



A Sharp Weak-Type (∞, ∞) Inequality for the Hilbert Transform

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Abstract The paper is devoted to sharp weak type (∞, ∞) estimates for $\mathcal{H}^{\mathbb{T}}$ and $\mathcal{H}^{\mathbb{R}}$, the Hilbert transforms on the circle and real line, respectively. Specifically, it is proved that

$$\left\| \mathcal{H}^{\mathbb{T}} f \right\|_{W(\mathbb{T})} \leq \|f\|_{L^\infty(\mathbb{T})}$$

and

$$\left\| \mathcal{H}^{\mathbb{R}} f \right\|_{W(\mathbb{R})} \leq \|f\|_{L^\infty(\mathbb{R})},$$

where $W(\mathbb{T})$ and $W(\mathbb{R})$ stand for the weak- L^∞ spaces introduced by Bennett, DeVore and Sharpley. In both estimates, the constant 1 on the right is shown to be the best possible.

Keywords Hilbert transform · Weak type inequality · Best constants

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1 Introduction

Our motivation comes from a very basic question about the Hilbert transform $\mathcal{H}^{\mathbb{T}}$ on the unit circle $\mathbb{T} \simeq (-\pi, \pi]$ equipped with a normalized uniform measure m . Recall that this operator is given by the singular integral

$$\mathcal{H}^{\mathbb{T}} f(x) = \text{p.v.} \int_{-\pi}^{\pi} f(t) \cot \frac{x-t}{2} m(dt), \quad x \in \mathbb{T},$$

when $f \in L^1(\mathbb{T})$. A classical result of Riesz [13] states that for any $1 < p < \infty$ there is a finite universal constant C_p such that

$$\left\| \mathcal{H}^{\mathbb{T}} f \right\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L^p(\mathbb{T})}, \quad f \in L^p(\mathbb{T}). \tag{1.1}$$

For $p = 1$ the above estimate does not hold with any $C_1 < \infty$, but, as Kolmogorov showed in [11], there is an absolute $c_1 < \infty$ such that

$$\left\| \mathcal{H}^{\mathbb{T}} f \right\|_{L^{1,\infty}(\mathbb{T})} := \sup_{\lambda > 0} \left[\lambda m \left(\left\{ x \in \mathbb{T} : |\mathcal{H}^{\mathbb{T}} f(x)| \geq \lambda \right\} \right) \right] \leq c_1 \|f\|_{L^1(\mathbb{T})}, \tag{1.2}$$

whenever $f \in L^1(\mathbb{T})$. The optimal values of the constants C_p and c_1 were determined in 1970s: Pichorides [12] and Cole (unpublished: see Gamelin [9]) proved that the best constant in (1.1) equals $\cot \frac{\pi}{2p^*}$, where $p^* = \max\{p, p/(p-1)\}$, and Davis [6] showed that the optimal choice for the constant c_1 in (1.2) is

$$\left(\frac{1}{\pi} \int_{\mathbb{R}} \frac{\frac{2}{\pi} \log |t|}{t^2 + 1} dt \right)^{-1} = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots} = 1.347\dots$$

The above results are of fundamental importance to harmonic analysis. Furthermore, the methods developed by Riesz [13] have had a profound influence on the shape of the contemporary mathematics. For numerous extensions and applications of the above statements, consult e.g. the works of Burkholder [3], Calderón and Zygmund [5], Essén [8], Gohberg and Krupnik [10], Stein [14] and Zygmund [15], and many more.

