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Birkhoff normalization, bifurcations of Hamiltonian systems and the Deprits formula

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Abstract. We consider Hamiltonian autonomous systems with n degrees of freedom near a singular point. In the case of absence of resonances of order less than or equal to 4 we present a direct computation of the Birkhoff normal form. In the case of two degrees of freedom, we study 1-parameter deformations of the $0:1$, $1:1$ and $2:1$ resonant singularities. The obtained results are used in a direct derivation of the Deprits formula for the isoenergetic degeneracy determinant in the restricted three-body problem.

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1. Introduction

Probably the most spectacular application of the Kolmogorov–Arnold–Moser (KAM) theory is in the restricted three-body problem (see [AKN, Mos, SiMo, Mar]). There one has a completely integrable Hamiltonian system with two degrees of freedom, which corresponds to fourth-degree Birkhoff normal form of the total energy near a Lagrangian libration point, and a perturbation, which corresponds to higher order terms. The invariant KAM tori, which fill a set of positive measure, guarantee the Lyapunov stability of this libration point. However, in order to use the KAM theorem, one has to check whether some determinant det does not vanish; this is the so-called *isoenergetic non-degeneracy condition* (see [Arn2]). This requires careful calculation of the

Birkhoff normal form and application of it to the restricted three-body problem. Recall also that the problem contains one parameter ζ defined by the ratio of masses of the heavy bodies.

The formula for the coefficients in the Birkhoff normal form was obtained in explicit form by Leontovich [Leo], but in the three-body case he was able only to show that the quantity det is not identically zero as a function of the parameter ζ. The complete formula for $\det(\zeta)$ was given by Deprit and Deprit-Bartholomé in [DDB] (see equation (4.9) below) and is cited in all sources about the subject. In practice, only few three-body systems are considered: Sun–Jupiter–Asteroid or Earth–Moon–Asteroid. However, more such situations in the celestial mechanics exist (with different values of ζ). Therefore, the formula for $\det(\zeta)$ is potentially useful.

It is rather hard to repeat the Deprits' derivation of their formula, because the paper [DDB] does not contain clear ideas about it. We tried to reprove the result of [DDB], but in [BaZo] we succeeded only to obtain the Leontovich formula and to apply it for some special values of the parameter ζ. Our calculations disagreed with the Deprits formula (see Remark 3 below). But later, at a conference in Siedlce (Poland), we have learned from Markeev and Prokopenya that they have checked the calculations (also using some computer programs) and obtained the same result as in [DDB].

Therefore, the principal reason for writing this paper was to fix our mistakes and eventually to find a direct derivation of the Deprits formula (with an explanation of its amazing simplicity). The idea was to show that $\det(\zeta)$, which is an algebraic function of ζ, is in fact a rational function of ζ^2 with possible poles corresponding to resonances of order 1, 2 and 3 and to $\zeta^2 = \infty$. We prove it in Section 4.2. The next step is to analyze the situation near the distinguished values of ζ^2.

In the case $\zeta^2 \approx \infty$, the eigenvalues of the corresponding linearization of the Hamiltonian system are not all imaginary, so a corresponding version of the Birkhoff normal form is needed. We do it in the next section. We do it in full generality, i.e., in the complex situation and with many degrees of freedom. Moreover, the derivation of the corresponding generalization of the Leontovich formula does not use iteration of the Birkhoff transformation. It is direct in the sense that it is reduced to a series of elementary substitutions. As a corollary, in Section 4.3 we find that the limit of $\det(\zeta)$ as $\zeta \to \infty$ is finite.

The case of $0:1$ resonance, $\zeta^2 = \zeta_0^2$, corresponds to the situation when a pair of eigenvalues is zero and a 2-dimensional Jordan cell arises. We obtain an analogue of the Birkhoff normal form for the vector field and for its 1-parameter perturbation corresponding to variation of one of the small eigenvalues. A general theory is described in Section 3.1, where we show that the order of the pole of $\det(\zeta)$ at ζ_0^2 is at most 2. In Section 4.4 we prove that in the three-body case $\det(\zeta)$ is regular here (no pole).

Analogous analysis is done near the two situations which correspond to the resonances of the type $1:1$ ($\zeta^2 = \zeta_1^2$) and $2:1$ ($\zeta^2 = \zeta_2^2$). The latter case is simple and is shortly discussed in Section 3.3. In the case of 1 : 1 resonance we have two 2-dimensional Jordan cells and the analysis is slightly more involved, but here we need only to show that the corresponding pole is simple. We do it in Section 3.2.

Section 4 is devoted to the restricted three-body problem. The function $\det(\zeta)$ is a ratio $P(\zeta^2)/Q(\zeta^2)$ of two quadratic polynomials, where $Q = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)$ defines the resonances of order 2 and 3. We easily find (in Section 4.6) the value of the residuum of $\det(\zeta)$ at ζ_2^2. By a direct calculation (in Section 4.7) of $\det(\zeta)$ at two special values of the parameter we find the three coefficients of P. In fact, we calculate $\det(\zeta)$ for three values of ζ.

2. Birkhoff normal form

Consider a linear autonomous Hamiltonian system in $\mathbb{C}^{2n}$,

$$\frac{dx_j}{dt} = \{x_j, G^{(2)}\} = \sum a_{jk}x_k, \quad j = 1, \ldots, 2n, \tag{2.1}$$

with respect to some linear symplectic structure on $\mathbb{C}^{2n}$ defined by a Poisson structure $\{\cdot,\cdot\}$. Assuming that the matrix $A = (a_{jk})$ has pairwise different and nonzero eigenvalues, which appear in pairs $\lambda_j, -\lambda_j$,[1] we can diagonalize this system:

$$\frac{dz_j}{dt} = \lambda_j z_j, \quad \frac{d\tilde{z}_j}{dt} = -\lambda_j \tilde{z}_j, \quad j = 1, \ldots, n; \tag{2.2}$$

here the functions (variables) $z = \{z_j, \tilde{z}_j : j = 1, \ldots, n\}$ are related with the variables $x = (x_1, \ldots, x_{2n})$ by means of a matrix B, i.e., $x = Bz$, such that the columns of B are the eigenvectors of A. The variables z satisfy the relations $\{z_j, z_k\} = \{\tilde{z}_j, \tilde{z}_k\} = \{z_j, \tilde{z}_k\} = 0$ for $j \neq k$, but the constants $\{z_j, \tilde{z}_j\}$ are not determined because the very variables z_j and $\tilde{z}_j$ are defined up to multiplicative constants. Therefore, the quadratic Hamiltonian $G^{(2)}$ from (2.1) can be written in the form

$$H^{(2)}(z) = G^{(2)}(Bz) = \lambda_1 \frac{z_1\tilde{z}_1}{\{z_1, \tilde{z}_1\}} + \cdots + \lambda_n \frac{z_n\tilde{z}_n}{\{z_n, \tilde{z}_n\}}. \tag{2.3}$$

Generally, with the Hamilton function (2.3) we can associate the *generalized actions* $J_k \in \mathbb{C}$ and *generalized angles* $\phi_k \in \mathbb{C}$ as follows:

$$J_k = \frac{z_k\tilde{z}_k}{\{z_k, \tilde{z}_k\}}, \qquad \phi_k = \frac{1}{2}\log\left(\frac{z_k}{\tilde{z}_k}\right). \tag{2.4}$$

Note that each function J_k is of rather special type: its zero locus consists of two transversal hyperplanes, but it has degenerate singularity at the origin (with infinite Milnor number).

[1] Recall that this follows from the invariance of the symplectic 2-form ω^2 under the Hamiltonian phase flow: $\omega^2(e^{tA}u, e^{tA}v) = \omega^2(u,v)$ or $\omega^2(Au,v) + \omega^2(u,Av) = 0$. Taking u and v as eigenvectors with the corresponding eigenvalues λ and μ, we get the identity $(\lambda+\mu)\omega^2(u,v) = 0$.

Assume now that we have a holomorphic Hamiltonian system with n degrees of freedom and with a singular point at the origin of the type considered above. Therefore, we have

$$H(z)=\sum_{j=1}^{n}\lambda_j\frac{\|z_j\|^2}{\kappa_j}+\sum_{|m|>2}h_m z^m, \tag{2.5}$$

where

$$\begin{aligned} z&=(z_1,\tilde{z}_1;\dots;z_n,\tilde{z}_n),\\ \|z_j\|^2&=z_j\tilde{z}_j,\quad \kappa_j=\{z_j,\tilde{z}_j\},\\ m&=(m_1,\tilde{m}_1;\dots;m_n,\tilde{m}_n),\\ z^m&=z_1^{m_1}\tilde{z}_1^{\tilde{m}_1}\cdots z_n^{m_n}\tilde{z}_n^{\tilde{m}_n},\\ |m_j|&=m_j+\tilde{m}_j,\quad |m|=|m_1|+\dots+|m_n|. \end{aligned}$$

We have

$$\begin{aligned} \frac{dz_j}{dt}&=\{z_j,H\}=\frac{\partial H}{\partial\tilde{z}_j}\{z_j,\tilde{z}_j\}=\lambda_j z_j+\kappa_j\sum\frac{\tilde{m}_j h_m z^m}{\tilde{z}_j},\\ \frac{d\tilde{z}_j}{dt}&=-\lambda_j\tilde{z}_j-\kappa_j\sum\frac{m_j h_m z^m}{z_j}. \end{aligned}$$

We will use also the following notations:

$$\begin{gathered} \tilde{m}=(\tilde{m}_1,m_1;\dots;\tilde{m}_n,m_n),\quad \tilde{h}_m=h_{\tilde{m}},\\ \|h_m\|^2=\|h_{\tilde{m}}\|^2=h_m\tilde{h}_m,\\ \mathrm{RE}(h_m h_n)=\frac{1}{2}\big(h_m h_n+\tilde{h}_m\tilde{h}_n\big),\\ [m,\lambda]=(m_1-\tilde{m}_1)\lambda_1+\dots+(m_n-\tilde{m}_n)\lambda_n. \end{gathered}$$

Additionally we assume the following:

$$\sum k_j\lambda_j\neq 0\quad \text{for } k_j\in\mathbb{Z},\qquad \sum|k_j|=1,2,3,4, \tag{2.6}$$

i.e., the absence of *resonances of order* 1, 2, 3 and 4. (Here the case $\sum|k_j|=1$ would correspond to a situation with a pair of zero eigenvalues and the case $\sum|k_j|=2$ would correspond to a situation with two equal eigenvalues.)

