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Degree sum conditions for hamiltonian index

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Abstract. In this note, we show a sharp lower bound of $\min\{\sum_{i=1}^{k} d_G(u_i) : u_1u_2\ldots u_k$ is a path of (2-)connected $G\}$ on its order such that (k-1)-iterated line graphs $L^{k-1}(G)$ are hamiltonian.

§1 Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only. Let $G = (V(G), E(G))$ be a connected graph and u be a vertex of G. We use $N_G(u)$ to denote the set of vertices which are adjacent with u (also called the *neighbors* of u) in the graph G. $d_G(u) = |N_G(u)|$ is the degree of u in G. Let S be a subset of $V(G)$(or $E(G)$). The *induced subgraph* of G is denoted by $G[S]$. We use K_n to denote the complete graph of order n. The *clique* C is a subset of $V(G)$ such that $G[C]$ is a complete graph.

The *line graph* $L(G)$ of $G = (V(G), E(G))$ has $E(G)$ as its vertex set, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges share a common end vertex in G. The *m-iterated line graph* $L^m(G)$ is defined recursively by $L^0(G) = G, L^1(G) = L(G)$ and $L^m(G) = L(L^{m-1}(G))$. The *hamiltonian index* of a graph G, denoted by $h(G)$, is the smallest integer m such that $L^m(G)$ is hamiltonian, i.e., it has a spanning cycle.

Chartrand [5] showed that the hamiltonian index for any graph other than a path always exists and that $L(G)$ of a hamiltonian graph G is hamiltonian. For a connected graph that is not a path, Ryjáček, Woeginger and Xiong [8] showed that the problem to decide whether the hamiltonian index of a given graph is less than or equal to a given constant is NP-complete.

Saražin [9] showed that $h(G) \le n - \Delta(G)$ if G is connected graph of order n, later, Xiong [11] improved this result and showed that $h(G) \le diam(G) - 1$ if G is a connected graph other than a path since $diam(G) - 1 \le n - \Delta(G)$, where $diam(G)$ denotes the diameter of a graph G. For its other sharp upper bounds and stability, see [4] and [12], and [14], respectively, while its sharp lower bound is also gave in [12]; in [15], you may see its survey paper.

Let $P \subseteq G$ be a path of order $k \ge 1$. By $d_G(P)$, we denote the degree of a path P. That is, $d_G(P) = d_G(v_1) + d_G(v_1) + \cdots + d_G(v_k)$, where $V(P) = \{v_1, v_2, \cdots, v_k\}$. By $\bar{\sigma}_k(G)$,

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we denote $\min\{d_G(P) : P$ is a path of G with $|V(P)| = k\}$. Obviously $\delta(G) = \bar{\sigma}_1(G)$ and $\delta(L(G)) = \bar{\sigma}_2(G) - 2$ for every nonempty graph G.

Dirac [7] showed a very famous result that every graph $G = L^0(G)$ of order n with $\delta(G) = \bar{\sigma}_1(G) \geq \frac{n}{2}$ is hamiltonian, while Brualdi and Shanny [3] gave a similar result on $L(G) = L^1(G)$ involving $\bar{\sigma}_2(G)$, which was later improved slightly by Clark [6] for graphs with large order.

Theorem 1 (Brualdi and Shanny, [3]). If G is a graph of order $n \geq 4$ and at least one edge such that $\bar{\sigma}_2(G) > n$, then $L(G)$ is hamiltonian.

Theorem 2 (Clark, [6]). If G is a connected graph of order $n \geq 6$ and if

$$\bar{\sigma}_2(G) \geq \begin{cases} n-1, & \text{if } n \text{ is even} \\ n-2, & \text{if } n \text{ is odd} \end{cases},$$

then $L(G)$ is hamiltonian.

For almost bridgeless graphs (i.e., graphs in which every cut edge is incident with vertex of degree one), Veldman improved the above result to the following theorem which settled a conjecture in [1].

Theorem 3 (Veldman, [10]). Let G be a connected almost bridgeless graph of sufficiently large order n such that $\bar{\sigma}_2(G) > 2(\lfloor \frac{n}{5} \rfloor - 1)$, then $L(G)$ is hamiltonian.

In this paper, we consider similar sufficient conditions for m-iterated line graphs $L^m(G)$ to be hamiltonian for $m \geq 2$ and get the following main results.

Theorem 4. Let $k \geq 3$ be an integer and let G be a connected graph of order $n > k+2$ such that $\bar{\sigma}_k(G) > n + k - 3$, then $L^{k-1}(G)$ is hamiltonian, i.e., $h(G) \leq k - 1$.