We will continue the research in this direction. We will be interested in a “dual” version of Kolmogorov’s result, i.e., in a weak- L^∞ estimate for $\mathcal{H}^{\mathbb{T}}$. To explain what the weak- L^∞ space is, we need more notation. For a given measurable function $f : \mathbb{T} \rightarrow \mathbb{R}$, we define f^* , the decreasing rearrangement of f , by

$$f^*(t) = \inf \{ \lambda \geq 0 : m(\{x \in \mathbb{T} : |f(x)| > \lambda\}) \leq t \}.$$

Then $f^{**} : (0, 1] \rightarrow [0, \infty)$, the maximal function of f^* , is given by the formula

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \quad t \in (0, 1].$$

One easily verifies that f^{**} can alternatively be defined by

$$f^{**}(t) = \frac{1}{t} \sup \left\{ \int_E |f| dm : E \subseteq \mathbb{T}, m(E) = t \right\}.$$

We are ready to introduce the weak- L^∞ space. Following Bennett, DeVore and Sharpley [1], we let

$$\|f\|_{W(\mathbb{T})} = \sup_{t \geq 0} (f^{**}(t) - f^*(t))$$

and define $W(\mathbb{T}) = \{f : \mathbb{T} \rightarrow \mathbb{R} : \|f\|_{W(\mathbb{T})} < \infty\}$. Some words explaining the meaning of this space are in order. For each $1 \leq p < \infty$, the usual weak space $L^{p,\infty}$ properly contains L^p , but for $p = \infty$, the two spaces coincide. Thus, there is no Marcinkiewicz interpolation theorem between L^1 and L^∞ for operators which are unbounded on L^∞ . The reason for introducing the space W was to fill this gap. It can be verified that this space contains L^∞ , can be understood as an appropriate limit of $L^{p,\infty}$ as $p \rightarrow \infty$, and enjoys the required interpolation property: if T is bounded as an operator from L^1 to $L^{1,\infty}$ and from L^∞ to W , then it has an extension which is bounded on L^p spaces, $1 < p < \infty$. See [1] for details. There is a further evidence, again rooted in the interpolation theory, that the space W can serve as a substitute for weak- L^∞ . Namely, the Peetre K -functional for the pair (L^1, L^∞) (cf. [4, p. 184]) is explicitly given by

$$K(f, t; L^1, L^\infty) = \int_0^t f^*(s) ds = t f^{**}(t), \quad t \in (0, 1].$$

Thus, the weak- L^1 norm can be expressed in terms of the K -functional by

$$\|f\|_{L^{1,\infty}(\mathbb{T})} = \sup_{t \in (0,1]} t f^*(t) = \sup_{t \in (0,1]} t \frac{d}{dt} K(f, t; L^1, L^\infty). \tag{1.3}$$

Now if we reverse the roles of L^1 and L^∞ , and make use of the identity $K(f, t; L^\infty, L^1) = tK(f, t^{-1}; L^1, L^\infty)$, we see that the expression on the right of (1.3) is precisely $\sup_{t \in (0,1]} [f^{**}(t) - f^*(t)]$. Hence this number can be understood as a substitute for the norm in the weak- L^∞ . For more on this interplay, the connections between W and BMO , as well as other interesting properties of W , we refer the reader to [1] and the monograph [2] by Bennett and Sharpley.

One of our main results is the identification of the norm of $\mathcal{H}^\mathbb{T}$ as an operator acting from $L^\infty(\mathbb{T})$ to $W(\mathbb{T})$. Here is the precise statement.

Theorem 1.1 *For any $f \in L^\infty(\mathbb{T})$ we have*

$$\left\| \mathcal{H}^\mathbb{T} f \right\|_{W(\mathbb{T})} \leq \|f\|_{L^\infty(\mathbb{T})}. \tag{1.4}$$

The inequality is sharp: for any $c < 1$ there is a function $f \in L^\infty(\mathbb{T})$ such that $\|\mathcal{H}^\mathbb{T} f\|_{W(\mathbb{T})} > c\|f\|_{L^\infty(\mathbb{T})}$.

