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Fixed point theorems for Kannan type mappings

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Abstract. In this note, we prove some fixed point theorems for Kannan type mappings. We will use the additional conditions as compactness or asymptotic regularity or involutions. Our proofs are inspired by the study of Lipschitzian mappings (Agarwal et al., Fixed point theory for Lipschitzian-type mappings with applications, 2009).

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1. Kannan fixed point theorem

In 1968, Kannan [10] proved the following fixed point theorem.

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that there exists $K < \frac{1}{2}$ satisfying*

$$d(Tx, Ty) \leqslant K[d(x, Tx) + d(y, Ty)] \quad \textit{for all} \quad x, y \in X. \tag{1}$$

Then, T has a unique fixed point $v \in X$, and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v and $d(T^{n+1}x, v) \leqslant K \cdot (\frac{K}{1-K})^n \cdot d(x, Tx)$, $n = 0, 1, 2, \ldots$

It is not difficult to see that Lipschitzian mappings and mappings satisfying (1) are independent, see [11]. Kannan's theorem is important because Subrahmanyam [14] proved that Kannan's theorem characterizes the metric completeness. That is, a metric space X is complete if and only if every mapping satisfying (1) on X with constant $K < \frac{1}{2}$ has a fixed point. Contractions (in the sense of Banach) do not have this property; Connell [5] gave an example of metric space X such that X is not complete and every contraction on X has a fixed point.

Here is an elementary proof of Kannan theorem. We will start from the following technical lemma, see [8].

Lemma 1.2. *Let C be a nonempty closed subset of a complete metric space (X, d) and let $T : C \to C$ be a mapping such that there exists $K < 1$ satisfying (1). Assume that there exist constants $a, b \in \mathbb{R}$ such that $0 \leqslant a < 1$ and $b > 0$. If for arbitrary $x \in C$ there exists $u \in C$ such that $d(u, Tu) \leqslant a \cdot d(x, Tx)$ and $d(u, x) \leqslant b \cdot d(x, Tx)$, then T has at least one fixed point.*

Proof. Let $x_0 \in C$ be an arbitrary point. Consider a sequence $\{x_n\} \subset C$ satisfies

$$d(Tx_{n+1}, x_{n+1}) \leqslant a \cdot d(Tx_n, x_n),$$

$$d(x_{n+1}, x_n) \leqslant b \cdot d(Tx_n, x_n), \quad n = 0, 1, 2, \ldots$$

Since

$$d(x_{n+1}, x_n) \leqslant b \cdot d(Tx_n, x_n) \leqslant b \cdot a^n \cdot d(Tx_0, x_0),$$

it is easy to see that $\{x_n\}$ is a Cauchy sequence in C. Because C is complete, there exists $v \in C$ such that $\lim_{n\to\infty} x_n = v$. Then

$$\begin{aligned} d(Tv, v) &\leqslant d(Tv, Tx_n) + d(Tx_n, x_n) + d(x_n, v) \\ &\leqslant K[d(v, Tv) + d(x_n, Tx_n)] + d(Tx_n, x_n) + d(x_n, v), \end{aligned}$$

and

$$\begin{aligned} d(Tv, v) &\leqslant \frac{K+1}{1-K} \cdot d(Tx_n, x_n) + \frac{1}{1-K} \cdot d(x_n, v) \\ &\leqslant \frac{K+1}{1-K} \cdot a^n \cdot d(Tx_0, x_0) + \frac{1}{1-K} \cdot d(x_n, v) \to 0 \end{aligned}$$

as $n \to \infty$. Hence, $Tv = v$. □

Proof of Theorem 1.1. For any $x \in X$ let $u = Tx$. Then

$$d(u, Tu) = d(Tx, Tu) \leqslant K[d(x, Tx) + d(u, Tu)],$$

$$d(u, Tu) \leqslant \frac{K}{1-K} \cdot d(x, Tx),$$

where by assumption $\frac{K}{1-K} < 1$, and $d(u, x) = d(Tx, x)$. Now, for arbitrary $x_0 \in X$ we can inductively define a sequence $\{x_{n+1} = Tx_n\}$. By Lemma 1.2, this sequence is convergent, $\lim_{n\to\infty} x_n = v$ and $Tv = v$. Suppose z is another fixed point of T. Then

$$0 < d(v, z) = d(Tv, Tz) \leqslant K[d(v, Tv) + d(z, Tz)] = 0,$$

a contradiction. Hence, T has unique fixed point $v \in X$.