Theorem 5. Let $k \geq 3$ be an integer and let G be a 2-connected graph of order $n > 6k+3$ such that $\bar{\sigma}_k(G) > \frac{2n}{5} - \frac{2k}{5} - \frac{1}{5}$, then $L^{k-1}(G)$ is hamiltonian, i.e., $h(G) \leq k - 1$.

§2 Preliminaries

Let G be a graph, define $V_i(G) = \{v \in V(G) : d_G(v) = i\}$ and $V_{\geq i}(G) = \{v \in V(G) : d_G(v) \geq i\}$. Let $P(u, v)$ denote a path between u and v. A *branch* in G is a nontrivial path with ends not in $V_2(G)$ and with internal vertices, if any, that have degree 2 in G. By $\mathcal{B}(G)$, we denote the set of branches of G. Define $\mathcal{B}_1(G) = \{B \in \mathcal{B}(G) : V(B) \cap V_1(G) \neq \emptyset\}$. Let H_1 and H_2 be two subgraphs of graph G. Define $H_1 \cup H_2 = G[E(H_1) \cup E(H_2)]$, $H_1 \cap H_2 = G[E(H_1) \cap E(H_2)]$, $H_1 - H_2 = G[E(H_1) \backslash E(H_2)]$, $H_1 \triangle H_2 = G[E(H_1) \triangle E(H_2)] = G[(E(H_1) \cup E(H_2)) \backslash (E(H_1) \cap E(H_2))]$, respectively. For any $S \subseteq V(G)$, define $H_1 \cup S$ is a graph with $V(H_1) \cup S$ and $E(H_1)$ as its vertex set and edge set, respectively. The distance $d_G(H_1, H_2)$ between H_1 and H_2 is defined to be $\min\{d_G(v_1, v_2) : v_1 \in V(H_1), v_2 \in V(H_2)\}$, where $d_G(v_1, v_2)$ denotes the number of edges of a shortest path between v_1 and v_2 in G.

Xiong and Liu [13] characterized the graphs for which the s-iterated line graph is hamiltonian for any integer $s \geq 2$.

Theorem 6 (Xiong and Liu, [13]). Let G be a connected graph that is not a 2-cycle and let $s \geq 2$ be an integer. Then $h(G) \leq s$ if and only if $EU_s(G) \neq \emptyset$, where $EU_s(G)$ denotes the set of those subgraphs H of a graph G that satisfy the following conditions:

(I) $d_H(x) \equiv 0 \pmod 2$ for every $x \in V(H)$;

(II) $V_0(H) \subseteq V_{\geq 3}(G) \subseteq V(H)$;

(III) $d_G(H_1, H - H_1) \le s - 1$ for every subgraph H_1 of H;

(IV) $|E(B)| \le s + 1$ for every branch $B \in \mathcal{B}(G)$ with $E(B) \cap E(H) = \emptyset$;

(V) $|E(B)| \le s$ for every branch $B \in \mathcal{B}_1(G)$.

§3 Proofs of main results

Proof of *Theorem* 4. Choose a subgraph H of G satisfying that:

$$\begin{aligned}&(1)\ d_H(x) \equiv 0(\bmod 2) \text{ for every } x \in V(H);\\&(2)\ V_0(H) \subseteq V_{\ge 3}(G) \subseteq V(H);\\&(3) \text{ subject to } (1),(2),\ |V(H)| \text{ is maximized.}\end{aligned} \tag{3.1}$$

By *Theorem* 6, for $s = k - 1$, it suffices to prove that $H \in EU_{k-1}(G)$. By the choice of H, H satisfies Conditions (I), (II) of *Theorem* 6.

We claim that H satisfies Condition (IV) of *Theorem* 6. That is, $|E(B)| \le k$ for every $B \in \mathcal{B}(G)$ with $E(B) \cap E(H) = \emptyset$. Suppose otherwise. Then G has a branch $B \in \mathcal{B}(G)$ such that $|E(B)| \ge k + 1$ and $E(B) \cap E(H) = \emptyset$. Hence, B contains a path P of order k with $V(P) \subseteq V_2(G)$. For $n > k + 2$, $\bar{\sigma}_k(G) \le d_G(P) = 2k < n + k - 2 \le \bar{\sigma}_k(G)$, a contradiction. We then claim that H satisfies Condition (V) of *Theorem* 6. That is, $|E(B)| \le k - 1$ for every $B \in \mathcal{B}_1(G)$. Suppose otherwise. Then G has a branch $B \in \mathcal{B}_1(G)$ with $|E(B)| \ge k$. Hence, B contains a path P of order k with $\bar{\sigma}_k(G) \le d_G(P) = 2k-1 < n+k-2 \le \bar{\sigma}_k(G)$, a contradiction.

