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A sufficient condition for the realizability of the least number of periodic points of a smooth map

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Abstract. There are two algebraic lower bounds of the number of n-periodic points of a self-map $f : M \to M$ of a compact smooth manifold of dimension at least 3:

$$NF_n(f) = \min\{\#\mathrm{Fix}(g^n);\ g \sim f;\ g \text{ continuous}\}$$

and

$$NJD_n(f) = \min\{\#\mathrm{Fix}(g^n);\ g \sim f;\ g \text{ smooth}\}.$$

In general $NJD_n(f)$ may be much greater than $NF_n(f)$. In the simply connected case, the equality of the two numbers is equivalent to the sequence of Lefschetz numbers satisfying restrictions introduced by Chow, Mallet-Parret and Yorke (1983). The last condition is not sufficient in the non-simply connected case. Here we give some conditions which guarantee the equality when $\pi_1 M = \mathbb{Z}_2$.

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1. Preliminaries

Let M be a smooth compact connected manifold, $\dim M = m \geq 3$, and let n be a fixed natural number. We consider the minimal number of n-periodic points in the homotopy class of f: $\min \#\{\mathrm{Fix}(g^n);\ g \sim f\}$. In turns out that this number depends on whether we consider all continuous, or only all smooth, maps g homotopic to the given f. Two algebraic invariants

$$NF_n(f) = \min\{\mathrm{Fix}(g^n);\ g \sim f;\ g \text{ continuous}\}$$

and

$$NJD_n(f) = \min \#\{\mathrm{Fix}(g^n);\ g \sim f;\ g \text{ smooth}\}$$

satisfy the natural inequality

$$NF_n(f) \leq NJD_n(f)$$

which is often sharp; see [8, 16]. In the present paper we ask when the equality holds.

It turns out that the equality holds for all self-maps of tori, compact nilmanifolds or solvmanifolds [13, 19]. Moreover, the problem is partially solved for self-maps of a Lie group [14].

For any self-map $f : M \to M$ of a simply connected m-manifold ($m \geq 3$) and a fixed natural number n, there is a deformation of f by a continuous map g with at most one n-periodic point: $\#\mathrm{Fix}(g^n) \leq 1$ (see [15, Theorem 5.3.2]). The map g may be chosen smooth if and only if the sequence of Lefschetz numbers $(L(f^k))_{k|n}$ is *smoothly realizable in* $\mathbb{R}^m$ (see [9]). Since the last condition is easier to verify, it is natural to ask, what does it imply if the fundamental group is not trivial? More precisely, does the assumption of smooth realizability of Lefschetz numbers imply that the invariants $NF_n(f)$ and $NJD_n(f)$ coincide? The example of even-dimensional projective spaces shows that it is not true in general (Section 5). We will show that the assumptions:

- $(L(f^k))_{k|n}$ is smoothly realizable in $\mathbb{R}^m$,
- $\pi_1 M = \mathbb{Z}_2$,
- all Reidemeister classes have the same index (Jiang condition),
- f is essentially reducible,

imply the equality $NF_n(f) = NJD_n(f)$ (Theorem 6.1). In particular, the equality holds for each compact Lie group with $\pi_1 M = \mathbb{Z}_2$ and smoothly realizable $(L(f^k))_{k|n}$.

2. Indices of iterations of a smooth map

This paper is based on [14] and we refer the reader to it for the details.

In this section we study the following problem. We are given a self-map of a smooth compact connected simply connected manifold $f : M \to M$ and an integer n. We ask if f is homotopic to a smooth map g with $\mathrm{Fix}(g^n)$ a point. We recall a necessary and sufficient condition for the existence of such a smooth map g.

Let us start with the remark that the same question in the class of continuous maps always has a positive answer (in dimensions greater than or equal to 3); see [15, Theorem 5.3.2]. However, the smooth case turned out to be quite different. The crucial difference is the sequence of Lefschetz numbers $L(f^k)$.

In 1983, Dold [5] noticed that a sequence of fixed point indices $A_k = \mathrm{ind}(f^k; x_0)$, where f is a continuous self-map of a Euclidean space $\mathbb{R}^m$ and x_0 is an isolated fixed point for each k, must satisfy some congruences. Namely, for each $n \in \mathbb{N}$,

$$\sum_{k|n} \mu\left(\frac{n}{k}\right) \cdot \operatorname{ind}\left(f^k; x_0\right) \equiv 0 \pmod{n},$$

where μ denotes the Möbius function.

It was shown in [1] that each sequence of integers (A_k) satisfying Dold congruences can be realized as $A_k = \operatorname{ind}(f^k; x_0)$, for a continuous self-map of $\mathbb{R}^m$, $m \geq 3$. In other words, Dold congruences are the only restrictions for the sequence of fixed point indices of a continuous map.

Surprisingly it turned out that there are much more restrictions on the sequences $A_k = \operatorname{ind}(f^k; x_0)$ when f is smooth [4, 18].

Definition 2.1. A sequence of integers (A_n) is called *smoothly realizable in* $\mathbb{R}^m$ (or *in dimension* m) if there exist a smooth self-map $f : \mathbb{R}^m \to \mathbb{R}^m$ and an isolated fixed point $x_0 \in \operatorname{Fix}(f)$, which is also an isolated fixed point of each iteration f^n, so that $A_k = \operatorname{ind}(f^k; x_0)$.

In Theorem 2.4 we recall all possible sequences which can be obtained as fixed point indices of a smooth self-map of $\mathbb{R}^m$ (for $m \geq 3$). Then, in Theorem 2.5, we give an equivalent but shorter list of smoothly realizable sequences.

It is convenient to present the sequences of integers as the sum of elementary periodic sequences as follows.