Since for each $x \in X$,

$$d(T^{n+1}x, T^n x) \leqslant K[d(T^n x, T^{n+1}x) + d(T^{n-1}x, T^n x)],$$

$$d(T^{n+1}x, T^n x) \leqslant \frac{K}{1-K} \cdot d(T^n x, T^{n-1}x),$$

we have

$$d(T^{n+1}x, v) \leqslant K[d(T^n x, T^{n+1}x) + d(v, Tv)] = K \cdot d(T^{n+1}x, T^n x)$$
$$\leqslant K \cdot \left(\frac{K}{1-K}\right)^n \cdot d(Tx, x), \quad n = 0, 1, 2, \ldots \qquad \square$$

Remark 1.3. The contraction of Kannan type with constant $K = \frac{1}{2}$ in complete metric space does not guarantee the existence of fixed points of T, [11].

Kannan's fixed point theorem and some of its generalizations are discussed in [9,12,13]. In particular, we have the following theorem (see [12]).

Theorem 1.4. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping with the following property:*

$$d(Tx, Ty) \leqslant Ad(x, Tx) + Bd(y, Ty) + Cd(x, y) \quad \textit{for all} \;\; x, y \in X, \qquad (2)$$

where A, B, C are nonnegative and satisfy $A + B + C < 1$. Then, T has a unique fixed point $v \in X$, and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.

Proof. It is analogous to the last one. $\square$

Note that this theorem is stronger than Banach's and Kannan's fixed point theorems. Let $X = [0, 1]$ be with usual metric and $T : [0, 1] \to [0, 1]$ be a mapping defined by $Tx = \frac{x}{3}$ for $0 \leqslant x < 1$ and $T1 = \frac{1}{6}$. T does not satisfy Banach's condition because it is not continuous at 1. Kannan's condition (1) also cannot be satisfied because $d(T0, T\frac{1}{3}) = \frac{1}{2}[d(0, T0) + d(\frac{1}{3}, T\frac{1}{3})]$. But it satisfies condition (2) if we put $A = \frac{1}{6}$, $B = \frac{1}{9}$ and $C = \frac{1}{3}$.

2. Compactness

Edelstein [6] proved the following.

Theorem 2.1. *Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping. Let us suppose that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then, T has a unique fixed point.*

Now, we prove the following theorem.

Theorem 2.2. *Let (X, d) be a compact metric space and let $T : X \to X$ be a continuous mapping. Let us suppose that*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)] \quad \textit{for all} \;\; x, y \in X \quad \textit{with} \;\; x \neq y.$$

Then, T has a unique fixed point $v \in X$ and for each $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.

Proof. Trivial example of such mapping is $T : [0, 1] \to [0, 1]$ defined by $Tx = c$, $c \in [0, 1]$, where the set $[0, 1]$ with the usual metric is a compact metric space.

The function $f : X \to [0, +\infty)$ defined by $f(x) = d(x, Tx)$ is continuous. In view of compactness, there exists a point $v \in X$ such that $f(v) = \inf\{f(x) : x \in X\}$. If $v \neq Tv$, then

$$d(Tv, T^2v) < \frac{1}{2}[d(v, Tv) + d(Tv, T^2v)],$$
$$d(Tv, T^2v) < d(v, Tv), \tag{3}$$

and hence

$$f(Tv) = d(Tv, T^2v) < d(v, Tv) = f(v),$$

a contradiction. Hence, $v = Tv$. It is obvious that v is unique.

