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Relative Reduction and Buchberger's Algorithm in Filtered Free Modules

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Abstract In this paper we develop a relative Gröbner basis method for a wide class of filtered modules. Our general setting covers the cases of modules over rings of differential, difference, inversive difference and difference–differential operators, Weyl algebras and multiparameter twisted Weyl algebras (the last class of rings includes the classes of quantized Weyl algebras and twisted generalized Weyl algebras). In particular, we obtain a Buchberger-type algorithm for constructing relative Gröbner bases of filtered free modules.

Keywords Filtered module · Admissible orders · Relative Gröbner basis · Gröbner reduction

Mathematics Subject Classification Primary 13P10; Secondary 12H05 · 12H10 · 68W30

1 Introduction

It is widely known that the classical Gröbner basis method, first introduced in [1], is a powerful algorithmic technique for solving problems in commutative algebra and algebraic geometry (in particular, problems that can be formulated in terms of systems of multivariate polynomial equations). Moreover, this method has various applications in geometric theorem proving, graph theory (e.g., in problems of coloring of graphs), linear programming, theory of error-correcting codes, robotics and many other areas. One of the important algebraic application of Gröbner bases is their use in the dimension theory, in particular, for the computation of Hilbert polynomials of graded and filtered modules. It turned out that the corresponding technique can be extended to the computation of differential, difference and difference–differential dimension polynomials via generalizations of the Gröbner basis method to differential, difference and difference–differential modules, respectively. Such generalizations were obtained in [18] (for modules over ring of differential operators with power series coefficients) and in [12, Chapter 4] (for difference and difference–differential modules).

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In the last fifteen years the Gröbner basis approach was applied to bifiltered and multifiltered modules over polynomial rings, rings of differential, difference and difference–differential polynomials, and Weyl algebras. The corresponding techniques use different types of reduction with respect to several term orderings; the resulting Gröbner-type bases are called *Gröbner bases with respect to several term orderings* ([13–15] and [5]) and *relative Gröbner bases* ([20,22] and [4]). As applications of the generalized Gröbner basis techniques, these works present proofs of the existence and methods of computation of multivariate difference–differential dimension polynomials, as well as bivariate Bernstein-type dimension polynomials of modules over Weyl algebras. Furthermore, a generalization of the relative Gröbner basis technique to the case of difference–differential modules with weighted basic operators obtained by Dönch in [3] allowed him to prove the existence and obtain methods of computation of Ehrhart-type dimension quasi-polynomials associated with filtrations of such modules.

In this paper we unify the theories of relative Gröbner bases and Gröbner bases with respect to several term orderings (including their ”weighted” versions) by developing a generalized relative Gröbner basis method for a wide class of filtered modules that includes modules over rings of differential, difference, inversive difference and difference–differential operators, Weyl algebras, and also multiparameter twisted Weyl algebras introduced and studied in [9]. (Note that the last class of algebras includes the classes of quantized Weyl algebras and twisted generalized Weyl algebras that play an important role in the quantum group covariant differential calculus, see, for example, [19].)

In the next section we describe basic settings, give a characterization of ring filtrations considered in the rest of the paper and present a concept of Gröbner reduction in a free module over a filtered ring with monomial filtration. The main results are presented in Sect. 3, where we introduce concepts of admissible orders and set-relative reduction in free modules over rings with monomial filtrations, define the notion of Gröbner basis in this setting and obtain a Buchberger-type algorithm for its construction.

2 Algebraic Setup

Throughout the paper $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$ denote the sets of integers, nonnegative integers and rational numbers, respectively. If p is a positive integer, then $\mathbb{N}^p$ is treated as a commutative semigroup with componentwise addition and as a partially ordered set with the product order $\leq_\pi$ such that

$$r = (r_1, \dots, r_p) \leq_\pi s = (s_1, \dots, s_p) \Longleftrightarrow r_i \leq s_i, \quad \forall i = 1, \dots, p.$$

In what follows, R denotes an arbitrary, left noetherian (non-commutative) ring containing a commutative ring $K \subseteq R$ as a subring. Unless the contrary is indicated, by an R-module, we always mean a left R-module.

Let M be a left R-module. A subset $\mathbb{M}$ of this module is said to be a *set of monomials of* M, if M is a free K-module with basis $\mathbb{M}$, that is, every element $f \in M$ has a unique representation of the form

$$f = \sum_{\mathfrak{m} \in \mathbb{M}} f_{\mathfrak{m}} \mathfrak{m}, \quad f_{\mathfrak{m}} \in K, \text{ only finitely many } f_{\mathfrak{m}} \neq 0.$$

We denote by

$$\mathrm{T}(f) = \mathrm{T}\left(\sum_{\mathfrak{m} \in \mathbb{M}} f_{\mathfrak{m}} \mathfrak{m}\right) := \{\mathfrak{m} \in \mathbb{M} : f_{\mathfrak{m}} \neq 0\},$$

the *support of* f, i.e. the set of monomials that appear in f with a non-zero coefficient. For example, if K is a field and Λ is a basis of R as a K-vector space (or K is a commutative ring and R is a free K-module with basis Λ), then Λ is a set of monomials of R. If F is a free R-module with basis E, then it is easy to see that $\Lambda E = \{\lambda e : \lambda \in \Lambda, e \in E\}$ is the set of monomials of F. In this case we write $R = K^{(\Lambda)}$ and $F = R^{(E)} = K^{(\Lambda E)}$.

Definition 2.1 A family of K-submodules $\{R_r : r \in \mathbb{N}^p\}$ of R is called a *p-fold filtration of a ring* R if

1. $R_r \subseteq R_s$, whenever $r \leq_\pi s \in \mathbb{N}^p$;
2. $R_r \cdot R_s \subseteq R_{r+s}, \quad r, s \in \mathbb{N}^p$;

3. $R = \bigcup_{r \in \mathbb{N}^p} R_r$;
4. $1 \in R_0$ where R_0 stands for $R_{(0,\ldots,0)}$.

In this case we say that R is a p-filtered ring. (If $p = 2$ we also use the term *bifiltration*). A p-fold filtration of R is called *monomial* (and R is said to be a *monomially p-filtered ring*) if and only if the inclusion $f \in R_r$ implies the inclusion $\mathrm{T}(f) \subseteq R_r$.

Definition 2.2 Let M be a left R-module. A *p-fold filtration of the module M* w.r.t. the p-fold filtered ring R is a family $\{M_r : r \in \mathbb{N}^p\}$ of K-submodules of M such that

- $M_r \subseteq M_s, \qquad r \leq_\pi s$;
- $R_r \cdot M_s \subseteq M_{r+s}, \quad$ for any $r, s \in \mathbb{N}^p$;
- $M = \bigcup_{r \in \mathbb{N}^p} M_r$.