Then we only need to prove that H satisfies Condition (III) of *Theorem* 6, that is, $d_G(H_1, H - H_1) \le k - 2$ for every subgraph H_1 of H. Suppose otherwise. Then H has a subgraph H_1 such that $d_G(H_1, H - H_1) \ge k - 1$. Since G is connected and $k \ge 3$, there is at least one path $B_0 = x_0x_1 \cdots x_l$ between H_1 and $H - H_1$ such that $l \ge k - 1$ and $x_0 \in V(H_1)$ and $x_l \in V(H - H_1)$. By the choice of H, B_0 is a branch of G. Without loss of generality, we assume that $|V(H_1)| \le |V(H - H_1)|$. Then $|V(H_1)| \le \lfloor \frac{|V(H)|}{2} \rfloor$.

Claim 1. For any vertex $y_0 \in V(G) - (V(H) \bigcup V(B_0))$, if $N_G(y_0) \subseteq V(H)$, then $N_G(y_0)$ is an independent set.

Proof. Suppose otherwise. Since $y_0 \notin V(H)$, $d_G(y_0) \le 2$. Since Claim 1 naturally holds when $d_G(y_0) = 1$, we only need to consider the case when $d_G(y_0) = 2$. Then v_1v_2 is an edge of G for $N_G(y_0) = \{v_1, v_2\} \subseteq V(H)$. Then $H^1 = H \triangle v_1v_2y_0v_1$ is a subgraph of G satisfying $(1), (2)$ of (3.1) and $|V(H^1)| > |V(H)|$, contradicting the choice of H in terms of (3) of (3.1). □

Claim 2. $|N_G(x_0) \bigcap N_G(x_l)| \le 1$.

Proof. Suppose otherwise. Then $|N_G(x_0) \bigcap N_G(x_l)| \ge 2$. Then there exists a vertex $x \in N_G(x_0) \bigcap N_G(x_l)$ and $x \notin B_0$. Then x_0xx_l is a branch of G, denoted by B_1. Then $C^2 = B_0 \bigcup B_1$ is a cycle. We have a subgraph $H^2 = H \triangle C^2$ of G satisfying $(1), (2)$ of (3.1) and $|V(H^2)| > |V(H)|$, contradicting the choice of H in terms of (3) of (3.1). □

Claim 3. $V(G) = V(H) \bigcup V(B_0)$.

Proof. Suppose otherwise. Since $d_G(x_0) \ge 3$, $N_G(x_0) \ne \emptyset$. We consider two cases.

Case 1. $N_G(x_0) \bigcap V(H_1) = \emptyset$.

Since $d_G(H_1, H - H_1) \ge k - 1 \ge 2$, $N_G(x_0) \subseteq V(G - H)$. Since $d_G(x_l) \ge 3$, $|N_G(x_l)| \ge 3$. By Claim 2, $d_G(x_0) \le n - (k - 3 + 3 + 1) = n - k - 1$. Note that $P = yx_0 \cdots x_{k-2}$ is a path of order k, where $y \in N_G(x_0)$. Then $d_G(y) \le 2$. However, $\bar{\sigma}_k(G) \le d_G(P_0) \le d_G(P) \le 2+n-k-1+2(k-2) = n+k-3 < \bar{\sigma}_k(G)$, a contradiction.

Case 2. $N_G(x_0) \bigcap V(H_1) \ne \emptyset$.

For $y \in N_G(x_0) \bigcap V(H_1)$, $P = yx_0 \cdots x_{k-2}$ is a path of order k. By Claim 1, $N_G(x_0) \bigcap N_G(y) \subseteq H_1$. Then $d_G(x_0)+d_G(y) \le 2(\lfloor \frac{|V(H)|}{2} \rfloor - 1) + n - (|V(H)|+k-3) \le n-k+1$. However, $\bar{\sigma}_k(G) \le d_G(P_0) \le d_G(P) \le n-k+1+2(k-2) \le n+k-3 < \bar{\sigma}_k(G)$, a contradiction. □

By Claim 3, $|V(H_1)| \le \lfloor \frac{n-k+2}{2} \rfloor$. We have a path $P_0 = yx_0 \cdots x_{k-2}$, where $y \in V(H_1)$. Note that $d_G(y) \le |V(H_1)| - 1 \le \lfloor \frac{n-k+2}{2} \rfloor - 1$ and $d_G(x_0) \le |V(H_1)| - 1 + 1 \le \lfloor \frac{n-k+2}{2} \rfloor$. However, $\bar{\sigma}_k(G) \le d_G(P_0) \le \lfloor \frac{n-k+2}{2} \rfloor + \lfloor \frac{n-k+2}{2} \rfloor - 1 + 2(k-2) \le n+k-3 < \bar{\sigma}_k(G)$, a contradiction. □

Proof of *Theorem* 5. Let $k \ge 3$ be an integer. For the convenience of proof, we define *k-tribe*. If H_0 is a maximal subgraph of G without any branch of length more than $k-2$ such that $d_G(H_1, H_0 - H_1) \le k-2$ for every subgraph H_1 of H_0, then we call H_0 a *k-tribe*. Furthermore, we use $f_k(\tilde{H})$ to denote the number of k-tribes of a subgraph $\tilde{H}$ of G.