Now take any $x \in X$ and define a sequence $\{x_n = T^n x\}$. If $x = v$, then $x_n = v$, $n = 1, 2, \dots$. Let $x \neq v$. Because

$$d(T^{n+1}x, T^n x) < \frac{1}{2}[d(T^n x, T^{n+1}x) + d(T^{n-1}x, T^n x)],$$

hence

$$d(T^{n+1}x, T^n x) < d(T^n x, T^{n-1}x) < \dots < d(Tx, x),$$

and the sequence of nonnegative numbers $b_n = d(T^{n+1}x, T^n x)$ is nondecreasing and thus convergent. Let $0 \leqslant b = \lim_{n\to\infty} b_n$. The assumption that $b > 0$ leads to the contradiction. Again by compactness of X the sequence $\{T^n x\}$ contains a converging subsequence $\{T^{n_i}x\}$ such that $T^{n_i}x \to z \in X$ as $i \to \infty$. Because T is continuous

$$0 < b = \lim_{i\to\infty} d(T^{n_i+1}x, T^{n_i}x) = d(Tz, z),$$

i.e. $z \neq v$. Moreover, by (3),

$$0 < b = \lim_{i\to\infty} d(T^{n_i+2}x, T^{n_i+1}x) = d(T^2z, Tz) < d(Tz, z) = b,$$

a contradiction. Thus, $b = 0$. Since

$$d(T^{n+1}x, v) = d(T^{n+1}x, Tv) < \frac{1}{2}[d(T^n x, T^{n+1}x) + d(v, Tv)]$$
$$= \frac{1}{2} \cdot d(T^{n+1}x, T^n x) \to b = 0$$

as $n \to \infty$, we have $\lim_{n\to\infty} d(T^{n+1}x, v) = 0$. □

The result holds also in the following case (see also [9]):

Theorem 2.3. *Let (X, d) be a compact metric space and let $T : X \to X$ be a continuous mapping. Let us suppose that*

$$d(Tx, Ty) < Ad(x, Tx) + Bd(y, Ty) + Cd(x, y)$$
$$\text{for all } \; x, y \in X \quad \text{and} \quad x \neq y,$$

where A, B, C are positive and satisfy $A + B + C = 1$. Then, T has a unique fixed point $v \in X$ and for each $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.

Proof. It is similar to the proof of Theorem 2.2. □

Question 2.4. Does there exist a complete but noncompact metric space (X, d) and a continuous mapping $T : X \to X$ such that

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)] \quad \text{for all } x, y \in X \text{ with } x \neq y,$$

and T is fixed point free?

3. Asymptotic regularity

Let (X, d) be a metric space. A mapping $T : X \to X$ satisfying the condition $\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0$ for all $x \in X$ is called *asymptotically regular* [3]. Asymptotically regular mappings are also studied in [2,4].

Let $X = \{0\} \cup [1, 2]$ with the usual metric. A mapping $T : X \to X$ defined by $T0 = 1$ and $Tx = 0$ for $1 \leqslant x \leqslant 2$ is satisfying (1) with $K = \frac{1}{2}$ and T is fixed point free. The iterative sequence $\{x_n = T^n 0\}$ is not convergent, so T is not asymptotically regular.

Now, we prove the following theorem.

Theorem 3.1. *If (X, d) is a complete metric space and $T : X \to X$ is an asymptotically regular mapping such that there exists $K < 1$ satisfying (1). Then, T has a unique fixed point $v \in X$.*

