ERRATUM

Erratum to: On the set-semidefinite representation of nonconvex quadratic programs over arbitrary feasible sets

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In this paper, an erratum is provided to the article "On the set-semidefinite representation of nonconvex quadratic programs over arbitrary feasible sets", published in Optim Lett, 2012. Due to precise observation of the first author, it has been found that the proof of Lemma 9 has a nontrivial gap, and consequently the main result (Theorem 10) is incorrect. In this erratum, we prove that Corollary 14 is still correct in the original setting while to fix the proof of Theorem 10 we need additional assumptions. We provide a list of different commonly used assumptions making this theorem to be true, and a new version of this theorem, which is now Theorem 17.

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1 The error

The second and the third author of this paper considered, in [4], the following quadratic optimization problem as an extension of that considered in [3], where $\mathcal{K} \subseteq \mathbb{R}^n$ is a nonempty set, $\mathcal{B} \subseteq \{1, \ldots, n\}$ is an index set, $Q \in \mathcal{S}^n$ is a symmetric matrix, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are vectors, and $A \in \mathbb{R}^{m \times n}$ is a matrix, with $\mathbf{a}_1^\top, \ldots, \mathbf{a}_m^\top \in \mathbb{R}^n$ denoting its rows:

$$
\begin{aligned}
\inf_{\mathbf{x} \in \mathbb{R}^n} \quad & \mathbf{x}^\top Q \mathbf{x} + 2\mathbf{c}^\top \mathbf{x} \\
\text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\
& x_j \in \{0, 1\} \qquad \text{for all } j \in \mathcal{B} \\
& \mathbf{x} \in \mathcal{K}.
\end{aligned}
\tag{QP}
$$

An assumption connected with this problem is as follows:

Assumption 1 If we have $\mathbf{x} \in \mathcal{K}$ such that $A\mathbf{x} = b$ then $x_j \in [0, 1]$ for all $j \in B$.

This assumption can be made to hold by either adding the constraints directly to $\mathcal{K}$ or by adding slack variables with nonnegative constraints.

It was claimed that if Assumption 1 holds then (QP_C), as given below, is a reformulation of (QP), by which we mean that the optimal values of both problems are equal.

$$
\begin{aligned}
\inf_{(\mathbf{x}, X) \in \mathbb{R}^n \times \mathcal{S}^n} \quad & \langle Q, X \rangle + 2\mathbf{c}^\top \mathbf{x} \\
\text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\
& \text{Diag}(AXA^\top) = \mathbf{b} \circ \mathbf{b} \\
& x_j = X_{jj} \qquad \text{for all } j \in \mathcal{B} \\
& \begin{pmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & X \end{pmatrix} \in \mathcal{C}^*_{\{1\} \times \mathcal{K}},
\end{aligned}
\tag{QP$_C$}
$$

where $(\mathbf{b} \circ \mathbf{b})^\top := (b_1^2, b_2^2, \ldots, b_m^2)^\top$, i.e. the Hadamard product. Also recall from [4, Lemma 4] that for a set $\mathcal{L} \subseteq \mathbb{R}^{n+1}$ we have

$$
\begin{aligned}
\mathcal{C}^*_{\mathcal{L}} &= \text{cl conv cone}\{\mathbf{x}\mathbf{x}^\top \mid \mathbf{x} \in \mathcal{L}\} \\
&= \text{conv}\{\mathbf{x}\mathbf{x}^\top \mid \mathbf{x} \in \text{cl cone}\,\mathcal{L}\}.
\end{aligned}
$$

The first author has constructed the following two counter examples showing that Assumption 1 holding is not a sufficient condition for (QP_C) being a reformulation of (QP):

Example 1 Let us consider the following example for (QP) with $n = 1$, $m = 0$, $\mathcal{B} = \{1\}$ and $\mathcal{K} = (0, 1]$:

$$
\inf\{x \in \mathbb{R} \mid x \in \{0, 1\}, \quad x \in \mathcal{K}\} = 1.
$$

Assumption 1 holds for the problem, and the corresponding version of (QP$_C$) is

$$\inf \left\{ x \in \mathbb{R} \ \middle| \ \begin{pmatrix} 1 & x \\ x & x \end{pmatrix} \in \mathcal{C}^*_{\{1\} \times \mathcal{K}} \right\}.$$