Definition 2.2. For a given $k \in \mathbb{N}$, we define

$$\operatorname{reg}_k(n) = \begin{cases} k & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}$$

In other words, reg_k is the periodic sequence

$$(0, \ldots, 0, k, 0, \ldots, 0, k, \ldots),$$

where the nonzero entries appear for indices divisible by k.

It turns out that each sequence of integers (A_n) can be uniquely written as a periodic expansion:

$$A_n = \sum_{k=1}^{\infty} a_k \operatorname{reg}_k(n),$$

where

$$a_n = \frac{1}{n} \sum_{k|n} \mu\left(\frac{n}{k}\right) A_k.$$

Moreover, all coefficients a_k are integers if and only if the sequence (A_n) satisfies Dold congruences.

Remark 2.3. Later we will be interested in iterations f^k, for k dividing a prescribed n, so we will consider only finite sequences labeled by $\{k \in \mathbb{N}; k|n\}$. We will say that a finite sequence $(B_k)_{k|n}$ is smoothly realizable in $\mathbb{R}^m$ if it is the restriction of a sequence smoothly realizable in $\mathbb{R}^m$.

For a finite subset $A \subset \mathbb{Z}$, we denote by $\mathrm{lcm}(A)$ the least common multiplicity of its elements and we denote

$$\mathrm{LCM}(A) = \{\mathrm{lcm}(B); B \subset A\}.$$

In the next theorem, $L(s)$ denotes

$$L(s) = \mathrm{LCM}(L),$$

where $L \subset \{3, 4, 5, \dots\}$ is any subset of cardinality s. Moreover, we denote

$$L_2(s) = \mathrm{LCM}(L \cup \{2\}),$$

where L is as above.

Theorem 2.4 (Main Theorem I in [9]**).** *Let $U \subset \mathbb{R}^m$, where $m \geq 3$, be an open neighborhood of 0 and let $f : U \to \mathbb{R}^m$ be a C^1 map having 0 as an isolated fixed point for each iteration. Then the sequence of local indices of iterations $\{\mathrm{ind}(f^n, 0)\}_{n=1}^{\infty}$ has one of the following forms.*

(I) *For m odd,*

$$(A^o) \quad \mathrm{ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-3}{2})} a_k \,\mathrm{reg}_k(n),$$

$$(B^o), (C^o), (D^o) \quad \mathrm{ind}(f^n, 0) = \sum_{k \in L(\frac{m-1}{2})} a_k \,\mathrm{reg}_k(n),$$

where

$$a_1 = \begin{cases} 1 & \text{in the case } (B^o), \\ -1 & \text{in the case } (C^o), \\ 0 & \text{in the case } (D^o), \end{cases}$$

$$(E^o), (F^o) \quad \mathrm{ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-1}{2})} a_k \,\mathrm{reg}_k(n),$$

where $a_1 = 1$ and

$$a_2 = \begin{cases} 0 & \text{in the case } (E^o), \\ -1 & \text{in the case } (F^o). \end{cases}$$

(II) *For m even,*

$$(A^e) \quad \mathrm{ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-4}{2})} a_k \,\mathrm{reg}_k(n),$$

$$(B^e) \quad \mathrm{ind}(f^n, 0) = \sum_{k \in L(\frac{m-2}{2})} a_k \,\mathrm{reg}_k(n),$$

$$(C^e), (D^e), (E^e) \quad \mathrm{ind}(f^n, 0) = \sum_{k \in L_2(\frac{m-2}{2})} a_k \,\mathrm{reg}_k(n),$$

where

$$a_1 = \begin{cases} 1 & \text{in the case } (C^e), \\ -1 & \text{in the case } (D^e), \\ 0 & \text{in the case } (E^e), \end{cases}$$

$$(F^e) \quad \operatorname{ind}(f^n, 0) = \sum_{k \in L(\frac{m}{2})} a_k \operatorname{reg}_k(n),$$

where $a_1 = 1$.

The next theorem is a more concise version of the above list of smoothly realizable sequences.

Theorem 2.5. *A sequence* (D_n) *is smoothly realizable in dimension* m *if and only if there exist natural numbers* $d_1, \ldots, d_s$ $(d_i \geq 3,\ 2s \leq m)$ *such that*

$$D_n = \sum_k \alpha_k \operatorname{reg}_k(n), \tag{2.1}$$

where the summation runs over the set $\mathrm{LCM}_2(\{d_1, \ldots, d_s\})$ *and the coefficients* α_k *are integers. If moreover* $m \leq 2s+2$*, then the following restrictions hold:*

[0] *if* $m = 2s$*, then* $\alpha_1 = 1$ *and* LCM*;*
[1] *if* $m = 2s+1$*, then* $(|\alpha_1| \leq 1$ *and* LCM$)$ *or* $(\alpha_1 = 1$ *and* $(\alpha_2 = 0$ *or* $1))$*;*
[2] *if* $m = 2s+2$*, then* $|\alpha_1| \leq 1$ *or* LCM*;*

where LCM *means that the summation in* (2.1) *must run over a set*

$$\mathrm{LCM}(\{d_1, \ldots, d_s\})$$

(*in other cases it may run over* $\mathrm{LCM}_2(\{d_1, \ldots, d_s\})$).

Proof. In the first two columns of Table 1 we list the cases of the theorem and in the last column we give the corresponding cases in Theorem 2.4.

