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DISCUSSION

Comments on: Process modeling for slope and aspect with application to elevation data maps

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It is a pleasure to comment on this interesting article by Wang, Bhattacharya and Gelfand. The article discusses formal Bayesian inference on topographic features such as slope and aspect using information from GIS. The authors have nicely exploited and extended some of the previously established results on spatial gradients to infer on topographic slopes and aspects. Bayesian inference, and sampling-based computation of the posterior predictive distribution, is very convenient here because the topographic functions of interest are fairly simple functions of directional spatial gradients. Therefore, posterior samples for slopes and aspects are immediately obtained from the posterior samples of the directional gradients.

The key underlying feature of such modeling is that finite difference increments of stationary Gaussian processes are again Gaussian processes and, hence, so are their limits as the increments become negligibly small (see, e.g., Parzen 1962). In fact, as the authors have correctly noted, since the smoothness of the process realizations is determined by the stationary covariance function, one only needs to specify an appropriate covariance function to ensure the existence of the gradient process. This leads to an elegant distribution theory for spatial gradients that can be embedded within hierarchical modeling contexts to carry out Bayesian inference for directional rates of change (e.g., Banerjee et al. 2003) or even spatiotemporal gradients (Quick et al. 2015).

This framework can, in fact, be generalized to infer on sufficiently smooth functionals of the process. Indeed, with an appropriate specification for a spatial process $Y(s)$, we can derive the multivariate process $\{Y(s), \mathcal{L}Y(s)\}$, where $\mathcal{L}Y(s)$ is a linear functional of $Y(s)$, and carry out posterior predictive inference on the posterior predictive distribution $[\mathcal{L}Y(s) \mid Y(S)]$, where $Y(S)$ are observations of $Y(s)$ over a finite set of locations S. One example is the estimation of gradients along curves to derive Bayesian detection rules for so-called curves of rapid change or "wombling

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boundaries" (Banerjee and Gelfand 2006). In fact, estimating maximal gradient processes (that are central in the current paper) is a central part of "wombling" or boundary detection problems (see, e.g., Figure 2c in Banerjee and Gelfand 2006). Since the joint process $\{Y(s), \mathcal{L}Y(s)\}$ is a well-defined stochastic process over the entire domain, one can predict either of $Y(s)$ or $\mathcal{L}Y(s)$ at arbitrary locations, including where none of the two processes have been observed. In full generality, we can compute the predictive densities $[Y(U_1), \mathcal{L}Y(U_2) \mid Y(S_1), \mathcal{L}Y(S_2)]$, where S_1 and S_2 are sets of locations yielding observations on $Y(s)$ and $\mathcal{L}Y(s)$, respectively, and U_1 and U_2 are sets of locations where predictions are sought for $Y(s)$ and $\mathcal{L}Y(s)$, respectively.

And while we are at it, why not extend even further to quantities of interest in vector analysis or differential geometry? For example, we can express the process in terms of the random position vector $r(u, v) = (u, v, Y(s))$, where $s = (u, v)$ and infer on the basis vectors $r_u(s) = \partial r(u, v)/\partial u$ and $r_v(s) = \partial r(u, v)/\partial v$, spanning so-called tangent plane. Subsequently, we can compute the "first fundamental form" $(E(s), F(s), G(s))$, defined as

$$E(s) = \langle r_u(s), r_u(s)\rangle; \;\; F(s) = \langle r_u(s), r_v(s)\rangle; \;\; G(s) = \langle r_v(s), r_v(s)\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes an appropriate inner-product. The first fundamental form completely determines several quantities of interest including curls, surface areas and arc lengths. For instance, the differential element of surface area, say $dA(s)$, is approximated by the area of the local patch on the tangent plane to the surface at (s) and is given by the cross-product of the fundamental vectors as

$$dA(s) = \|\partial r/\partial u \times \partial r/\partial v\| ds.$$

This can be shown as equivalent to $dA(s) = E(s)G(s) - F(s)^2$. The surface area of $r(s)$ is the continuous sum (integral) of the areas of these infinitesimal parallelograms on the surface and is defined as

$$A(D) = \int dA(u, v) = \int_{s \in D} \sqrt{E(s)G(s) - F(s)^2} ds.$$

The first fundamental form depends only upon the gradient (components of ∇r), so statistical inference on the above quantities and their functions (such as $A(D)$) fits well within the framework provided in the paper. In fact, in the sampling-based framework one would only need to compute the gradients of $r(s)$, which would immediately provide the posterior distributions of the components of the first fundamental form. The inference on physical quantities in classical vector analysis involving higher-order differentials such as the Laplacians, curvatures and divergences can also be formulated using appropriate hypersurface parametrizations.

As is apparent from the above, one could indulge oneself with the distribution theory available for linear functionals of Gaussian processes. But are such problems scientifically relevant? What types of applications would demand such inference? This is not yet clear to me. In spatial statistics, inferring on $\mathcal{L}Y(s)$ given observations on the response seems to have been relevant for understanding zones and boundaries of rapid change. However, I have not seen many applications involving information on gradients or linear functionals. This is perhaps because reliable information on $\mathcal{L}Y(s)$ is difficult to gather. We usually observe elevation, not gradients. In computer experiments, where Gaussian processes are widely used as response surface emulators for complex functional outputs, the computer program (or the physical system) can provide information on gradients. Such information can, then, be used as part of the data to interpolate the response surface (see, e.g., Morris et al. 1993). I think similar explorations on predictive performances when gradient information is available, perhaps from DEMs or DTMs, will be an interesting exercise to pursue. The authors state that they never observe the slope or aspect process. This is correct, but can DEMs provide such information? Will incorporating such information help enhance inferential performance?

I conclude with a few remarks specific to the paper's contribution. This is relevant since a casual reader may erroneously conclude that much of the paper rehashes established results on the theory of spatial gradients. The authors have derived two novel results, both of which play a central role in the inference of slope and aspect processes. The first establishes conditions for the independence between the slope and the aspect processes. The second outlines conditions for the aspect to have, a priori, a circular uniform distribution. The proofs are elegant, and indeed, the results help in building intuition behind these processes. I do note that the authors have focused on inference for the mean $\mathrm{E}[Y(s)] = \mu(s) + w(s)$, which will require the mean function $\mu(s)$ to be smooth and admit gradients. This may not always be true: consider settings where $\mu(s) = x(s)^\top \beta$ and some of the explanatory variables are categorical. Inference for $\nabla w(s)$ is still permissible. Finally, while slopes and aspects on the mean may be most relevant in purely spatial contexts, estimating slopes and aspects in spatiotemporal latent processes $w(s, t)$ and testing for their dynamic behavior for the underlying process may still be useful in understanding local behavior of latent processes.

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