We have that $\begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^{\mathsf{T}} \in \mathcal{C}^*_{\{1\} \times \mathcal{K}}$ for all $x \in (0, 1]$, therefore, as $\mathcal{C}^*_{\{1\} \times \mathcal{K}}$ is closed, we have that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}^*_{\{1\} \times \mathcal{K}}$. From this it can be seen that the optimal value of the corresponding version of (QP$_C$) equals zero, and thus the optimal values of these two problems are not equal.

Example 2 We let

$$\mathcal{K}_1 = \text{conv cone} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \{ \mathbf{x} \in \mathbb{R}^3 \mid x_1 - x_3 \geq 0, \quad x_3 - x_2 \geq 0, \quad x_2 \geq 0 \},$$

$$\mathcal{K}_2 = \text{conv cone} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$= \{ \mathbf{x} \in \mathbb{R}^3 \mid x_1 - x_3 \geq 0, \quad x_1 - x_2 \geq 0, \quad -x_1 + x_2 + x_3 \geq 0 \}.$$

We now consider the following example for (QP) with $n = 3$, $m = 2$, $\mathcal{B} = \emptyset$ and $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$:

$$\inf \{ -x_1^2 \mid x_2 = 3, \quad x_3 = 2, \quad \mathbf{x} \in \mathcal{K} \}.$$

It can be seen that

$$\{ \mathbf{x} \in \mathcal{K}_1 \mid x_2 = 3, \quad x_3 = 2 \} = \emptyset,$$

$$\{ \mathbf{x} \in \mathcal{K}_2 \mid x_2 = 3, \quad x_3 = 2 \} = \text{conv} \left\{ \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \right\}.$$

Therefore the optimal value for the problem (QP) is equal to -25. One can however verify that the corresponding version of (QP$_C$) has the optimal value $-\infty$ and thus the optimal values of these two problems are not equal.

The error in the paper in fact comes from [4, Lemma 9], in the last paragraph of the proof. In this, the mistake was effectively in assuming that for a set $\mathcal{Y}_1$ defined by linear equality constraints and a convex set $\mathcal{Y}_2$, we have that $\mathcal{Y}_1 \cap (\text{cl } \mathcal{Y}_2) \subseteq \text{cl } (\mathcal{Y}_1 \cap \mathcal{Y}_2)$, which is not true in general.

2 Repairing the error

In this section we essentially provide two new versions of Theorem 10 from [4], where we decided to provide a detailed study of two different situations where Theorem 10 still holds. We do this using the asymptotic cone, where we recall that for a nonempty set $\mathcal{K} \subseteq \mathbb{R}^n$, the asymptotic cone is defined as follows and we note that this is always a closed cone:

$$\mathcal{K}_\infty := \{\mathbf{x} \in \mathbb{R}^n \mid \exists \{(\lambda_i, \mathbf{x}_i) \mid i \in \mathbb{N}\} \subseteq \mathbb{R}_+ \times \mathcal{K} \quad \text{such that} \quad \lim_{i \to \infty} \lambda_i = 0, \quad \lim_{i \to \infty} \lambda_i \mathbf{x}_i = \mathbf{x}\}.$$

We begin by presenting the following lemma from [1], where they used an equivalent definition of the asymptotic cone.

Lemma 3 [1, Lemma 2.1.1] *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\text{cl cone}(\{1\} \times \mathcal{K}) = \text{cone}(\{1\} \times \mathcal{K}) \cup (\{0\} \times \mathcal{K}_\infty).$$

Together with Lemma 4 in [4] we obtain the following.

Corollary 4 *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\mathcal{C}^*_{\{1\} \times \mathcal{K}} = \text{conv}\{\mathbf{z}\mathbf{z}^\mathsf{T} \mid \mathbf{z} \in \text{cone}(\{1\} \times \mathcal{K}) \cup (\{0\} \times \mathcal{K}_\infty)\}.$$

By applying Carathéodory's Theorem we get the following corollary.