Choose a subgraph H of G satisfying that:

(1) $d_H(x) \equiv 0(\bmod 2)$ for every $x \in V(H)$;

(2) $V_0(H) \subseteq V_{\ge 3}(G) \subseteq V(H)$;

(3) subject to (1), (2), $|(G;H)|$ is minimized, where $(G;H) = \{H_1 \subseteq H : d_G(H_1, H-H_1) \ge k-1\}$; (3.2)

(4) subject to (1), (2), (3), $|V(H)|$ is maximized.

By *Theorem* 6, for $s = k-1$, it suffices to prove that $H \in EU_{k-1}(G)$. By the choice of H, H satisfies Conditions (I), (II) in *Theorem* 6. Since G is 2-connected, H satisfies Condition (V). Besides, H satisfies Condition (IV). Suppose otherwise. Then G has a branch $B \in \mathcal{B}(G)$ such that $|E(B)| \ge k+1$ and $E(B) \cap E(H) = \emptyset$. Then there is a path $P \subseteq B$ of order k with $V(P) \subseteq V_2(G)$. However, since $n > 6k+3$, $\bar{\sigma}_k(G) \le d_G(P) = 2k < \frac{2n}{5} - \frac{2k}{5} - \frac{1}{5} < \bar{\sigma}_k(G)$, a contradiction. Next, we prove that H satisfies Condition (III), that is, $(G;H) = \emptyset$.

We then assume that $(G;H) \ne \emptyset$, i.e., there is a subgraph H_1 of H such that $d_G(H_1, H-H_1) \ge k-1$. Since G is 2-connected, there are at least two paths $B_1(x_1,y_1)$, $B_2(x_2,y_2)$ between H_1 and $H-H_1$ such that $x_1, x_2 \in V(H_1)$, $y_1, y_2 \in V(H-H_1)$ and $x_1 \ne x_2$, $y_1 \ne y_2$. Since $d_G(H_1, H-H_1) \ge k-1$, $E(B_i) \bigcap E(H) = \emptyset$ and $|E(B_i)| \ge k-1$, $i = 1, 2$. By the choice of H, both H_1 and $H-H_1$ are the union of connected even subgraphs of G and $B_1(x_1,y_1)$, $B_2(x_2,y_2) \in \mathcal{B}(G)$.

Since G is 2-connected and $B_1, B_2 \in \mathcal{B}(G)$, there is a cycle C with minimum order containing B_1 and B_2. Furthermore, we claim that $E(C) \bigcap E(H_1) \ne \emptyset$ or $E(C) \bigcap E(H-H_1) \ne \emptyset$. Otherwise, we have a subgraph $H^1 = H \triangle C$ of G satisfying (1), (2) of (3.2) and $|(G:H^1)| < |(G:H)|$, contradicting the choice of H in terms of (3) of (3.2).

Let $\tilde{H} = H - \{B \in \mathcal{B}(G) : |E(B)| \ge k-1\}$. $\tilde{H}$ is the union of some k-tribes. In the following text, we investigate $f_k(\tilde{H})$. Since $(G;H) \ne \emptyset$, $f_k(\tilde{H}) \ge 2$.

Claim 4. Either $\{x_1, x_2\}$ or $\{y_1, y_2\}$ is in distinct k-tribes.

Proof. Suppose otherwise. Then $H^1 = (H \triangle C) \cup V_{\ge 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G;H^1)| < |(G;H)|$, contradicting the choice of H in terms of (3) of (3.2). □

By Claim 4, we know $f_k(\tilde{H}) > 2$. By symmetry, we always assume that $H - H_1$ contains more k-tribes than H_1 and $\{y_1, y_2\}$ is in distinct k-tribes in the following text.

Claim 5. $f_k(\tilde{H}) \geq 5$

Proof. Suppose otherwise. Then we consider the following three cases.

Case 1. $f_k(\tilde{H}) = 3$.