TABLE 1

Theorem 2.5		Theorem 2.4
$m = 2s$	LCM, $\alpha_1 = 1$	(F^e)
$m = 2s+1$	LCM, $\|\alpha_1\| \leq 1$	(B^o), (C^o), (D^o)
$m = 2s+1$	$\alpha_1 = 1$, $\alpha_2 = 0, -1$	(E^o), (F^o)
$m = 2s+2$	$\|\alpha_1\| \leq 1$	(C^e), (D^e), (E^e)
$m = 2s+2$	LCM	(B^e)

We notice that $\sum_{L(s)}$ and $\sum_{L_2(s)}$ in Theorem 2.4 mean LCM and LCM_2, respectively. It remains to check that the data in the last column coincide with the conditions of the theorem and to notice that the two missing cases (A^o), (A^e) correspond to $m \geq 2s+3$, hence they need no restrictions. ☐

Corollary 2.6. *If the above sequence* (D_n) *is smoothly realizable in dimension* m*, then*

(1) $|\alpha_1| \geq 2$ *implies* $m \geq 2s+3$ *or* $(m = 2s+2$ *and* LCM$)$*;*
(2) $\alpha_1 = 0$ *implies* $m \geq 2s+2$ *or* $(m \geq 2s+1$ *and* LCM$)$.

It is easy to notice that if $f : M \to M$ is a self-map of a compact manifold and $\mathrm{Fix}(f^n)$ is a point, then the sequence $(L(f^k))_{k|n}$ is smoothly realizable in $\mathbb{R}^m$, where $m = \dim M$. It turns out that if M is simply connected and $\dim(M) \geq 3$, then the inverse implication is also true.

Lemma 2.7 (See [14, Lemma 2.5]**).** *Let f be a self-map of a compact smooth connected simply connected manifold of dimension $m \geq 3$ and $n \in \mathbb{N}$. Then f is homotopic to a smooth map h so that $\mathrm{Fix}(h^n)$ is a point (or empty set) if and only if the sequence of Lefschetz numbers $(L(f^k))_{k|n}$ is smoothly realizable in $\mathbb{R}^m$.*

3. Nielsen fixed point theory

Now we consider spaces with nontrivial fundamental groups. It will be impossible to reduce the set of fixed (or periodic) points to a single point when the Nielsen-type invariants introduced below are greater than 1.

We consider a self-map of a compact connected polyhedron $f : X \to X$ and its fixed point set $\mathrm{Fix}(f)$. We define the *Nielsen relation* on this set by: $x \sim y$ if and only if there is a path ω joining x and y such that $f\omega$ and ω are fixed-endpoint homotopic.

This relation splits $\mathrm{Fix}(f)$ into *Nielsen classes*. We denote the set of Nielsen classes by $\mathcal{N}(f)$. We say that a Nielsen class A is *essential* if its fixed point index is nonzero: $\mathrm{ind}(f; A) \neq 0$. The number of essential Nielsen classes is called *Nielsen number* and is denoted by $N(f)$. This is a homotopy invariant and, moreover, it is the lower bound of the number of fixed points in the (continuous) homotopy class: $N(f) \leq \min_{h \sim f} \# \mathrm{Fix}(h)$; see [3, 15, 16].

On the other hand, we define the set of *Reidemeister classes* of the map f as the quotient space of the action of the fundamental group $\pi_1 M$ on itself given by

$$\omega * \alpha = \omega \cdot \alpha \cdot (f_\# \omega)^{-1}.$$

Here we take as the basepoint a fixed point of f. We denote the quotient space by $\mathcal{R}(f)$. There is a natural injection from the set of the Nielsen classes to the set of Reidemeister classes $\mathcal{N}(f) \subset \mathcal{R}(f)$ defined as follows. We choose a point x in the given Nielsen class A and a path ω from the basepoint x_0 to x. Then the loop $\omega * (f\omega)^{-1}$ represents the corresponding Reidemeister class.

Now we consider iterations of the map f. For fixed natural numbers $l|k$, there is a natural inclusion

$$\mathrm{Fix}(f^l) \subset \mathrm{Fix}(f^k)$$

which induces the map $\mathcal{N}(f^l) \to \mathcal{N}(f^k)$ (which may be not injective). This map extends to the map $i_{kl} : \mathcal{R}(f^l) \to \mathcal{R}(f^k)$ defined by

$$i_{kl}[x] = \big[x \cdot f^l(x) \cdot f^{2l}(x) \cdots f^{k-l}(x)\big].$$

The functorial equalities $i_{kl}\, i_{lm} = i_{km}$ and $i_{kk} = \mathrm{id}$ are satisfied and, moreover, the diagram

$$\begin{tikzcd} \mathcal{N}(f^l) \arrow[r, "i_{kl}"] \arrow[d] & \mathcal{N}(f^k) \arrow[d] \\ \mathcal{R}(f^l) \arrow[r, "i_{kl}"] & \mathcal{R}(f^k) \end{tikzcd}$$

commutes.

The group $\mathbb{Z}_k$ acts on $\mathrm{Fix}(f^k)$ by

$$\mathrm{Fix}(f^k) \ni [x] \to [fx] \in \mathrm{Fix}(f^k)$$

and on $\mathcal{R}(f^k)$ by

$$\mathcal{R}(f^k) \ni [a] \to [f_{\#}(a)] \in \mathcal{R}(f^k).$$

Then the diagram

$$\begin{tikzcd} \mathrm{Fix}(f^k) \arrow[r] \arrow[d] & \mathrm{Fix}(f^k) \arrow[d] \\ \mathcal{R}(f^k) \arrow[r] & \mathcal{R}(f^k) \end{tikzcd}$$

commutes. We denote by $\mathcal{OR}(f^k)$ the set of orbits of the above action (*orbits of Reidemeister classes*).

Consider the Reidemeister classes $A \in \mathcal{R}(f^k)$ and $B \in \mathcal{R}(f^l)$, $l|k$, satisfying $i_{kl}(B) = A$. Then we say that A *reduces* to B, or B *precedes* A and we write $B \preceq A$. The class A is called *reducible* if $B \preceq A$ for $B \neq A$. A similar definition holds for orbits.

Let $\mathcal{ER}(f^k)$ and $\mathcal{IR}(f^k)$ denote the sets of essential and irreducible classes, respectively.