Corollary 5 *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a nonempty closed set, let $Y \in \mathcal{C}^*_{\{1\} \times \mathcal{K}}$ and let M equal the dimension of the space containing $\mathcal{C}^*_{\{1\} \times \mathcal{K}}$. Then there exists $\lambda_1, \ldots, \lambda_p > 0$ and $\mathbf{x}_1, \ldots, \mathbf{x}_p \in \mathcal{K}$ and $\mathbf{d}_1, \ldots, \mathbf{d}_q \in \mathcal{K}_\infty$ such that $p + q = M$ and*

$$Y = \sum_{i=1}^p \lambda_i \begin{pmatrix} 1 \\ \mathbf{x}_i \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{x}_i \end{pmatrix}^\mathsf{T} + \sum_{i=1}^q \begin{pmatrix} 0 \\ \mathbf{d}_i \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{d}_i \end{pmatrix}^\mathsf{T}.$$

We define the following sets related to the problems (QP) and (QP$_C$)

$$\text{Feas}^+(\text{QP}) := \text{conv} \left\{ \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}^\mathsf{T} \,\middle|\, \mathbf{x} \in \text{Feas}(\text{QP}) \right\},$$

$$\text{Feas}^+(\text{QP}_C) := \left\{ \begin{pmatrix} 1 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & X \end{pmatrix} \,\middle|\, (\mathbf{x}, X) \in \text{Feas}(\text{QP}_C) \right\},$$

$$L_\infty := \{\mathbf{d} \in \mathcal{K}_\infty \mid \mathbf{a}_j^\mathsf{T} \mathbf{d} = 0 \quad \text{for all } j = 1, \ldots, m\},$$

$$L_\infty^+ := \text{conv} \left\{ \begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix}^\mathsf{T} \,\middle|\, \mathbf{d} \in L_\infty \right\}.$$

Note that this notation is the same as that used in [2–4] with one difference that the definition of L_∞ uses the asymptotic cone.

We now define the following assumptions. Although these assumptions are currently quite technical, later we shall show that some combinations of them can be implied by simpler assumptions.

Assumption 2 $\mathcal{K} \subseteq \mathbb{R}^n$ is closed, and, if there exists an $\mathbf{x} \in \mathcal{K}$ such that $A\mathbf{x} = \mathbf{b}$, then $(\mathbf{d})_j = 0$ for all $\mathbf{d} \in L_\infty$, $j \in \mathcal{B}$.

Assumption 3 The optimal value of (QP$_C$), denoted Val (QP$_C$), is not equal to $-\infty$.

Assumption 4 For all $\mathbf{d} \in L_\infty$, $\mathbf{y} \in \text{Feas(QP)}$, $\widetilde{\lambda} \in \mathbb{R}_+$, there exists a $\lambda \in \mathbb{R} \backslash (-\widetilde{\lambda}, \widetilde{\lambda})$, such that $\mathbf{y} + \lambda \mathbf{d} \in \text{Feas(QP)}$, where $\mathbb{R} \backslash (-\widetilde{\lambda}, \widetilde{\lambda}) = (-\infty, -\widetilde{\lambda}] \cup [\widetilde{\lambda}, \infty)$.

We now present the following results connected to these assumptions. In these, we consider the Minkowski sum, i.e. for two sets $\mathcal{X}, \mathcal{Y}$, we define $\mathcal{X} + \mathcal{Y} := \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \mathcal{X}, \ \mathbf{y} \in \mathcal{Y}\}$, and we note that $\emptyset + \mathcal{X} = \emptyset$.

Lemma 6 *If Assumption 2 holds then* $\text{Feas}^+(\text{QP}) + L_\infty^+ \subseteq \text{Feas}^+(\text{QP}_C)$.

Proof This follows immediately from Corollary 4. □

Lemma 7 *Consider problems* (QP) *and* (QP$_C$) *such that Assumptions 1 and 2 hold. Then* $\text{Feas}^+(\text{QP}_C) = \text{Feas}^+(\text{QP}) + L_\infty^+$.