We assume that H_1 contains one k-tribe $\tilde{H}_1$ and $H - H_1$ contains two k-tribes $\tilde{H}_2$ and $\tilde{H}_3$. Then $x_1, x_2 \in \tilde{H}_1$. By Claim 4, we assume that $y_1 \in \tilde{H}_2$ and $y_2 \in \tilde{H}_3$ in Case 1.

Subcase 1.1. $H - H_1$ does not contain any branch connecting $\tilde{H}_2$ and $\tilde{H}_3$. By Symmetry, $|(G;H)| = 2\binom{3}{1} = 6$. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G;H^1)| < |(G;H)|$, a contradiction.

Subcase 1.2. $H - H_1$ contains a branch B_3 connecting $\tilde{H}_2$ and $\tilde{H}_3$.

Since $H - H_1$ is an even subgraph, $H - H_1$ contains another branch B_4 connecting $\tilde{H}_2$ and $\tilde{H}_3$. Then $(G;H) = \{H_1, H - H_1\}$. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2). Since H^1 contains B_3 or B_4, $|(G;H^1)| < |(G;H)|$, a contradiction.

Case 2. H_1 contains one k-tribe $\tilde{H}_1$ and $H - H_1$ contains three k-tribes $\tilde{H}_2$, $\tilde{H}_3$ and $\tilde{H}_4$.

We assume that $y_1 \in \tilde{H}_2$ and $y_2 \in \tilde{H}_4$. Then we claim that G has no branch between $\tilde{H}_2$ and $\tilde{H}_4$. Otherwise, G contains a branch B_3 connecting $\tilde{H}_2$ and $\tilde{H}_4$. Since G is 2-connected, there is a path $P(y_1, y_2)$ with minimum order such that $B_3 \subseteq P$ and $E(P) \bigcap E(B_1 \bigcup B_2 \bigcup H_1) = \emptyset$. Note that cycle C contain a path $C(y_1, y_2)$ such that $E(C(y_1, y_2)) \bigcap E(B_1 \bigcup B_2 \bigcup H_1) = \emptyset$. By replacing $C(y_1, y_2)$ with $P(y_1, y_2)$, we have a cycle C_1 containing B_1, B_2 and B_3. Then $H^2 = (H \triangle C_1) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G;H^2)| < |(G;H)|$, a contradiction. Therefore, any (y_1, y_2)-path containing common vertex with $H - H_1$ has to contain common vertex with $\tilde{H}_3$. Then we distinguish the following two subcases.

Subcase 2.1. $H - H_1$ does not contain any branch between $\tilde{H}_2$ and $\tilde{H}_3$ or any branch between $\tilde{H}_3$ and $\tilde{H}_4$.

Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G;H^1)| < |(G;H)|$, a contradiction.

Subcase 2.2. $H - H_1$ contains both some branch between $\tilde{H}_2$ and $\tilde{H}_3$ and some branch between $\tilde{H}_3$ and $\tilde{H}_4$.

Since $H - H_1$ is an even subgraph, $H - H_1$ contains at least two branches between $\tilde{H}_2$ and $\tilde{H}_3$ and two branches between $\tilde{H}_3$ and $\tilde{H}_4$. Then $(G;H) = \{H_1, H - H_1\}$. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2). Since H^1 contains at least one branch connecting $\tilde{H}_2$ and $\tilde{H}_3$ and one branch connecting $\tilde{H}_3$ and $\tilde{H}_4$, $|(G;H^1)| < |(G;H)|$, a contradiction.

Case 3. Both H_1 and $H - H_1$ contain exactly two k-tribes, say $\tilde{H}_1$ and $\tilde{H}_2$, $\tilde{H}_3$ and $\tilde{H}_4$, respectively.

We claim that x_1 and x_2 lie in distinct k-tribes. Otherwise, $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G;H^1)| < |(G;H)|$, a contradiction. Then we assume that $x_1 \in \tilde{H}_1$, $x_2 \in \tilde{H}_2$, $y_1 \in \tilde{H}_3$ and $y_2 \in \tilde{H}_4$, respectively.

Subcase 3.1. H contains a branch connecting $\tilde{H}_1$ and $\tilde{H}_2$ or connecting $\tilde{H}_3$ and $\tilde{H}_4$.

Since H is an even subgraph, $H - H_1$ contains at least two such branches. But cycle C contains at most one of these branches. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G;H^1)| < |(G;H)|$, a contradiction.

Subcase 3.2. H does not contain any branch between $\tilde{H}_1$ and $\tilde{H}_2$ or between $\tilde{H}_3$ and $\tilde{H}_4$.

