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On local fixed or periodic point properties

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Abstract. A space X has the local fixed point property LFPP (resp., local periodic point property LPPP) if it has an open basis $\mathcal{B}$ such that, for each $B \in \mathcal{B}$, the closure $\overline{B}$ has the fixed (resp., periodic) point property. Weaker versions wLFPP, wLPPP are also considered and examples of metric continua that distinguish all these properties are constructed. We show that for planar or one-dimensional locally connected metric continua the properties are equivalent.

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1. Introduction

A topological space X has the *fixed point property* (denoted $X \in$ FPP) if each continuous map $f : X \to X$ has a fixed point; X has the *periodic point property* ($X \in$ PPP) if each continuous map $f : X \to X$ has a periodic point.

We can localize these properties in two ways as follows.

Definition 1.1.

- A topological space X has the *local fixed point property* ($X \in$ LFPP) if X has an open basis $\mathcal{B}$ (a basis for LFPP) such that $\overline{B} \in$ FPP for each $B \in \mathcal{B}$.
- X has the *weak local fixed point property* ($X \in$ wLFPP) if X has an open basis $\mathcal{B}$ (a basis for wLFPP) such that, for each $B \in \mathcal{B}$ and each continuous map $f : X \to X$, whenever $f(\overline{B}) \subset B$, then f has a fixed point in B [6].
- X has the *local periodic point property* ($X \in$ LPPP) if X has an open basis $\mathcal{B}$ (a basis for LPPP) such that $\overline{B} \in$ PPP for each $B \in \mathcal{B}$.
- X has the *weak local periodic point property* ($X \in$ wLPPP) if X has an open basis $\mathcal{B}$ (a basis for wLPPP) such that, for each $B \in \mathcal{B}$ and each continuous map $f : X \to X$, whenever $f(\overline{B}) \subset B$, then f has a periodic point in B [6].

Obviously, LFPP $\Rightarrow$ wLFPP and LPPP $\Rightarrow$ wLPPP. Examples 3.5 and 3.6 show that the converse implications do not hold.

The main motivation for studying local fixed or periodic point properties is that they play an essential role in the dynamics of continuous maps: if a perfect compact metric absolute neighborhood retract (ANR-compactum) X has the wLPPP, then the set of periodic points of a generic map $f : X \to X$ has no isolated points and is dense in the set of chain recurrent points of f [6].

All spaces that are locally compact absolute retracts (ARs) (i.e., they have a basis $\mathcal{B}$ such that $\overline{B} \in AR$ for each $B \in \mathcal{B}$) have the LFPP. In particular, all polyhedra and all Euclidean manifolds have the LFPP. A harmonic fan has the LPPP but it does not have the wLFPP [6]. It is natural to ask for what continua properties LFPP, LPPP and their weak versions differ.

2. When (w)LFPP = (w)LPPP?

In this section we show that for all planar or one-dimensional locally connected metric continua the properties are equivalent.

The proof for planar continua is based on two classic theorems.

Proposition 2.1 (See [1, Theorem 13.1, p. 132]**).** *If a locally connected continuum X in the plane does not separate the plane, then X is an absolute retract. In particular, $X \in$ FPP.*

Proposition 2.2 (See [7, Theorem 5, p. 513]**).** *If a locally connected continuum X in the plane separates the plane between two points, then X contains a simple closed curve S which separates the plane between these points. In particular, S is a retract of X.*

Theorem 2.3. *Let X be a locally connected metric continuum in the plane. Then*

(1) $X \in$ FPP *if and only if* $X \in$ PPP*,*
(2) *the* wLFPP*,* LFPP*,* wLPPP *and* LPPP *for X are mutually equivalent.*

Proof. To show the nontrivial implications, we first assume that $X \in$ PPP. In view of Proposition 2.1, it suffices to observe that X does not separate the plane $\mathbb{R}^2$. Indeed, if X separates the plane, then, by Proposition 2.2, there exists a retraction $r : X \to S$ onto a simple closed curve S. Then the composition θr, where θ is a "rotation" of S without periodic points, would be a self-map of X without periodic points, contrary to the assumption.