Proof From Lemma 6 we have $\text{Feas}^+(\text{QP}) + L_\infty^+ \subseteq \text{Feas}^+(\text{QP}_C)$.

We shall now show that $\text{Feas}^+(\text{QP}) + L_\infty^+ \supseteq \text{Feas}^+(\text{QP}_C)$. If $\text{Feas}^+(\text{QP}_C) = \emptyset$ then $\emptyset = \text{Feas}^+(\text{QP}) + L_\infty^+$ and we are done. From now on we assume that $\text{Feas}^+(\text{QP}_C) \neq \emptyset$ and consider an arbitrary $\begin{pmatrix} 1 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & X \end{pmatrix} \in \text{Feas}^+(\text{QP}_C)$, which we will show to be in $\text{Feas}^+(\text{QP}) + L_\infty^+$.

From Corollary 5, there exists $\lambda_1, \ldots, \lambda_p > 0$ and $\mathbf{x}_1, \ldots, \mathbf{x}_p \in \mathcal{K}$ and $\mathbf{d}_1, \ldots, \mathbf{d}_q \in \mathcal{K}_\infty$ with $p + q = M$, where M equals the dimension of the space containing $\mathcal{C}^*_{\{1\} \times \mathcal{K}}$, such that

$$\begin{pmatrix} 1 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & X \end{pmatrix} = \sum_{i=1}^{p} \lambda_i \begin{pmatrix} 1 \\ \mathbf{x}_i \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{x}_i \end{pmatrix}^\mathsf{T} + \sum_{i=1}^{q} \begin{pmatrix} 0 \\ \mathbf{d}_i \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{d}_i \end{pmatrix}^\mathsf{T}.$$

We have $1 = \sum_{i=1}^{p} \lambda_i$ and thus $p > 0$. For all $j = 1, \ldots, m$ we have

$$\begin{aligned}
0 &= b_j^2 - 2b_j^2 + b_j^2 \\
&= \mathbf{a}_j^\mathsf{T} X \mathbf{a}_j - 2b_j(\mathbf{a}_j^\mathsf{T}\mathbf{x}) + b_j^2 \\
&= \sum_{i=1}^{p} \lambda_i(\mathbf{a}_j^\mathsf{T}\mathbf{x}_i)^2 + \sum_{i=1}^{q}(\mathbf{a}_j^\mathsf{T}\mathbf{d}_i)^2 - 2b_j \sum_{i=1}^{p} \lambda_i(\mathbf{a}_j^\mathsf{T}\mathbf{x}_i) + b_j^2 \sum_{i=1}^{p} \lambda_i \\
&= \sum_{i=1}^{q} \underbrace{(\mathbf{a}_j^\mathsf{T}\mathbf{d}_i)^2}_{\geq 0} + \sum_{i=1}^{p} \underbrace{\lambda_i}_{>0} \underbrace{(\mathbf{a}_j^\mathsf{T}\mathbf{x}_i - b_j)^2}_{\geq 0} \geq 0.
\end{aligned}$$

Therefore $\mathbf{d}_i \in L_\infty$ for all $i = 1, \ldots, q$, and $\mathbf{a}_j^\mathsf{T}\mathbf{x}_i = b_j$ for all $i = 1, \ldots, p$, $j = 1, \ldots, m$. From Assumption 1, we have that $(\mathbf{x}_i)_j \in [0, 1]$ for all $i = 1, \ldots, p$, $j \in \mathcal{B}$, and from Assumption 2, we have that $(\mathbf{d}_i)_j = 0$ for all $i = 1, \ldots, q, j \in \mathcal{B}$. Using this, for all $j \in \mathcal{B}$ we have

$$0 = (\mathbf{x})_j - (X)_{jj}$$

$$= \sum_{i=1}^p \lambda_i (\mathbf{x}_i)_j - \sum_{i=1}^p \lambda_i (\mathbf{x}_i)_j^2 - \sum_{i=1}^q (\mathbf{d}_i)_j^2$$

$$= \sum_{i=1}^p \underbrace{\lambda_i}_{>0} \underbrace{(\mathbf{x}_i)_j (1 - (\mathbf{x}_i)_j)}_{\geq 0} \geq 0.$$