Example 1. If G is *real* and the eigenvalues are *imaginary*, $\lambda_k=-\sqrt{-1}\omega_k=-i\omega_k$, $\omega_k>0$, (i.e., when the origin is the so-called *elliptic singular point*), then there are natural symplectic variables $\{(q_k,p_k)\}$ such that $z_k=q_k+ip_k$, $\tilde{z}_k=q_k-ip_k$ and $\{z_k,\tilde{z}_k\}=-2i$ or $z_k=q_k-ip_k$, $\tilde{z}_k=\bar{z}_k=q_k+ip_k$ and $\{z_k,\bar{z}_k\}=2i$. In this case there exist so-called *action-angle variables* $I_k=(p_k^2+q_k^2)/2=|z_k|^2/2$ and $\varphi_k=\arg z_k$. Then $J_k=iI_k$ and $H^{(2)}=\pm\omega_1 I_1\pm\dots\pm\omega_n I_n$. Note that using representation (2.3) we avoid the problem of signs before $\omega_j I_j$.

If G is *real* and some eigenvalue λ_k is *real*, then one can choose $z_k=q_k+p_k$, $\tilde{z}_k=q_k-p_k$ with $\{z_k,\tilde{z}_k\}=-2$ (or $z_k=q_k-p_k$, $\tilde{z}_k=q_k+p_k$ with $\{z_k,\tilde{z}_k\}=2$). Here the analogue of the corresponding action-angle variables $J_k=(q_k^2-p_k^2)/2$, ϕ_k are such that $z_k=\sqrt{2I_k}e^{\phi_k}$, $\tilde{z}_k=\sqrt{2I_k}e^{-\phi_k}$.

If $n = 2$, G is real and we have nonreal and nonimaginary eigenvalues $\lambda = \lambda_1 = \nu + i\omega$, $-\lambda$, $\lambda_2 = \bar{\lambda} = \nu - i\omega$ and $-\bar{\lambda}$, then we can choose $z_1 = q_1 + iq_2$, $\tilde{z}_1 = p_1 - ip_2$, $z_2 = \bar{z}_1$, $\tilde{z}_2 = \overline{(\tilde{z})}$ (thus $\{z_1, \tilde{z}_1\} = \{z_2, \tilde{z}_2\} = 2$). Then $J_1 = J = \frac{1}{2} z_1 \tilde{z}_1$ and $J_2 = \bar{J}_1$ are nonreal and $H^{(2)} = 2\,\mathrm{Re}(\lambda J)$. (Here the notation Re has the standard meaning, the real part, in contrary to the notation RE.)

The *fourth order Birkhoff normal form* [Bir] in the above situation is a symplectic change of variables

$$Z_j = z_j + \cdots , \qquad \tilde{Z}_j = \tilde{z}_j + \cdots \tag{2.7}$$

such that

$$H(z, \tilde{z}) = H^{\mathrm{Bir}}(Z, \tilde{Z}) + \cdots = \sum \lambda_j J_j + \sum D_{jk} J_j J_k + \cdots , \tag{2.8}$$

where

$$J_k = \frac{Z_k \tilde{Z}_k}{\{Z_k, \tilde{Z}_k\}} = \frac{\|Z_k\|^2}{\kappa_k}. \tag{2.9}$$

The symplecticity of this change of coordinates implies preservation of the Poisson brackets: $\{Z_k, \tilde{Z}_k\} = \{z_k, \tilde{z}_k\} = \kappa_k$ and the other brackets vanish.

The main result of this section is the following theorem which generalizes a result by Leontovich [Leo].

Theorem 1. *The coefficients in equation* (2.8) *are the following:*

$$\begin{aligned} D_{jj} = {} & \kappa_j^2 h^j_{22} - 3\kappa_j^3 \frac{\|h^j_{30}\|^2 + \|h^j_{21}\|^2}{\lambda_j} \\ & - \kappa_j^2 \sum_{l \neq j} \kappa_l \left\{ \frac{\|h^{jl}_{11;10}\|^2}{\lambda_l} + \frac{\|h^{jl}_{20;10}\|^2}{\lambda_l + 2\lambda_j} + \frac{\|h^{jl}_{02;10}\|^2}{\lambda_l - 2\lambda_j} \right\}, \end{aligned}$$

$$\begin{aligned} D_{jk} = {} & \kappa_j \kappa_k h^{jk}_{11;11} - 4\kappa_j^2 \kappa_k \left\{ \frac{2\,\mathrm{RE}\left(h^{jk}_{21;00} h^{jk}_{01;11}\right)}{2\lambda_j} + \frac{\|h^{jk}_{20;10}\|^2}{2\lambda_j + \lambda_k} + \frac{\|h^{jk}_{20;01}\|^2}{2\lambda_j - \lambda_k} \right\} \\ & - 4\kappa_j \kappa_k^2 \left\{ \frac{2\,\mathrm{RE}\left(h^{jk}_{00;21} h^{jk}_{11;01}\right)}{2\lambda_k} + \frac{\|h^{jk}_{10;20}\|^2}{2\lambda_k + \lambda_j} + \frac{\|h^{jk}_{10;02}\|^2}{2\lambda_k - \lambda_j} \right\} \\ & - \kappa_j \kappa_k \sum_{l \neq j,k} \kappa_l \left\{ \frac{\|h^{jkl}_{10;10;10}\|^2}{\lambda_l + \lambda_j + \lambda_k} + \frac{\|h^{jkl}_{01;10;10}\|^2}{\lambda_l - \lambda_j + \lambda_k} \right. \\ & \qquad \left. + \frac{\|h^{jkl}_{10;01;10}\|^2}{\lambda_l + \lambda_j - \lambda_k} + \frac{\|h^{jkl}_{01;01;10}\|^2}{\lambda_l - \lambda_j - \lambda_k} \right\} \end{aligned}$$

for $j \neq k$. Here $h^j_{\rho\sigma}$ (resp., $h^{jk}_{\rho\sigma;\varsigma\tau}$ or $h^{jkl}_{\rho\sigma;\varsigma\tau;\upsilon\phi}$) denotes h_m with $(m_j, \tilde{m}_j) = (\rho, \sigma)$ (resp., with given distinguished indices) and with zero other indices.

The proof we give below is essentially new and does not use generating function for the symplectic change; the standard (rather involved) proof for $n = 2$ can be found in [BaZo].

Lemma 1. *The Birkhoff transformation is of the following form:*

$$Z_j = z_j\left(1 + \sum a_j^k \|z_k\|^2\right) + \sum_{|m|=3} \alpha_m^j \frac{z^m}{\tilde{z}_j} + \cdots,$$

$$\tilde{Z}_j = \tilde{z}_j\left(1 + \sum \tilde{a}_j^k \|z_k\|^2\right) + \sum_{|m|=3} \tilde{\alpha}_m^j \frac{z^m}{z_j} + \cdots,$$

where

$$\alpha_m^j = -\frac{\kappa_j \tilde{m}_j}{[m,\lambda]} h_m, \qquad \tilde{\alpha}_m^j = \frac{\kappa_j m_j}{[m,\lambda]} h_m, \tag{2.10}$$

the $a_j^k, \tilde{a}_j^k$ are constants (calculated below) and the dots denote inessential terms (of degree greater than or equal to 3).

The above form for the quadratic terms follows from the Poincaré–Dulac theorem and is unique (see [Arn1]). The distinguished cubic terms, like $a_k^j z_j \|z_k\|^2$, are resonant in the Poincaré–Dulac sense. Therefore, they are not determined here.[2]

Lemma 2. *The action parts of the action-angle variables*

$$J_j = \frac{Z_j \tilde{Z}_j}{\{Z_j, \tilde{Z}_j\}} = \frac{\|Z_j\|^2}{\kappa_j}$$

take the form

$$\begin{aligned} J_j = {} & \frac{1}{\kappa_j}\|z_j\|^2 + \sum_{|m|=3} \frac{m_j - \tilde{m}_j}{[m,\lambda]} h_m z^m \\ & + \frac{1}{\kappa_j}\left\{A_j\|z_j\|^4 + \sum_{k\neq j} B_j^k \|z_j\|^2\|z_k\|^2 + \sum_{k,l\neq j} C_j^{kl}\|z_k\|^2\|z_l\|^2\right\} \end{aligned} \tag{2.11}$$

(plus inessential quartic and higher order terms) where

$$\begin{aligned} A_j &= \sum_{|m_j|=3} \left\|\alpha_m^j\right\|^2 + \left(a_j^j + \tilde{a}_j^j\right), \\ B_j^k &= \sum_{|m_j|=2, |m_k|=1} \left\|\alpha_m^j\right\|^2 + 2\,\mathrm{RE}\left(\alpha_{12;00}^{j;jk}\, \tilde{\alpha}_{10;11}^{j;jk}\right) + \left(a_j^k + \tilde{a}_j^k\right), \\ C_j^{kk} &= \sum_{|m_j|=1, |m_k|=2} \left\|\alpha_m^j\right\|^2, \qquad C_j^{kl} = \sum_{|m_j|=|m_k|=|m_l|=1} \left\|\alpha_m^j\right\|^2 \end{aligned} \tag{2.12}$$

[2] The Poincaré–Dulac theorem says that a system $\dot{x}_1 = \lambda_1 x_1 + \sum a_k^1 x^k, \ldots, \dot{x}_n = \lambda_n x_n + \sum a_k^n x^k$ can be reduced to the so-called *Poincaré–Dulac normal form* $\dot{y}_1 = \lambda_1 y_1 + \sum b_k^1 y^k, \ldots, \dot{y}_n = \lambda_n y_n + \sum b_k^n y^k$, where only resonant terms $b_k^j y^k$ (for which $\lambda_j \neq \lambda_1 k_1 + \cdots + \lambda_n k_n$) in the jth equation remain.

