.

Historically, it was Regge and Wheeler [9] who first derived Eq. (3) and the exact expression for the potential for axial perturbations. And rightfully so, the potential is known as the *Regge–Wheeler potential*. And it was Zerilli [10] who solved the problem of polar perturbations. But in this article, we don't follow the procedure by either Regge and Wheeler or Zerilli, but our method is akin to what is followed by Chandrasekhar [1].

2.1 *Regge–Wheeler Equation*

A small perturbation of the Einstein equation (1) will give[9], $\delta R_{\mu\nu} = 0$. The equations governing the three variables (Ω, q_1, q_2) in an unperturbed Schwarzschild metric are, $R_{\phi r} = 0 = R_{\phi\theta}$. The equations governing their perturbation are

$$\left(e^{3\zeta+\nu-\mu_1-\mu_2} Q_{12}\right)_{,2} = -e^{3\zeta-\nu+\mu_2-\mu_1} Q_{01,0} \qquad (\delta R_{\phi r} = 0), \qquad (18)$$

$$\left(e^{3\zeta+\nu-\mu_1-\mu_2} Q_{12}\right)_{,1} = e^{3\zeta-\nu+\mu_1-\mu_2} Q_{02,0} \qquad (\delta R_{\phi\theta} = 0), \qquad (19)$$

where

$$Q_{12} = q_{1,2} - q_{2,1}, \qquad (20)$$

$$Q_{0i} = \Omega_{,i} - q_{i,0} \qquad i = 1, 2. \qquad (21)$$

We have used a shorthand notation for writing derivatives. $\gamma_{\alpha,\beta}$ means partial derivative of γ_α with respect to x^β where γ can be any variable, and x^β corresponds to t, r, θ, ϕ for $\beta = 0, 1, 2, 3$ respectively. For example, in equation (20), $q_{1,2}$ refers to partial derivative of q_1 with respect to x^2 $(= \theta)$.

[9]Recall that $R_{\mu\nu} = 0$ for Schwarzschild background. And since we are only interested in gravitational perturbations, $\delta T_{\mu\nu} = 0$.

We now introduce a new variable, $Q(t, r, \theta) = \kappa Q_{12} \sin^3 \theta$, where κ is defined in (15) and we get,

$$\frac{1}{r^4 \sin^3 \theta} \frac{\partial Q}{\partial \theta} = -\left(\Omega_{,1} - q_{1,0}\right)_{,0} , \tag{22}$$

$$\frac{\kappa}{r^4 \sin^3 \theta} \frac{\partial Q}{\partial r} = \left(\Omega_{,2} - q_{2,0}\right)_{,0} . \tag{23}$$

We assume that the perturbations (Ω and q_i) have time dependence of the form, $e^{i\omega t}$. This corresponds to a single Fourier mode of frequency ω. The equations now become

$$\frac{1}{r^4 \sin^3 \theta} \frac{\partial Q}{\partial \theta} = -i\omega\Omega_{,1} - \omega^2 q_1, \tag{24}$$

$$\frac{\kappa}{r^4 \sin^3 \theta} \frac{\partial Q}{\partial r} = i\omega\Omega_{,2} + \omega^2 q_2. \tag{25}$$

Eliminating Ω from above equations, we get,

$$r^4 \frac{\partial}{\partial r}\left(\frac{\kappa}{r^4}\frac{\partial Q}{\partial r}\right) + \sin^3 \theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin^3 \theta}\frac{\partial Q}{\partial \theta}\right) + \omega^2 \frac{r^4}{\kappa} Q = 0. \tag{26}$$

Observe that the above equation is separated between the radial (r) and the angular part (θ). Writing $Q(r, \theta) = R(r)\Theta(\theta)$, we can decouple the equations as

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\sin^3 \theta}\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}\right) + \sin^3 \theta (l + 2)(l - 1)\Theta(\theta) = 0, \tag{27}$$

$$\kappa \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\kappa}{r^4}\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right) - (l + 2)(l - 1)\frac{\kappa}{r^4} R(r) + \omega^2 R(r) = 0. \tag{28}$$