Therefore $(\mathbf{x}_i)_j \in \{0, 1\}$ for all $i = 1, \ldots, p, j \in \mathcal{B}$, which implies that $\mathbf{x}_i \in \mathrm{Feas(QP)}$ for all $i = 1, \ldots, p$. This along with our earlier observation that $\mathbf{d}_i \in L_\infty$ for all $i = 1, \ldots, q$ completes the proof. □

We now consider Assumption 3 in addition to Assumptions 1 and 2.

Theorem 8 *Consider problems* (QP) *and* (QP$_\mathrm{C}$) *such that Assumptions 1, 2 and 3 hold. Then* $\mathrm{Val(QP)} = \mathrm{Val\,(QP_C)}$. *Furthermore, if we let* $\mathrm{Opt(QP)}$ *and* $\mathrm{Opt\,(QP_C)}$ *be the set of optimal solutions of* (QP) *and* (QP$_\mathrm{C}$) *respectively, then*

$$\mathrm{Opt}^+ (\mathrm{QP_C}) := \left\{ \begin{pmatrix} 1 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & X \end{pmatrix} \middle| (\mathbf{x}, X) \in \mathrm{Opt\,(QP_C)} \right\}$$

$$= \mathrm{conv} \left\{ \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}^\mathsf{T} \middle| \mathbf{x} \in \mathrm{Opt(QP)} \right\} + \mathrm{conv} \left\{ \begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix}^\mathsf{T} \middle| \begin{matrix} \mathbf{d} \in L_\infty, \\ \mathbf{d}^\mathsf{T} Q \mathbf{d} = 0 \end{matrix} \right\}.$$

Proof From the previous lemma we have that $\mathrm{Feas}^+ (\mathrm{QP_C}) = \mathrm{Feas}^+(\mathrm{QP}) + L_\infty^+$. We now split this proof into two cases:

Case 1 $\mathrm{Feas}^+(\mathrm{QP}) = \emptyset$: Then $\mathrm{Feas}^+ (\mathrm{QP_C}) = \emptyset$ and the results immediately follow.

Case 2 $\mathrm{Feas}^+(\mathrm{QP}) \neq \emptyset$: We shall show that $\langle Q, D \rangle \geq 0$ for all $\begin{pmatrix} 0 & \mathbf{0}^\mathsf{T} \\ \mathbf{0} & D \end{pmatrix} \in L_\infty^+$, from which the required results immediately follow.

We consider an arbitrary $Z \in \mathrm{Feas}^+(\mathrm{QP})$ and $\begin{pmatrix} 0 & \mathbf{0}^\mathsf{T} \\ \mathbf{0} & D \end{pmatrix} \in L_\infty^+$. As L_∞^+ is a cone, for all $\lambda \geq 0$ we have that $Z + \lambda \begin{pmatrix} 0 & \mathbf{0}^\mathsf{T} \\ \mathbf{0} & D \end{pmatrix} \in \mathrm{Feas}^+(\mathrm{QP}) + L_\infty^+ = \mathrm{Feas}^+ (\mathrm{QP_C})$. Therefore we must have that $\langle Q, D \rangle \geq 0$, otherwise $\mathrm{Val\,(QP_C)} = -\infty$. □

We next consider Assumption 4 in addition to Assumptions 1 and 2.

Lemma 9 *Consider problems* (QP) *and* (QP$_\mathrm{C}$) *such that Assumptions 1, 2 and 4 hold. Then* $\mathrm{Feas}^+(\mathrm{QP}) + L_\infty^+ = \mathrm{Feas}^+ (\mathrm{QP_C}) = \mathrm{cl\,(Feas}^+(\mathrm{QP}))$.