(for $k \neq l$) and

$$\left\|\alpha^j_m\right\|^2 = \alpha^j_m \tilde{\alpha}^j_{\tilde{m}} = \left(\frac{\kappa_j \tilde{m}_j}{[m,\lambda]}\right)^2 h_m h_{\tilde{m}} = \left(\frac{\kappa_j \tilde{m}_j}{[m,\lambda]}\right)^2 \|h_m\|^2,$$

$$2\,\mathrm{RE}(\alpha^j_m \tilde{\alpha}^j_n) = \alpha^j_m \tilde{\alpha}^j_n + \alpha^j_{\tilde{n}} \tilde{\alpha}^j_{\tilde{m}}.$$

From this we arrive at the following lemma.

Lemma 3. *We have*

$$\begin{aligned} D_{jj} &= \kappa_j^2 h^j_{22} - \lambda_j \kappa_j A_j - \sum_{l\neq j} \lambda_l \left(\frac{\kappa_j^2}{\kappa_l}\right) C^{jj}_l, \\ D_{jk} &= \kappa_j \kappa_k h^{jk}_{11;11} - \lambda_j \kappa_k B^k_j - \lambda_k \kappa_j B^j_k - \sum_{l\neq j,k} \lambda_l \left(\frac{\kappa_j \kappa_k}{\kappa_l}\right) C^{jk}_l. \end{aligned} \tag{2.13}$$

Indeed, from (2.11) we find

$$\begin{aligned} H = \sum \lambda_j \Bigg\{ J_j - \left(\frac{A_j}{\kappa_j}\right) \|z_j\|^4 - \sum \left(\frac{B^k_j}{\kappa_j}\right) \|z_j\|^2 \|z_k\|^2 \\ - \sum \left(\frac{C^{kl}_j}{\kappa_j}\right) \|z_k\|^2 \|z_l\|^2 \Bigg\} \\ + \sum h^j_{22} \|z_j\|^4 + \sum_{j<k} h^{jk}_{11;11} \|z_j\|^2 \|z_k\|^2 + \cdots \end{aligned}$$

because the cubic terms contribute completely to $\sum \lambda_j J_j$. But this equals

$$\sum \lambda_j J_j + \sum_{j\leq k} D_{jk} \left(\frac{\|z_j\|^2}{\kappa_j}\right) \left(\frac{\|z_k\|^2}{\kappa_k}\right) + \cdots .$$

Next, we replace $\|z_j\|^2/\kappa_j$ with $J_j + \cdots$.

Let us pass to the determination of the coefficients a^k_j and $\tilde{a}^k_j$. Since the change from Lemma 1 must be symplectic, the sums $a^k_j + \tilde{a}^k_j, \ldots$ follow from the following formula:

$$\begin{aligned} \frac{\{Z_j, \tilde{Z}_j\}}{\{z_j, \tilde{z}_j\}} = 1 + 2\big(a^j_j + \tilde{a}^j_j\big) \|z_j\|^2 + \sum_{k\neq j} \left(a^k_j + \tilde{a}^k_j\right) \|z_k\|^2 \\ + \frac{1}{\kappa_j} \sum \left\|\alpha^j_m\right\|^2 \left\{\frac{z^m}{\tilde{z}_j}, \frac{z^{\tilde{m}}}{z_j}\right\}. \end{aligned}$$

(The complete calculation of all the coefficients uses other Poisson brackets, but we skip it.)

Lemma 4. *We have*

$$a_j^j + \tilde{a}_j^j = 2\left\|\alpha_{03}^{j;j}\right\|^2 - 2\left\|\alpha_{21}^{j;j}\right\|^2 + \frac{1}{2}\sum_{k\neq j}\frac{\kappa_k}{\kappa_j}\left(\left\|\alpha_{02;01}^{j;jk}\right\|^2 - \left\|\alpha_{02;10}^{j;jk}\right\|^2\right),$$

$$\begin{aligned} a_j^k + \tilde{a}_j^k = {} & \left\|\alpha_{02;01}^{j;jk}\right\|^2 + \left\|\alpha_{02;10}^{j;jk}\right\|^2 - 2\left\|\alpha_{11;10}^{j;jk}\right\|^2 \\ & + 4\frac{\kappa_k}{\kappa_j}\left(\left\|\alpha_{01;02}^{j;jk}\right\|^2 - \left\|\alpha_{01;20}^{j;jk}\right\|^2\right) \\ & - \sum_{l\neq j,k}\frac{\kappa_l}{\kappa_j}\Big(\left\|\alpha_{01;10;10}^{j;jkl}\right\|^2 + \left\|\alpha_{01;01;10}^{j;jkl}\right\|^2 \\ & \qquad - \left\|\alpha_{01;10;01}^{j;jkl}\right\|^2 - \left\|\alpha_{01;01;01}^{j;jkl}\right\|^2\Big), \end{aligned}$$

where $\alpha_{\rho\sigma}^{j;j} = \alpha_m^j$ for $m_j = \rho$, $\tilde{m}_j = \sigma$ and zero other indices. Similarly, $\alpha_{\rho\sigma;\varsigma\tau}^{j;jk}$ (resp., $\alpha_{\rho\sigma;\varsigma\tau;\upsilon\phi}^{j;jkl}$) denotes α_m^j with distinguished indices at the jth and kth places (resp., at the jth, kth and lth places).

Proof of Theorem 1. Using equations (2.10)–(2.13) we find contributions arising from different products $h_n h_m$. We begin with D_{11}.

The term

$$-\lambda_1\kappa_1 A_1 = -\lambda_1\kappa_1\left(\sum_{|m_1|=3}\left\|\alpha_m^1\right\|^2 + \left(a_1^1 + \tilde{a}_1^1\right)\right)$$

in (2.13) contains the terms

$$\begin{aligned} -\lambda_1\kappa_1\,(1+2)\left\|\alpha_{03;0\ldots0}^1\right\|^2 &= -3\lambda_1\kappa_1\left(\frac{\kappa_1}{\lambda_1}\right)^2\left\|h_{30;0\ldots0}\right\|^2 \\ &= -3\left(\frac{\kappa_1^3}{\lambda_1}\right)\left\|h_{30;0\ldots0}\right\|^2 \end{aligned}$$

and

$$\begin{aligned} & -\lambda_1\kappa_1\left[(1-2)\left\|\alpha_{21;0\ldots0}^1\right\|^2 + \left\|\alpha_{12;0\ldots0}^1\right\|^2\right] \\ & \quad = \lambda_1\kappa_1\left[\left(\frac{\kappa_1}{\lambda_1}\right)^2 - \left(\frac{2\kappa_1}{\lambda_1}\right)^2\right]\left\|h_{21;0\ldots0}\right\|^2 \\ & \quad = -3\left(\frac{\kappa_1^3}{\lambda_1}\right)\left\|h_{21;0\ldots0}\right\|^2. \end{aligned}$$

The term $-\lambda_2(\kappa_1^2/\kappa_2)C_2^{11}$ in (2.13) includes, in particular, the term

$$\begin{aligned} -\lambda_2\left(\frac{\kappa_1^2}{\kappa_2}\right)\left\|\alpha_{11;01\ldots}^2\right\|^2 &= -\lambda_2\left(\frac{\kappa_1^2}{\kappa_2}\right)\left(\frac{\kappa_2}{\lambda_2}\right)^2\left\|h_{11;10\ldots}\right\|^2 \\ &= -\left(\frac{\kappa_1^2\kappa_2}{\lambda_2}\right)\left\|h_{11;10\ldots}\right\|^2. \end{aligned}$$

The sum $-\lambda_1\kappa_1 A_1 - \lambda_2(\kappa_1^2/\kappa_2)C_2^{11}$ includes also the term

$$\begin{aligned} &-\lambda_1\kappa_1 \frac{1}{2}\frac{\kappa_2}{\kappa_1}\left\|\alpha^1_{02;01\ldots}\right\|^2 - \lambda_2\left(\frac{\kappa_1^2}{\kappa_2}\right)\left\|\alpha^2_{02;01\ldots}\right\|^2 \\ &\quad = -\frac{1}{2}\lambda_1\kappa_1\left(\frac{2\kappa_1}{2\lambda_1+\lambda_2}\right)^2\left\|h_{20;10\ldots}\right\|^2 \\ &\qquad - \lambda_2\left(\frac{\kappa_1^2}{\kappa_2}\right)\left(\frac{\kappa_2}{2\lambda_1+\lambda_2}\right)^2\left\|h_{20;10\ldots}\right\|^2 \\ &\quad = -\left(\frac{\kappa_1^2\kappa_2}{2\lambda_1+\lambda_2}\right)\left\|h_{20;10\ldots}\right\|^2; \end{aligned}$$

similarly we find the contribution to D_{11} which contains $\left\|h_{20;01\ldots}\right\|^2$.

Other terms in D_{11} arising from $\left\|h^{1l}_{\rho\sigma;01\ldots}\right\|^2$ are analogous to the terms with $l = 2$. The formulas for general D_{jj} are the same as for D_{11}, only with indices changed.

In a similar way the coefficients D_{jk} are computed. We omit this. □

Remark 1. For given j we can make the change $z_j \mapsto \tilde{z}_j$, $\tilde{z}_j \mapsto z_j$ which implies $\lambda_j \mapsto -\lambda_j$, $\kappa_j \mapsto -\kappa_j$ and which evidently leaves $H^{(2)}$ invariant. But it implies the changes $h_{\ldots;m_j\tilde{m}_j;\ldots} \mapsto h_{\ldots;\tilde{m}_j m_j;\ldots}$ and $J_j \mapsto -J_j$ (compare equation (2.4)). Therefore, we have the following implication:

$$\begin{aligned} \lambda_j \longmapsto -\lambda_j \Longrightarrow D_{jl} &\longmapsto -D_{jl}\ (l \neq j), \\ D_{kl} &\longmapsto D_{kl}\ \text{(otherwise)}. \end{aligned} \tag{2.14}$$

Analogously, for fixed $j \neq k$, we have the implication

$$\begin{aligned} \lambda_j \longleftrightarrow \lambda_k \Longrightarrow D_{jj} &\longleftrightarrow D_{kk}, \\ D_{kl} &\longleftrightarrow D_{jl}\ (l \neq j,k), \\ D_{lm} &\longleftrightarrow D_{lm}\ \text{(otherwise)}. \end{aligned} \tag{2.15}$$

These implications should be understood as some monodromy transformations. Namely, we can consider a big space $\mathcal{G}$ of germs of holomorphic Hamiltonian systems in $\left(\mathbb{C}^{2n},0\right)$ with the Hamilton function $G = G^{(2)} + \cdots$. There is a natural projection $\pi : \mathcal{G} \mapsto \mathcal{G}_2$ onto the space $\mathcal{G}_2 \simeq \mathbb{C}^{2n+1}$ of linear Hamiltonian systems. In $\mathcal{G}_2$ we can distinguish a hypersurface Σ (of codimension 1) consisting of the Hamiltonian systems for which the nonresonance condition (2.6) fails. The hypersurface Σ has four components: Σ_0 corresponding to a zero eigenvalue (when some $\lambda_j = 0$), Σ_1 corresponding to a $1:1$ resonance, Σ_2 corresponding to a $2:1$ resonance and Σ_3 corresponding to $3:1$ resonance (we skip components corresponding to higher order resonances). Of course, we have also corresponding *bifurcational hypersurfaces* $\pi^{-1}(\Sigma)$ and $\pi^{-1}(\Sigma_j)$ in $\mathcal{G}$.