We will also study an analogue of the above result in the nonperiodic case. Recall that the Hilbert transform $\mathcal{H}^{\mathbb{R}}$ on the real line is defined by the principal value integral

$$\mathcal{H}^{\mathbb{R}} f(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(t)}{x-t} dt, \quad x \in \mathbb{R},$$

when $f \in L^1(\mathbb{R})$. The above strong and weak-type inequalities (1.1), (1.2) can be extended to analogous statements for $\mathcal{H}^{\mathbb{R}}$ and the optimal constants remain unchanged (see e.g. [13, 15]). It is natural to ask about a sharp weak-type (∞, ∞) inequality in this setting. To study this problem, define the weak space $W(\mathbb{R})$ in the same manner as above:

$$W(\mathbb{R}) = \left\{ f : \mathbb{R} \rightarrow \mathbb{R} : \|f\|_{W(\mathbb{R})} := \sup_{t>0} [f^{**}(t) - f^*(t)] < \infty \right\},$$

where, as previously, f^* denotes the decreasing rearrangement of f and f^{**} stands for the maximal function of f^* . Here is the nonperiodic version of Theorem 1.1. It is well known that some technical problems arise when one defines the action of the Hilbert transform on $L^\infty(\mathbb{R})$; to avoid these, we impose a slightly stronger integrability on functions.

Theorem 1.2 *If f belongs to $L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ for some $1 \leq p < \infty$, then we have the sharp bound*

$$\left\| \mathcal{H}^{\mathbb{R}} f \right\|_{W(\mathbb{R})} \leq \|f\|_{L^\infty(\mathbb{R})}. \quad (1.5)$$

The paper is organized as follows. In the next section we establish Theorem 1.1. In the proof of (1.4) we make use of Bellman function method: the estimate is deduced from the existence of a certain special superharmonic function. In the final part of the paper we present the proof of Theorem 1.2, which follows from Theorem 1.1 by certain transference-type arguments.

2 Periodic Case

For any $c \geq 0$, define the function $V^{(c)} : [-1, 1] \times [0, \infty) \rightarrow \mathbb{R}$ by $V^{(c)}(x, y) = (y - c)\chi_{\{y>0\}}$ (here and below, χ_A denotes the indicator function of a set A). Furthermore, let $U^{(c)} : (-1, 1) \times (0, \infty) \rightarrow \mathbb{R}$ be given by the formula

$$\begin{aligned} U^{(c)}(x, y) = & y - c + \frac{2c}{\pi} \arctan \left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} - \tan \frac{\pi x}{2} \right) \\ & + \frac{2c}{\pi} \arctan \left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} + \tan \frac{\pi x}{2} \right). \end{aligned}$$

It is easy to check that $U^{(c)}$ is a harmonic function. Actually, it can be regarded as a harmonic lift of $V^{(c)}$, in the sense explained in the first part of the lemma below.

Lemma 2.1 *The function $U^{(c)}$ has the following properties.*

1. *If $Y > 0$, then $\lim_{(x,y) \rightarrow (\pm 1, Y)} U^{(c)}(x, y) = V^{(c)}(\pm 1, Y)$; if $X \in (-1, 1)$, then $\lim_{(x,y) \rightarrow (X, 0)} U^{(c)}(x, y) = V^{(c)}(X, 0)$.*
2. *For any $x \in (-1, 1)$, we have*

$$\lim_{y \downarrow 0} U^{(c)}(x, y)/y = 1 - c \left(\cos \frac{\pi x}{2} \right)^{-1}.$$

3. *For any $(x, y) \in (-1, 1) \times (0, \infty)$, we have $U^{(c)}(x, y) \geq V^{(c)}(x, y)$.*

Proof The properties (1) and (2) are straightforward and left to the reader. The majorization (3) is also easy: we must show that

$$\arctan \left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} - \tan \frac{\pi x}{2} \right) + \arctan \left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} + \tan \frac{\pi x}{2} \right) \geq 0.$$

This follows from the estimate

$$\left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} - \tan \frac{\pi x}{2} \right) + \left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} + \tan \frac{\pi x}{2} \right) \geq 0$$

and the fact that the arctangent function is odd and increasing on the real line. □

It will be convenient for us to extend $U^{(c)}$ to the halfstrip $[-1, 1] \times [0, \infty)$ by the requirement that $U^{(c)}$ and $V^{(c)}$ match at the boundary of this set. Then $U^{(c)}$ becomes a harmonic majorant of $V^{(c)}$ on the whole $[-1, 1] \times [0, \infty)$, and it is continuous except for the points $(\pm 1, 0)$. In addition, part (2) of the above lemma implies that for $c \geq 1$, the one-sided partial derivative $U_{y+}^{(c)}$ satisfies $U_{y+}^{(c)}(x, 0) \leq 0$ for all $x \in (-1, 1)$.