Proof If $\mathrm{Feas(QP)} = \emptyset$ then both $\mathrm{Feas}^+(\mathrm{QP})$ and $\mathrm{Feas}^+ (\mathrm{QP_C})$ are empty. Therefore, in this case, $\emptyset = \mathrm{Feas}^+(\mathrm{QP}) + L_\infty^+ = \mathrm{Feas}^+ (\mathrm{QP_C}) = \mathrm{cl\,(Feas}^+(\mathrm{QP}))$, and we are done. From now on we shall assume that $\mathrm{Feas(QP)} \neq \emptyset$.

From Lemma 7 we have $\text{Feas}^+(\text{QP}) + L_\infty^+ = \text{Feas}^+(\text{QP}_C)$, and in the remainder of this proof we shall show that we also have $\text{Feas}^+(\text{QP}) + L_\infty^+ \subseteq \text{cl}(\text{Feas}^+(\text{QP})) \subseteq \text{Feas}^+(\text{QP}_C)$.

$\text{Feas}^+(\text{QP}_C)$ is the intersection of closed sets, and so is itself closed. Therefore, we have $\text{Feas}^+(\text{QP}_C) = \text{cl}(\text{Feas}^+(\text{QP}) + L_\infty^+) \supseteq \text{cl}(\text{Feas}^+(\text{QP}))$.

We now consider an arbitrary $Y \in \text{Feas}^+(\text{QP})$ and $\begin{pmatrix} 0 & \mathbf{0}^\mathsf{T} \\ \mathbf{0} & D \end{pmatrix} \in L_\infty^+$. By Carathéodory's Theorem we have that for M equal to the dimension of the space containing $\mathcal{C}_{\{1\}\times\mathcal{K}}^*$, there exists $\mathbf{d}_1, \ldots, \mathbf{d}_M \in L_\infty$ such that $D = \sum_{i=1}^M \mathbf{d}_i \mathbf{d}_i^\mathsf{T}$. We now consider an arbitrary $\mathbf{x} \in \text{Feas}(\text{QP})$ and note from Assumption 4, that there exists a set $\{\lambda_{i,j} \mid j \in \mathbb{N}, \ i = 1, \ldots, M\}$ such that for all i, j we have $\lambda_{i,j}^2 \geq M\exp(j)$ and $\mathbf{x} + \lambda_{i,j}\mathbf{d}_i \in \text{Feas}(\text{QP})$. We then have the following for all i, j:

$$0 \leq \lambda_{i,j}^{-2}, \qquad \sum_{i=1}^M \lambda_{i,j}^{-2} \leq \exp(-j) \leq 1.$$

From the definition of $\text{Feas}^+(\text{QP})$ we have $\begin{pmatrix} 1 \\ \mathbf{x} + \lambda_{i,j}\mathbf{d}_i \end{pmatrix}\begin{pmatrix} 1 \\ \mathbf{x} + \lambda_{i,j}\mathbf{d}_i \end{pmatrix}^\mathsf{T} \in \text{Feas}^+(\text{QP})$ for all i, j. Now, using the fact that $\text{Feas}^+(\text{QP})$ is convex, we have the following for all $j \in \mathbb{N}$:

$$\left[\sum_{i=1}^M \lambda_{i,j}^{-2}\begin{pmatrix} 1 \\ \mathbf{x} + \lambda_{i,j}\mathbf{d}_i \end{pmatrix}\begin{pmatrix} 1 \\ \mathbf{x} + \lambda_{i,j}\mathbf{d}_i \end{pmatrix}^\mathsf{T} + \left(1 - \sum_{i=1}^M \lambda_{i,j}^{-2} \right)Y \right] \in \text{Feas}^+(\text{QP}).$$

Finally, considering $j \to \infty$, we get $\lambda_{i,j}^{-1} \to 0$, $\lambda_{i,j}^{-2} \to 0$ and $\left(1 - \sum_{i=1}^M \lambda_{i,j}^{-2}\right) \to 1$, which implies $\left[Y + \begin{pmatrix} 0 & \mathbf{0}^\mathsf{T} \\ \mathbf{0} & D \end{pmatrix} \right] \in \text{cl}(\text{Feas}^+(\text{QP}))$. $\square$

We now need the following lemma from [4].