An R-module equipped with a p-fold filtration is said to be a *p-filtered module*.

Note that if R is a ring with p-fold filtration $\{R_r : r \in \mathbb{N}^p\}$ and M is a finitely generated R-module with generators $\{h_1, \ldots, h_q\}$, then M can be naturally treated as a p-filtered module with the p-fold filtration

$$\left\{ M_r = \sum_{i=1}^{q} R_r h_i \, : \, r \in \mathbb{N}^p \right\}.$$

Clearly, this filtration is monomial if the filtration of R is monomial.

Given any mapping $u : R \to \mathbb{N}$, one can consider a family of additive subgroups $\{R_k^{(u)} : k \in \mathbb{N}\}$ of the ring R such that $R_k^{(u)} := \{r \in R : u(r) \leq k\}$ for every $k \in \mathbb{N}$. Clearly, $\bigcup_k R_k^{(u)} = R$. The following statement tells when a family of this kind is a (onefold) filtration of R.

Lemma 2.3 (Characterization of one-dimensional filtrations) *With the above notation, the family $\{R_k^{(u)} : k \in \mathbb{N}\}$ is a (onefold) filtration of R if and only if the mapping u satisfies the following three conditions.*

(i) If $x \in R$, then $u(x) = 0$ if and only if $x \in K$;
(ii) $u(x + y) \leq \max\{u(x), u(y)\}$ for all $x, y \in R$;
(iii) $u(xy) \leq u(x) + u(y)$ for all $x, y \in R$;

Furthermore, for any onefold filtration $\{R_r : r \in \mathbb{N}\}$, there exists a mapping $u : R \to \mathbb{N}$ satisfying conditions (i)–(iii) such that $R_r = R_r^{(u)}$ for all $r \in \mathbb{N}$.

Proof Clearly, if $u : R \to \mathbb{N}$ is a mapping satisfying the above conditions and $R_k^{(u)} = \{x \in R : u(x) \leq k\}$ $(k \in \mathbb{N})$, then the family $\{R_k^{(u)} : k \in \mathbb{N}\}$ satisfies conditions 1–4 of Definition 2.1 (with $p = 1$). (Note that if $x \in R_k^{(u)}$ and $c \in K$, then $u(cx) \leq u(c) + u(x) = u(x)$; this observation and property (ii) imply that every $R_k^{(u)}$ is a K-module.). For the converse, suppose that u is a mapping from R to $\mathbb{N}$ such that the family $R_k^{(u)} = \{x \in R : u(x) \leq k\}, k \in \mathbb{N}$, satisfies conditions 1–4 of Definition 2.1. Since $R_0^{(u)} = K$ and $u(x) \geq 0$ for any $x \in R$, we obtain that $x \in K$ is equivalent to $u(x) = 0$. The other properties of the map u follow from the fact that every $R_k^{(u)}$ is a K-module and from the first two conditions of Definition 2.1.

In order to prove the last part of the statement, consider a onefold filtration $\{R_r : r \in \mathbb{N}\}$ of R and define the mapping $u : R \to \mathbb{N}$ by setting $u(x) = \min\{k : x \in R_k\}$. It is easy to check that u satisfies conditions (i)–(iii). Indeed, since $R_0 = K$, we have that $u(a) = 0$ for any $a \in K$ and, conversely, the equality $u(x) = 0$ $(x \in R)$ implies that $x \in R_0 = K$. Furthermore, the fact that every R_k is a K-module and the first two properties of a filtration (see Definition 2.1) imply that the mapping u satisfies conditions (ii) and (iii).

It remains to show that $R_r^{(u)} = R_r$ for any $r \in \mathbb{N}$. Let $x \in R_r^{(u)}$ and $u(x) = k_0$. Then $k_0 = \min\{k \in \mathbb{N} : x \in R_k\} \leq r$, hence $x \in R_{k_0} \subseteq R_r$. Conversely, if $y \in R_r$, then $u(y) = \min\{k \in \mathbb{N} : y \in R_k\} \leq r$, so $y \in R_r^{(u)}$. Thus, $R_r^{(u)} = R_r$. □

Remark 2.4 The first part of Lemma 2.3 can be generalized to p-fold filtrations ($p > 1$) as follows. Let us consider a mapping $u : R \to \mathbb{N}^p$ and let $u_i = \pi_i \circ u : R \to \mathbb{N}$ ($1 \le i \le p$) where π_i is the projection of $\mathbb{N}^p$ onto its i-th component: $(a_1, \dots, a_p) \mapsto a_i$. For any $r = (r_1, \dots, r_p) \in \mathbb{N}^p$, let $R_r^{(u)} = \{x \in R :\ u_i(x) \le r_i \text{ for } 1 \le i \le p\}$. Then, one can mimic the corresponding part of the proof of Lemma 2.3 to obtain that $\{R_r^{(u)} :\ r \in \mathbb{N}^p\}$ is a p-fold filtration of R if and only if the mapping u satisfies the following conditions:

(i) If $x \in R$, then $u(x) = 0$ if and only if $x \in K$;
(ii) $u(x + y) \le_\pi (\max\{u_1(x), u_1(y)\}, \dots, \max\{u_p(x), u_p(y)\})$ for all $x, y \in R$;
(iii) $u(xy) \le_\pi (u_1(x) + u_1(y), \dots, u_p(x) + u_p(y))$ for all $x, y \in R$.

At the same time, if $p > 1$, then not every p-fold filtration is of the form $\{R_r^{(u)} :\ r \in \mathbb{N}^p\}$ with a mapping $u : R \to \mathbb{N}^p$ satisfying the above conditions. It follows from the fact that the same element of R can belong to different components R_r and R_s with incomparable (with respect to $\le_\pi$) p-tuples $r, s \in \mathbb{N}^p$. For example, let $R = K[x_1, x_2]$ be a polynomial ring in two variables over a field K, equipped with a natural twofold filtration

$$R_{r_1,r_2} = \{f \in K[x_1, x_2] :\ \deg_{x_1}(f) \le r_1 \wedge \deg_{x_2}(f) \le r_2\}, \quad (r_1, r_2) \in \mathbb{N}^2,$$

and let the factor ring $\overline{R} = K[x_1, x_2]/\langle x_1^3 - x_2^2\rangle$ be equipped with the canonical image $\overline{R}_{r_1,r_2}$ of the filtration $\{R_r :\ r \in \mathbb{N}^2\}$. Denoting the coset of a polynomial $f \in K[x_1, x_2]$ by $\overline{f}$, we obtain that, say, the element $t = \overline{x_1^3} = \overline{x_2^2}$ lies in $\overline{R}_{3,0} \cap \overline{R}_{0,2}$. If there is a function $u : \overline{R} \to \mathbb{N}^2$ such that $\overline{R}_{r_1,r_2} = \overline{R}^{(u)}_{r_1,r_2}$ for every $r = (r_1, r_2) \in \mathbb{N}^2$, then one would have $u_1(t) \le 0$ and $u_2(t) \le 0$ (we use the notation of Remark 2.4). These inequalities imply that $t \in K$, contrary to the obvious fact that $x_1^3 - a \notin \langle x_1^3 - x_2^2\rangle$ and $x_2^2 - b \notin \langle x_1^3 - x_2^2\rangle$ for any $a, b \in K$.