Equation (27) is known as the Gegenbauer equation and its solutions are called the Gegenbauer functions and is well known in mathematical literature. Equation (28) can be further simplified by introducing tortoise coordinate,

$$r_* = r + 2M \ln(r/2M - 1), \qquad \frac{\mathrm{d}}{\mathrm{d}r_*} = \frac{\kappa}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}, \tag{29}$$

and let,

$$R(r) = r\Psi(r). \tag{30}$$

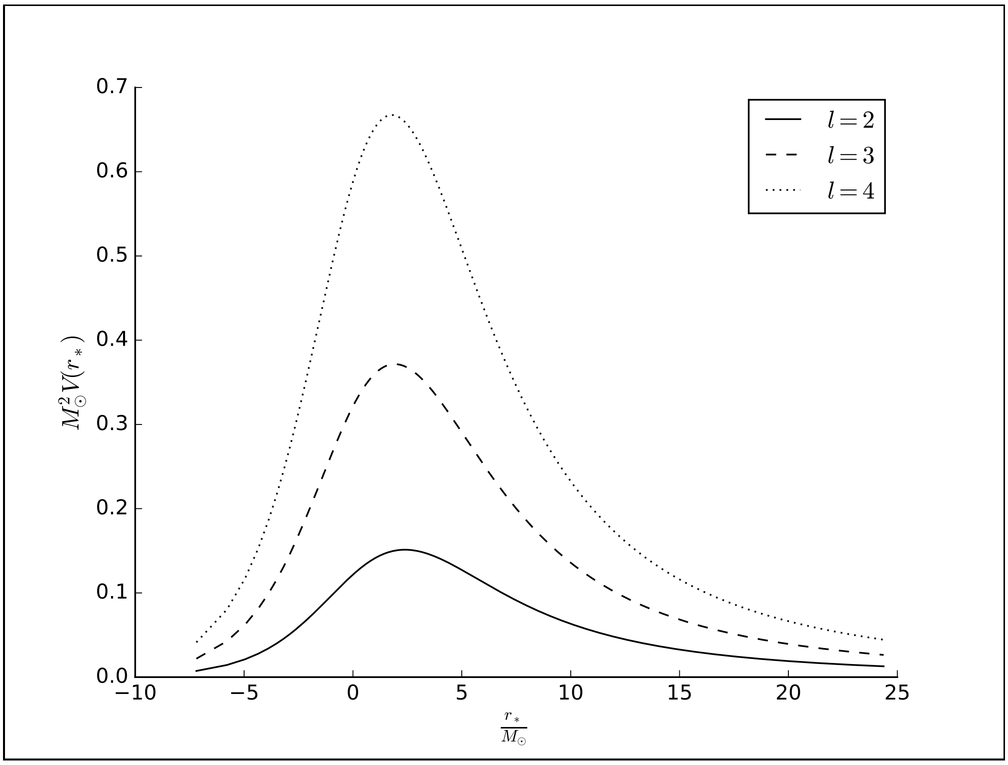

Figure 1. The potential barrier for Schwarzschild black holes is shown for modes, $l = 2, 3, 4$.

Using substitutions (29) and (30), we find that (28) reduces to

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r_*^2} + \omega^2\Psi = V\Psi, \tag{31}$$

where potential, V is given by

$$V(r) = \frac{\kappa}{r^5}\left(l(l+1)r - 6M\right), \tag{32}$$

or,

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right). \tag{33}$$

Equation (31) is known as the Regge–Wheeler equation and the potential (33) is known as the Regge–Wheeler potential. The potential for a solar mass black hole versus the tortoise coordinate is plotted in *Figure* 1. $V(r_*)$ is implicitly defined as, $V(r_*) \equiv$

$V(r(r_*))$, where $r(r_*)$ is found by inverting equation (29).