By symmetry, $|(G;H)| = 2(\binom{4}{1} + \binom{4}{2}) = 20$. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G;H^1)| < |(G;H)|$, a contradiction. □

Claim 6. Either each (x_1, x_2)-path in H_1 or each (y_1, y_2)-path in $H - H_1$ contains two branches with length at least $k-1$ such that these two branches have end vertices in a same k-tribe which has not any end vertex of other branches connecting other k-tribes in H.

Proof. Firstly, we prove that either each (x_1, x_2)-path in H_1 or each (y_1, y_2)-path in $H - H_1$ contains at least two branches with length at least $k-1$. Suppose otherwise. Let $P(x_1, x_2) \subseteq H_1$, $Q(y_1, y_2) \subseteq H - H_1$ be two paths and both P and Q contain exactly one branch with length at least $k-1$, say B_3 and B_4, respectively. Since G is 2-connected, there exists a cycle C_1 containing B_3 and B_4 with minimum order. Note that if $B_3 \subseteq H$ or $B_4 \subseteq H$, since H is an even subgraph, H must contain another branch connecting the same two k-tribes. Then $H^2 = (H \triangle C_1) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G;H^2)| < |(G;H)|$, a contradiction. Hence, either each (x_1, x_2)-path in H_1 or each (y_1, y_2)-path in $H - H_1$ contains at least two branches with length at least $k-1$. Without loss of generality, we assume that each (y_1, y_2)-path in $H - H_1$ contains at least two branches with length at least $k-1$. Let $P(y_1, y_2)$ be a path in $H - H_1$ containing two distinct branches with length at least $k-1$ $B_3(y_3, y_4)$ and $B_4(y_5, y_6)$. Let y_4 and y_5 lie in the same k-tribe $\tilde{H}_0$. We prove that there is not any branch connecting $\tilde{H}_0$ and other k-tribes in H except B_3 and B_4. Suppose otherwise. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2). Since cycle C contains at most two of these branches connecting $\tilde{H}_0$ and other k-tribes in H, H^1 contains at least one. Then $|(G;H^1)| < |(G;H)|$, a contradiction. □

In the following text, we always assume that each (y_1, y_2)-path in $H - H_1$ contains two branches with length at least $k-1$ such that these two branches have end vertices in a same k-tribe which has not any end vertex of other branches connecting other k-tribes in $H - H_1$.

Claim 7. For any vertex $y_0 \in V(G) - (V(H) \bigcup V(B_0))$, if $N_G(y_0) \subseteq V(H)$, then $N_G(y_0)$ is an independent set.

Proof. Suppose otherwise. Let $N_G(y_0) = \{v_1, v_2\}$ and $C^1 = v_1v_2y_0v_1$. We have a subgraph $H^1 = H \triangle C^1$ of G satisfying (1), (2), (3) of (3.2) and $|H^1| > |H|$, contradicting the choice of H in terms of (4) of (3.2). □

In the following text, we will consider two cases, $f_k(\tilde{H}) = 5$ and $f_k(\tilde{H}) > 5$, respectively.

Firstly, we consider the case when $f_k(\tilde{H}) = 5$. We first consider the case when H_1 contains two k-tribes $\tilde{H}_1$ and $\tilde{H}_2$, and that $H - H_1$ contains three k-tribes $\tilde{H}_3$, $\tilde{H}_4$ and $\tilde{H}_5$. By Claims 4 and 5, we assume that $x_1 \in \tilde{H}_1$, $x_2 \in \tilde{H}_2$, $y_1 \in \tilde{H}_3$ and $y_2 \in \tilde{H}_4$. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G;H^1)| < |(G;H)|$, a contradiction.

Therefore, it remains to consider the case when H_1 contains one k-tribe $\tilde{H}_1$ and $H - H_1$ contains four k-tribes $\tilde{H}_2$, $\tilde{H}_3$, $\tilde{H}_4$ and $\tilde{H}_5$. We assume that $y_1 \in \tilde{H}_2$ and $y_2 \in \tilde{H}_4$. By Claim 6, H contains four branches, say B_3, B_4, B_5, B_6, connecting $\tilde{H}_2$ and $\tilde{H}_3$, $\tilde{H}_2$ and $\tilde{H}_5$, $\tilde{H}_4$ and $\tilde{H}_3$, $\tilde{H}_2$ and $\tilde{H}_5$, respectively. We claim that G does not contain any other branch connecting two different k-tribes. Suppose otherwise. Firstly, we consider the case when G contains another branch B_0 between H_1 and $H - H_1$. Since G is 2-connected, there exist two cycles C_1 and C_2 such that $B_0, B_1 \subseteq C_1$ and $B_0, B_2 \subseteq C_2$. Without loss of generality, we assume that $|C_1| \geq |C_2|$. Then $H^2 = (H \triangle C_1) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G;H^2)| < |(G;H)|$, a contradiction. By symmetry, it remains to consider the case when G contains another branch connecting two different k-tribes of $H - H_1$. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying $(1), (2)$ of (3.2) and $|(G; H^1)| < |(G; H)|$, a contradiction.