Let R be a p-fold filtered ring with a p-fold filtration $\{R_r :\ r \in \mathbb{N}^p\}$, and let F be a free R-module with set of free generators E. Assume that R is a free K-module with basis Λ (we still use the notation and conventions introduced at the beginning of this section), then the support $\mathrm{T}(f)$ of an element $f \in F$ is defined as its support with respect to the set of monomials $\Lambda E = \{\lambda e :\ \lambda \in \Lambda, e \in E\}$.

In what follows, a binary relation $\rho \subseteq F \times F$ is said to be a *reduction* and the inclusion $(f, g) \in \rho$ is written as $f \longrightarrow g$. Furthermore, for every $f, h \in F$, we write $f \longrightarrow^\star h$ if there exists a finite chain

$$f = f_0 \longrightarrow f_1 \longrightarrow \cdots \longrightarrow f_k = h.$$

Finally, I_ρ will denote the set of ρ-irreducible elements of F, that is

$$I_\rho := \{f \in F :\ \text{there is no } h \in F \text{ such that } f \longrightarrow h \wedge h \ne f\}.$$

With the above notation, one can consider the following concept of Gröbner reduction first introduced in [7].

Definition 2.5 *(Gröbner Reduction)* With the above notation, let the p-fold filtration of the ring R be monomial and let N be an R-submodule of the free R-module F. A reduction $\rho \subseteq F \times F$ is said to be a *Gröbner reduction for N* if and only if it satisfies the following conditions

1. every reduction sequence $f_1 \longrightarrow f_2 \longrightarrow \cdots$ terminates in a finite number of steps;
2. ${}_K\langle I_\rho\rangle \subseteq I_\rho$ and $f \in I_\rho \Rightarrow \mathrm{T}(f) \subseteq I_\rho$ (where ${}_K\langle I_\rho\rangle$ denotes the K-module generated by I_ρ);
3. $f \longrightarrow h$ implies that $f \equiv h \pmod N$
4. $I_\rho \cap N = 0$, that is, every non-zero element in N is reducible,
5. $f \in F_r \wedge f \longrightarrow h \Rightarrow h \in F_r, \quad r \in \mathbb{N}^p$.

Remark 2.6 Examples of Gröbner reductions can be found in [6, Chapter 3] and [8]. If K is a field of zero characteristic, then the theory of Gröbner reduction provides an algorithmic computation of the dimensions of components of p-fold filtrations of finitely generated R-modules. In the cases of p-dimensional filtrations of differential, difference and difference–differential modules (where R is the ring of the corresponding operators), the dimensions of the components are expressed by multivariate polynomials in p variables with rational coefficients (see [12, Theorem 4.3.39] and [17, Theorems 3.3.16]). Similar result for modules over Weyl algebras and rings of

Ore polynomials are obtained in [14] and [16]. (Note that by a ring of Ore polynomials we mean a ring defined as follows. Let K be a field and $\Delta = \{\delta_1, \dots, \delta_n\}$, $\sigma = \{\alpha_1, \dots, \alpha_n\}$ sets of derivations and injective endomorphisms of K, respectively, such that any two mappings from the set $\Delta \cup \sigma$ commute. Let $\Theta = \Theta(X)$ be a free commutative semigroup generated by a set $X = \{x_1, \dots, x_n\}$ and let $\mathbb{O}$ denote the vector K-space with the basis Θ (elements of $\mathbb{O}$ are of the form $\sum_{\theta \in \Theta} a_\theta \theta$ where $a_\theta \in K$ and only finitely many coefficients a_θ are different from zero). Then $\mathbb{O}$ can be treated as a ring if one introduces the multiplication according to the rule $x_i a = \alpha_i(a) x_i + \delta_i(a)$ ($a \in K$, $1 \leq i \leq n$) and the distributive laws. Then we say that $\mathbb{O}$ is a *ring of Ore polynomials* in the variables $x_1, \dots, x_n$ over K.) In all these cases the results on the multivariate dimension polynomials were proved with the use of certain types of Gröbner bases whose constructions are based on the corresponding reductions. All these reductions satisfy the conditions of Definition 2.5 and therefore are Gröbner reductions. Note that the reduction with respect to several orderings defined in [16], which is a special instance of the Gröbner reduction, can be naturally applied to algebras of a certain subclass of the class of algebras of solvable type (algebras of solvable type were introduced and studied in [11]). This subclass consists of algebras of solvable type $R = K\{X_1, \dots, X_n\}$ (K is a field) where the term ordering $<$ is degree-respecting (that is, for any two power products $t = X_1^{k_1} \dots X_n^{k_n}$ and $t' = X_1^{l_1} \dots X_n^{l_n}$, the inequality $\deg t = \sum_{i=1}^n k_i < \deg t' = \sum_{i=1}^n l_i$ implies that $t < t'$) and the axiom 1.2(3) in [11, Section 1] is strengthened by the requirement that for all $1 \leq i \leq j \leq n$, there exist $0 \neq c_{ij} \in K$ and $p_{ij} \in R$ such that $X_j * X_i = c_{ij} X_i X_j + p_{ij}$ and $\deg p_{ij} \leq 1$. (Algebras of this kind are considered in [6, Section 1.4] where, however, the above requirements on the term order and commutation of the generators are not explicitly formulated. Note also that one can mimic the proof of [16, Theorem 4.2] and obtain a theorem on a multivariate dimension polynomials for finitely generated modules over such algebras.) Finally, using the weight relative Gröbner basis technique developed in [3, Section 2.3] (the corresponding reduction satisfies the conditions of Definition 2.5 as well) C. Dönch showed that the dimensions of components of a multidimensional filtration of a finitely generated difference-skew-differential module over a difference-skew-differential field (that is a field K with the action of finitely many mutually commuting injective endomorphisms and skew derivations of K, see [3, Definition 2.1.1]) can be expressed by a multivariate quasi-polynomial (see [3, Theorem 3.1.22]).

3 Relative Reduction and Buchberger's Algorithm

Buchberger's algorithm, introduced in [1] and formulated for multivariate polynomial rings over a field of characteristic zero, is an algorithm for computing a generating set of an ideal, with the property that every non-zero element contained in the ideal can be reduced to zero in finitely many steps.