Assume now that $X \in$ wLPPP and let $\mathcal{B}$ be a basis for wLPPP. We are going to define a neighborhood basis $\mathcal{V}_p$ at each point $p \in X$ such that $\bigcup_{p\in X} \mathcal{V}_p$ is a basis for LFPP.

For any $\varepsilon > 0$ there are $B \in \mathcal{B}$ and an open connected subset $V \subset B$ such that $p \in V \subset \overline{V} \subset B$ and $\operatorname{diam}(B) < \varepsilon/3$. It is also well known that V can be taken to be uniformly locally connected [10, Theorem 3.3, p. 77], so the continuum $A = \overline{V}$ is locally connected [10, Theorem 3.6, p. 79].

If A does not separate $\mathbb{R}^2$, then $A \in \mathrm{FPP}$ by Proposition 2.1 and then we include V in $\mathcal{V}_p$.

Suppose A separates $\mathbb{R}^2$. We claim that

$$\textit{Each bounded component of } \mathbb{R}^2 \setminus A \textit{ is contained in } X. \tag{2.1}$$

In fact, if a bounded component D of $\mathbb{R}^2 \setminus A$ satisfies $D \setminus X \neq \emptyset$, then, by Proposition 2.2, there is a simple closed curve $S \subset A$ that separates $\mathbb{R}^2$ between a point $a \in D \setminus X$ and a point $b \in \mathbb{R}^2 \setminus X$. By the Jordan curve theorem, there exists a retraction $r : X \to S$. If θ is a “rotation” of S without periodic points, then the composition $\theta r : X \to X$ has no periodic points and maps $\overline{B}$ into B, contrary to the property of the wLPPP-basic set B.

Put

$$A' = A \cup \bigcup \{D : D \text{ is a bounded component of } \mathbb{R}^2 \setminus A\}$$

and

$$V' = V \cup \bigcup \{D : D \text{ is a bounded component of } \mathbb{R}^2 \setminus A\}.$$

Notice that $\overline{V'} = A'$. By (2.1), A' is a subcontinuum of X. Obviously, A' is locally connected and it does not separate $\mathbb{R}^2$, so $A' \in \mathrm{FPP}$ by Proposition 2.1. Moreover, since the diameter of each bounded component D of $\mathbb{R}^2 \setminus A$ does not exceed the diameter of A, we have $\operatorname{diam} A' \leq \varepsilon$. So, in the case when A separates $\mathbb{R}^2$, we include V' in $\mathcal{V}_p$. □

Recall that a continuum is a *local dendrite* if each of its points has a closed neighborhood which is a dendrite and that dendrites have the FPP.

In the proof of the next theorem we use the following well-known facts.

Proposition 2.4 (See [7, Theorem 1, p. 354]**).** *Each simple closed curve contained in a one-dimensional metric continuum X is a retract of X.*

Proposition 2.5 (See [7, Theorems 4 and 5, pp. 303–304]**).** *If a locally connected metric continuum is not a local dendrite, then it contains a sequence of simple closed curves with diameters converging to* 0.

Theorem 2.6. *The following conditions are equivalent for a locally connected one-dimensional metric continuum X:*

(1) $X \in \mathrm{LPPP}$,
(2) $X \in \mathrm{wLPPP}$,
(3) X *is a local dendrite,*
(4) $X \in \mathrm{LFPP}$,
(5) $X \in \mathrm{wLFPP}$.

Proof. (2) $\Rightarrow$ (3). Suppose X is not a local dendrite. Then, by Proposition 2.5, there is a sequence of simple closed curves $S_n \subset X$ converging (in the Hausdorff metric) to a singleton $\{p\}$. If $\mathcal{B}$ is an arbitrary open basis in X and $B \in \mathcal{B}$ contains p, then B contains a simple closed curve $S = S_n$ for some $n \in \mathbb{N}$. By Proposition 2.4, there is a retraction $r : X \to S$. As before, compose r with a map $\theta : S \to S$ without periodic points to get a map $f : X \to X$ without periodic points such that $f(\overline{B}) \subset B$.