The above function $U^{(c)}$ is a “building block” for a larger class of superharmonic functions. For a fixed parameter $\lambda \geq 0$, introduce the functions $\mathcal{U}_\lambda^{(c)}, \mathcal{V}_\lambda^{(c)}$ on the strip $[-1, 1] \times \mathbb{R}$ by the formulas

$$\mathcal{U}_\lambda^{(c)}(x, y) = U^{(c)}(x, (|y| - \lambda)_+) = \begin{cases} U^{(c)}(x, y - \lambda) & \text{if } y \geq \lambda, \\ 0 & \text{if } |y| < \lambda, \\ U^{(c)}(x, -\lambda - y) & \text{if } y < -\lambda \end{cases}$$

and $\mathcal{V}_\lambda^{(c)}(x, y) = V^{(c)}(x, (|y| - \lambda)_+) = (|y| - \lambda)_+ - c \chi_{\{|y| > \lambda\}}$.

Lemma 2.2 *For each $\lambda \geq 0$ and $c \geq 1$, the function $\mathcal{U}_\lambda^{(c)}$ is a superharmonic majorant of $\mathcal{V}_\lambda^{(c)}$.*

Proof Assume first that $c > 1$. The inequality $\mathcal{U}_\lambda^{(c)} \geq \mathcal{V}_\lambda^{(c)}$ follows immediately from the majorization $U^{(c)} \geq V^{(c)}$ established above; hence all we need is the superharmonicity of $\mathcal{U}_\lambda^{(c)}$. Observe that this function is harmonic on each of the domains

$(-1, 1) \times (-\infty, -\lambda)$, $(-1, 1) \times (-\lambda, \lambda)$ and $(-1, 1) \times (\lambda, \infty)$. Consequently, it is enough to check that $\mathcal{U}_\lambda^{(c)}$ satisfies the mean value property at each point of the form $(x, \pm\lambda)$. But this follows at once from the inequality $U_{y\pm}^{(c)}(x, 0) < 0$ (here the strictness is due to $c > 1$). To get the claim for $c = 1$, note that $U^{(1)}$ is a pointwise limit of $U^{(c)}$ as $c \downarrow 1$. □

In the next lemma we establish an intermediate result which is of its own interest.

Lemma 2.3 *For any $f : \mathbb{T} \rightarrow [-1, 1]$ and any $\lambda \geq 0$, we have*

$$\int_{\mathbb{T}} \left(\left| \mathcal{H}^{\mathbb{T}} f \right| - \lambda \right)_+ dm \leq m \left(\left\{ x \in \mathbb{T} : \left| \mathcal{H}^{\mathbb{T}} f(x) \right| > \lambda \right\} \right). \tag{2.1}$$

Proof Let u, v denote the harmonic extensions of f and $\mathcal{H}^{\mathbb{T}} f$ to the unit disc, obtained via the Poisson kernel. Then u, v satisfy Cauchy-Riemann equations and $v(0, 0) = 0$ (cf. Riesz [13]). Consequently, the function $\mathcal{U}_\lambda^{(1)}(u, v)$ is superharmonic (being the composition of a superharmonic $\mathcal{U}^{(1)}$ and the analytic $u + iv$) and it majorizes $\mathcal{V}_\lambda^{(1)}(u, v)$. Therefore, by the mean value property,

$$\begin{aligned} \int_{\mathbb{T}} \mathcal{V}_\lambda^{(1)}(u, v) dm &\leq \int_{\mathbb{T}} \mathcal{U}_\lambda^{(1)}(u, v) dm \\ &\leq \mathcal{U}_\lambda^{(1)}(u(0, 0), v(0, 0)) = \mathcal{U}_\lambda^{(1)}(u(0, 0), 0) = 0. \end{aligned}$$

This is precisely (2.1). □

We turn our attention to Theorem 1.1.