Proof. Let $x \in X$ and define a sequence $\{x_n = T^n x\}$. According to asymptotic regularity, we get for $m > n$,

$$d(T^{n+1}x, T^{m+1}x) \leqslant K \cdot \{d(T^n x, T^{n+1}x) + d(T^m x, T^{m+1}x)\} \to 0$$

as $n \to \infty$. This shows that $\{T^n x\}$ is a Cauchy sequence in X. Because X is complete, there exists $v \in X$ such that $\lim_{n\to\infty} T^n x = v$. Then

$$\begin{aligned} d(v, Tv) &\leqslant d(v, T^{n+1}x) + d(T^{n+1}x, Tv) \\ &\leqslant d(v, T^{n+1}x) + K \cdot \{d(T^n x, T^{n+1}x) + d(v, Tv)\}, \end{aligned}$$

and hence

$$d(v, Tv) \leqslant \frac{K}{1-K} \cdot d(T^{n+1}x, T^n x) + \frac{1}{1-K} \cdot d(v, T^{n+1}x) \to 0$$

as $n \to \infty$, it follows that $Tv = v$. Of course such the fixed point is exactly one, so for each $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v. □

A Kannan type mapping $T : X \to X$ such that

$$d(Tx, Ty) < d(x, Tx) + d(y, Ty) \quad \text{for all } x, y \in X \quad \text{with } x \neq y,$$

and asymptotically regular may not have a fixed point. It can be seen from the following example.

Example 3.2. Let $X = [0, 1]$ be with usual metric and $T : [0, 1] \to [0, 1]$ be a mapping defined by $T0 = \frac{1}{2}$ and $Tx = \frac{x}{2}$ for $0 < x \leqslant 1$. Then for $0 < x < y \leqslant 1$,

$$|Tx - Ty| = \frac{1}{2}(y - x) < \frac{1}{2}(x + y) = |x - Tx| + |y - Ty|.$$

Moreover, for $0 < x \leqslant 1$,

$$|T0 - Tx| = \frac{1}{2} - \frac{x}{2} < \frac{1}{2} + \frac{x}{2} = |0 - T0| + |x - Tx|.$$

Thus

$$|Tx - Ty| < |x - Tx| + |y - Ty| \quad \text{for all} \;\; x, y \in [0,1], \; x \neq y.$$

Of course T is an asymptotically regular and fixed point free.

Similarly to Theorem 3.1 we can prove the following:

Theorem 3.3. *If (X, d) is a complete metric space and $T : X \to X$ is an asymptotically regular mapping such that there exists $M < 1$ satisfying*

$$d(Tx, Ty) \leqslant M[d(x, Tx) + d(y, Ty) + d(x, y)] \quad \textit{for all} \;\; x, y \in X.$$

Then, T has a unique fixed point $v \in X$.

Proof. Let $x \in X$ and define a sequence $\{x_n = T^n x\}$. According to asymptotic regularity, we get for $m > n$,

$$\begin{aligned} d(T^{n+1}x, T^{m+1}x) &\leqslant M \cdot \{d(T^n x, T^{n+1}x) + d(T^m x, T^{m+1}x) + d(T^n x, T^m x\} \\ &\leqslant 2M \cdot \{d(T^n x, T^{n+1}x) + d(T^m x, T^{m+1}x)\} \\ &\quad + M \cdot d(T^{n+1}x, T^{m+1}x), \end{aligned}$$

so

$$d(T^{n+1}x, T^{m+1}x) \leqslant \frac{2M}{1-M} \cdot \{d(T^n x, T^{n+1}x) + d(T^m x, T^{m+1}x)\} \to 0$$

as $n \to \infty$. This shows that $\{T^n x\}$ is a Cauchy sequence in X. Because X is complete, there exists $v \in X$ such that $\lim_{n\to\infty} T^n x = v$. Then

$$\begin{aligned} d(v, Tv) &\leqslant d(v, T^{n+1}x) + d(T^{n+1}x, Tv) \\ &\leqslant d(v, T^{n+1}x) + M \cdot \{d(T^n x, T^{n+1}x) + d(v, Tv) + d(T^n x, v)\}, \end{aligned}$$

and hence

$$d(v, Tv) \leqslant \frac{M}{1-M} \cdot \{d(T^{n+1}x, T^n x) + d(T^n x, v)\} + \frac{1}{1-M} \cdot d(v, T^{n+1}x) \to 0$$

as $n \to \infty$, it follows that $Tv = v$. Finally, we prove that there is only one fixed point. Let v, u be two different fixed points. Then

$$d(u, v) = d(Tu, Tv) \leqslant M \cdot \{d(u, Tu) + d(v, Tv) + d(u, v)\} = M \cdot d(u, v)$$

and we would have $1 \leqslant M$, a contradiction. Hence T has a unique fixed point, so for each $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v. □