All the remaining implications are obvious. □

3. Examples and questions

Next we present an example of a plane continuum $X \in \mathrm{wLPPP} \setminus \mathrm{wLFPP}$ having an open basis $\mathcal{B}$ such that, for each map $f : X \to X$ and each $B \in \mathcal{B}$, f has a periodic point of period at most 2 in B. It was observed in [6] that a harmonic fan F is a plane continuum in $\mathrm{LPPP} \setminus \mathrm{wLFPP}$, but it is easy to see that in F it is not possible to find a bound for the periods, in the described sense.

Example 3.1. We use the Maćkowiak's example of a chainable, hereditarily decomposable continuum M which is rigid; i.e., it admits no nonconstant, nonidentity maps between subcontinua [8]. We locate M in the plane semi-annulus

$$\{z = (x, y) : 1 \leq |z| \leq 2,\ 0 \leq y\}$$

so that there are symmetric points

$$a \in M \cap ([1, 2] \times \{0\}) \quad \text{and} \quad -a \in M \cap ([-2, -1] \times \{0\}).$$

We may also ask that

$$\{a\} = M \cap ([1, 2] \times \{0\}) \quad \text{and} \quad \{-a\} = M \cap ([-2, -1] \times \{0\}).$$

In fact, we may assume that $a = (2, 0)$.

Define $Y = M \cup (-M)$. Let $h : Y \to Y$ be the homeomorphism given by $h(y) = -y$. Then h has no fixed points in Y and $h \circ h$ is the identity on Y.

Fix an open set U in Y such that $a \in U$ and $V = -U$ is disjoint from U.

Our example X is an infinite wedge of a null-sequence of homeomorphic copies Y_n of Y, $n \in \mathbb{N}$, intersecting at the point $v = (0, 0) \in \mathbb{R}^2$.

To describe X more precisely, consider an auxiliary sequence of convex triangles T_n in $\mathbb{R}^2$ such that v is a vertex of each T_n, $T_n \cap T_m = \{v\}$ if $n \neq m$ and $\lim T_n = \{v\}$ in the Hausdorff metric. For each $n \in \mathbb{N}$, let Y_n be a homeomorphic copy of Y located in T_n such that the point a_n corresponding to a equals v. Let $-a_n$ be the point in Y_n that corresponds to $-a$. We may assume that

$$d(v, -a_n) = \max \{d(v, x) : x \in Y_n\},$$

where d is a metric in Y_n. Let M_n (resp., $-M_n$, U_n and V_n) be the subcontinuum of Y_n that corresponds to M (resp., $-M$, U and V) and let $h_n : Y_n \to Y_n$ be the homeomorphism that corresponds to h. Then

$$U_n \cap V_n = \emptyset, \quad -M_n = h_n(M_n), \quad M_n = h_n(-M_n),$$
$$-a_n = h_n(a_n), \quad V_n = h_n(U_n),$$

$h_n \circ h_n$ is the identity on M_n and h_n is a fixed-point-free map.

Define

$$X = \bigcup \{Y_n : n \in \mathbb{N}\}.$$

In order to see that $X \notin \mathrm{wLFPP}$, take any open basis $\mathcal{B}$ of X. Fix $B \in \mathcal{B}$ with $v \in B$. Let $n \in \mathbb{N}$ be such that $Y_n \subset B$. Let $f : X \to X$ be defined as

$$f(x) = \begin{cases} h_n(v) & \text{if } x \notin Y_n, \\ h_n(x) & \text{if } x \in Y_n. \end{cases}$$

Clearly, f is a continuous fixed-point-free map such that $f(\overline{B}) \subset B$. Therefore, $X \notin$ wLFPP.