Proof of (1.4) By homogeneity, we may and do assume that $\|f\|_{L^\infty(\mathbb{T})} = 1$. By the definition of $(\mathcal{H}^{\mathbb{T}} f)^{**}$, we may write

$$\begin{aligned} & \left(\mathcal{H}^{\mathbb{T}} f \right)^{**} (t) - \left(\mathcal{H}^{\mathbb{T}} f \right)^* (t) \\ &= \sup \left\{ \frac{1}{m(E)} \int_E \left[\left| \mathcal{H}^{\mathbb{T}} f(x) \right| - \left(\mathcal{H}^{\mathbb{T}} f \right)^* (t) \right] m(dx) : E \subseteq \mathbb{T}, m(E) = t \right\}. \end{aligned}$$

It is clear that when computing this supremum, we may restrict ourselves to those E , which satisfy

$$\left\{ x \in \mathbb{T} : \left| \mathcal{H}^{\mathbb{T}} f(x) \right| > \lambda \right\} \subseteq E \subseteq \left\{ x \in \mathbb{T} : \left| \mathcal{H}^{\mathbb{T}} f(x) \right| \geq \lambda \right\}$$

for some $\lambda \geq 0$. Actually, since $m(E) = t$, this λ must be equal to $(\mathcal{H}^{\mathbb{T}} f)^*(t)$. For such E , it is clear that

$$\begin{aligned} & \frac{1}{m(E)} \int_E \left[\left| \mathcal{H}^{\mathbb{T}} f(x) \right| - \left(\mathcal{H}^{\mathbb{T}} f \right)^*(t) \right] m(dx) \\ & \leq \frac{1}{m(\{x \in \mathbb{T} : |\mathcal{H}^{\mathbb{T}} f(x)| > \lambda\})} \int_{\{|\mathcal{H}^{\mathbb{T}} f| > \lambda\}} \left[\left| \mathcal{H}^{\mathbb{T}} f \right| - \lambda \right] dm \leq 1, \end{aligned}$$

where the latter bound is due to (2.1). This establishes (1.4). □

Sharpness Let $c \in (0, 1)$ and fix an arbitrary $\lambda \in (0, c/2)$. Consider the region $\mathcal{C} = [-1, 1] \times [-\lambda, \infty)$ and let F be the conformal mapping which sends the unit disc \mathbb{D} onto \mathcal{C} and $(0, 0) \in \mathbb{D}$ to $(0, 0) \in \mathcal{C}$. Then F transports m , the harmonic measure on \mathbb{T} with respect to $(0, 0)$, to μ , the harmonic measure on $\partial\mathcal{C}$ with respect to $(0, 0)$. Finally, put $u = \operatorname{Re} F$ and $v = \operatorname{Im} F$; clearly, the restrictions $f = u|_{\mathbb{T}}$ and $g = v|_{\mathbb{T}}$ satisfy the relation $g = \mathcal{H}^{\mathbb{T}} f$. The function $(x, y) \mapsto U^{(c)}(x, y + \lambda)$ is harmonic in the interior of \mathcal{C} , so by the mean-value property,

$$\begin{aligned} U^{(c)}(0, \lambda) &= \int_{\partial\mathcal{C}} U^{(c)}(x, y + \lambda) d\mu(x, y) \\ &= \int_{\mathbb{T}} U^{(c)}(u, v + \lambda) dm \\ &= \int_{\mathbb{T}} (v + \lambda - c) \chi_{\{v + \lambda > 0\}} dm \\ &\leq \int_{\mathbb{T}} (v + \lambda - c) \chi_{\{v - \lambda > 0\}} dm \\ &= \int_{\mathbb{T}} \mathcal{H}^{\mathbb{T}} f \chi_{\{\mathcal{H}^{\mathbb{T}} f > \lambda\}} dm - (c - \lambda) m(\{x \in \mathbb{T} : \mathcal{H}^{\mathbb{T}} f(x) > \lambda\}). \end{aligned}$$