This was the starting point for generalizations of this algorithm towards many (non-commutative) ground domains, for example, to the ring of multivariate Ore-polynomials in [15], Weyl-Algebras in [14] and [5], or to the ring of difference–differential operators (see [12, Chapter 4], [13,20] and [22]).

With our considerations, we want to cover modules over rings, whose elements are K-linear combinations of monomials. The monomials should reflect the commutation properties of the considered operators, for example, generalized versions of derivations and automorphisms. Therefore, it is reasonable and appropriate to restrict our view to monomials in finite sets of symbols $A := \{a_1, \dots, a_m\}$ and $B := \{b_1, \dots, b_n\}$ where all elements in $A \cup B$ are pairwise commutative (which is not necessarily the case for elements in K). Then the monomials are defined as power products of the form $\Lambda = A^k \cdot B^l$ where $k \in \mathbb{N}^m$ and $l \in \mathbb{Z}^n$, using obvious multi-index notation.

Recall (see [22, Definition 2.3]) that a family of subsets $\{\mathbb{Z}_j^{(n)} : j = 1, \dots, k\}$ of $\mathbb{Z}^n$ (n is a positive integer) is called an *orthant decomposition* of $\mathbb{Z}^n$ if it satisfies the following conditions:

(i) For any $j = 1, \dots, k$, $(0, \dots, 0) \in \mathbb{Z}_j^{(n)}$ and $\mathbb{Z}_j^{(n)}$ does not contain any pair of nonzero mutually opposite elements of the form $(c_1, \dots, c_n)$ and $(-c_1, \dots, -c_n)$.
(ii) Every $\mathbb{Z}_j^{(n)}$ is a finitely generated subsemigroup of the additive group of $\mathbb{Z}^n$ which is isomorphic to $\mathbb{N}^n$ as a semigroup.
(iii) For any $j = 1, \dots, k$, the group generated by $\mathbb{Z}_j^{(n)}$ is $\mathbb{Z}^n$.

Given such an orthant decomposition of $\mathbb{Z}^n$ and $m \in \mathbb{N}$, the family $\{\mathbb{N}^m \times \mathbb{Z}_j^{(n)} : j = 1, \dots, k\}$ is said to be an orthant decomposition of $\mathbb{N}^m \times \mathbb{Z}^n$. A standard example of an orthant decomposition of $\mathbb{Z}^n$ is a family $\{\mathbb{Z}_1^{(n)}, \dots, \mathbb{Z}_{2^n}^{(n)}\}$ of all distinct Cartesian products of n sets each of which is either $\mathbb{N}$ or $\mathbb{Z}_- = \{a \in \mathbb{Z} : a \le 0\}$.

To extend our concept to free modules $F = R^{(E)} = K^{(\Lambda E)}$ with finite generating set $E := \{e_1, \dots, e_t\}$, let $\{\mathbb{Z}_j^{(n)} : j = 1, \dots, k\}$ be an orthant decomposition of $\mathbb{Z}^n$. A total order $\prec$ on the set $\mathbb{N}^m \times \mathbb{Z}^n \times E$ (where $m \in \mathbb{N}$) is said to be a *generalized term order* on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ if the following conditions hold:

(a) For every $i = 1, \dots, t$, $(0, \dots, 0, e_i)$ is the smallest element of $\mathbb{N}^m \times \mathbb{Z}^n \times \{e_i\}$.
(b) If $a, b, c \in \mathbb{N}^m \times \mathbb{Z}^n$, $(a, e_i) \prec (b, e_j)$ $(1 \le i, j \le t)$ and c and b lie in the same orthant $\mathbb{Z}_l^{(n)}$, then $(a + c, e_i) \prec (b + c, e_j)$.

As we have seen, $\Lambda E = \{\lambda e_i : \lambda \in \Lambda,\ 1 \le i \le t\}$ is a set of monomials of F, which is in natural one-to-one correspondence with the set $\mathbb{N}^m \times \mathbb{Z}^n \times E$ (obviously, $a_1^{k_1} \dots a_m^{k_m} b_1^{l_1} \dots b_n^{l_n} e_i \leftrightarrow (k_1, \dots, k_m, l_1, \dots, l_n, e_i)$). A total order $\prec$ of the set of monomials ΛE is called a *generalized term order* on ΛE if the corresponding order of the set $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is a generalized term order in the above sense.

Let $\mu = A^k \cdot B^l$ and $\nu = A^{k'} \cdot B^{l'}$, where $k, k' \in \mathbb{N}^m$ and $l, l' \in \mathbb{Z}^n$, and let j be such that $l \in \mathbb{Z}_j^{(n)}$. Then, we say that μ divides ν if and only if $(k', l') \in (k, l) + \mathbb{N}^m \times \mathbb{Z}_j^{(n)}$. If $t_1 = \mu e_i$ and $t_2 = \nu e_j$ are elements of ΛE, we say that t_1 divides t_2 and write $t_1 | t_2$ if and only if $\mu | \nu$ and $i = j$.

Since ΛE is a free basis of F as a K-module, every element $f \in F$ has a unique representation of the form

$$f = a_1 \lambda_1 e_{j_1} + \dots + a_d \lambda_d e_{j_d}, \quad a_i \in K, 1 \le i \le d,$$

where $\lambda_1 e_{j_1}, \dots, \lambda_d e_{j_d}$ are distinct elements of ΛE. Given a generalized term order $\prec$ on ΛE, the greatest monomial with respect $\prec$ among $\lambda_1 e_{j_1}, \dots, \lambda_d e_{j_d}$ is called the *leading monomial* of f; it is denoted by $\mathrm{LT}_\prec(f)$. The coefficient of the leading monomial is called the *leading coefficient* of f and denoted by $\mathrm{LC}_\prec(f)$. It is easy to see, that if $\mathrm{LT}_\prec(f) = \mu e_i$ and the monomials μ and ν lie in the same orthant, then $\mathrm{LT}_\prec(\nu \cdot f) = \nu \cdot \mathrm{LT}_\prec(f)$, while the converse does not to hold in general.