A map f is called *essentially reducible* if $A \preceq B$ and B is essential implies A is essential.

We denote by $\mathcal{IEOR}(f)$, or simply $\mathcal{IEOR}$, the set of irreducible essential orbits of the Reidemeister classes of f.

4. The Reidemeister graph

We fix a map $f : M \to M$. It is very convenient to consider a directed graph whose vertices are orbits of Reidemeister classes and the fixed point indices are their weights. Then the problem of minimizing the number of n-periodic points is reduced to finding some minimal subsets in the graph. We denote the *graph of orbits of Reidemeister classes* by $\mathcal{GOR}(f)$ and we define it as follows:

- vertices are elements of the disjoint sum

$$\mathrm{Vert}\left(\mathcal{GOR}(f)\right) = \bigcup_{k\in\mathbb{N}} \mathcal{OR}(f^k);$$

- there is a unique directed edge from $A \in \mathcal{OR}(f^l)$ to $B \in \mathcal{OR}(f^k)$ if $i_{kl}(A) = B$;
- for each $B \in \mathrm{Vert}(\mathcal{GOR}(f))$, *Dold congruences* are satisfied (see [8, Lemma 3.3]):

$$\sum_{C \preceq B} \mu\left(\frac{b}{c}\right) \cdot \mathrm{ind}(f^c(C)) \equiv 0 \pmod{b},$$

where μ denotes the Mobius function.

The fixed point index defines the map $\mathrm{ind} : \mathrm{Vert}(\mathcal{GOR}(f)) \to \mathbb{Z}$ by

$$\mathrm{ind}(A) = \mathrm{ind}(f^a; A)$$

for $A \in \mathcal{OR}(f^a)$. We will decompose this function into summands coming from smoothly realizable sequences.

Definition 4.1. For a fixed Dold sequence c and an orbit $H \in \mathcal{OR}(f^h)$, we define the function $C_H : \mathrm{Vert}(\mathcal{GOR}(f)) \to \mathbb{Z}$ by

$$C_H(A) = \begin{cases} c(a/h)h & \text{for } H \preceq A,\ A \in \mathcal{OR}(f^a), \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Then we say that the sequence c is attached to the orbit $H \in \mathcal{OR}(f^h)$. We also say that C_H comes from c.

We denote by

$$\mathrm{Reg}_H^d : \mathrm{Vert}(\mathcal{GOR}(f)) \to \mathbb{Z}$$

the result of attaching reg_d to the orbit $H \in \mathcal{OR}(f^h)$; i.e.,

$$\mathrm{Reg}_H^d(A) = \begin{cases} dh & \text{when } i_{dh,h}(H) \preceq A, \\ 0 & \text{otherwise.} \end{cases}$$

The sequence

$$c(n) = \sum_{d \in \mathbb{N}} a_d \, \mathrm{reg}_d(n), \tag{4.2}$$

attached to the orbit H, then gives

$$C_H(A) = \sum_{B} a_d \, \mathrm{Reg}_B^d(A), \tag{4.3}$$

where the summation runs over the set $\mathrm{Vert}(\mathcal{GOR}(f))$.

On the other hand, any Dold function $\mathcal{D} : \mathrm{Vert}(\mathcal{GOR}(f)) \to \mathbb{Z}$ may be uniquely represented as

$$\mathcal{D} = \sum_{B} a_B \, \mathrm{Reg}_B^1,$$

where the summation runs over the set $\mathrm{Vert}(\mathcal{GOR}(f))$ and a_B are integers (see [8]). Now we may reformulate the problem of realizing the least number of periodic points of a smooth map as an algebraic question.

Theorem 4.2 (See [14, Theorem 7.3]**).** *An essentially reducible map $f : M \to M$ is homotopic to a smooth map g realizing the least number of n-periodic points if and only if one can attach to each orbit $\mathcal{A} \in \mathcal{IEOR}(f^k)$, $k|n$, a sequence $C_{\mathcal{A}}$, smoothly realizable in $\mathbb{R}^m$, so that*

$$\mathrm{ind}(f^k; \mathcal{B}) = \sum_{\mathcal{A} \in \mathcal{IEOR}} C_{\mathcal{A}}(\mathcal{B})$$

for each $\mathcal{B} \in \mathcal{OR}(f^k)$, $k|n$.

Now we illustrate the above theorem in the case $\pi_1 M = \mathbb{Z}_2$ (the simplest nontrivial non-simply connected case). Compare [8, Section 6].

We consider a connected manifold M with $\pi_1 M = \mathbb{Z}_2$. Then the induced homotopy homomorphism $f_\# : \mathbb{Z}_2 \to \mathbb{Z}_2$ may be either the zero map or the identity map.

First, we consider the case when $f_\# = 0$. Then the set of Reidemeister classes $\mathcal{R}(f) = \mathbb{Z}_2/(im(\mathrm{id} - 0)) = 0$ and the same holds for each iteration: $\mathcal{R}(f^n) = 0$. Now the Reidemeister graph is reduced to the graph of natural numbers.

Let $f_\# = \mathrm{id}_{\mathbb{Z}_2}$. Then

$$\mathcal{R}(f) = \mathbb{Z}_2/(im(\mathrm{id} - \mathrm{id})) = \mathbb{Z}_2$$

and similarly

$$\mathcal{R}(f^n) = \mathbb{Z}_2$$

for any iteration n. On the other hand, since $f_\# = \mathrm{id}_{\mathbb{Z}_2}$, the action of $\mathbb{Z}_n$ on $\mathcal{R}(f^n) = \mathbb{Z}_2$ is trivial for each n. This means that each orbit reduces to a point and $\mathcal{OR}(f^n) = \mathcal{R}(f^n) = \mathbb{Z}_2$ for each n.