Without loss of generality, we assume that $\tilde{H}_2$ has the minimum order among the five k-tribes. Let $B_3 = y_0 y_1 \cdots y_l$, where end vertex $y_0 \in H_2$ and $l \geq k-1$. Since G is 2-connected, y_0 belongs to at most two branches with length at least 2. Then $d_G(y_0) \leq |V(\tilde{H}_2)| - 1 + 2 \leq \frac{n-6(k-2)}{5} - 1 + 2$. We claim that $N_G(y_0) \bigcap V_2(G) = \{y_1\}$. Suppose otherwise. Then there exists a vertex $y \in N_G(y_0) - \{y_1\}$ with $d_G(y) = 2$, and that $P = y y_0 \cdots y_{k-2}$ a path of order k. Since $n > 6k+3$, $\bar{\sigma}_k(G) \leq d_G(P) \leq \frac{n-6(k-2)}{5} + 1 + 2(k-1) < \frac{2n}{5} - \frac{2k}{5} + \frac{4}{5} \leq \bar{\sigma}_k(G)$, a contradiction. Then $d_G(y_0) \leq \frac{n-6(k-2)}{5}$. We consider the path $Q = x y_0 \cdots y_{k-2}$, where $x \in V(\tilde{H}_2)$. By Claim 7, $d_G(x) \leq |V(\tilde{H}_2)| - 1 \leq \frac{n-6(k-2)}{5} - 1$. Then $\bar{\sigma}_k(G) \leq d_G(P) \leq \frac{n-6(k-2)}{5} + \frac{n-6(k-2)}{5} - 1 + 2(k-2) = \frac{2n}{5} - \frac{2k}{5} - \frac{1}{5} < \bar{\sigma}_k(G)$, a contradiction.

Therefore, $f_k(\tilde{H}) \neq 5$. Next, we consider the case when $f_k(\tilde{H}) > 5$.

By Claim 4, we assume that $y_1 \subseteq \tilde{H}_2$ and $y_2 \subseteq \tilde{H}_4$. Let $H_2 \supseteq \tilde{H}_2$ be a maximal subgraph of H satisfying the condition that if H_2 contains a k-tribe $\tilde{H}^0 \neq \tilde{H}_2$, then H_2 contains another k-tribe $\tilde{H}^1$ such that there are at least two branches between $\tilde{H}^0$ and $\tilde{H}^1$ in G. And let $H_4 \supseteq \tilde{H}_4$ be a maximal subgraph of H satisfying the condition that if H_4 contains a k-tribe $\tilde{H}^0 \neq \tilde{H}_2$, then H_4 contains another k-tribe $\tilde{H}^1$ such that there are at least two branches between $\tilde{H}^0$ and $\tilde{H}^1$ in G. We claim that $V(H_2) \bigcap V(H_4) = \emptyset$. Suppose otherwise. Then $H_2 = H_4$. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying $(1), (2)$ of (3.2) and $|(G; H^1)| < |(G; H)|$, a contradiction. Since G is 2-connected, there are at least two paths connecting H_2 and H_4.

Claim 8. Each path connecting H_2 and H_4 contains two branches with length at least $k-1$ such that these two branches have end vertices in a same subgraph which has not any end vertex of other branches connecting other k-tribes in G.

Proof. We first prove that each path connecting H_2 and H_4 contains at least two branches with length at least $k-1$. Suppose otherwise. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying $(1), (2)$ of (3.2) and $|(G; H^1)| < |(G; H)|$, a contradiction.

Hence, let P be a path in $H - H_1$ containing two distinct branches $B_3(y_3, y_4)$, $B_4(y_5, y_6)$ and $\{y_4, y_5\}$ lies in the same subgraph H_3. We prove that there is not any branch connecting H_2 and H_3 or H_3 and H_4 in G except B_3 or B_4. Suppose otherwise. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying $(1), (2)$ of (3.2) and $|(G; H^1)| < |(G; H)|$, a contradiction. □