4. Involutions

A mapping $T : X \to X$ is called an *involution* if $T^2 = I$, where I denotes the identity map. Not much is known even about Lipschitz involution. For example, it is known the following result [7] (for simple proof see [8]).

Theorem 4.1. *Let C be a nonempty closed convex subset of a Banach space and let $T : C \to C$ be a mapping. If $T^2 = I$ and if for arbitrary $x, y \in C$ we have $\|Tx - Ty\| \leqslant L \cdot \|x - y\|$ where L is constant such that $L < 2$, then T has a fixed point in C.*

Question 4.2. Does there exist 2-Lipschitz involution of a nonempty closed convex set in a Banach space which has no fixed point?

Probably for Kannan type mappings, the situation is more difficult. Here is an example Kannan's type mapping for which $T^2 = I$.

Example 4.3. Let $X = [0, 1]$ be with usual metric and let $T : [0, 1] \to [0, 1]$ be a mapping defined by $Tx = 1 - x$. If $|x - Tx| = a$, $|y - Ty| = b$, then (it can be easily visualized through a simple drawing)

$$|Tx - Ty| = \frac{1}{2}(a + b) = \frac{1}{2}[|x - Tx| + |y - Ty|], \; x, y \in [0, 1],$$

hence, T is a Kannan type mapping with constant $K = \frac{1}{2}$. Of course T is an involution.

Let X be a Banach space, $T : X \to X$ be a mapping such that there exists $K < 1$ satisfying (1). Assume that for a given element $x \in X$ the equation $2u - Tu = x$ has a solution in X (a similar case is described in [3]). For example, if $Tu = 1 - u$, $u \in [0, 1]$ (see Example 4.3), then for any $x \in [0, 1]$ a solution of the equation is $u = \frac{x+1}{3}$.

Theorem 4.4. *Let C be a nonempty closed convex subset of a Banach space X and let $T : C \to C$ be a mapping such that $T^2 = I$ and that there exists $K < 1$ satisfying (1). If for each $x \in C$ the equation $2u - Tu = x$ has a solution in C, then T has a unique fixed point in C.*

Proof. We will use a slightly modified version of Lemma 1.2. For any $x \in C$ let $u = \frac{1}{2}(x + Tu)$. Then

$$\|u - Tu\| = \frac{1}{2}\|T^2 x - Tu\| \leqslant \frac{K}{2}(\|x - Tx\| + \|u - Tu\|)$$

and

$$\|u - Tu\| \leqslant \frac{\frac{K}{2}}{1 - \frac{K}{2}} \cdot \|x - Tx\|,$$

where by assumption $a = \frac{\frac{K}{2}}{1-\frac{K}{2}} < 1$. Using the triangle inequality,

$$\|u - x\| = \frac{1}{2}\|Tu - x\| \leqslant \frac{1}{2} \cdot (\|Tu - u\| + \|u - x\|),$$

and we get

$$\|u - x\| \leqslant \|u - Tu\| \leqslant a \cdot \|x - Tx\|.$$

Now, for arbitrary $x_0 \in C$ we can inductively define a sequence $\{x_n\} \subset C$ in the following manner $x_{n+1} = \frac{1}{2}(x_n + Tx_{n+1})$, $n = 0, 1, 2, \ldots$ By Lemma 1.2, this sequence is convergent: $\lim_{n\to\infty} x_n = v$ and $Tv = v$. It is obvious that v is unique. □

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