We are going to define a neighborhood basis $\mathcal{V}_p$ at each point $p \in X$ such that $\bigcup_{p\in X} \mathcal{V}_p$ is a basis for wLPPP. In every case $\mathcal{V}_p$ is of the form

$$\mathcal{V}_p = \{B(p,\varepsilon) \cap X : 0 < \varepsilon < \varepsilon_p\},$$

where $B(p,\varepsilon)$ is the Euclidean ε-ball in $\mathbb{R}^2$. So, we need to say how to choose ε_p. For $p = v$, let $\varepsilon_p = 1$. If $p \in Y_n \backslash \{a_n, -a_n\}$ for some $n \in \mathbb{N}$, we choose $\varepsilon_p > 0$ such that

$$\begin{aligned} &B(p,2\varepsilon_p) \cap X \subset Y_n \setminus \{a_n, -a_n\},\\ &B(p,2\varepsilon_p) \cap -M_n = \emptyset \quad \text{for } p \in M_n,\\ &B(p,2\varepsilon_p) \cap M_n = \emptyset \quad \text{for } p \in -M_n. \end{aligned}$$

Finally, if $p = -a_n$ for some n, we choose $\varepsilon_p > 0$ such that $B(p,\varepsilon_p) \cap X \subset V_n$.

We will need the following property of X.

Property 3.1.1. Let $f : X \to X$ be a continuous map. If there exists a nondegenerate subcontinuum A of M_n (resp., $-M_n$) such that $f(A) = \{q\}$ for some $q \neq v$, then $f(M_n) = \{q\}$ (resp., $f(-M_n) = \{q\}$).

Proof. To prove Property 3.1.1, consider the hyperspace of subcontinua $C(X)$ of X with the Hausdorff metric and use the fact that, given subcontinua E and B of X such that $E \subset B$, there exists a continuous map $\alpha : [0,1] \to C(X)$ such that $\alpha(0) = E$, $\alpha(1) = B$ and $\alpha(s) \subset \alpha(t)$ if $0 \le s \le t \le 1$ (see [5, Theorem 14.6]).

In our setting, let $\alpha : [0,1] \to C(X)$ be such that

$$\begin{aligned} &\alpha(0) = A, \quad \alpha(1) = M_n,\\ &\alpha(s) \subset \alpha(t) \quad \text{if } 0 \le s \le t \le 1. \end{aligned}$$

Let $r \in \mathbb{N}$ be such that $q \in Y_r = M_r \cup (-M_r)$. We may assume that $q \in M_r$. Let $t_0 = \max\{t \in [0,1] : f(\alpha(t)) \subset M_r\}$. Notice that

$$f|_{\alpha(t_0)} : M_n \supset \alpha(t_0) \to M_r.$$

By the rigidity of M, the map $f|_{\alpha(t_0)}$ is either an embedding or a constant map. Since $A \subset \alpha(t_0)$, $f|_{\alpha(t_0)}$ cannot be one-to-one, so $f|_{\alpha(t_0)}$ is a constant map. Thus, $f(\alpha(t_0)) = \{q\}$.

We claim that $t_0 = 1$. Suppose to the contrary that $t_0 < 1$. If $q \in M_r \setminus \{-a_r\}$, then, since $q \neq v$ and $M_r \setminus \{-a_r, v\}$ is open in X, there exists $t_0 < t < 1$ such that

$$f(\alpha(t)) \subset M_r \setminus \{-a_r, v\},$$

contradicting the choice of t_0. Hence, $q = -a_r$. Since q belongs to the open subset $Y_r \setminus \{v\}$ of X, there exists $t_0 < t < 1$ such that

$$f(\alpha(t)) \subset Y_r \setminus \{v\}.$$

Now,

$$\big(M_r \cap f(\alpha(t))\big) \cap \big(-M_r \cap f(\alpha(t))\big) = \{-a_r\},$$

so both sets $M_r \cap f(\alpha(t))$ and $-M_r \cap f(\alpha(t))$ are subcontinua of $f(\alpha(t))$ and each of them is a retract of $f(\alpha(t))$. Let

$$\varphi : f(\alpha(t)) \to M_r \cap f(\alpha(t))$$

be the retraction that sends $-M_r \cap f(\alpha(t))$ to q. By the rigidity of M, the map $\varphi \circ f : \alpha(t) \to M_r \cap f(\alpha(t))$ is either an embedding or a constant map. Since $A \subset \alpha(t)$ and $\varphi(f(A)) = \{q\}$, $\varphi \circ f$ is the constant map that sends $\alpha(t)$ to q. Thus, for each $x \in \alpha(t)$ satisfying $f(x) \in M_r$, we have $f(x) = q$. Similarly,

$$f\big(\alpha(t) \cap f^{-1}(-M_r)\big) = \{q\}.$$

Thus $f(\alpha(t)) = \{q\}$. This contradicts the choice of t_0 and shows that $t_0 = 1$. Hence, $f(M_n) = f(\alpha(1)) = \{q\}$ and the proof of Property 3.1.1 is finished. $\square$