We may look for a solution of the form, $\Psi = \exp\left(i\int_0^{r_*}\phi dr_*\right)$ with,

$$\phi \to -\omega \text{ as } r_* \to \infty \text{ and } \phi \to \omega \text{ as } r_* \to -\infty \qquad (34)$$

as dictated by the boundary conditions (6) and (7). Using this form in (31), we get a first order equation for perturbations,

$$i\phi_{,r_*} + \omega^2 - \phi^2 - V = 0. \qquad (35)$$

Solutions of (35) exist only when ω takes discrete values as shown by Chandrasekhar [1]. The number of such frequencies are infinite, as proved by Bachelot and Motet-Bachelot [6].

In the next section, we will explain one method to compute quasinormal frequencies.

2.2 *WKB Approximation*

Inspired by the quantum mechanics (QM), we can try to solve (31) by the Wentzel–Kramers–Brillouin (WKB) approximation method, which is the theme of this section. In 1-dimensional QM, when a particle scatters off a potential (see *Figure* 2), some part of the incident beam is reflected and the other part is transmitted. The amplitudes of the transmitted (Region III), and reflected (Region I) waves are $e^{-\gamma}$, and $\sqrt{1-e^{-2\gamma}}$ times the amplitude of the incoming wave from region I respectively. Thus, the amplitude of the transmitted wave is $e^{-\gamma}$ times the amplitude of the reflected wave (for $\gamma \gg 1$). In general, $\gamma = \int_{r_1}^{r_2} dr_* \sqrt{V(r_*) - E}$, where $V(r_*)$ is the potential, E is the energy of wave, and r_1 and r_2 are the turning points[10]. And if the potential is slowly varying, we can approximate the potential as a Taylor series to first order in r_* near the turning point r_1, $V(r_*) \approx V(r_1) + V'(r_1)(r_* - r_1)$. For an elaborate introduction to the topic, the reader is referred to [2].

[10]The points at which the potential is same as the energy of the particle or radiation are known as turning points as indicated in *Figure* 2.

But for the case of black holes, there is no analogue of the incident wave. And, the waves are purely ingoing at the event horizon and purely outgoing at the spatial horizon. Therefore, the transmitted and reflected waves have comparable amplitudes.

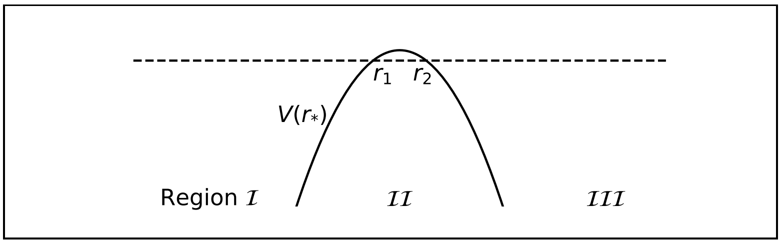

Figure 2. Dashed line represents the energy of the radiation. Two turning points r_1 and r_2 are shown. Three regions $\mathcal{I}$, $\mathcal{II}$, and $\mathcal{III}$ are marked.

The only way we can have comparable amplitudes for transmitted and reflected waves is when the turning points are close to each other. The transmitted and reflected waves will have the same amplitude when there is only one turning point.

But if the turning points are very close or degenerate, the first order approximation of the potential breaks down. Instead, we should include higher order terms to the approximation of the potential. In this section, we model the potential by a parabola and discuss the results as first derived by Schutz and Will [11].