However, if λ is sufficiently close to 0, then $U^{(c)}(0, \lambda) > 0$: this follows from Lemma 2.1 (2). Hence, for such λ ,

$$\frac{1}{m(\{x \in \mathbb{T} : \mathcal{H}^{\mathbb{T}} f(x) > \lambda\})} \int_{\{\mathcal{H}^{\mathbb{T}} f > \lambda\}} \left| \mathcal{H}^{\mathbb{T}} f(x) \right| m(dx) \geq c - \lambda.$$

Now take $t = m(\{x \in \mathbb{T} : \mathcal{H}^{\mathbb{T}} f(x) > \lambda\})$. The above inequality implies that

$$\left(\mathcal{H}^{\mathbb{T}} f \right)^{**}(t) \geq c - \lambda. \tag{2.2}$$

In addition, since $\mathcal{H}^{\mathbb{T}} f \geq -\lambda$ on \mathbb{T} , we actually have $t = m(\{x \in \mathbb{T} : |\mathcal{H}^{\mathbb{T}} f(x)| > \lambda\})$. Hence, from the very definition of the decreasing rearrangement, we infer that $(\mathcal{H}^{\mathbb{T}} f)^*(t) \leq \lambda$. Combining this with (2.2), we obtain

$$\left\| \mathcal{H}^{\mathbb{T}} f \right\|_{W(\mathbb{T})} \geq c - 2\lambda.$$

It remains to observe that the right-hand side can be made arbitrarily close to 1, by choosing c appropriately close to 1 and then picking λ sufficiently small. This proves that the constant 1 cannot be replaced in (1.4) by a smaller number. □

3 The Non-Periodic Case

Proof of (1.5) To deduce the weak-type estimate for the Hilbert transform on the real line, we use a standard argument known as “blowing up the circle”, which is due to Zygmund ([15], Chapter XVI, Theorem 3.8). Let $f \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a fixed function. For a given positive integer n and $x \in \mathbb{R}$, put

$$g_n(x) = \frac{1}{2\pi n} \text{p.v.} \int_{-\pi n}^{\pi n} f(t) \cot \frac{x-t}{2n} dt.$$

As shown in [15], we have $g_n \rightarrow \mathcal{H}^{\mathbb{R}} f$ almost everywhere as $n \rightarrow \infty$. On the other hand, the function

$$x \mapsto g_n(nx) = \text{p.v.} \int_{-\pi}^{\pi} f(nt) \cot \frac{x-t}{2} m(dt)$$

is precisely the periodic Hilbert transform of the function $f_n : x \mapsto f(nx)$, $|x| \leq \pi$. Consequently, by (2.1), we may write

$$\begin{aligned} |\{x \in (-\pi n, \pi n) : |g_n(x)| > \lambda\}| &= 2\pi n m \left(\left\{ x \in \mathbb{T} : \left| \mathcal{H}^{\mathbb{T}} f_n(x) \right| > \lambda \right\} \right) \\ &\geq 2\pi n \int_{\mathbb{T}} \left(\left| \mathcal{H}^{\mathbb{T}} f_n(x) \right| - \lambda \right)_+ m(dx) \\ &= \int_{-\pi n}^{\pi n} (|g_n(x)| - \lambda)_+ dx. \end{aligned}$$

Now we let $n \rightarrow \infty$; using some routine limiting arguments, we get

$$\left| \{x \in \mathbb{R} : \mathcal{H}^{\mathbb{R}} f(x) > \lambda\} \right| \geq \int_{\mathbb{R}} \left(\left| \mathcal{H}^{\mathbb{R}} f(x) \right| - \lambda \right)_+ dx.$$

It remains to repeat the reasoning from the periodic case to obtain, for any $t > 0$,

$$\begin{aligned} & \left(\mathcal{H}^{\mathbb{R}} f \right)^{**}(t) - \left(\mathcal{H}^{\mathbb{R}} f \right)^*(t) \\ & \leq \frac{1}{|\{x \in \mathbb{R} : |\mathcal{H}^{\mathbb{R}} f(x)| > \lambda\}|} \int_{\{|\mathcal{H}^{\mathbb{R}} f| > \lambda\}} \left[\left| \mathcal{H}^{\mathbb{R}} f(x) \right| - \lambda \right] dx \leq 1. \end{aligned}$$