Example A difference–differential ring is a commutative ring K equipped with two sets of operators

$$\Delta := \{\delta_1, \dots, \delta_m\}, \quad \Sigma := \{\sigma_1, \dots, \sigma_n\},$$

consisting of derivations and automorphisms of K respectively, such that every two operators from the set $\Delta \cup \Sigma$ commute. The corresponding ring of difference–differential operators D is defined as a free K-module whose free basis consists of all monomials of the form $\delta_1^{k_1} \dots \delta_m^{k_m} \sigma_1^{l_1} \dots \sigma_n^{l_n}$, where $(k_1, \dots, k_m) \in \mathbb{N}^m$ and $(l_1, \dots, l_n) \in \mathbb{Z}^n$. The set of all such monomials is denoted by $\Lambda_{m,n}$. The multiplication in D is defined by the relationships

$$\delta_i a = a \delta_i + \delta_i(a), \quad \sigma_j a = \sigma_j(a) \sigma_j, \quad a \in K, 1 \le i \le m, 1 \le j \le n,$$

extended by distributivity. Furthermore, for any $\nu := \delta_1^{k_1} \dots \delta_m^{k_m} \sigma_1^{l_1} \dots \sigma_n^{l_n} \in \Lambda_{m,n}$, we define the orders of ν relative to Δ and Σ as

$$|\nu|_1 = k_1 + \dots + k_m, \quad |\nu|_2 = |l_1| + \dots + |l_n|, \quad \text{respectively.}$$

Let F be the free D-module generated by a finite set $E := \{e_1, \dots, e_t\}$. Then, the orders of a monomial $\nu e_i \in F$ $(\nu \in \Lambda_{m,n},\ 1 \le i \le t)$ with respect to Δ and Σ are defined as $|\nu|_1$ and $|\nu|_2$ respectively. In this case F can be considered as a bifiltered D-module with twofold filtration $\{F_{r,s} : (r, s) \in \mathbb{N}^2\}$ defined as follows:

$$F_{r,s} := \{f \in F : |f|_1 \le r \wedge |f|_2 \le s\}, \quad r, s \in \mathbb{N},$$

where for any $f \in F$, $|f|_i := \max\{|\nu|_i;\ \exists j \in \{1, \dots, t\} : \nu e_j \in \mathrm{T}(f)\}$, $(i = 1, 2)$. Clearly, if $F = D$, then the functions $u_i = |\cdot|_i : D \to \mathbb{N}$ satisfy the conditions of Lemma 2.3 and $D_{r,s} = D_{r,s}^{(u)}$ where $u = (u_1, u_2) : D \to \mathbb{N}^2$.

The proof of the following theorem can be obtained by mimicking the proof of [22, Theorem 3.1] that states a similar result for rings of difference–differential operators.

Theorem 3.1 (Relative reduction) *Let $R = K^{(\Lambda)}$ be a ring, $E := \{e_1, \ldots, e_t\}$ be a set of free generators of the free R-module $F = R^{(E)}$, and let $\prec_1$ and $\prec_2$ be two generalized term orders on ΛE. Let $G := \{g_1, \ldots, g_q\} \subseteq F \backslash \{0\}$ and $f \in F$. Then f can be represented as*

$$f = h_1 g_1 + \cdots + h_q g_q + r, \tag{3.1}$$

for some elements $h_1, \ldots, h_q \in R$ and $r \in F$ such that

1. *$h_i = 0$ or $\mathrm{LT}_{\prec_1}(h_i g_i) \preceq_1 \mathrm{LT}_{\prec_1}(f)$, $i = 1, \ldots, q$;*
2. *$r = 0$ or $r \neq 0 \wedge \mathrm{LT}_{\prec_1}(r) \prec_1 \mathrm{LT}_{\prec_1}(f)$ such that*

$$\mathrm{LT}_{\prec_1}(r) \notin \left\{\mathrm{LT}_{\prec_1}(\lambda g_i) : \ \mathrm{LT}_{\prec_2}(\lambda g_i) \preceq_2 \mathrm{LT}_{\prec_2}(r), \ \lambda \in \Lambda, \ i = 1, \ldots, q\right\}. \tag{3.2}$$

We say that f $\prec_1$-reduces modulo G relative to $\prec_2$ to r, and the transition from f to r is said to be the relative reduction.

With this modified reduction, we can now consider the notion of *relative Gröbner basis*.

Definition 3.2 *(Relative Gröbner basis [22, Definition 3.3])* As above, let F be a finitely generated free R-module, and $N \subseteq F$ a submodule. Further, let $\prec_1$ and $\prec_2$ be a pair of generalized term orders, $G := \{g_1, \ldots, g_q\} \subseteq N \backslash \{0\}$. The set G is called a *$\prec_1$-Gröbner basis relative to $\prec_2$* if and only if every $f \in N \backslash \{0\}$ can be $\prec_1$-reduced modulo G relative to $\prec_2$ to zero. Then, if no confusion is possible, G is called a *relative Gröbner basis* for N.

We consider two generalized term orders $\prec_1$ and $\prec_2$ and relative reduction for monomially filtered rings R. It is obvious that G is a (relative) Gröbner basis for N if and only if $N = {}_R\langle G \rangle$ and every non-zero element in N can be $\prec_1$-reduced modulo G relative to $\prec_2$ to zero, which is equivalent to $I_\rho \cap N = 0$, i.e. axiom 4 of Gröbner reduction.

It is shown in [7, Section 3], that for the bivariate filtration $F_{r,s}$, using an appropriate choice of term orders $\prec_1$ and $\prec_2$ it is possible to ensure condition $f \in F_{r,s}$ and $f \longrightarrow h \Rightarrow h \in F_{r,s}$. In particular, this property holds with respect to the orders given in [22, Section 3].

To ensure that every non-zero element can be reduced to zero, we need to exploit Buchberger's algorithm, as presented in [22, Theorem 3.4]. At the same time, Dönch [4] gave an example of a certain pair of term-orders such that the Buchberger's algorithm, which takes into account relative reduction, does not terminate. This termination property was further investigated in [10] where one can find a condition on the considered generalized orders that guarantees the termination of the Buchberger's algorithm (it is called the *difference–differential degree compatibility* condition, see [10, Definition 3.1]).

In what follows, a ring R is assumed to be a free K-module with a basis Λ. Furthermore, we suppose that R is equipped with a bivariate filtration induced by a mapping $u : R \to \mathbb{N}^2$ satisfying conditions (i)–(iii) of Remark 2.4. In other words,

$$R_{r,s} = \{f \in R : \ u_1(f) \leq r \wedge u_2(f) \leq s\}, \quad (r, s) \in \mathbb{N}^2,$$

where the mappings $u_i := \pi_i \circ u$ ($i = 1, 2$) satisfy the conditions of Lemma 2.3. Let F be a free R-module with basis E and let $\{F_{r,s} = R_{r,s}E : \ (r, s) \in \mathbb{N}^2\}$ be the induced filtration of F.

Definition 3.3 *(Admissible orders)* With the above notation, a pair of generalized term orders $\prec_1$ and $\prec_2$ on the set ΛE is said to be *admissible*, if for any two terms $t_1, t_2 \in \Lambda E$

- $t_1 \prec_1 t_2$ when $u_1(t_1) < u_1(t_2)$, or $u_1(t_1) = u_1(t_2)$ and $u_2(t_1) < u_2(t_2)$;
- $t_1 \prec_2 t_2$ when $u_2(t_1) < u_2(t_2)$, or $u_2(t_1) = u_2(t_2)$ and $u_1(t_1) < u_1(t_2)$;
- $t_1 \prec_1 t_2 \Leftrightarrow t_1 \prec_2 t_2$ if $u_1(t_1) = u_1(t_2)$ and $u_2(t_1) = u_2(t_2)$.