The changes (2.14) and (2.15) describe the change of the Birkhoff normal form as we move along a small loop around $\pi^{-1}(\Sigma_0)$ (resp., around $\pi^{-1}(\Sigma_1)$). Such a loop lies at a small disc intersecting transversally $\pi^{-1}(\Sigma_0)$

(resp., $\pi^{-1}(\Sigma_1)$) at a point separated from other components of the bifurcational hypersurface $\pi^{-1}(\Sigma)$ and from its singular locus (which correspond to multiple vanishing of eigenvalues and to multiple resonances). Therefore, they are the *monodromy transformations* corresponding to such loops.

Remark 2. The coefficients D_{ij} are independent of the choice of the diagonalization coordinates, i.e., with respect to the action of the torus $(\mathbb{C}^*)^{2n}$:

$$z_j \longmapsto \mu_j z_j, \qquad \tilde{z}_j \longmapsto \tilde{\mu}_j \tilde{z}_j.$$

3. Bifurcations near resonant cases

In this section we consider Hamiltonian systems in $\mathbb{C}^4$ (two degrees of freedom) near a singular point with a low-degree resonance and their 1-parameter deformations. We assume that the Hamilton functions depend analytically on the (complex) coordinates and on a complex parameter. These are deformations of Hamiltonian systems from the bifurcational hypersurfaces $\pi^{-1}(\Sigma_0)$, $\pi^{-1}(\Sigma_1)$ and $\pi^{-1}(\Sigma_2)$ defined in Remark 1.

3.1. Jordan cell with zero eigenvalues

Here we consider the situation when $n = 2$ and

$$\lambda_2 = 0 \qquad \text{and} \qquad \lambda_1 \neq 0,$$

i.e., the $0:1$ *resonance*.

We have two possibilities: (i) the linear part, the matrix A, is diagonalizable, or (ii) the matrix A contains a 2-dimensional Jordan cell. Since the case (i) is rather straightforward and not needed for our aims, we assume the second possibility.

From [Arn2, Appendix 6] we learn that the quadratic part of the Hamilton function can be reduced to

$$H^{(2)}(z, \tilde{z}, x, y) = \frac{\mu}{\kappa} \|z\|^2 - \frac{1}{2} x^2, \tag{3.1}$$

where $\mu = \lambda_1$, $z = z_1$, $\tilde{z} = \tilde{z}_1$, $\kappa = \{z, \tilde{z}\}$ (as before) and x, y are coordinates with $\{x, y\} = 1$. Indeed, such a Hamiltonian is a limit of a family of generic Hamiltonians such that the 2-dimensional subspaces corresponding to two pairs of eigenvalues $\pm\lambda_1 \to \pm\mu$ and $\pm\lambda_2 \to 0$ are symplectic and skew orthogonal. Therefore, the (x, y)-plane which supports the Jordan cell is symplectic and skew orthogonal to the z-plane. (In the real case the sign before $\frac{1}{2}x^2$ is invariant with respect to real linear symplectic changes which preserve the form (3.1), but here we can put -1.)

Assume the following expansion of the Hamiltonian:

$$\begin{aligned} H = H^{(2)} &+ A_1 y^3 + A_2 y \|z\|^2 + A_3 y^4 + A_4 y^2 \|z\|^2 + A_5 \|z\|^4 \\ &+ \left(a x^3 + b x^2 y + c x y^2 + d x \|z\|^2 \right) \\ &+ z P + \tilde{z} \tilde{P} + z^2 Q + \tilde{z}^2 \tilde{Q} + R + \cdots, \end{aligned} \tag{3.2}$$

where $P = p_{20}x^2 + p_{11}xy + p_{02}y^2$, $\tilde{P} = \tilde{p}_{20}x^2 + \tilde{p}_{11}xy + \tilde{p}_{02}y^2$, $Q = q_1 x + q_2 y$,

$\tilde{Q} = \tilde{q}_1 x + \tilde{q}_2 y$, $R = \sum_{j+k=3} r_{jk} z^j \tilde{z}^k$ and the dots denote inessential quartic and higher order terms.

We apply the following symplectic change:

$$\begin{aligned} x &= X + aX^2 + bXY + cY^2 + d\,\|Z\|^2 + \cdots, \\ y &= Y - 2aXY - \frac{b}{2}Y^2 + \cdots, \\ z &= Z\left(1 + \gamma\,\|Z\|^2\right) - \frac{\kappa}{\mu}\tilde{P} - \frac{\kappa}{\mu}\tilde{Z}\tilde{Q} - S(Z,Z) + \cdots, \\ \tilde{z} &= \tilde{Z}\left(1 + \tilde{\gamma}\,\|Z\|^2\right) - \frac{\kappa}{\mu}P - \frac{\kappa}{\mu}ZQ - \tilde{S}(Z,Z) + \cdots, \end{aligned} \tag{3.3}$$

where γ, $\tilde{\gamma}$, S, $\tilde{S}$ are analogous like in Lemma 1 for $n = 1$. Note that the change $(X,Y) \mapsto (x,y)$ is a time 1 flow map g_F^1 generated by the Hamiltonian

$$F = aX^2Y + \frac{b}{2}XY^2 + \frac{c}{3}Y^3 + dY\,\|Z\|^2 + \cdots$$

(with the parameter $\|Z\|^2$) and the map $(Z,\tilde{Z}) \mapsto (z,\tilde{z})$ is an analogous map g_G^1 generated by the Hamiltonian

$$G = -\frac{1}{\mu}\left(2\,\mathrm{RE}(ZP) + \mathrm{RE}(Z^2Q)\right) + G_0(Z,\tilde{Z}).$$

After this change we arrive at the following analogue of the Birkhoff normal form:

$$\begin{aligned} H^{\mathrm{Nor}} = {} & \frac{\mu}{\kappa}\,\|Z\|^2 - \frac{1}{2}X^2 + B_1Y^3 + B_2Y\,\|Z\|^2 + B_3Y^4 \\ & + B_4Y^2\,\|Z\|^2 + B_5\,\|Z\|^4 + \cdots, \end{aligned} \tag{3.4}$$

where[3]

$$\begin{aligned} B_1 &= A_1, \quad B_2 = A_2, \\ B_3 &= A_3 + \frac{1}{2}c^2 - \frac{\kappa}{\mu}\,\|p_{02}\|^2 - \frac{3}{2}A_1 b - 2A_2\,\mathrm{RE}\,p_{02}, \\ B_4 &= A_4 + cd - 3\,\frac{\kappa}{\mu}\,\|q_2\|^2 - \frac{1}{2}A_2 b - 4\,\frac{\kappa}{\mu}\,\mathrm{RE}\,r_{21}p_{02}, \\ B_5 &= A_5 + \frac{1}{2}d^2 - 3\,\frac{\kappa}{\mu}\left(\|r_{03}\|^2 + \|r_{21}\|^2\right) \end{aligned} \tag{3.5}$$

and we adopt the same notations for $\|\cdot\|$ and RE as in the previous section. Here the last term in B_5 comes from Theorem 1 for $n = 1$.

Consider now a deformation H_ε, $\varepsilon \in (\mathbb{C}, 0)$, of the Hamiltonian (3.2) such that the origin $x = y = z = \tilde{z} = 0$ is critical[4] and for $\varepsilon \neq 0$ the corresponding matrix A_ε is nondegenerate. The natural deformation of the Jordan

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[3] Baider and Sanders [BaSa] proved that the unique normal form of a Hamiltonian with one degree of freedom and with nilpotent linear part of the corresponding differential system is $H = \pm\frac{1}{2}x^2 + a_k y^k + \sum a_l y^l$, where $a_k \neq 0$ and the sum runs over $l > k$ such that $l \neq -1$ (mod k).

[4] This assumption is restrictive, because it is possible to deform H in a way that the critical point splits into two (or more) singular points. But it is what we need in Section 4.

cell is $\left(\begin{smallmatrix} 0 & \varepsilon \\ -1 & 0 \end{smallmatrix}\right)$; this is achieved by some genericity assumption ($\frac{d}{d\varepsilon}\det A_\varepsilon|_{\varepsilon=0} \neq 0$) and eventual change of the parameter. In this case we can apply a symplectic change of the form (3.3) where the coefficients depend on ε and the terms of degree greater than or equal to 3 can be reduced to the form (3.4). Thus we arrive at the following proposition.

Proposition 1. *The family H_ε can be reduced to the following normal form:*

$$H_\varepsilon^{\mathrm{Nor}} = \frac{\mu}{\kappa}\|Z\|^2 - \frac{1}{2}\left(X^2 + \varepsilon Y^2\right) + B_1 Y^3 + B_2 Y\|Z\|^2 + B_3 Y^4 + B_4 Y^2 \|Z\|^2 + B_5 \|Z\|^4 + \cdots,$$

where the coefficients $\mu = \mu(\varepsilon)$, $\kappa = \kappa(\varepsilon)$, $B_j = B_j(\varepsilon)$ depend analytically on the parameter and $B_j(0)$ are given in equations (3.5).