Lemma 10 [4, Lemma 8] *We have that the optimal value of* (QP) *is equal to*

$$\inf\left\{ \left\langle \begin{pmatrix} 0 & \mathbf{c}^\mathsf{T} \\ \mathbf{c} & Q \end{pmatrix}, Y \right\rangle \ \middle| \ Y \in \text{cl}(\text{Feas}^+(\text{QP})) \right\}.$$

Using the obvious fact that the set of optimal solutions of an unbounded problem is empty and $\emptyset + \mathcal{X} = \emptyset$ for every set $\mathcal{X}$, we immediately get the following theorem from the previous two lemmas and the proof of Theorem 8.

Theorem 11 *Consider problems* (QP) *and* (QP$_C$) *such that Assumptions* 1, 2 *and* 4 *hold. Then* $\text{Val}(\text{QP}) = \text{Val}(\text{QP}_C)$, *and*

$$\text{Opt}^+(\text{QP}_C) = \text{conv}\left\{ \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}^\mathsf{T} \ \middle| \ \mathbf{x} \in \text{Opt}(\text{QP}) \right\} + \text{conv}\left\{ \begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix}^\mathsf{T} \ \middle| \ \begin{matrix} \mathbf{d} \in L_\infty, \\ \mathbf{d}^\mathsf{T}Q\mathbf{d} = 0 \end{matrix} \right\}.$$

3 Main theorem

We consider the following assumptions, which are simpler to verify and are more commonly used in the literature. Recall for these that $\mathcal{K} \subseteq \mathbb{R}^n$ is nonempty.

Assumption 5 $\mathcal{K}$ is closed and $\mathcal{B} = \emptyset$.

Assumption 6 $\mathcal{K}$ is closed and bounded.

Assumption 7 $\mathcal{K}$ is closed and for all $\mathbf{d} \in \mathcal{K}_\infty$, $\mathbf{x} \in \mathcal{K}$, $\widetilde{\lambda} \in \mathbb{R}_+$, there exists $\lambda \in \mathbb{R} \setminus (-\widetilde{\lambda}, \widetilde{\lambda})$ such that $\mathbf{x} + \lambda \mathbf{d} \in \mathcal{K}$.

Assumption 8 $\mathcal{K}$ is closed and $\mathcal{K}_\infty = \operatorname{recc}(\mathcal{K})$, where

$$\operatorname{recc}(\mathcal{K}) := \{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{x} + \omega \mathbf{d} \in \mathcal{K} \quad \text{for all } \mathbf{x} \in \mathcal{K}, \ \omega \in \mathbb{R}_+\}.$$

(Note that, from the definition, we have $\mathcal{K}_\infty \supseteq \operatorname{recc}(\mathcal{K})$.)

Assumption 9 $\mathcal{K}$ is closed and convex.

We now consider the following lemmas connected to these assumption.

Lemma 12 *If Assumption* 5 *holds then Assumption* 2 *also holds.*

Proof This is trivial to see. □

Lemma 13 *If Assumption* 6 *holds then Assumptions* 2 *and* 4 *also hold.*

Proof If $\mathcal{K}$ is bounded then $\mathcal{K}_\infty = \{\mathbf{0}\}$ and the implication immediately follows. □

Lemma 14 *If Assumptions* 1 *and* 7 *hold then Assumptions* 2 *and* 4 *also hold.*

Proof Suppose there exists $\mathbf{x} \in \mathcal{K}$ such that $\mathbf{a}_i^\mathsf{T} \mathbf{x} = b_i$ for all i. Then for all $\widetilde{\lambda} \in \mathbb{R}_+$, $\mathbf{d} \in L_\infty \subseteq \mathcal{K}_\infty$, there exists $\lambda \in \mathbb{R} \setminus (-\widetilde{\lambda}, \widetilde{\lambda})$ such that $\mathbf{x} + \lambda \mathbf{d} \in \mathcal{K}$. We then also have that $\mathbf{a}_i^\mathsf{T}(\mathbf{x} + \lambda \mathbf{d}) = b_i$ for all i. Now combining this observation with Assumption 1 and $\mathcal{K}$ being closed, we get that Assumption 2 holds, from which we get that Assumption 4 also holds. □

Lemma 15 *If Assumption* 8 *holds then Assumption* 7 *also holds.*

Proof Trivial from the definition of the recession cone. □

Lemma 16 *If Assumption* 9 *holds then* 8 *also holds.*

Proof This is a well known result. □

Combining these lemmas with Theorems 8 and 11 gives us the following theorem, which is the best approximation to the disproved Theorem 10 from [4].