Example Let $\varphi : \Lambda E \to \mathbb{N}^s$ (s is some appropriate chosen positive integer) uniquely identify a monomial. An example is given in [8, Paragraph after Proposition 6.]. The pair

$$t_1 \prec_1 t_2 :\Leftrightarrow (u_1(t_1), u_2(t_1), \varphi(t_1)) <_{\text{lex}} (u_1(t_2), u_2(t_2), \varphi(t_2)),$$
$$t_1 \prec_2 t_2 :\Leftrightarrow (u_2(t_1), u_1(t_1), \varphi(t_2)) <_{\text{lex}} (u_2(t_2), u_1(t_2), \varphi(t_2)),$$

is an example for a pair of admissible orders on the monomials ΛE in F.

The following Lemma is proven in [13, Lemma 4.1].

Lemma 3.4 *Let $R = K^{(\Lambda)}$, where $\Lambda = A^k \cdot B^l$ with $(k, l) \in \mathbb{N}^m \times \mathbb{Z}^n$. Further, let S be an infinite sequence of monomials ΛE (where $E := \{e_1, \ldots, e_t\}$). Then, there exists an index $1 \le j \le t$, and an infinite subsequence*

$$\{\lambda_1 e_j, \lambda_2 e_j, \ldots, \lambda_k e_j, \ldots\} \subseteq S,$$

such that λ_k divides λ_{k+1} for $k \ge 1$.

As next step, we generalize [10, Lemma 3.2] to arbitrary rings with that particular kind of monomials.

Lemma 3.5 *Let F be a free R-module, $\prec_1$ and $\prec_2$ be a pair of admissible term orders on ΛE, and $G_i := \{g_1, \ldots, g_q, r_1, \ldots, r_i\} \subseteq F \backslash \{0\}$. If r_{i+1} is $\prec_1$-reduced modulo G_i relative to $\prec_2$ (see Theorem 3.1), and if*

$$\text{for any } \lambda \in \Lambda, h \in G_i : \mathrm{LT}_{\prec_1}(r_{i+1}) \neq \mathrm{LT}_{\prec_1}(\lambda \cdot h), \tag{3.3}$$

then the ascending chain $G_1 \subseteq G_2 \subseteq \cdots$ stabilizes.

Proof Since for all $\lambda \in \Lambda$ and $h \in G_i$ we have $\mathrm{LT}_{\prec_1}(r_{i+1}) \neq \mathrm{LT}_{\prec_1}(\lambda \cdot h)$, the element r_{i+1} is irreducible with respect to G_i. The second condition (3.2) involving the order $\prec_2$ would apply only if r_{i+1} would be reducible, hence, we are considering the usual notion of reduction with respect to G_i for the order $\prec_1$. If the chain $G_1 \subseteq G_2 \subseteq \cdots$ does not stabilize, then there would be an infinite sequence of monomials $\{\mathrm{LT}(r_i) : i = 1, 2, \ldots\}$ in ΛE such that $\mathrm{LT}(r_i)$ does not divide $\mathrm{LT}(r_{i+1})$ for all i, contrary to the statement of Lemma 3.4. □

Assume given a fixed orthant decomposition of $\mathbb{N}^m \times \mathbb{Z}^n$ consisting of k orthants and Λ_j $(1 \le j \le k)$ is a subset of Λ (we use the notation of Lemma 3.4) consisting of all power products whose exponent vectors lie in the j-th orthant. Furthermore, $K[\Lambda_j]$ will denote the subring of R generated by the set Λ_j. If f and g are non-zero elements of a free R-module F with a finite set of free generators E and $\prec$ a generalized term order of ΛE, then $V(j, f, g)$ will denote a finite system of generators of the $K[\Lambda_j]$-module

$$_{K[\Lambda_j]}\langle \mathrm{LT}_{\prec}(\lambda f) \in \Lambda_j E : \lambda \in \Lambda\rangle \cap {}_{K[\Lambda_j]}\langle \mathrm{LT}_{\prec}(\eta g) \in \Lambda_j E : \eta \in \Lambda\rangle.$$

(The idea of considering such modules and their systems of generators is due to F. Winkler and M. Zhou, see [22].) As it is shown in [21, Lemma 3.5], for any $h \in F$ and $j = 1, \ldots, k$, there exists some $\lambda \in \Lambda$ and a monomial u_j in h such that $\mathrm{LT}_{\prec}(\lambda h) = \lambda u_j \in \Lambda_j E$. Moreover, this term u_j in h is unique; it is denoted by $\mathrm{LT}_{j,\prec}(h)$. With the above notation, for every generator $v \in V(j, f, g)$, the element

$$S_{\prec}(j, f, g, v) = \frac{v}{\mathrm{LT}_{j,\prec}(f)} \frac{f}{\mathrm{LC}_{j,\prec}(f)} - \frac{v}{\mathrm{LT}_{j,\prec}(g)} \frac{f}{\mathrm{LC}_{j,\prec}(g)}$$

is said to be an S-polynomial of f and g with respect to j, v and $\prec$. The following algorithm is applied for each orthant $\mathbb{Z}_j^{(n)}$.

Algorithm 1 Buchberger Algorithm for bifiltered Rings, [22, Algorithm 1]

Require: F is a free R-module, $G := \{f_1, \ldots, f_r\} \subseteq F\backslash\{0\}$;
The ring R is 2-fold filtered;
$\prec_1$ and $\prec_2$ are generalized term orders on ΛE.
Ensure: $G'' := \{g_1, \ldots, g_s\} \subseteq F\backslash\{0\}$ where ${}_R\langle G''\rangle = {}_R\langle G\rangle$ such that $I_\rho \cap (RG'') = 0$.
$G' \leftarrow G$;
while there exist $f, g \in G'$ and $v \in V(j, f, g)$ such that
$S_{\prec_2}(j, f, g, v)$ is $\prec_2$-reduces relative to $\prec_2$ to $r \neq 0$ by G'; **do**
$G' \leftarrow G' \cup \{r\}$;
$G'' \leftarrow G'$;
while there exist $f, g \in G''$ and $v \in V(j, f, g)$ such that
$S_{\prec_1}(j, f, g, v)$ $\prec_1$-reduces relative to $\prec_2$ to $r \neq 0$ by G''; **do**
$G'' \leftarrow G'' \cup \{r\}$;
return G''.