By Claim 8, we assume that H_3 is a subgraph of $H - H_1$, and that B_3 and B_4 are two branches connecting H_2 and H_3, H_3 and H_4, respectively. We claim that $B_3, B_4 \subseteq H - H_1$. Suppose otherwise. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying $(1), (2)$ of (3.2) and $|(G; H^1)| < |(G; H)|$, a contradiction. Since $H - H_1$ is an even subgraph, $H - H_1$ contains a subgraph H_5 and two branches B_5 and B_6 such that B_5 connects H_2 and H_5, and B_6 connects H_4 and H_5, respectively. We claim $V(H_3) \bigcap V(H_5) = \emptyset$. Otherwise, $H_3 = H_5$ is the same subgraph and then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying $(1), (2)$ of (3.2), hence $|(G; H^1)| < |(G; H)|$, a contradiction. Furthermore, we claim that G does not contain any other branch between subgraphs H_1, H_2, H_3, H_4 and H_5. Suppose otherwise. Firstly, we consider the case when G contains another branch B_0 between H_1 and $H - H_1$. Since G is 2-connected, there exist two cycles C_1 and C_2 such that $B_0, B_1 \subseteq C_1$ and $B_0, B_2 \subseteq C_2$. By symmetry, we assume that $|C_1| \geq |C_2|$. Then $H^2 = (H \triangle C_1) \cup V_{\geq 3}(G)$ is a subgraph satisfying $(1), (2)$ of (3.2) and $|(G; H^2)| < |(G; H)|$, a contradiction. It remains to consider the case when G contains another branch connecting two different subgraphs of $H - H_1$. Then $H^1 = (H \triangle C) \cup V_{\geq 3}(G)$ is a subgraph satisfying (1), (2) of (3.2) and $|(G; H^1)| < |(G; H)|$, a contradiction.

Without loss of generality, we assume that H_2 has the minimum order among the five subgraphs. Let $B_3 = y_0 y_1 \cdots y_l$, where end vertex $y_0 \in H_2$ and $l \geq k-1$. Since G is 2-connected, y_0 belongs to at most two branches with length at least 2. Then $d_G(y_0) \leq |V(H_2)| - 1 + 2 \leq \frac{n-6(k-2)}{5} - 1 + 2$. Furthermore, we claim that $N_G(y_0) \bigcap V_2(G) = \{y_1\}$. suppose otherwise. Then there exists a vertex $y \in N_G(y_0) - \{y_1\}$ with $d_G(y) = 2$, and that $P = yy_0 \cdots y_{k-2}$ is a path of order k. Since $n > 6k+3$, $\bar{\sigma}_k(G) \leq d_G(P) \leq \frac{n-6(k-2)}{5} + 1 + 2(k-1) < \frac{2n}{5} - \frac{2k}{5} + \frac{4}{5} \leq \bar{\sigma}_k(G)$, a contradiction. Then $d_G(y_0) \leq \frac{n-6(k-2)}{5}$. We consider the path $Q = xy_0 \cdots y_{k-2}$, where $x \in V(H_2)$. By Claim 7, $d_G(x) \leq |V(H_2)| - 1 \leq \frac{n-6(k-2)}{5} - 1$. Then $\bar{\sigma}_k(G) \leq d_G(P) \leq \frac{n-6(k-2)}{5} + \frac{n-6(k-2)}{5} - 1 + 2(k-2) = \frac{2n}{5} - \frac{2k}{5} - \frac{1}{5} < \bar{\sigma}_k(G)$, a contradiction. □

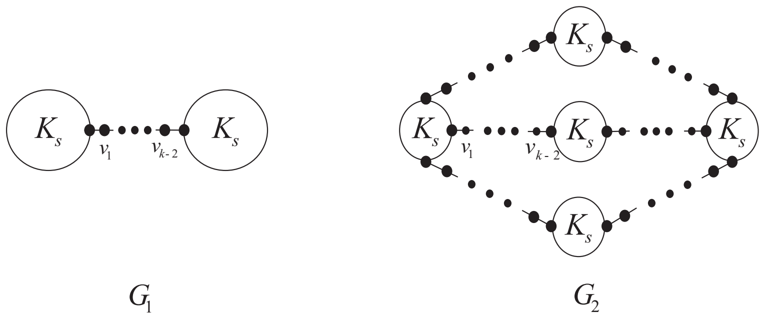

Figure 1. G_1 and G_2.

§4 Conclusion: Sharpness

Both *Theorems* 4 and 5 are best possible, this may be seen by G_1 and G_2 in Figure 1, respectively.

Comparing *Theorem* 2 with *Theorem* 4, one might think that they would have a unified bound. Unfortunately, this is not true: *Theorem* 4 is not direct promotion of *Theorem* 2. However, taking the Dirac result and Theorems 2, 3, 4 and 5 into consideration, we conclude that we completely know the sharp lower bounds of $\bar{\sigma}_k(G)$ involving its order for the graph $L^{k-1}(G)$ of a (2-)connected graph G to be hamiltonian for all $k \geq 1$.

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