□

Sharpness As we have shown in the previous section, for any $c \in (0, 1)$ and λ sufficiently close to 0, there is a function $\varphi : \mathbb{T} \rightarrow [-1, 1]$ such that $\int_{\mathbb{T}} \varphi dm = 0$ and

$$\frac{1}{|\{x \in \mathbb{T} : |\mathcal{H}^{\mathbb{T}} \varphi(x)| > \lambda\}|} \int_{\{|\mathcal{H}^{\mathbb{T}} \varphi| > \lambda\}} \left| \mathcal{H}^{\mathbb{T}} \varphi(x) \right| dx \geq c - \lambda. \tag{3.1}$$

We will expand this function onto the real line. We will use Davis’ argument from [6]. For the sake of clarity, we have divided the reasoning into three parts.

1. *A conformal mapping and its properties* Let H denote the closed upper halfplane of \mathbb{C} and consider the conformal mapping $K(z) = -(1 - z)^2/4z$. This function maps the halfdisc $\mathbb{D} \cap H$ onto H , and the boundary of $\mathbb{D} \cap H$ onto \mathbb{R} . Let L be the inverse of K . Then L maps $[0, 1]$ onto the halfcircle $\{e^{i\theta} : 0 \leq \theta \leq \pi\}$, and $\mathbb{R} \setminus [0, 1]$ onto $(-1, 1)$. Specifically, for $x \in [0, 1]$ we have $L(x) = \exp(2i \arcsin(\sqrt{x}))$, while for $x \notin [0, 1]$,

$$L(x) = \begin{cases} 1 - 2x - 2\sqrt{x^2 - x} & \text{if } x < 0, \\ 1 - 2x + 2\sqrt{x^2 - x} & \text{if } x > 1. \end{cases}$$

We will also need the property

$$L(z) \rightarrow 0 \quad \text{as } z \rightarrow \infty. \tag{3.2}$$

Next, for a positive integer n , let d_n be the density of $L^n([0, 1])$ on \mathbb{T} with respect to m , i.e. for any $-\pi < \alpha < \beta < \pi$,

$$\int_{\alpha}^{\beta} d_n(e^{i\theta}) m(d\theta) = \left| \left\{ r \in [0, 1] : L^n(r) \in \left\{ e^{i\theta} : \alpha < \theta < \beta \right\} \right\} \right|.$$

Then it is easy to prove that

$$d_n \rightarrow 1 \text{ uniformly on } \mathbb{T}, \tag{3.3}$$

see Lemma 3 in [6].

2. *Expansion of φ* Let Φ denote the holomorphic extension of $\varphi + i\mathcal{H}^{\mathbb{T}}\varphi$ to the unit disc. Then Φ satisfies $\Phi(0) = 0$: indeed, $\text{Re } \Phi(0) = 0$ is due to the condition $\int_{\mathbb{T}} \varphi dm = 0$, while $\text{Im } \Phi(0) = 0$ follows from the normalization property of the periodic Hilbert transform. Combining this with (3.2), we see that the analytic function $F_n = \Phi(L^n(z))$ ($n = 1, 2, \dots$), given on the halfplane H , satisfies $\lim_{z \rightarrow \infty} F_n(z) = 0$. Put $f_n(x) = \text{Re } F_n(x)$ for any $x \in \mathbb{R}$. This function is bounded in absolute value by 1, since so is φ . Furthermore, f_n is integrable when $n \geq 2$. Indeed, for any $x \notin [-1, 1]$ we have

$$|f_n(x)| = |\text{Re } \Phi(L^n(x))| \leq \kappa_1 |L^n(x)| \leq \kappa_2 |x|^{-n},$$

for some universal constants κ_1, κ_2 . Thus, we may speak of $\mathcal{H}^{\mathbb{R}} f_n$. Furthermore, by the aforementioned property $\lim_{z \rightarrow \infty} F_n(z) = 0$, we have $\mathcal{H}^{\mathbb{R}} f_n = \text{Im } F_n|_{\mathbb{R}}$.