Theorem 3.6 *If R denotes a bifiltered ring, where R is built from monomials of the form $A^k \cdot B^l$ with $(k, l) \in \mathbb{N}^m \times \mathbb{Z}^n$, and the orders $\prec_1$ and $\prec_2$ are chosen to be admissible, then Buchberger's algorithm for filtered rings terminates in a finite number of steps.*

Proof We start with $G := \{g_1, \ldots, g_q\}$ and already assume that it is a Gröbner basis with respect to $\prec_2$. Suppose that the relative reduction proceeds by generating the sequence of sets $G_i := \{g_1, \ldots, g_q, r_1, \ldots, r_i\}$ for $i \geq 1$ and let r_{i+1} be reduced with respect to G_i. Then, either

- $\mathrm{LT}_{\prec_1}(r_{i+1}) \neq \mathrm{LT}_{\prec_1}(\lambda h)$ for any $\lambda \in \Lambda$ and $h \in G_i$, or
- $\mathrm{LT}_{\prec_1}(r_{i+1}) = \mathrm{LT}_{\prec_1}(\lambda h)$ for some $\lambda \in \Lambda$, $h \in G_i$ such that $\mathrm{LT}_{\prec_2}(r_{i+1}) \prec_2 \mathrm{LT}_{\prec_2}(\lambda h)$.

By Lemma 3.5, the first case cannot occur infinitely many times (that is, if all G_{i+1} are obtained from G_i via a transition of the first type, then the ascending chain $G_1 \subseteq G_2 \subseteq \cdots$ stabilizes). For the second case, we have that

$$\begin{aligned}&\mathrm{LT}_{\prec_1}(r_{i+1}) = \mathrm{LT}_{\prec_1}(\lambda h),\\ &\mathrm{LT}_{\prec_2}(r_{i+1}) \prec_2 \mathrm{LT}_{\prec_2}(\lambda h) \Leftrightarrow u_2(\mathrm{LT}_{\prec_2}(r_{i+1})) < u_2(\mathrm{LT}_{\prec_2}(\lambda h)).\end{aligned}$$

If the algorithm does not terminate, then $(G_i)_{i\geq 1}$ is a strictly increasing sequence. Therefore, we can assume that there are infinitely many pairs $(i, j) \in \mathbb{N}^2$ with $i > j$ such that

$$\mathrm{LT}_{\prec_1}(r_i) = \mathrm{LT}_{\prec_1}(\lambda r_j) \wedge \mathrm{LT}_{\prec_2}(r_i) \prec_2 \mathrm{LT}_{\prec_2}(\lambda r_j).$$

We obtain a strictly descending (with respect to the order $\prec_2$) infinite sequence of monomials in ΛE that contradicts the fact that ΛE is well-ordered with respect to $\prec_2$. □

Consider now the following situation: Let $u_i \in \mathbb{N}^R$ satisfy the conditions of Lemma 2.3, let R be p-fold filtered by

$$R_r := \bigcap_{i=1}^{p} \{f \in R : u_i(f) \leq r_i\}, \quad r = (r_1, \ldots, r_p) \in \mathbb{N}^p,$$

We are going to give a formulation of Buchberger's algorithm that corresponds to the Gröbner reduction in this case.

Let us consider two monomial orders $\prec_n^m$ and $\prec_n$ on the set ΛE defined as follows. If $t_1, t_2 \in \Lambda E$, then

$$\begin{aligned}&t_1 \prec_n^m t_2 :\Leftrightarrow (u_n(t_1), u_m(t_1), \varphi(t_1)) <_{\mathrm{lex}} (u_n(t_2), u_m(t_2), \varphi(t_2)),\\ &t_1 \prec_n t_2 :\Leftrightarrow (u_n(t_1), u(t_1), \varphi(t_1)) <_{\mathrm{lex}} (u_n(t_2), u(t_2), \varphi(t_2)),\end{aligned}$$

where $u(t_i) = u_1(t_i) + \cdots + u_p(t_i)(1 \le m, n \le p)$ and φ is a fixed map $\varphi : \Lambda \to \mathbb{N}^s$ that uniquely identifies the monomials (as demonstrated in [8, Paragraph after Proposition 6.]). Obviously, the pair of generalized term orders $\prec_n^m$ and $\prec_m^n$ are admissible.

Theorem 3.7 (Set-relative reduction) *Let F be the free R-module, where R is a (not necessarily commutative) noetherian ring and the fixed commutative subring K of R is a field. Let $f \in F$ and $G = \{g_1, \dots, g_q\} \subseteq F\backslash\{0\}$. Let $\mathcal{A}$ be a subset of the orders $\{\prec_1, \dots, \prec_p\}$, and an order $\prec_m^n$ $(1 \le m, n \le p)$ defined above be fixed. Then there exist elements $h_1, \dots, h_q \in R$ and $r \in F$ such that*

$f = h_1 g_1 + \cdots + h_q g_q + r$ *and*

- $h_i = 0$ *or for all* $\prec$ *in* $\mathcal{A}$, $1 \le i \le q$: $\mathrm{LT}_\prec(h_i g_i) \preceq \mathrm{LT}_\prec(f)$
- $r = 0$ *or for all* $\prec$ *in* $\mathcal{A}$: $\mathrm{LT}_\prec(r) \preceq \mathrm{LT}_\prec(f)$ *such that*

  $\mathrm{LT}_{\prec_m^n}(r) \notin \{\mathrm{LT}_{\prec_m^n}(\lambda g_i) : 1 \le i \le q, \lambda \in \Lambda, \mathrm{LT}_{\prec_n^m}(\lambda g_i) \preceq_n^m \mathrm{LT}_{\prec_n^m}(f)\}$.

  In this case we say that the element f $\mathcal{A}$-reduces to r by G relative to $\prec_m^n$.

Proof We give a constructive proof, along the lines of [22, Theorem 3.1]. First, we initialize $r = f$ and $h_i = 0$ for $1 \le i \le q$. The next steps are repeated until $r = 0$ or there exists no λ and g_i that satisfy the conditions of the theorem. If there exists $\lambda \in \Lambda$ such that for all $\prec$ in $\mathcal{A}$ we have $\mathrm{LT}_\prec(r) \preceq \mathrm{LT}_\prec(f)$ and

$$\mathrm{LT}_{\prec_m^n}(r) = \mathrm{LT}_{\prec_m^n}(\lambda g_i) \quad \wedge \quad \mathrm{LT}_{\prec_n^m}(\lambda g_i) \preceq_n^m \mathrm{LT}_{\prec_n^m}(r),$$

we are allowed to perform the reduction step, and update the quantities r to r', respectively h_i to h_i', as follows:

$$r' = r - \frac{\mathrm{LC}_{\prec_m^n}(r)}{\mathrm{LC}_{\prec_m^n}(\lambda g_i)}\lambda g_i \quad h_i' = h_i + \frac{\mathrm{LC}_{\prec_m^n}(r)}{\mathrm{LC}_{\prec_m^n}(\lambda g_i)} \cdot \lambda.$$

Obviously, we have

$$\mathrm{LT}_{\prec_m^n}(r') \prec_m^n \mathrm{LT}_{\prec_m^n}(r), \quad \text{while for all } \prec in\mathcal{A}\text{we have } \mathrm{LT}_\prec(\lambda g_i) \preceq \mathrm{LT}_\prec(f).$$

Since the set of monomials $\Lambda E \subseteq F$ is well-ordered, this can only be repeated finitely often. Summing up the λs to h_i we obtain that for all $\prec \in \mathcal{A}$, one has $\mathrm{LT}_\prec(h_i g_i) \preceq \mathrm{LT}_\prec(f)$. □

Based on set-relative reduction, we can now consider a p-step procedure for computing Gröbner bases in this setting.