Assume $\varepsilon \neq 0$. Let

$$Z_2 = X + i\sqrt{\varepsilon}\,Y, \qquad \tilde{Z}_2 = \tilde{Z}_2 = X - i\sqrt{\varepsilon}\,Y.$$

Thus we have

$$Y = \frac{Z_2 + \tilde{Z}_2}{2i\sqrt{\varepsilon}}, \tag{3.6}$$

$$\lambda_2 = i\sqrt{\varepsilon}, \qquad \kappa_2 = -2i\sqrt{\varepsilon}. \tag{3.7}$$

The quadratic part of the Hamiltonian $H_\varepsilon^{\mathrm{Nor}}$ equals

$$\frac{\mu}{\kappa}\|Z\|^2 + \frac{\lambda_2}{\kappa_2}\|Z_2\|^2 = \frac{\lambda_1}{\kappa_1}\|Z_1\|^2 + \frac{\lambda_2}{\kappa_2}\|Z_2\|^2 = \lambda_1 J_1 + \lambda_2 J_2.$$

The cubic and quartic terms come from the substitution of (3.7) to $H_\varepsilon^{\mathrm{Nor}}$.

After applying Theorem 1 we arrive at the Birkhoff normal form (2.8) with

$$\begin{aligned} D_{11} &= \kappa^2\left(-\frac{B_2^2}{2\varepsilon} + B_5\right), \\ D_{12} &= \frac{i\kappa\left(-3B_1B_2/\varepsilon + B_4\right)}{\sqrt{\varepsilon}}, \\ D_{22} &= \frac{15B_1^2}{4\varepsilon^2} - \frac{3B_3}{2\varepsilon}. \end{aligned} \tag{3.8}$$

These formulas imply the monodromy transformation (2.14): the loop in the parameter space is $\{\varepsilon = \varepsilon_0 e^{i\tau}, \tau \in [0, 2\pi]\}$ for some small $\varepsilon_0 > 0$.

3.2. Pair of Jordan cells with nonzero eigenvalues

Here we assume

$$\lambda_1 = \lambda_2 = \mu \neq 0$$

and that the matrix A is not diagonalizable. (The case with diagonal A is the same as in Section 2.)

From [Arn2] (see also [Mar, Dui, vdM]) we learn that, when the real matrix A has two imaginary eigenvalues $\pm i\omega$ of multiplicity 2 and A is not diagonalizable, then we can write $H^{(2)} = \pm\frac{1}{2}\left(q_1^2 + q_2^2\right) + \omega(q_1p_2 - q_2p_1)$, or

$H^{(2)} = \pm\frac{1}{2}\|v_1\|^2 + \frac{\omega}{2i}(\bar{v}_1 v_2 - v_1\bar{v}_2)$, where $v_1 = q_1 + iq_2$, $v_2 = p_1 + ip_2$ (with $\{v_j, \bar{v}_j\} = \{v_1, v_2\} = 0$ and $\{v_1, \bar{v}_2\} = 2$). (Like in the previous section, in the real case the sign before $\frac{1}{2}\|v_1\|^2$ is invariant under symplectic changes.) This suggests that we should take

$$H^{(2)} = \frac{1}{2}\|v_1\|^2 + \frac{\mu}{2}(v_1\tilde{v}_2 - \tilde{v}_1 v_2), \tag{3.9}$$

where the variables v_j, v_k obey the following Poisson brackets:

$$\{v_j, \tilde{v}_j\} = \{v_j, v_k\} = \{\tilde{v}_j, \tilde{v}_k\} = 0, \qquad \{v_1, \tilde{v}_2\} = \{\tilde{v}_1, v_2\} = 2. \tag{3.10}$$

Then we get the system

$$\frac{dv_1}{dt} = \mu v_1, \quad \frac{dv_2}{dt} = \mu v_2 - v_1, \quad \frac{d\tilde{v}_1}{dt} = -\mu\tilde{v}_1, \quad \frac{d\tilde{v}_2}{dt} = -\mu\tilde{v}_2 - \tilde{v}_1, \tag{3.11}$$

i.e., with two Jordan cells.

To obtain the form (3.9) one should treat this case as a limit of a generic case H_ε, with two distinct pairs $\pm\lambda_1(\varepsilon)$, $\pm\lambda_2(\varepsilon)$ such that $\lambda_{1,2}(\varepsilon) \to \mu$ as $\varepsilon \to 0$. Since the invariant 2-dimensional subspaces corresponding to the eigenvalues λ_1, λ_2 and $-\lambda_2, -\lambda_2$, respectively, are Lagrangian (with vanishing restricted symplectic form), this holds also for $\varepsilon = 0$. Thus $\{v_j, v_k\} = 0$. Now the condition

$$0 = \frac{d}{dt}\{v_1, \tilde{v}_2\} = \left\{\frac{dv_1}{dt}, \tilde{v}_2\right\} + \left\{v_1, \frac{d\tilde{v}_2}{dt}\right\}$$

(due to the Hamiltonian equations) implies $\{v_1, \tilde{v}_1\} = 0$. Thus $\{v_1, \tilde{v}_2\} \neq 0$ and $\{\tilde{v}_1, v_2\} \neq 0$ (by the nondegeneracy of the symplectic structure). We can normalize the variables in a way the second pair of equations (3.10) holds.

(Note also that $H^{(2)} = I_1 + \mu I_2$, where I_1, I_2 are commuting first integrals for the differential system.)

Let us consider the problem of normalizations of higher order terms. We use the method of generating function (see [Arn2]). The symplectic 2-form corresponding to the brackets (3.10) is

$$\frac{1}{2}dv_1 \wedge d\tilde{v}_2 + \frac{1}{2}d\tilde{v}_1 \wedge dv_2 = \frac{1}{2}d\left(v_1 d\tilde{v}_2 + \tilde{v}_1 dv_2\right).$$

If the change is $(v) \mapsto (V)$, then the 1-form $v_1 d\tilde{v}_2 + \tilde{v}_1 dv_2 - V_1 d\tilde{V}_2 - \tilde{V}_1 dV_2$ is closed, hence exact. It follows that there exists a generating function S such that $v_1 d\tilde{v}_2 + \tilde{v}_1 dv_2 + \tilde{V}_2 dV_1 + V_2 d\tilde{V}_1 = dS$. Here the generating function

$$S = V_1\tilde{v}_2 + \tilde{V}_1 v_2 + \sum s_m V_1^{m_1}\tilde{V}_1^{\tilde{m}_1} v_2^{m_2}\tilde{v}_2^{\tilde{m}_2} = V_1\tilde{v}_2 + \tilde{V}_1 v_2 + S_1$$

depends on $V_1, \tilde{V}_1, v_2, \tilde{v}_2$ and satisfies the equations

$$v_1 = \frac{\partial S}{\partial \tilde{v}_2} = V_1 + \sum \frac{\tilde{m}_2 s_m V_1^{m_1}\tilde{V}_1^{\tilde{m}_1} v_2^{m_2}\tilde{v}_2^{\tilde{m}_2}}{\tilde{v}_2},$$
$$\tilde{v}_1 = \tilde{V}_1 + \frac{\partial S_1}{\partial v_2}, \quad V_2 = v_2 + \frac{\partial S_1}{\partial \tilde{V}_1}, \quad \tilde{V}_2 = \tilde{v}_2 + \frac{\partial S_1}{\partial V_1}.$$

By construction this change is symplectic and the corresponding change in the Hamilton function is the following:

$$H(v) = H(V) + \sum s_m \left\{ \frac{1}{2}\left(\tilde{m}_2 \frac{\tilde{V}_1}{\tilde{V}_2} + m_2 \frac{V_1}{V_2} \right) - \frac{\mu}{2}(m_1 + m_2 - \tilde{m}_1 - \tilde{m}_2) \right\} V^m$$

plus nonlinear terms with respect to s_m's.

We see that the corresponding (homological) linear operator $\mathcal{H} : S_1 \mapsto H(V) - H(v)$ acting on the space $\mathcal{S}$ of homogeneous polynomials of fixed degree $|m| = m_1 + \tilde{m}_1 + m_2 + \tilde{m}_2$ is of the form: diagonal with the eigenvalues $-\frac{\mu}{2}(m_1 + m_2 - \tilde{m}_1 - \tilde{m}_2)$ plus upper triangular (with respect to suitable ordering of the multi-indices m). It is clear that for $d = 3$ this operator is invertible.

For $d = 4$ the subspace $\mathcal{S}_0 \subset \mathcal{S}$ consisting of s_m's with $m_1 + m_2 = \tilde{m}_1 + \tilde{m}_2 = 2$ (i.e., with zero eigenvalues of $\mathcal{H}$) is invariant (and on the complementary to $\mathcal{S}_0$ subspace the operator $\mathcal{H}$ is invertible). We have

$$\begin{aligned} 0 &\longmapsto 0 \cdot s_{00;22}, \\ s_{00;22} &\longmapsto s_{01;21} + s_{10;12}, \\ s_{01;21} &\longmapsto \frac{1}{2} s_{02;20} + s_{11;11}, \\ s_{10;12} &\longmapsto s_{11;11} + \frac{1}{2} s_{20;02}. \end{aligned}$$

These maps describe block operators between some subspaces of $\mathcal{S}_0$ with fixed $m_2 + \tilde{m}_2$ (the sum of these indices decreases by 1). The distinguished operators are not surjective, because of the dimension counting (other block operators are surjective). It is easy to see that the subspaces complementary to images of the distinguished block operators are generated by $\|V_2\|^4$, $\|V_2\|^2 (V_1\tilde{V}_2 - \tilde{V}_1 V_2) = \|V_2\|^2 I_2$ and $(V_1\tilde{V}_2 - \tilde{V}_1 V_2)^2 = I_2^2$, respectively.

This implies the following normal form:[5]

$$H^{\text{Nor}} = H^{(2)} + A_1 \|V_2\|^4 + A_2^2 \|V_2\|^2 I_2 + A_3 I_2^2 + \cdots . \tag{3.12}$$

Because the homological operator $\mathcal{H}$ is nondiagonal, the expressions for the coefficients A_j are quite complicated, so we do not provide corresponding formulas.