Rewriting (31) as

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r_*^2} + p(r_*)\Psi = 0, \tag{36}$$

where

$$p(r_*) = \omega^2 - V(r_*). \tag{37}$$

Approximating $p(r_*)$ as:

$$p(r_*) = p(r_0) + \frac{1}{2}p_0''(r_* - r_0)^2 + O(r_* - r_0)^3, \tag{38}$$

where r_0 is the maximum of the potential barrier and $p_0'' = d^2p/dr_*^2|_{r_0}$. Further let r_1 and r_2 be the two turning points, that is, roots of the equation $p(r_*) = 0$. We define some new quantities,

$$k = \frac{1}{2}p_0'', \qquad t = (4k)^{1/4}e^{i\pi/4}(r_* - r_0), \tag{39}$$

$$\rho + \frac{1}{2} = -i\frac{p(r_0)}{(2p_0'')^{1/2}}. \tag{40}$$

Using (38)–(40) in (31), we obtain,

$$\frac{d^2\Psi}{dt^2} + \left(\rho + \frac{1}{2} - \frac{1}{4}t^2\right)\Psi = 0. \tag{41}$$

The solutions of this equation are the parabolic cylinder functions. The asymptotic form for $r_* \gg r_2$ is given by

$$\Psi \approx Be^{-3i\pi(\rho+1)/4}(4k)^{-(\rho+1)/4}(r_* - r_0)^{-(\rho+1)}e^{ik^{1/2}(r_*-r_0)^2/2}$$
$$+ \left(A + \frac{B(2\pi)^{1/2}e^{-i\rho\pi}/2}{\Gamma(\rho+1)}\right)e^{i\pi\rho/4}(4k)^{\rho/4}(r_* - r_0)^{\rho}e^{-ik^{1/2}(r_*-r_0)^2/2}, \tag{42}$$

and for $r_* \ll r_1$,

$$\Psi \approx Ae^{-3i\pi\rho/4}(4k)^{\rho/4}(r_* - r_0)^{\rho}e^{-ik^{1/2}(r_*-r_0)^2/2}$$
$$+ \left(B - \frac{iA(2\pi)^{1/2}e^{i\rho\pi}/2}{\Gamma(-\rho)}\right)e^{i\pi(\rho+1)/4}(4k)^{-(\rho+1)/4}$$
$$(r_* - r_0)^{-(\rho+1)}e^{ik^{1/2}(r_*-r_0)^2/2}, \tag{43}$$

where $\Gamma(x)$ is the usual gamma function given by,

$$\Gamma(x) = \int_0^\infty z^{x-1}e^{-z}dz.$$

Since there is no incident radiation, the coefficient of $\exp\left(ik^{1/2}(r_* - r_0)^2/2\right)$ must be zero. From these conditions, we find $B = 0$ and $1/\Gamma(-\rho) = 0$. The latter implies that ρ must be a non-negative integer. We get, from (40),

$$\frac{p(r_0)}{(2p_0'')^{1/2}} = i\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \ldots \tag{44}$$

The quasinormal modes are summarized in *Table* 1 for different l. To convert into kHz, multiply by $2\pi(5142)\times(M_\odot/M)$. A graphical comparison between $l = 2$ quasinormal mode obtained via WKB method [11] and numerical method used by Chandrasekhar and Detweiler [5] is shown in *Figure* 3.

The formula works great for $n = 0$ but there is a large deviation from the numerically obtained values for higher overtones ($n \geq 1$). But there is another method, called the Continued Fraction Method ([7], [8]), using which we can find QNMs for $n \geq 1$ and they match well with numerical results.