We recall that the map $i_{kl} : \mathcal{OR}(f^l) \to \mathcal{OR}(f^k)$ is given by

$$i_{kl}(x) = x + f_\#^l(x) + f_\#^{2l}(x) + \cdots + f_\#^{k-l}(x),$$

so in the case $f_\# = \mathrm{id}$ we get $i_{kl}(x) = k/l \cdot x$. In particular, for $\pi_1 M = \mathbb{Z}_2$ we get

$$i_{kl} = \begin{cases} \mathrm{id} & \text{when } k/l \text{ is odd,} \\ 0 & \text{when } k/l \text{ is even.} \end{cases} \tag{4.4}$$

We introduce the following notation. We denote

$$\mathcal{OR}(f) = \{1', 1''\}.$$

The element $1'$ corresponds to the choice of a point in the definition of the homotopy group $\pi_1 M = \mathbb{Z}_2$. Then we denote

$$\mathcal{OR}(f^k) = \{k', k''\},$$

where $k' = i_{k1}(1')$.

Let us recall that the orbit $A \in \mathcal{OR}(f^l)$ precedes $B \in \mathcal{OR}(f^k)$ if $l|k$ and $i_{kl}(A) = B$. An irreducible orbit is the orbit which is preceded only by itself.

Formula (4.4) gives *irreducible* orbits of Reidemeister classes for $f_\# = \mathrm{id}_{\mathbb{Z}_2}$. We notice that $i_{kl}(l') = (k')$ for all $l|k$ and

$$i_{kl}(l'') = \begin{cases} (k'') & \text{for } k/l \text{ odd}, \\ (k') & \text{for } k/l \text{ even}. \end{cases}$$

Now the set of irreducible classes is given by

$$\{1'; 1'', 2'', (2^2)'', \dots, (2^k)'', \dots\}.$$

5. Self-maps of odd-dimensional projective spaces

Each odd self-map $\tilde{f} : S^m \to S^m$ (i.e., $\tilde{f}(-x) = -\tilde{f}(x)$) defines a self-map f of the projective space $\mathbb{R}P^m$ which induces the identity homomorphism of the fundamental group $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$. We assume, moreover, that m is also odd and $|\deg(\tilde{f})| \geq 3$. We show that then the sequence of Lefschetz numbers is constant $L(f^k) = 1$, hence it is smoothly realizable in each dimension, but there exist natural numbers l such that the least number of l-periodic points in the homotopy class of f cannot be realized by a smooth map (in fact it is enough to take an odd l that is a product of sufficiently many different primes, where the number of the primes depends only on the dimension m).

Since $f_\# = \mathrm{id}$, $\mathcal{OR}(f^k) = \mathbb{Z}_2$, Section 4 implies that

$$i_{kl}(l') = k' \quad \text{and} \quad i_{kl}(l'') = k''$$

for odd $l|k$. On the other hand, since the dimension m is odd, $\mathbb{R}P^n$ is not a Jiang space, it turns out that

$$\mathrm{Fix}(f) = p\big(\mathrm{Fix}(\tilde{f})\big) \cup p\big(\mathrm{Fix}(-\tilde{f})\big)$$

gives the splitting of $\mathrm{Fix}(f)$ into Nielsen classes (the same holds for all iterations). Now

$$\mathrm{ind}\Big(f^k; p\big(\mathrm{Fix}(\tilde{f})\big)\Big) = \frac{1-d^k}{2}$$

and

$$\mathrm{ind}\Big(f^k; p\big(\mathrm{Fix}(-\tilde{f})\big)\Big) = \frac{1+d^k}{2}.$$

We assume that for each r, the least number of r-periodic points in the homotopy class of f can be realized by a smooth map. We get a contradiction.

We recall that

$$\mathcal{IEOR} = \{1'; 1'', 2'', (2^2)'', \dots, (2^k)'', \dots\}$$

and we notice that the only elements in $\mathcal{IEOR}$ preceding orbits l^* (for odd l) may be only $1'$ or $1''$. Now by Theorem 4.2 there exist expressions, coming from sequences smoothly realizable in $\mathbb{R}^m$, so that

$$\mathrm{ind}(f^l; l^*) = C_{1'}(l^*) + C_{1''}(l^*)$$

for all l^*. But

$$\big|\operatorname{ind}(f^l;l^*)\big| = \left|\operatorname{ind}\left(f^k;p\big(\operatorname{Fix}(\pm\tilde{f})\big)\right)\right| = \left|\frac{1\mp d^k}{2}\right|$$

tends to infinity. On the the hand, the smoothly realizable expressions C_1' and C_1'' in dimension m can take at most $2^{[(m+1)/2]}$ values (see [1]). Now the number of values on the right-hand side is bounded by a number which depends only on m.

We have obtained a contradiction, since the indices of Reidemeister orbits were growing exponentially but their sum was always equal to 1. This explains why in the main theorem of the paper (Theorem 6.1) some extra assumptions for the equality will be necessary.

6. The main result

The main result of the paper is the following.

Theorem 6.1. *Let $f: M \to M$ be a self-map of a smooth compact connected manifold satisfying*

1. *f is essentially reducible;*
2. *$\pi_1 M = \mathbb{Z}_2$;*
3. *each iteration f^k is Jiang, i.e., the indices of all Reidemeister classes in $\mathcal{R}(f^k)$ are the same;*
4. *the sequence of Lefschetz numbers $(L(f^k))_{k\in\mathbb{N}}$ is smoothly realizable in $\mathbb{R}^m$.*

Then, for each k, f is homotopic to a smooth map g_k having the minimal possible number of k-periodic points in the homotopy class of the map f.