Consider now a 1-parameter deformation H_ε, $\varepsilon \in (\mathbb{C}, 0)$, of the above Hamiltonian. Under some genericity assumption the quadratic part can be transformed to the following form:

$$H_\varepsilon^{(2)} = \frac{1}{2}\left(\|v_1\|^2 - \varepsilon \|v_2\|^2 \right) + \frac{\mu}{2}(v_1\tilde{v}_2 - \tilde{v}_1 v_2), \tag{3.13}$$

[5] In [BaSa] it was proved that (in the case of imaginary eigenvalues) a unique normal form is $H = H_2 + f(I_2, \|V_2\|^2)$, where f is a formal power series. In [Mar, Ch. 4, Sect. 4] a slightly different normal form is given.

where $\mu = \mu(\varepsilon)$ may depend on the parameter. The genericity condition is expressed in terms of the characteristic polynomial

$$\det\left(\lambda I + A_\varepsilon\right) = \lambda^4 - 2a_2(\varepsilon)\lambda^2 + a_4(\varepsilon)$$

such that

$$\frac{d}{d\varepsilon}\left(a_2^2 - a_4\right)|_{\varepsilon=0} \neq 0.$$

(In the normal form (3.13) we have $a_2 = \mu^2 + \varepsilon$ and $a_4 = \left(\mu^2 - \varepsilon\right)^2$.)

Indeed, then system (3.12) becomes perturbed to

$$\frac{dv_1}{dt} = \mu v_1 - \varepsilon v_2, \quad \frac{dv_2}{dt} = \mu v_2 - v_1, \quad \frac{d\tilde{v}_1}{dt} = -\mu\tilde{v}_1 - \varepsilon\tilde{v}_2, \quad \frac{d\tilde{v}_2}{dt} = -\mu\tilde{v}_2 - \tilde{v}_1.$$

Assume $\varepsilon \neq 0$. In the variables

$$\begin{aligned} z_1 &= v_1 - \sqrt{\varepsilon}\, v_2, \qquad & \tilde{z}_1 &= \tilde{v}_1 + \sqrt{\varepsilon}\,\tilde{v}_2, \\ z_2 &= v_1 + \sqrt{\varepsilon}\, v_2, \qquad & \tilde{z}_2 &= \tilde{v}_1 - \sqrt{\varepsilon}\,\tilde{v}_2, \end{aligned} \tag{3.14}$$

the latter system is diagonalizable with the corresponding eigenvalues

$$\lambda_1 = \lambda_1(\varepsilon) = \mu + \sqrt{\varepsilon}, \quad -\lambda_1, \quad \lambda_2 = \mu - \sqrt{\varepsilon}, \quad -\lambda_2. \tag{3.15}$$

We note also that

$$\begin{aligned} &\{z_1, z_2\} = \{\tilde{z}_1, \tilde{z}_2\} = \{z_1, \tilde{z}_2\} = \{\tilde{z}_1, z_2\} = 0, \\ &\{z_1, \tilde{z}_1\} = \{z_2, \tilde{z}_2\} = 2\sqrt{\varepsilon}. \end{aligned} \tag{3.16}$$

Like in the previous section, we find that the above reduction of cubic and quartic terms for $\varepsilon = 0$ can be extended to the case $\varepsilon \neq 0$ (but small). We arrive at the following proposition.

Proposition 2. *The family H_ε can be reduced to the following normal form:*

$$H_\varepsilon^{\mathrm{Nor}} = H_\varepsilon^{(2)} + A_1\left\|V_2\right\|^4 + A_2\left\|V_2\right\|^2 I_2 + A_3 I_2^2 + \cdots,$$

where the constants $A_j = A_j(\varepsilon)$ depend analytically on the parameter.

Let us reduce the Hamiltonian from Proposition 2 for $\varepsilon \neq 0$ to the Birkhoff normal form. From equations (3.14) and (3.16) (with capital V and Z) we find

$$\left\|V_2\right\|^2 = -\frac{1}{4\varepsilon}\left\|Z_1 - Z_2\right\|^2, \qquad I_2 = \frac{1}{2\sqrt{\varepsilon}}\left(\left\|Z_1\right\|^2 - \left\|Z_1\right\|^2\right).$$

It follows that

$$\begin{aligned} D_{11} &= \frac{A_1}{4\varepsilon} - \frac{A_2}{\sqrt{\varepsilon}} + 4A_3, \\ D_{12} &= \frac{3A_1}{2\varepsilon} - 8A_3, \\ D_{22} &= \frac{A_1}{4\varepsilon} + \frac{A_2}{\sqrt{\varepsilon}} + 4A_3 \end{aligned} \tag{3.17}$$

in the Birkhoff normal form (2.8). As before these formulas agree with the monodromy transformation (2.15), where the loop in the parameter space is $\{\varepsilon = \varepsilon_0 e^{i\tau}, \tau \in [0, 2\pi]\}$.[6]

3.3. Deformation of the 2 : 1 resonance

Here we assume that for $\varepsilon = 0$ we have

$$\lambda_1 = 2\lambda_2 = 2\mu \neq 0.$$

This case is easy and we formulate only the final result. (More detailed analysis of this case with examples is given in [BHLV].)

Proposition 3. *The family H_ε can be reduced to the following normal form:*

$$\begin{aligned} H_\varepsilon^{\text{Nor}} = {} & \frac{2\mu + \varepsilon}{\kappa_1} \|Z_1\|^2 + \frac{\mu}{\kappa_2} \|Z_2\|^2 + A_1 Z_1 \tilde{Z}_2^2 + A_2 \tilde{Z}_1 Z_2^2 \\ & + A_3 \|Z_1\|^4 + A_4 \|Z_1\|^2 \|Z_2\|^2 + A_5 \|Z_2\|^4 + \cdots , \end{aligned}$$

where the constants $A_j = A_j(\varepsilon)$ and $\kappa_j = \kappa_j(\varepsilon)$ depend on the parameter.

For $\varepsilon \neq 0$ we can reduce $H_\varepsilon^{\text{Nor}}$ to the Birkhoff normal form with

$$\begin{aligned} D_{11} &= \kappa_1^2 A_3, \\ D_{12} &= \frac{4\kappa_1\kappa_2^2 A_1 A_2}{\varepsilon} + \kappa_1\kappa_2 A_4, \\ D_{22} &= -\frac{\kappa_1\kappa_2^2 A_1 A_2}{\varepsilon} + \kappa_2^2 A_5. \end{aligned} \tag{3.18}$$

4. The restricted three-body problem

4.1. The Hamiltonian and the KAM theory

In the restricted three-body problem (see [Mar]) one deals with the Hamiltonian (expressed in local coordinates q_1, q_2 near Lagrangian libration point and corresponding momenta p_1, p_2)

$$G = G(q_1, p_1, q_2, p_2; \zeta) = G^{(2)} + G^{(3)} + \cdots ,$$

where

$$\begin{aligned} G^{(2)} &= \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + p_1 q_2 - p_2 q_1 + \frac{1}{8} q_1^2 - \zeta q_1 q_2 - \frac{5}{8} q_2^2, \\ G^{(3)} &= -\frac{7\sqrt{3}\,\zeta}{36} q_1^3 + \frac{3\sqrt{3}}{16} q_1^2 q_2 + \frac{11\sqrt{3}\,\zeta}{12} q_1 q_2^2 + \frac{3\sqrt{3}}{16} q_2^3, \\ G^{(4)} &= \frac{37}{128} q_1^4 + \frac{25\zeta}{24} q_1^3 q_2 - \frac{123}{64} q_1^2 q_2^2 - \frac{15\zeta}{8} q_1 q_2^3 - \frac{3}{128} q_2^4 \end{aligned} \tag{4.1}$$

[6]Duistermaat [Dui] also considered monodromy transformations in this situation, but of different kind. He considered projection π from $\mathbb{R}^4$ to $\mathbb{R}^2$ corresponding to taking the pair $(S, G) = (iI_2, I_1 + \varepsilon \|V_2^2\| + A_1 \|V_2\|^4)$. He studied the change of the topology of the fiber $\pi^{-1}(S, G)$ as the point (S, G) varies along a loop in the in the base $\mathbb{R}^2$.

and $\zeta = 3\sqrt{3}(1-2\mu)/4$ is a parameter (related with the ratio $\mu/(1-\mu)$) of the masses of heavy bodies, like Jupiter and Sun. The matrix of the linear part of the differential system equals

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1/4 & 0 & \zeta & 1 \\ -1 & 0 & 0 & 1 \\ \zeta & -1 & 5/4 & 0 \end{pmatrix}. \tag{4.2}$$

Its characteristic polynomial and the eigenvalues are the following:

$$P(\lambda,\zeta) = \lambda^4 + \lambda^2 + \frac{27}{16} - \zeta^2, \tag{4.3}$$

$$\pm\lambda_{1,2} = \pm\sqrt{-\frac{1 \pm \sqrt{4\zeta^2 - 23/4}}{2}} = \pm\sqrt{-\frac{1 \pm \sqrt{\Delta}}{2}}. \tag{4.4}$$

Here equation (4.4), or $P(\lambda,\zeta) = 0$, defines an algebraic function

$$\mathbb{C} \ni \zeta \longmapsto \lambda(\zeta) \in \mathbb{C}$$

which has four sheets and four ramification points at

$$\pm\zeta_0 = \pm\frac{\sqrt{27}}{4} \qquad \text{and} \qquad \pm\,\zeta_1 = \pm\frac{\sqrt{23}}{4}; \tag{4.5}$$

they correspond to $\lambda_2 = 0$ and $\lambda_1 = \lambda_2$, respectively. Of course, λ depends only on ζ^2. When ζ^2 approaches ζ_0^2, the eigenvalues $\lambda_1(\zeta)$ and $-\lambda_1(\zeta)$ approach the value $\lambda = 0$ and exchange their positions as ζ^2 makes a full turn around ζ_0^2. When ζ^2 approaches ζ_1^2, the eigenvalues $\lambda_1(\zeta)$ and $\lambda_2(\zeta)$ (resp., $\lambda_3(\zeta)$ and $\lambda_4(\zeta)$) approach the value $\lambda = \sqrt{-1/2}$ (resp., $\lambda = -\sqrt{-1/2}$) and exchange their positions as ζ^2 makes a full turn around ζ_1^2. We note also that the 2 : 1 and 3 : 1 resonances correspond respectively to

$$\zeta_2^2 = \frac{611}{400} \qquad \text{and} \qquad \zeta_3^2 = \frac{639}{400}. \tag{4.6}$$