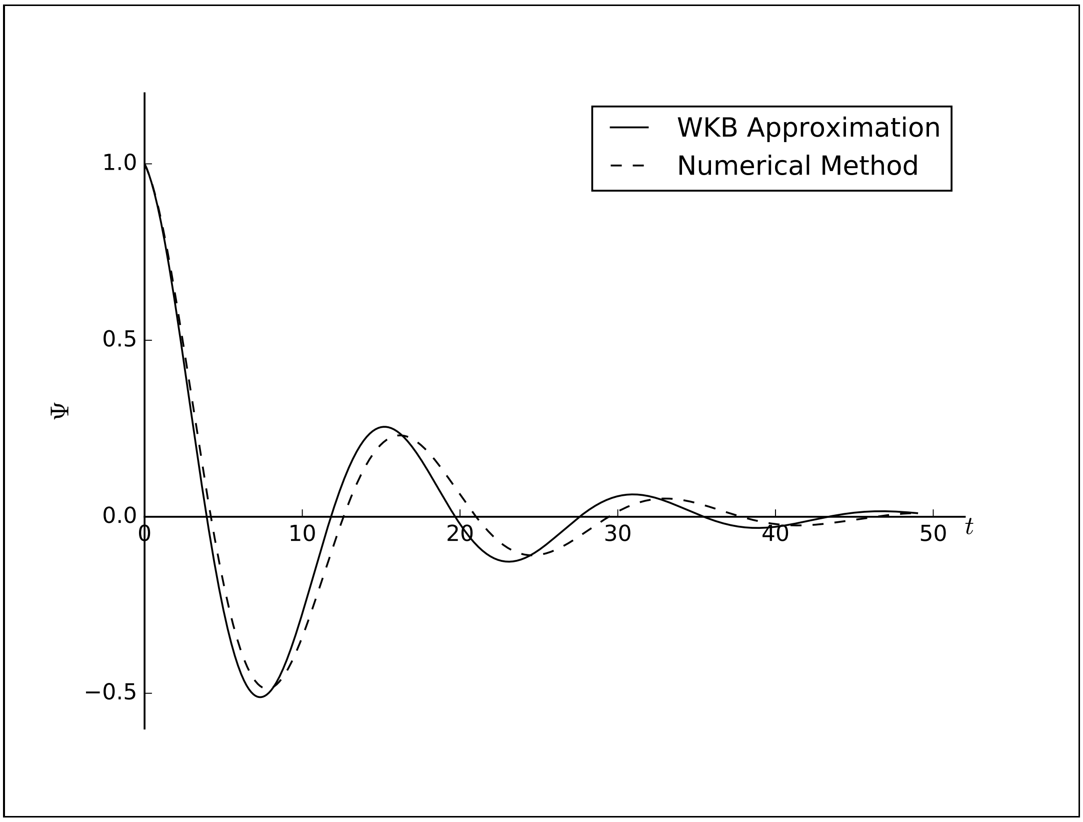

Figure 3. Comparison between the $l = 2$ quasinormal mode found from the WKB method and the Numerical method [5].

3. Conclusion

In this article, we have discussed the behavior of the black holes when they are disturbed from their equilibrium. We saw that black holes settle to a stable state by radiating out energy in some characteristic frequencies, called the quasinormal modes. This bell-like ringing down behavior is observed in gravitational wave detectors. Here, we have focused our attention on uncharged and non-rotating black holes. Figuratively, a black hole can be

$n = 0$	WKB	Numerical
$l = 2$	$0.3988 + i0.08828$	$0.3737 + i0.0889$
$l = 3$	$0.6165 + i0.09232$	$0.5994 + i0.0927$
$l = 4$	$0.8223 + i0.09392$	$0.8092 + i0.0941$

Table 1. Comparison of QNM obtained via WKB approximation and numerical method. Numerical results are from Chandrasekhar and Detweiler [5].

slightly perturbed either perpendicularly which results in a non-zero change in the parameters of the black hole or it can be disturbed tangentially which imparts a rotation to the black hole. We have discussed this tangential perturbation in Section 2.

We confined ourselves to the study of perturbations of Schwarzschild black holes and discussed a method of computing the lowest quasi-normal mode ($n = 0$) using WKB approximation. To find the QNMs for higher n's, we need more sophisticated methods like the continued fraction method [8].

The procedure we discussed for perturbations of Schwarzschild black holes can be extended to charged black holes which lead to a similar eigenvalue equation (3) with a different potential. We refer the readers to Chandrasekhar's book [1] for a thorough treatment of the perturbations of charged black holes.

We observe that calculating QNMs is tedious even for the simplest case of a black hole, but they can be tackled by semi-analytical methods such as WKB approximation. However, finding QNM spectrum for other black holes like Kerr (rotating) black holes require the tools of numerical relativity.

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