Algorithm 2 Buchberger Algorithm for Filtered Rings

```
Require: F is a free R-module, V := {f_1, ..., f_r} ⊆ F\{0};
  The ring R is p-fold filtered.
Ensure: G := {g_1, ..., g_s} ⊆ F\{0} where _R⟨G⟩ = _R⟨V⟩ such that I_ρ ∩ (RG) = 0.
  (I_ρ is the set of all irreducible elements with respect to the set-relative reduction)
  A ← {≺_p};
  G^(0) ← {f_1, ..., f_r};
  while there exist j ∈ {1, ..., k}, f, g ∈ G^(0) and v ∈ V(j, f, g) such that
     S_{≺_p}(j, f, g, v) A-reduces to r ≠ 0 relative to ≺_p by G^(0) do
     G^(0) ← G^(0) ∪ {r};
  G^(1) ← G^(0);
  for ℓ = p − 1, ..., 1 do
     (A, ≺) ← ({≺_p, ..., ≺_{ℓ+1}}, ≺_ℓ^{ℓ+1});
     while there exist j ∈ {1, ..., k}, f, g ∈ G^(p−ℓ) and v ∈ V(j, f, g) such that
              S_{≺_ℓ}(j, f, g, v) is A-reduces to r ≠ 0 relative to ≺ by G^(p−ℓ) do
        G^(p−ℓ) ← G^(p−ℓ) ∪ {r};
     G^(p−ℓ+1) ← G^(p−ℓ);
  return G^(p)
```

The termination of the algorithm is justified by the following facts. First, a generalized term order is a well-order and the first loop terminates by Buchberger's Theorem (it is proven in [1]). For the second loop, in each step we have a pair of admissible orders $\prec^{l}_{\ell-1}$ and $\prec^{\ell-1}_{\ell}$, and hence one can apply Theorem 3.6. The set-relative reduction ensures condition 5 of Definition 2.5. Finally, property 4 in Definition 2.5 follows from a straight-forward generalization of [22, Theorem 3.3]. An implementation of a the algorithm for the ring of difference–differential operators and a bi-filtered ring has been obtained in [2].

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References

1. Buchberger, B.: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. Ph.D. Thesis, University of Innsbruck (1965)
2. Dönch, C., Winkler, F.: Bivariate difference-differential polynomials and their computation in Maple. In: Proceedings of 8th International Conference on Applied Informatics, pp. 211–218. Eger, Hungary (2010)
3. Dönch, C.: Standard bases in finitely generated difference-skew-differential modules and their application to dimension polynomials. Ph.D. Thesis, Research Institute for Symbolic Computation (RISC), Linz, Austria (2012)
4. Dönch, C.: Characterization of relative Gröbner Bases. J. Symb. Comput. **55**(1), 19–29 (2013)
5. Dönch, C., Levin, A.: Bivariate dimension polynomials and new invariants of finitely generated D-modules. Internat. J. Algebra Comput. **23**(7), 1625–1651 (2013)
6. Fürst, C.: Axiomatic description of Gröbner reduction. Ph.D. Thesis, Research Institute for Symbolic Computation (RISC), Linz, Austria (2016)
7. Fürst, C., Landsmann, G.: Computation of dimension in filtered free modules by Gröbner reduction. In: Proceedings of ISSAC, Bath, UK, pp. 181–188 (2015)
8. Fürst, C., Landsmann, G.: Three examples of Gröbner reduction over noncommutative rings. RISC report series 15–16, Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz, Schloss Hagenberg, 4232 Hagenberg, Austria (2016)
9. Futorny, V., Hartwig, J.: Multiparameter twisted Weyl algebras. J. Algebra **357**, 69–93 (2012)
10. Huang, G., Zhou, M.: Termination of algorithm for computing relative Gröbner bases and difference differential dimension polynomials. Front. Math. China **10**(3), 635–648 (2015)
11. Kandri-Rodi, A., Weispfenning, V.: Non-commutative Gröbner bases in algebras of solvable type. J. Symb. Comput. **9**(1), 1–26 (1990)
12. Kondrateva, M., Levin, A., Mikhalev, A., Pankratev, E.: Differential and Difference Dimension Polynomials. Kluwer Academic Publishers, Massachusetts (1998)
13. Levin, A.: Reduced Gröbner bases, free difference-differential modules and difference-differential dimension polynomials. J. Symb. Comput. **30**(4), 357–382 (2000)
14. Levin, A.: Characteristic polynomials of finitely generated modules over Weyl algebras. Bull. Austral. Math. Soc. **61**, 387–403 (2000)
15. Levin, A.: Gröbner bases with respect to several term orderings and multivariate dimension polynomials. In: Proceedings of ISSAC, pp. 251–260. Waterloo, Ontario, Canada (2007)
16. Levin, A.: Gröbner bases with respect to several orderings and multivariable dimension polynomials. J. Symb. Comput. **42**(5), 561–578 (2007)
17. Levin, A.: Difference Algebra. Springer, New York (2008)
18. Oaku, T., Shimoyama, T.: A Gröbner basis method for modules over rings of differential operators. J. Symb. Comput. **18**(3), 223–248 (1994)
19. Pusz, W., Woronowicz, S.: Twisted second quantization. Rep. Math. Phys. **27**(2), 231–257 (1989)
20. Winkler, F., Zhou, M.: Gröbner bases in difference–differential modules. Proc. ISSAC. ACM Press, New York (2006)
21. Winkler, F., Zhou, M.: Gröbner bases in difference–differential modules and difference–differential dimension polynomials. Sci. China. Ser. A Math. **51**(9), 1732–1752 (2008)
22. Winkler, F., Zhou, M.: Computing difference–differential dimension polynomials by relative Gröbner bases in difference–differential modules. J. Symb. Comput. **43**(10), 726–745 (2008)