The proof is based on Theorem 4.2: we must find for each essential irreducible orbit

$$k^* \in \{1';1'',2'',(2^2)'',\ldots,(2^k)'',\ldots\},$$

an expression C_{k^*} smoothly realizable in $\mathbb{R}^m$ so that for each orbit $B \in \mathcal{R}(f^b)$,

$$\operatorname{ind}\big(f^b;B\big) = \left(C_{1'} + \sum_{k=0}^{\infty} C_{(2^k)''}\right)(B). \tag{6.1}$$

The following lemma will be useful.

Lemma 6.2. *Let $f: M \to M$ be a self-map of a compact connected manifold M with $\pi_1 M = \mathbb{Z}_2$. If, moreover, all iterations of f are Jiang maps and in Dold's expansion $L(f^n) = \sum_k a_k \cdot \operatorname{reg}_k(n)$, the coefficients $a_k = 0$ for all even k, then $a_k = 0$ also for all odd numbers k. In particular, the case* LCM *in Theorem* 2.5 *occurs only for all $a_k = 0$.*

Proof. Let us assume that l_0 is the smallest number satisfying $a_{l_0} \neq 0$. We will inductively prove the formula

$$a_{(2^{r_0}l_0)'} = \frac{-a_{l_0}}{2^{r_0+1}} \tag{6.2}$$

for all $r_0 \geq 1$. Since the number $a_{(2^{r_0}l_0)'}$ must be integer, $a_{l_0} = 0$ and we get a contradiction.

We start with some general remarks. Let us notice that $a_{l'} = a_{l''} = 0$ for all $l < l_0$, since then $a_l = 0$ by the assumption and the Jiang property. Moreover,

$$L\big(f^{l_0 2^{r_0}}\big) = \sum_{k|l_0 2^{r_0}} a_k \cdot k = \sum_{k|l_0} a_k \cdot k = a_{l_0} l_0 = L\big(f^{l_0}\big)$$

for all $r_0 \geq 0$.

The following equalities follow from the Reidemeister graph:

$$\begin{aligned}\operatorname{ind}\big(f^{2l_0}, (2l_0)'\big) &= a_{(l_0)'} l_0 + a_{(l_0)''} l_0 + a_{(2l_0)'} 2l_0 \\ &= a_{l_0} l_0 + a_{(2l_0)'} 2l_0 = L\big(f^{l_0}\big) + a_{(2l_0)'} 2l_0, \\ \operatorname{ind}\big(f^{2l_0}, (2l_0)''\big) &= \sum_{l|l_0} a_{(2l)''} \cdot 2l = a_{(2l_0)''} \cdot 2l_0\end{aligned}$$

since $l|l_0$ and $l \neq l_0$ imply $2l < l_0$.

On the other hand, by the Jiang property, the above indices are equal to $\frac{1}{2}L(f^{2l_0})$ and hence to

$$\frac{1}{2}\, L\big(f^{l_0}\big) = \frac{1}{2}\, a_{l_0} \cdot l_0.$$

Moreover, $a_{(2l_0)'} + a_{(2l_0)''} = 0$, since $a_{2l_0} = 0$ by the assumption. Now

$$\begin{aligned}&L\big(f^{l_0}\big) + a_{(2l_0)'} \cdot 2l_0 = a_{(2l_0)''} 2l_0, \\ &L\big(f^{l_0}\big) = a_{(2l_0)''} \cdot 2l_0 - a_{(2l_0)'} 2l_0 = -a_{(2l_0)'} 4l_0, \\ &a_{l_0} \cdot l_0 = -a_{(2l_0)'} 4l_0,\end{aligned}$$

which implies that

$$a_{(2l_0)'} = \frac{a_{l_0}}{-2^2},$$

hence we get the desired formula for $r_0 = 1$.

In a similar way we perform the inductive step. We assume that formula (6.2) holds for all $r < r_0$.

The following equalities follow from the Reidemeister graph and the inductive assumption:

$$\begin{aligned}\operatorname{ind}\big(f^{2^{r_0+1}l_0}, (2^{r_0+1}l_0)'\big) &= L\big(f^{2^{r_0+1}l_0}\big) + a_{(2^{r_0+1}l_0)'} \cdot 2^{r_0+1} l_0, \\ \operatorname{ind}\big(f^{2^{r_0+1}l_0}, (2^{r_0+1}l_0)''\big) &= \sum_{l|l_0} a_{(2^{r_0+1}l)''} \cdot 2^{r_0+1} l \\ &= -\sum_{l|l_0} a_{(2^{r_0+1}l)'} \cdot 2^{r_0+1} l \\ &= -a_{(2^{r_0+1}l)'} \cdot 2^{r_0+1} l_0\end{aligned}$$

since $a_{k'} + a_{k''} = 0$ for even k and $a_{(2^{r_0+1}l)'} = 0$ for $l < l_0$.

On the other hand, by the Jiang property, the above indices are equal, hence

$$L\big(f^{2^{r_0} l_0}\big) + a_{(2^{r_0+1} l_0)'} \cdot 2^{r_0+1} l_0 = -a_{(2^{r_0+1} l_0)'} \cdot 2^{r_0+1} l_0.$$

Since $L(f^{2^{r_0} l_0}) = L(f^{l_0}) = a_{l_0} \cdot l_0$, we get

$$a_{(2^{r_0+1} l_0)'} = \frac{-a_{l_0}}{2^{r_0+2}}$$

which ends the inductive step. □

7. Dold decomposition

We give the Dold decomposition of the index function of the map satisfying the assumption of Theorem 6.1; i.e., for each orbit k^* we exhibit the number a_{k^*} such that the equality

$$\operatorname{ind}(f^n; B) = \sum_{k=1}^{\infty} \big(a_{k'} \operatorname{Reg}^1_{k'}(B) + a_{k''} \operatorname{Reg}^1_{k''}(B)\big)$$

holds for each orbit B.