The corresponding diagonalizing coordinates (z) can be chosen as follows:

$$\begin{aligned} q_1 &= \sum_{j=1}^{2} 2\,\mathrm{RE}\,(\zeta + 2\lambda_j)\, z_j, & p_1 &= \sum_{j=1}^{2} 2\,\mathrm{RE}\left(\frac{3}{4} + \zeta\lambda_j + \lambda_j^2\right) z_j, \\ q_2 &= \sum_{j=1}^{2} 2\,\mathrm{RE}\left(\lambda_j^2 - \frac{3}{4}\right) z_j, & p_2 &= \sum_{j=1}^{2} 2\,\mathrm{RE}\left(\zeta + \frac{5\lambda_j}{4} + \lambda_j^3\right) z_j, \\ \kappa_1 &= -\left[2\lambda_1\left(\frac{3}{4} - \lambda_1^2\right)\sqrt{\Delta}\right]^{-1}, & \kappa_2 &= \left[2\lambda_2\left(\frac{3}{4} - \lambda_2^2\right)\sqrt{\Delta}\right]^{-1}, \end{aligned} \tag{4.7}$$

where RE is understood like in the previous section with the agreement that $\tilde{\lambda}_j = -\lambda_j$, $\tilde{\zeta} = \zeta$ and $\sqrt{\Delta} = \lambda_2^2 - \lambda_1^2$.[7]

When $\zeta^2 \neq \zeta_j^2$, $j = 0, 1, 2, 3$, we can reduce the Hamiltonian G to the Birkhoff normal form (2.8). In the study of the Lyapunov stability of the three-body problem, the following *isoenergetic degeneracy determinant* is important:[8]

$$\det = \begin{vmatrix} \partial^2 H^{\mathrm{Bir}}/\partial J_j \partial J_k & \partial H^{\mathrm{Bir}}/\partial J_k \\ \partial H^{\mathrm{Bir}}/\partial J_j & 0 \end{vmatrix} = -2\lambda_2^2 D_{11} + 2\lambda_1\lambda_2 D_{12} - 2\lambda_1^2 D_{22} \tag{4.8}$$

evaluated at $J_1 = J_2 = 0$.

(When $23/16 < \zeta^2 < 27/16$, the linear part of the system has purely imaginary eigenvalues $\pm\lambda_1 = \mp i\omega_1$ and $\pm\lambda_2 = \mp i\omega_2$, where $\omega_{1,2} > 0$, and we have $H_0 + H_1$, where $H_0 = \omega_1 I_1 - \omega_2 I_2 - D_{11} I_1^2 + D_{12} I_1 I_2 - D_{22} I_2^2$, H_1 is a perturbation and $I_{1,2} = \mp i J_{1,2}$. Therefore, the corresponding linear system is Lyapunov stable. In order to prove the genuine Lyapunov stability one uses the KAM theory (see [Mar]). That theory requires that the frequencies $\omega_1(I) = \partial H_0/\partial I_1$ and $-\omega_2(I) = \partial H_0/\partial I_2$ vary regularly at the level hypersurfaces $\{H_0 = \mathrm{const}\}$ of the unperturbed completely integrable Hamiltonian H_0. This is the nondegeneracy condition $\det \neq 0$.)

Theorem 2 (See [DDB]**).** *In the restricted three-body problem we have*

$$\det(\zeta) = \frac{36 - 541\lambda_1^2\lambda_2^2 + 644\lambda_1^4\lambda_2^4}{-8\left(1 - 4\lambda_1^2\lambda_2^2\right)\left(4 - 25\lambda_1^2\lambda_2^2\right)}$$

$$= \frac{41\,216\zeta^4 - 104\,480\zeta^2 + 61\,245}{-8\left(16\zeta^2 - 23\right)\left(400\zeta^2 - 611\right)}.$$

Our aim is to calculate $\det = \det(\zeta)$ as a function of ζ using only analytic properties of the Birkhoff normal form and bifurcations near singular points of the function $\lambda(\zeta)$.

4.2. Symmetries

Due to the formulas from Theorem 1 we see that $\det(\zeta)$ is an algebraic function of $\zeta \in \mathbb{C}$ with possible singular points at $\pm\zeta_0$, $\pm\zeta_1$, $\pm\zeta_2$ and at ∞. But we can say more.

Lemma 5. *The function* $\det(\zeta)$ *is a rational function which depends only on* ζ^2 *with possible poles at* ζ_0^2 *(of order at most* 2*),* ζ_1^2 *(of order at most* 1*),* ζ_2^2 *(of order at most* 1*) and at* $\zeta^2 = \infty$*.*

---

[7]These coordinates are directly related with the change used in [BaZo]. In [Mar, Pro] other choices are taken: in particular, following [Pro] one can take $q_1 = \sum 2\,\mathrm{RE}(4\lambda_j^2 - 9)z_j$, $q_2 = \sum 2\,\mathrm{RE}(4\zeta - 8\lambda_j)z_j$, $p_1 = \sum 2\,\mathrm{RE}(4\lambda_j^3 - \lambda_j - 4\zeta)z_j$, $p_2 = \sum 2\,\mathrm{RE}(4\lambda_j^2 - 4\zeta\lambda_j + 9)z_j$.

[8]In [DDB] the determinant $\det(\zeta)$ is denoted by D, in [BaZo] we used the quantity $\Gamma = \det(\zeta)/8$ whereas in [Mar, Pro] the authors use $-\det(\zeta)/2$.

Proof. Firstly, we have to show that the function $\det(\zeta)$ is single valued near the points $\pm\zeta_0$, $\pm\zeta_1$, $\pm\zeta_2$; the orders of the corresponding poles follow from equations (3.8), (3.17) and (3.18). Indeed, then also the monodromy of $\det(\zeta)$ generated by a loop around $\zeta=\infty$ must be trivial. Finally, $\det(\zeta)$ has a monomial growth (as an algebraic function).

By the results of Section 3.3 only the first two cases are under question.

The family G_ζ of Hamiltonians defines a complex curve in the space $\mathcal{G}$ of Hamiltonians defined in Remark 1. This family meets the bifurcational surfaces $\pi^{-1}(\Sigma_0)$, $\pi^{-1}(\Sigma_1)$ and $\pi^{-1}(\Sigma_2)$ at the corresponding points $\pm\zeta_0$, $\pm\zeta_1$ and $\pm\zeta_2$.

The monodromy map $\mathcal{M}_0$ generated by a loop around ζ_0 corresponds to the change $\lambda_1 \leftrightarrow \tilde{\lambda}_1 = -\lambda_1$ and the changes (2.14) in the coefficients D_{jk}. Of course, $\det(\zeta)$ is invariant under such change and the same is true in the case the point $-\zeta_0$ is surrounded.

The monodromy $\mathcal{M}_1$ generated by a loop around ζ_1 corresponds to the changes $\lambda_1 \leftrightarrow \lambda_2$, $\tilde{\lambda}_1 \leftrightarrow \tilde{\lambda}_2$ and the changes (2.15). Also here $\det(\zeta)$ remains invariant.

To prove that the function $\det(\zeta)$ is even, we note that the low order terms (in fact, all terms) of G_ζ are invariant under the change

$$\zeta \longmapsto -\zeta, \quad (q_1,p_1,q_2,p_2) \longmapsto (-q_1,p_1,q_2,-p_2).$$

The second change corresponds to the composition of changes:

$$\lambda_j \longleftrightarrow -\lambda_j, \quad z_j \longleftrightarrow \tilde{z}_j \quad (j=1,2)$$

and

$$z_j \longmapsto -z_j, \quad \tilde{z}_j \longmapsto -\tilde{z}_j \quad (j=1,2)\,.$$

The first change is induced by the transformation $\mathcal{M}_0 \circ \mathcal{M}_1^{-1}\mathcal{M}_0\mathcal{M}_1$, under which $\det(\zeta)$ is invariant, and the invariance of $\det(\zeta)$ under the second change follows from Remark 2. □

4.3. Asymptotic at infinity

Here we assume that

$$\zeta \longrightarrow \infty.$$

Let us apply the following symplectic normalization:

$$(q_1,q_2) = \zeta^{-1/4}\,(Q_1,Q_2)\,, \qquad (p_1,p_2) = \zeta^{1/4}\,(P_1,P_2)\,.$$

Then we get

$$G^{(2)} = \zeta^{1/2}\left\{\frac{P_1^2+P_2^2}{2} - Q_1Q_2\right\} + O(1) = \zeta^{1/2}G_0^{(2)} + \cdots\,,$$

$$G^{(3)} = \zeta^{1/4}G_0^{(3)} + \cdots\,,$$

$$G^{(4)} = G_0^{(4)} + \cdots\,.$$

The reduction of $G_0^{(2)}$ to the normal form (2.3) is the following:

$$Q_1 = z_1+\tilde{z}_1+z_2+\tilde{z}_2, \qquad Q_2 = z_2+\tilde{z}_2-z_1-\tilde{z}_1,$$
$$P_1 = z_2-\tilde{z}_2 - i\,(z_1-\tilde{z}_1)\,, \qquad P_2 = z_2-\tilde{z}_2 + i\,(z_1-\tilde{z}_1)\,,$$

$$H^{(2)} = 4\zeta^{1/2}\left(z_1\tilde{z}_1 - z_2\tilde{z}_2\right),$$

$$\kappa_1 = -1/4, \quad \kappa_2 = -i/4, \qquad \lambda_1 = -i\zeta^{1/2}, \quad \lambda_2 = \zeta^{1/2}.$$

It is not difficult to see that the leading part of $\det(\zeta)$ is proportional to ζ^1 and the coefficient before ζ is calculated using the part

$$\zeta^{1/2}G_0^{(2)} + \zeta^{1/4}G_0^{(3)} + G_0^{(4)}$$

of the Hamiltonian. But we know that $\det(\zeta)$ is an even function. So this coefficient must equal zero. Therefore,

$$\det(\zeta) \longrightarrow \det(\infty) \neq \infty \quad \text{as } \zeta \longrightarrow \infty. \tag{4.9}$$

This property is also confirmed by direct calculations.