In order to show that $\mathcal{B} = \bigcup_{p \in X} \mathcal{V}_p$ is a basis for wLPPP, let $f : X \to X$ be a continuous map, and let $p \in X$ and $0 < \varepsilon < \varepsilon_p$ be such that $f(\overline{B}) \subset B$, where $B = B(p, \varepsilon) \cap X$.

We are going to show that f has a periodic point of period at most 2. We analyze three cases.

Case 1. $p \in (M_n \setminus \{v, -a_n\}) \cup (-M_n \setminus \{v, -a_n\})$ for some $n \in \mathbb{N}$.

In this case, we suppose that $p \in M_n \setminus \{v, -a_n\}$, the other case is similar. By the choice of ε_p, $\overline{B} \subset M_n \setminus \{v, -a_n\}$. By [5, Theorem 14.6], there exists a nondegenerate subcontinuum A of X such that $p \in A \subset B$. By the rigidity of M_n, $f|_A$ is either the natural embedding or a constant map. If $f(a) = a$ for each $a \in A$, we are done. If $f(a) = q \in M_n$ for each $a \in A$, then $f(M_n) = \{q\}$ by Property 3.1.1. Hence, q is a fixed point of f in B.

Case 2. $p = -a_n$ for some $n \in \mathbb{N}$.

By the choice of ε_p, $B \subset V_n \subset Y_n \setminus \{v\}$. We consider two subcases.

Subcase 2.1. There exists a nondegenerate subcontinuum E of B such that $f(E)$ is a one-point set.

Suppose that $\{q\} = f(E)$. Note that $q \in B$. There exists a nondegenerate subcontinuum G of $E \setminus \{v, -a_n\}$. We may assume that $G \subset M_n$. By Property 3.1.1, $f(M_n) = \{q\}$, in particular, $f(-a_n) = q$. If $q \in M_n$, then q is a fixed point for f in B and we are done. Suppose then that $q \in -M_n \setminus \{v, -a_n\}$. Let $\eta > 0$ be such that

$$B(-a_n, \eta) \cap B(q, \eta) = \emptyset \quad \text{and} \quad B(q, \eta) \cap X \subset -M_n \setminus \{v, -a_n\}.$$

Let

$$0 < \delta < \min\{\varepsilon, \eta\} \quad \text{and} \quad f(B(-a_n, \delta) \cap X) \subset B(q, \eta).$$

Since $B(-a_n, \delta) \cap -M_n$ is a nonempty open subset of the continuum $-M_n$, there exists a nondegenerate subcontinuum K of $-M_n$ such that

$$K \subset B(-a_n, \delta).$$

Then

$$f(K) \subset -M_n \setminus \{v, -a_n\}.$$

By the rigidity of $-M_n$, $f|_K$ is the identity on K or $f(K)$ is a singleton. Since $K \subset B(-a_n, \delta)$ and $f(K) \subset B(q, \eta)$, $f|_K$ cannot be the identity on K. Thus, $f(K) = \{w\}$ for some $w \in -M_n \setminus \{v, -a_n\}$ and, by Property 3.1.1,

$$f(-M_n) = \{w\} = \{q\}.$$

Hence, q is a fixed point for f in B.

Subcase 2.2. For every nondegenerate subcontinuum E of B, $f(E)$ is nondegenerate.