We present the natural numbers as $k_0 = 2^{r_0} l_0$, where l_0 is an odd number. We define (for $k_0^* = k_0''$)

$$a_{(2^{r_0} l_0)''} = \frac{1}{2^{r_0+1}} \left(\sum_{r=0}^{r_0} a_{2^r l_0} \cdot 2^r \right).$$

Now we consider $k_0^* = k_0'$. Then we put

$$a_{k_0'} = \begin{cases} \frac{1}{2}\, a_{k_0} & \text{if } k_0 \text{ is odd,} \\ \frac{1}{2}\, a_{k_0} - \frac{1}{2}\, a_{(k_0/2)''} & \text{if } k_0 \text{ is even.} \end{cases}$$

Now we show that equality (6.1) is satisfied for each n^*. Let us recall that f is a Jiang map, so

$$\operatorname{ind}(f^k; n^*) = \frac{1}{2}\, L(f^n)$$

for $n^* = n'$ or n'', and it remains to show that

$$\frac{1}{2}\, L(f^n) = \sum_{k=1}^{\infty} \big(a_{k'} \operatorname{Reg}^1_{k'}(n^*) + a_{k''} \operatorname{Reg}^1_{k''}(n^*)\big). \tag{7.1}$$

We may rewrite the right-hand side of equality (7.1) as

$$\begin{aligned} &\Bigg(\sum_{k \text{ odd}} \frac{1}{2}\, a_k \operatorname{Reg}^1_{k'} + \sum_{k \text{ even}} \left(\frac{1}{2}\, a_k - \frac{1}{2}\, a_{(k/2)''} \right) \operatorname{Reg}^1_{k'} \\ &\qquad + \sum_{k=1}^{\infty} a_{k''} \operatorname{Reg}^1_{k''} \Bigg)(n^*) = S_1 + S_2 + S_3 = (*). \end{aligned} \tag{7.2}$$

We check equality (7.1) for $n^* = n''$ and $n = 2^{r_0} \cdot l_0$. Since $2^{r_0} \cdot l_0$ is preceded only by elements of the set $\{(2^{r_0} \cdot l)''; l|l_0\}$,

$$
\begin{aligned}
(*) = S_3 &= \sum_{l|l_0} a_{(2^{r_0}l)''} \operatorname{Reg}_{(2^{r_0}l)''}(2^{r_0}l_0)'' \\
&= \sum_{l|l_0} \frac{1}{2^{r_0+1}} \left(\sum_{r=0}^{r_0} a_{2^r l} 2^r \right) \cdot 2^{r_0} l \\
&= \sum_{l|l_0} \sum_{r=0}^{r_0} \frac{1}{2}\, a_{2^r l} 2^r \cdot l \\
&= \sum_{k|n} \frac{1}{2}\, a_k \cdot k = \frac{1}{2}\, L(f^n).
\end{aligned}
$$

Now let $n^* = (l_0)'$, where l_0 is an odd number. Then $(l_0)'$ is preceded by elements of the set $\{l : l|l_0\}$. Now

$$
(*) = S_1 = \sum_{k \text{ odd}} \frac{1}{2}\, a_l \operatorname{Reg}^1_{l'} \big((l_0)'\big) = \sum_{l|l_0} \frac{1}{2}\, a_l \cdot l = \frac{1}{2}\, L\big(f^{l_0}\big).
$$

Finally, we consider $n^* = n' = (2^{r_0} l_0)'$. Then $(2^{r_0} l_0)'$ is preceded by elements of the union

$$
\big\{(2^r l)'; 0 \le r \le r_0, l|l_0\big\} \cup \big\{(2^r l)''; \le r < r_0, l|l_0\big\}.
$$

Now

$$
\begin{aligned}
(*) = &\left(\sum_{k \text{ odd}} \frac{1}{2}\, a_k \operatorname{Reg}^1_{k'} + \sum_{k \text{ even}} \frac{1}{2} \left(a_k - \frac{1}{2}\, a_{(k/2)''} \right) \operatorname{Reg}^1_{k'} \right. \\
&\qquad\qquad \left. + \sum_{k=1}^{\infty} a_{k''} \operatorname{Reg}^1_{k''} \right)(n') \\
= &\left(\frac{1}{2} \sum_{k \text{ odd}} a_k \operatorname{Reg}^1_{k'} + \frac{1}{2} \sum_{k \text{ even}} a_k \operatorname{Reg}^1_{k'} \right. \\
&\qquad\qquad \left. - \frac{1}{2} \sum_{k;2k|n} a_{k''} \operatorname{Reg}^1_{(2k)'} + \sum_{k;2k|n} a_{k''} \operatorname{Reg}^1_{k''} \right)(n') \\
= &\left(\frac{1}{2} \sum_{k=1}^{n} a_k \cdot k - \frac{1}{2} \sum_{k;2k|n} a_{k''} 2 \cdot k + \sum_{k;2k|n} a_{k''} \cdot k \right) \\
= &\frac{1}{2} \sum_{k=1}^{n} a_k \cdot k = \frac{1}{2} L(f^n).
\end{aligned}
$$

8. Proof of Theorem 6.1

Lemma 8.1. *Under the assumptions of Theorem* 6.1*, if the fundamental group homomorphism* $f_\# = \mathrm{id}$ *and the sequence* $L(f^n)$ *is nonzero, then* $m \ge 2s+3$.