We do not compute here the value $\det(\infty)$, because it is rather involved.

4.4. Asymptotic at the 0 : 1 resonance

For $\zeta = 3\sqrt{3}/4$ we find the following change of variables leading to the quadratic Hamiltonian of the form (3.1):

$$q_1 = -\frac{x}{2\sqrt{3}} + \frac{3y}{2} - 2\,\mathrm{RE}\,\big(3\sqrt{3} - 8i\big)z,$$

$$p_1 = \frac{\sqrt{3}y}{2} + 2\,\mathrm{RE}\,\big(1 + 3i\sqrt{3}\big)z,$$

$$q_2 = \frac{3x}{2} - \frac{\sqrt{3}y}{2} + 7z + 7\tilde{z},$$

$$p_2 = -\frac{2x}{\sqrt{3}} + \frac{3y}{2} - 2\,\mathrm{RE}\,\big(3\sqrt{3} - i\big)z,$$

where $\tilde{w} = w$ for a complex number w. Moreover,

$$\mu = -i, \qquad \kappa = -\frac{i}{56}.$$

Substituting into $G^{(3)}$ and $G^{(4)}$ we find

$$a = -\frac{\sqrt{3}}{9}, \quad b = 6, \quad c = -3\sqrt{3}, \quad d = -112\sqrt{3},$$
$$P = \big(-11\sqrt{3} + 34i\big)x^2 + \big(66 - 12i\sqrt{3}\big)xy - \big(9\sqrt{3} + 18i\big)y^2,$$
$$Q = \big(-120\sqrt{3} + 264i\big)x + \big(288 + 24i\sqrt{3}\big)y,$$
$$R = 2\,\mathrm{RE}\left\{\big(-504\sqrt{3} + 560i\big)z^3 - \big(168\sqrt{3} + 366i\big)z^2\tilde{z}\right\},$$
$$A_1 = A_2 = 0, \quad A_3 = -\frac{27}{8}, \quad A_4 = -1008, \quad A_5 = -14112$$

in equation (3.2). The perturbation parameter is

$$\varepsilon = \frac{27}{16} - \zeta^2.$$

Since $A_j(\varepsilon)$ are analytic in ε, we have $B_{1,2} = A_{1,2}(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. Also calculation of B_3 in (3.5) shows that $B_3 = 0$ for $\varepsilon = 0$. Therefore, equations (3.8) give

$$D_{22} = O\left(\frac{1}{\varepsilon}\right), \quad D_{12} = O\left(\frac{1}{\sqrt{\varepsilon}}\right), \quad D_{11} = O(1).$$

Since $\lambda_1 = O(1)$ and $\lambda_2 = O(\sqrt{\varepsilon})$, equation (4.8) implies that

$$\det(\zeta) \longrightarrow \det(\zeta_0) \neq \infty \quad \text{as } \zeta \longrightarrow \pm\zeta_0. \tag{4.10}$$

Like in the previous case we omit the calculation of the constant $\det(\zeta_0)$.

4.5. Asymptotic at the 1 : 1 resonance

Here we only check the "transversality" of the deformation relying on changing $\zeta_1^2 = 23/16$ to $\zeta^2 = 23/16 + \epsilon$. Then from (3.17) and (4.8) it follows that $D(\zeta)$ has simple poles at $\zeta = \pm\zeta_1$:

$$\det(\zeta) \sim \text{const}\left(\zeta^2 - \zeta_1^2\right)^{-1}. \tag{4.11}$$

Firstly, one has to check the behavior of the discriminant of the characteristic polynomial in (4.3), but this is obvious. Secondly, one has to check that for $\epsilon = 0$ the eigenspace corresponding to the double eigenvalue $\lambda_1 = \lambda_2 = \sqrt{-1/2}$ is one dimensional. This is easy and we refer the reader to [vdM].

Also we do not compute here the constant in (4.11); in fact, such calculation (using Theorem 1) was done in the master's thesis of Wiliński [Wil] for ζ close to ζ_2.

4.6. Asymptotic at the 2 : 1 resonance

For $\zeta = \zeta_2 = \sqrt{611}/20$ we have $\sqrt{\Delta} = 3/5$, $\lambda_1 = -2i/\sqrt{5}$, $\lambda_2 = \mu = -i/\sqrt{5}$. Next we use equation (4.7). Thus we have

$$\begin{aligned} q_1 &= \left(\frac{\sqrt{611}}{20} - \frac{4i}{\sqrt{5}}\right) z_1 + \left(\frac{\sqrt{611}}{20} + \frac{2i}{\sqrt{5}}\right) \tilde{z}_2 + \cdots, \\ q_2 &= -\frac{31}{20} z_1 - \frac{19}{20} \tilde{z}_2 + \cdots, \\ \kappa_1 &= -\frac{25i\sqrt{5}}{93}, \qquad \kappa_2 = \frac{50i\sqrt{5}}{57}, \end{aligned}$$

where we have skipped the terms with $\tilde{z}_1$ and z_2. Substituting it into $G^{(3)}$, expanding and collecting terms, we find the following two resonant terms $h_{10;02} z_1 \tilde{z}_2^2$ and $\bar{h}_{10;02} \tilde{z}_1 z_2$ with

$$h_{10;02} = A_1 = \bar{A}_2 = \frac{\sqrt{3}\left(-20\,226 - 204i\sqrt{5 \cdot 611}\right)}{90\,000}.$$

Moreover, for $\zeta^2 - \zeta_2^2$ small we have

$$\varepsilon = \lambda_1 - 2\lambda_2 = \frac{125i(\zeta^2 - \zeta_2^2)}{12\sqrt{5}} + \cdots.$$

Now from (3.18) and (4.8) we obtain

$$\det(\zeta) \sim 24\mu^2\kappa_1\kappa_2^2 \left|A_1\right|^2 \varepsilon^{-1} \sim -\frac{2662}{5625}\left(\zeta^2 - \zeta_2^2\right)^{-1}. \tag{4.12}$$

This agrees with the Deprits formula.

4.7. Calculations for three special values of the parameter

The first value is taken as $\zeta_3 = \sqrt{639}/20$, i.e., corresponding to the 3 : 1 resonance. We have

$$\begin{aligned}
&\sqrt{\Delta} = \frac{4}{5}, \quad \lambda_1 = -\frac{3i}{\sqrt{10}}, \quad \lambda_2 = -\frac{i}{\sqrt{10}\,i}, \quad (\lambda_1\lambda_2)^2 = \frac{9}{100}, \\
&\kappa_1 = -\frac{25}{99}i\sqrt{\frac{5}{2}}, \quad \kappa_2 = \frac{25}{17}i\sqrt{\frac{5}{2}}, \\
&q_1 = 2\,\mathrm{RE}\left(\frac{\sqrt{639}}{20} - 3i\sqrt{\frac{2}{5}}\right) z_1 + 2\,\mathrm{RE}\left(\frac{\sqrt{639}}{20} - i\sqrt{\frac{2}{5}}\right) z_2, \\
&q_2 = -2\,\mathrm{RE}\,\frac{33}{20} z_1 - 2\,\mathrm{RE}\,\frac{17}{20} z_2
\end{aligned}$$

(see equations (4.7)). The calculations give

$$D_{11} = -\frac{309}{2240}, \quad D_{12} = -\frac{1219}{560}, \quad D_{22} = -\frac{79}{320}$$

and finally

$$\det(\zeta_3) = \frac{4671}{5600},$$

like in the Deprits formula.

The other two values correspond to the 4 : 1 resonance and the 3 : 2 resonance, i.e., $\zeta_4 = \sqrt{7547}/68$ (with $\lambda_1 = -4i/\sqrt{17}$, $\lambda_2 = -i/\sqrt{17} - i$, $(\lambda_1\lambda_2)^2 = 16/289$) and $\zeta_5 = 3\sqrt{443}/52$ (with $\lambda_1 = -3i/\sqrt{13}$, $\lambda_2 = -2i/\sqrt{13}$, $(\lambda_1\lambda_2)^2 = 36/169$). The calculations give

$$\det(\zeta_4) = -\frac{167\,509}{340\,200} \qquad \text{and} \qquad \det(\zeta_5) = -\frac{89\,289}{2800},$$

in agreement with equation (4.9).

Remark 3. In [BaZo] the isoenergetic degeneracy determinant $\det(\zeta)$ was studied near the points $\zeta = \infty$, $\zeta = \zeta_0$ and $\zeta = \zeta_2$. In the third case the difference between our asymptotic and equation (4.12) relied only on another definition of this determinant (see Note 8). In the first case we have found $\det(\zeta) \to \infty$ (compare equation (4.9)), because we have made improper choice of branches of some multivalued functions (like $\lambda(\zeta)$). In the case $\zeta \to \zeta_0$ we have committed a mistake in calculations and obtained $\det(\zeta) \to \infty$ (compare equation (4.10)); the same wrong asymptotic was given in [Leo].

4.8. The speciality of the restricted three-body problem

The above analysis indicates that the 1-parameter family of Hamiltonians associated with the restricted three-body problem is somewhat special. This speciality is related with the way it meets the bifurcational values of the parameter ζ. Whereas the bifurcations at $\zeta = \infty$, $\zeta = \zeta_3$, $\zeta = \zeta_2$ and at $\zeta = \zeta_1$ are of generic type, the bifurcation at $\zeta = \zeta_0$ is highly degenerate. Three coefficients in the normal form (3.4) vanish: B_1, B_2 and B_3. This explains the astonishing simplicity of the Deprits formula.

There are works devoted to generalization of the restricted three-body problem, like the $(N+1)$-body problems where a configuration of N "heavy" bodies forms a special central planar configuration and the "light" body moves in the gravitational field formed by the heavy bodies (in works of Grebenikov and his students [GKJ]). Special case (studied numerically by Prokopenya [Pro]) is when the $N = 3$ heavy bodies form the triangular Lagrange configuration. These models contain several parameters. It would be interesting to study bifurcations of resonant singular points in the spirit it is done in our paper.

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