Since B is open and nonempty in X, there exists a nondegenerate subcontinuum $E \subset B$. Since $E \setminus \{v, -a_n\}$ is a nonempty open subset of E, we may assume that $E \cap \{v, -a_n\} = \emptyset$. Moreover, the set $f(E)$ being nondegenerate, $E \setminus f^{-1}(\{v, -a_n\})$ is a nonempty open subset of E. Thus, we may also assume that $E \cap f^{-1}(\{v, -a_n\}) = \emptyset$. Then $E \subset M_n$ or $E \subset -M_n$ and $f(E) \subset M_n$ or $f(E) \subset -M_n$. We may assume that $E \subset M_n$. Recall that

$$h_n(-M_n) = M_n \quad \text{and} \quad h_n(M_n) = -M_n.$$

Thus, $f(E) \subset M_n$ or $h_n(f(E)) \subset M_n$. Since $f(E)$ is nondegenerate, the rigidity of M_n implies that $f(e) = e$ for each $e \in E$ or $h_n(f(e)) = e$ for each $e \in E$. In the first case, there are fixed points of f in B. Hence, we may assume that $h_n(f(E)) \subset M_n$ and $h_n(f(e)) = e$ for each $e \in E$. Fix a point $e_0 \in E \subset B \subset V_n$. Since $h_n(f(e_0)) = e_0$, we get

$$f(e_0) = h_n(e_0) \in h_n(V_n) = U_n.$$

But, $f(e_0) \in f(B) \subset B \subset V_n$. Thus, $f(e_0) \in U_n \cap V_n$, a contradiction. This completes Subcase 2.2.

Case 3. $p = v$.

Here, we may assume that $f(p) \neq p$ and consider two subcases.

Subcase 3.1. $f(p) \in Y_n \setminus \{v, -a_n\}$ for some $n \in \mathbb{N}$.

In this subcase we will see that f is a constant map. We may assume that $f(p) \in M_n$, the case $f(p) \in -M_n$ being similar. Let $m \in \mathbb{N}$. Since

$$f^{-1}\big(M_n \setminus \{v, -a_n\}\big)$$

is an open subset of X containing p, there exists a nondegenerate subcontinuum A of M_m such that $p \in A \subset f^{-1}(M_n \setminus \{v, -a_n\})$. Then

$$f(A) \subset M_n \setminus \{v, -a_n\} \quad \text{and} \quad f|_A : M_m \supset A \to M_n.$$

By the rigidity of M, $f|_A$ is either a constant map or the natural embedding. Since $f(a_m) = f(p) \neq v = a_n$, $f|_A$ is not the natural embedding. Hence, $f(a) = f(p)$ for each $a \in A$. By Property 3.1.1, $f(M_m) = \{f(p)\}$. By an analogous argument, we see that $f(-M_m) = \{f(p)\}$. Therefore,

$$f(Y_m) = \{f(p)\}$$

for each $m \in \mathbb{N}$, so f is constant and $f(p)$ is a fixed point of f in B.

Subcase 3.2. $f(p) = -a_n$ for some $n \in \mathbb{N}$.

In this case $-a_n \in B$ and since

$$d(v, -a_n) = \max\{d(v, x) : x \in Y_n\},$$

we have $Y_n \subset B$. Let $\alpha : [0,1] \to C(M_n)$ be a continuous map such that $\alpha(0) = \{p\}$, $\alpha(1) = M_n$ and $\alpha(s) \subsetneq \alpha(t)$ if $0 \le s < t \le 1$. If $\alpha(t)$ contains a nondegenerate subcontinuum A such that $p \in A$ and $f(A)$ is a one-point set, then $f(M_n) = \{-a_n\}$ by Property 3.1.1. So, $f(-a_n) = -a_n$ is a fixed point for f in B. Suppose then that for each $t > 0$ and, for each nondegenerate subcontinuum A of $\alpha(t)$ containing p, $f(A)$ is nondegenerate. Let

$$t_0 = \max\{t \in [0,1] : f(\alpha(t)) \subset -M_n\}.$$

Then $f(\alpha(t_0)) \subset -M_n$.