Proof. Suppose that $m \le 2s+2$. Then $2s \le m \le 2s+2$ and condition [0], [1] or [2] in Theorem 2.5 is satisfied. We notice that $f_{\#} = \mathrm{id}$ implies $\mathcal{R}(f^1) = \mathbb{Z}_2$ and by the Jiang property, $\alpha_1 = L(f^1)$ must be even. Moreover, $\alpha_1 \neq 0$, since otherwise $L(f^1) = 0$ and essential reducibility would imply that all the classes are inessential. Thus $|\alpha_1| \ge 2$ which eliminates all extra conditions in Theorem 2.5 with the exception of $2s+2$ and LCM. But then Lemma 6.2 implies $L(f^n) \equiv 0$. □

Now we notice that under the assumption $m \ge 2s+3$ in Theorem 2.5, the only restriction is that the summation in $\sum_k \alpha_k \operatorname{reg}_k(n)$ runs over the set $\mathrm{LCM}(\{2; d_1, \dots, d_s\})$. Now, by Theorem 4.2, it remains to show that one can attach to each essential irreducible Reidemeister class an expression coming from an expression of the type as above so that their sum realizes the fixed point index of each orbit (Theorem 4.2).

We may rewrite (7.2) as

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{2}\, a_k \operatorname{Reg}^1_{k'} + \sum_{k=1}^{\infty} a_{k''} \operatorname{Reg}^1_{k''} - \frac{1}{2} \sum_{k=1}^{\infty} a_{k''} \operatorname{Reg}^1_{(2k)'} \\
&\quad = \sum_{k=1}^{\infty} \frac{1}{2}\, a_k \operatorname{Reg}^1_{k'} + \sum_{r=0}^{\infty} \sum_{l \text{ odd}} \left(a_{(2^r l)''} \cdot \operatorname{Reg}^1_{(2l)''} - \frac{1}{2}\, a_{(2^r l)''} \operatorname{Reg}^1_{(2^{r+1} l)'} \right) \\
&\quad = C_{1'} + \sum_{r=0}^{\infty} C_{(2^r)''}.
\end{aligned}$$

We will show that the expressions

$$C_{1'} = \sum_{k=1}^{\infty} \frac{1}{2}\, a_k \operatorname{Reg}^1_{k'} \tag{8.1}$$

and

$$C_{(2^r)''} = \sum_{l \text{ odd}} \left(a_{(2^r l)''} \cdot \operatorname{Reg}^1_{(2^r l)''} - \frac{1}{2}\, a_{(2^r l)''} \operatorname{Reg}^1_{(2^{r+1} l)'} \right) \tag{8.2}$$

come from expressions smoothly realizable in $\mathbb{R}^m$ attached to the orbits $1'$ and $(2^r)''$, respectively.

Consider the sequence $C_{1'}$. We notice that it comes from

$$\sum_{k=1}^{\infty} \frac{1}{2}\, a_k \operatorname{reg}_k .$$

We recall that the sum $\sum_k a_k \operatorname{reg}_k$ satisfies

(a) the summation runs over $\mathrm{LMC}(2; d_1, \dots, d_s)$,
(b) $2s + 3 \ge m$.

Now the sum $\sum_k \frac{1}{2} a_k \operatorname{reg}_k$ also satisfies (a) and (b), but (a) and (b) form a sufficient condition for the sum to be smoothly reducible in dimension m (Theorem 2.5).

Now we consider $C_{(2^{r_0})''}$, for a fixed number $r_0 \geq 0$. We notice that the sum (8.2) comes from

$$\sum_{l \text{ odd}} \left(a_{(2^r l)''} \cdot \mathrm{reg}_l - \frac{1}{2}\, a_{(2^r l)''}\, \mathrm{reg}_{2l} \right) \tag{8.3}$$

attached to the orbit $(2^{r_0})''$. It remains to show that (8.3) is smoothly realizable in dimension $m = \dim M$. This follows from the following lemma.

Lemma 8.2. *The sum* (8.3) *is smoothly realizable in dimension* $m = \dim M$.

Proof. By Lemma 8.1 we may assume that $m \geq 2s + 3$. Now it is enough to show that the summation in (8.3) runs over the set $\mathrm{LCM}(2; d'_1, \ldots, d'_s)$ for some $d'_1, \ldots, d'_s \geq 3$.

The assumption that the sequence $L(f^n)$ is smoothly realizable in $\mathbb{R}^m$ implies the existence of $d_1, \ldots, d_s$ so that in (8.3),

$$a_i \neq 0 \text{ implies } i \in \mathrm{LCM}(2; d_1, \ldots, d_s).$$

Now we define d'_i as the largest odd number dividing d_i, i.e., $d_i = 2^r d'_i$. Assume that $a_{(2^{r_0} l_0)''} \neq 0$. Since

$$a_{(2^{r_0} l_0)''} = \frac{1}{2^{r_0+1}} \left(\sum_{r=0}^{r_0} a_{2^r l_0} \cdot 2^r \right),$$

$a_{2^r l_0} \neq 0$ for an $r \leq r_0$. This implies that

$$2^{r_0} l_0 = \mathrm{lcm}(d_{i_1}, \ldots, d_{i_{\bar{s}}})$$

for some $1 \leq i_1 < \cdots < i_{\bar{s}} \leq s$. Now we eliminate the powers of 2 and we get

$$l_0 = \mathrm{lcm}\left(d'_{i_1}, \ldots, d'_{i_{\bar{s}}}\right)$$

which implies that $l_0 \in \mathrm{LCM}(d'_1, \ldots, d'_s)$. □

Corollary 8.3. *The assumptions*

- *f is essentially reducible,*
- *each iteration f^k is Jiang, i.e., the indices of all the Reidemeister classes in $\mathcal{R}(f^k)$ are the same*

hold for all self-maps of the so-called DJ spaces (*introduced in* [14], *see also* [6]), *so to get the equality*

$$NF_n(f) = NJD_n(f),$$

it is enough to assume then that $\pi_1 M = \mathbb{Z}_2$ and the smooth realizability of the sequence of Lefschetz numbers $(L(f^k))_{k \in \mathbb{N}}$. Since the class of DJ spaces contains compact Lie groups, we get the similar implication for compact Lie groups with $\pi_1 M = \mathbb{Z}_2$.

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