3. *Computations* If $x \notin [0, 1]$, then $L(x) \in (-1, 1)$ and hence $L^n(x) \rightarrow 0$ as $n \rightarrow \infty$. Consequently, we have

$$\left| \left\{ x \notin [0, 1] : \left| \mathcal{H}^{\mathbb{R}} f_n(x) \right| > \lambda \right\} \right| \rightarrow 0$$

and, by Lebesgue's dominated convergence theorem,

$$\int_{\{x \notin [0, 1] : |\mathcal{H}^{\mathbb{R}} f_n(x)| > \lambda\}} \left| \mathcal{H}^{\mathbb{R}} f_n(x) \right| dx \xrightarrow{n \rightarrow \infty} 0.$$

Next, observe that by (3.3),

$$\begin{aligned} \left| \left\{ x \in [0, 1] : \left| \mathcal{H}^{\mathbb{R}} f_n(x) \right| > \lambda \right\} \right| &= |\{x \in [0, 1] : |\operatorname{Im} \Phi(L^n(x))| > \lambda\}| \\ &= \left| \left\{ x \in [0, 1] : \left| \mathcal{H}^{\mathbb{T}} \varphi(L^n(x)) \right| > \lambda \right\} \right| \\ &\xrightarrow{n \rightarrow \infty} m \left(\left\{ x \in \mathbb{T} : \left| \mathcal{H}^{\mathbb{T}} \varphi(x) \right| > \lambda \right\} \right) \end{aligned}$$

and

$$\begin{aligned} \int_{\{x \in [0, 1] : |\mathcal{H}^{\mathbb{R}} f_n(x)| > \lambda\}} \left| \mathcal{H}^{\mathbb{R}} f_n(x) \right| dx &= \int_{\{x \in [0, 1] : |\mathcal{H}^{\mathbb{T}} \varphi(L^n(x))| > \lambda\}} \left| \mathcal{H}^{\mathbb{T}} \varphi(L^n(x)) \right| dx \\ &\xrightarrow{n \rightarrow \infty} \int_{\mathbb{T}} \left| \mathcal{H}^{\mathbb{T}} \varphi(x) \right| m(dx). \end{aligned}$$

Let us put all the above facts together and combine them with (3.1). We get that for an arbitrary $\eta < 1$ we have

$$\frac{1}{|\{x \in \mathbb{R} : |\mathcal{H}^{\mathbb{R}} f_n(x)| > \lambda\}|} \int_{\{|\mathcal{H}^{\mathbb{R}} f_n| > \lambda\}} \left| \mathcal{H}^{\mathbb{R}} f_n(z) \right| dz \geq \eta(c - \lambda),$$

provided n is sufficiently large. Now, set $t = |\{x \in \mathbb{R} : |\mathcal{H}^{\mathbb{R}} f(x)| > \lambda\}|$. Arguing as above, we prove that

$$\left| \left\{ x \notin [0, 1] : \left| \mathcal{H}^{\mathbb{R}} f_n(x) \right| > 2\lambda \right\} \right| \rightarrow 0$$

and

$$\left| \left\{ x \in [0, 1] : \left| \mathcal{H}^{\mathbb{R}} f_n(x) \right| > 2\lambda \right\} \right| \rightarrow m \left(\left\{ x \in \mathbb{T} : \left| \mathcal{H}^{\mathbb{T}} \varphi(x) \right| > 2\lambda \right\} \right) < t.$$

This shows that if n is sufficiently large, then $(\mathcal{H}^{\mathbb{R}} f_n)^*(t) \leq 2\lambda$. Hence, for large n ,

$$\left\| \mathcal{H}^{\mathbb{R}} f_n \right\|_{W(\mathbb{R})} \geq \left(\mathcal{H}^{\mathbb{R}} f_n \right)^{**}(t) - \left(\mathcal{H}^{\mathbb{R}} f_n \right)^*(t) \geq \eta(c - \lambda) - 2\lambda.$$

The latter constant can be made arbitrarily close to 1, by choosing appropriate values for the parameters η , c and λ . This proves that the constant 1 is indeed the best possible in (1.5). \square

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