We claim that

$$f(x) = h_n(x) \quad \text{for each } x \in \alpha(t_0). \tag{3.1}$$

If $t_0 = 0$, then

$$\alpha(t_0) = \{p\} \quad \text{and} \quad f(p) = -a_n = h_n(a_n) = h_n(p).$$

Now, if $t_0 > 0$, $\alpha(t_0)$ is a nondegenerate subcontinuum of M_n such that

$$f|_{\alpha(t_0)} : M_n \supset \alpha(t_0) \to -M_n,$$

so $h_n \circ f|_{\alpha(t_0)} : M_n \supset \alpha(t_0) \to M_n$. The rigidity of M_n implies that $h_n \circ f|_{\alpha(t_0)}$ is either a constant map or the identity on $\alpha(t_0)$. Since $t_0 > 0$, $f(\alpha(t_0))$ is non-degenerate, so $h_n(f(\alpha(t_0)))$ is nondegenerate. Thus, $h_n \circ f|_{\alpha(t_0)}$ is the identity on $\alpha(t_0)$. Hence, for each $x \in \alpha(t_0)$, $h_n \circ f(x) = x$ and $f(x) = h_n(x)$.

Now, we will see that $p \in f(\alpha(t_0))$. Otherwise,

$$-a_n = h(a_n) = h_n(p) \notin h_n\big(f(\alpha(t_0))\big) = \alpha(t_0) \quad \text{and} \quad t_0 < 1.$$

Thus,

$$\alpha(t_0) \cap f(\alpha(t_0)) \subset (M_n \cap -M_n) \setminus \{p, -a_n\} = \emptyset.$$

Let W_1 and W_2 be disjoint open subsets of X such that

$$\alpha(t_0) \subset W_1 \quad \text{and} \quad f(\alpha(t_0)) \subset W_2 \subset Y_n \setminus \{p\}.$$

Then there exists $t_0 < t_1 < 1$ such that $\alpha(t_1) \subset W_1$ and $f(\alpha(t_1)) \subset W_2$. By the choice of t_0, there exists a point

$$y_0 \in f(\alpha(t_1)) \setminus -M_n \subset M_n \setminus -M_n.$$

Let $x_0 \in \alpha(t_1)$ be such that $y_0 = f(x_0)$. Since $\alpha(t_1) \setminus f^{-1}(-M_n)$ is an open subset of the nondegenerate continuum $\alpha(t_1)$ and it contains x_0, there exists a nondegenerate subcontinuum A of $\alpha(t_1)$ such that

$$x_0 \in A \subset \alpha(t_1) \setminus f^{-1}(-M_n).$$

Then

$$A \subset \alpha(t_1), \quad f(A) \subset W_2 \subset Y_n \setminus \{p\}, \quad f(A) \cap -M_n = \emptyset.$$

Thus,

$$f(A) \subset M_n \quad \text{and} \quad f|_A : M_n \supset A \to M_n.$$

By the rigidity of M_n, $f(a) = a$ for each $a \in A$ or $f(A)$ is a singleton. In the latter case, by Property 3.1.1, $f(M_n) = \{-a_n\}$ and $-a_n$ is a fixed point for f in B. Suppose now that $f(a) = a$ for each $a \in A$. In particular,

$$x_0 = f(x_0) \in \alpha(t_1) \cap f(\alpha(t_1)) \subset W_1 \cap W_2,$$

a contradiction. We have shown that $p \in f(\alpha(t_0))$.

Let $x_1 \in \alpha(t_0)$ be such that $p = f(x_1) = h_n(x_1)$ (by (3.1)). Then

$$x_1 = h_n(p) = -a_n \quad \text{and} \quad f(f(p)) = f(-a_n) = f(x_1) = p.$$

Therefore, p is a periodic point for f in B of order 2.

This completes the proof that for each continuous map $f : X \to X$ and for each $B \in \mathcal{B}$ such that $f(\overline{B}) \subset B$, f has a periodic point in B of order at most 2.

In particular, we have shown that $X \in \mathrm{wLPPP} \setminus \mathrm{wLFPP}$.

It is easy to check that continuum Y employed in Example 3.1 does not have the FPP, but any continuous self-map of Y has a point of period at most 2. The example shows that the infinite wedge of a null-sequence of copies of Y localize the properties. Looking for a respective locally connected example, we can recall that all connected compact polyhedra X with trivial odd-dimensional homology groups $H_{2i+1}(X, \mathbb{Q})$ have the PPP [2, Proposition 4.4, p. 232]. Specifically, for any fixed-point-free self-map of a 2-sphere there is a point of period 2. Hence, we pose the following questions.

Question 3.2. Is there a locally connected continuum $X \in \mathrm{LPPP} \setminus \mathrm{LFPP}$ (resp., $X \in \mathrm{wLPPP} \setminus \mathrm{wLFPP}$)? Is the infinite wedge of a null-sequence of 2-spheres such a continuum?

Question 3.3. Are the LPPP and LFPP (resp., wLPPP and wLFPP) equivalent for ANR-continua?

Question 3.4. Are the LPPP and wLPPP (resp., LFPP and wLFPP) equivalent for ANR-continua?

Example 3.5. There exists a continuum

$$X \in \mathrm{wLFPP} \setminus \mathrm{LPPP} \subset \mathrm{wLFPP} \setminus \mathrm{LFPP}\,.$$

We consider the Cook's continuum X constructed in [2, Theorem 8], which is one-dimensional, nonplanar, hereditarily indecomposable and it is also rigid (it admits no nonconstant, nonidentity maps between subcontinua).

In order to see that $X \in \mathrm{wLFPP}$, take any open basis $\mathcal{B}$ of nonempty sets. Let $B \in \mathcal{B}$ and let $f : X \to X$ be a continuous map such that $f(\overline{B}) \subset B$. By the rigidity of X, f is either the identity map or it is a constant map. In the first case, any point in B is a fixed point for f. In the second case, there exists $q \in X$ such that $f(x) = q$ for each $x \in X$. Notice that $q \in B$, so q is a fixed point for f in B.

Now we check that $X \notin$ LPPP. Take any open basis $\mathcal{B}$ for X. By [9], there exists a Cantor set $C_0 \subset X$ which contains at most one point of each composant of X. Fix a point $p_0 \in C_0$ and let $B \in \mathcal{B}$ be such that $p_0 \in B$ and $\overline{B} \neq X$. Since B is open, there exists a Cantor set $C \subset B \cap C_0$. Then C has also the property that it contains at most one point of each composant of X. Observe that there is a retraction $r : \overline{B} \to C$. Indeed, if D is the quotient space of the decomposition of $\overline{B}$ into components and $q : \overline{B} \to D$ is the quotient map, then $q|C : C \to q(C)$ is a homeomorphism and $q(C)$ is a nonempty closed subset of a zero-dimensional space D, so there is a retraction $f : D \to q(C)$ [3, Problem 1.3.C, p. 22]. Then put $r = (q|C)^{-1} \circ f \circ q$. Let $\sigma : C \to C$ be a periodic-point-free map (any minimal map of C, say the adding machine on $C = \{0,1\}^\infty$, can be used). Then the composition $\sigma \circ r : \overline{B} \to \overline{B}$ is periodic-point free.

Example 3.6. There exists a (nonplanar) continuum

$$Z \in \mathrm{wLPPP} \setminus (\mathrm{LPPP} \cup \mathrm{wLFPP}).$$

Consider the Cook's continuum X, that we used in Example 3.5. Fix two different points $p, q \in X$. Let X' be a disjoint topological copy of X, with corresponding points p', q', and let Y be a continuum obtained from $X \cup X'$ by identifying p with q' and q with p'. Take an infinite wedge

$$Z = \bigcup \{Y_n : n \in \mathbb{N}\}$$

of copies Y_n of Y, joined at the point $v = p_n$ for each n, where $p_n \in Y_n$ corresponds to p and $\lim \operatorname{diam} Y_n = 0$.

We can mimic the argument from Example 3.1 to obtain

$$Z \in \mathrm{wLPPP} \setminus \mathrm{wLFPP}\,.$$

In order to show that $Z \notin$ LPPP, take an open basis $\mathcal{B}$ of Z. Let $B \in \mathcal{B}$ be such that

$$\emptyset \neq \overline{B} \subset Y_1 \setminus \{v\}.$$

Now, the proof from Example 3.5 that $\overline{B} \notin$ PPP can be repeated step by step.

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