Theorem 17 *Consider problems* (QP) *and* (QP$_C$) *such that Assumption* 1 *holds and at least one of the following is true:*

1. *Assumptions 2 and 3 hold,*
2. *Assumptions 2 and 4 hold,*
3. *Assumptions 3 and 5 hold,*
4. *Assumptions 4 and 5 hold,*
5. *Assumption 6 holds,*
6. *Assumption 7 holds,*
7. *Assumption 8 holds,*
8. *Assumption 9 holds.*

Then $\mathrm{Val(QP)} = \mathrm{Val}\,(\mathrm{QP_C})$ *and we have that*

$$
\mathrm{Opt}^+\,(\mathrm{QP_C}) = \mathrm{conv}\left\{ \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}^\mathsf{T} \;\middle|\; \mathbf{x} \in \mathrm{Opt(QP)} \right\} + \mathrm{conv}\left\{ \begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix}^\mathsf{T} \;\middle|\; \begin{array}{l} \mathbf{d} \in L_\infty, \\ \mathbf{d}^\mathsf{T} Q\mathbf{d} = 0 \end{array} \right\}.
$$

Note from this theorem that, in such cases as described, $\mathrm{Opt(QP)} = \emptyset$ if and only if $\mathrm{Opt}\,(\mathrm{QP_C}) = \emptyset$. In other words, the optimal solution in (QP) is attained if and only if the optimal solution to $(\mathrm{QP_C})$ is attained.

4 Dealing with one quadratic inequality constraint

In this section we show that Corollary 14 from [4], stating that Theorem 10 from the same paper is valid for $\mathcal{K}$ defined by a (possibly non-convex) quadratic inequality, is still true even though Theorem 10 is disproved. Recall that we consider $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^\mathsf{T} P\mathbf{x} + 2\mathbf{p}^\mathsf{T}\mathbf{x} + p_0 \leq 0\}$ with $P \in \mathcal{S}^n$, $\mathbf{p} \in \mathbb{R}^n$ and $p_0 \in \mathbb{R}$, such that $\mathcal{K} \neq \emptyset$.

First we provide an explicit description of $\mathcal{K}_\infty$.

Theorem 18 *For* $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^\mathsf{T} P\mathbf{x} + 2\mathbf{p}^\mathsf{T}\mathbf{x} + p_0 \leq 0\}$ *we have*

1. $\mathcal{K}_\infty \subseteq \{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{d}^\mathsf{T} P\mathbf{d} \leq 0\}$,
2. *If* $P \notin \mathcal{S}_+$ *then* $\mathcal{K}_\infty = \{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{d}^\mathsf{T} P\mathbf{d} \leq 0\} = \mathrm{cl}\,\{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{d}^\mathsf{T} P\mathbf{d} < 0\}$,
3. *If* $P \in \mathcal{S}_+$ *and* $\mathcal{K} \neq \emptyset$, *then* $\mathcal{K}_\infty = \mathrm{recc}(\mathcal{K}) = \{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{d}^\mathsf{T} P\mathbf{d} = 0,\ \mathbf{p}^\mathsf{T}\mathbf{d} \leq 0\}$.

Proof We shall prove each point in turn.

1. We consider an arbitrary $\mathbf{d} \in \mathcal{K}_\infty$. There exists a sequence $\{(\lambda_i, \mathbf{x}_i) \mid i \in \mathbb{N}\} \subseteq \mathbb{R}_+ \times \mathcal{K}$ such that $\lim_{i\to\infty} \lambda_i = 0$ and $\lim_{i\to\infty} \lambda_i \mathbf{x}_i = \mathbf{d}$. We then have that

$$
\begin{aligned}
0 &\geq \lim_{i\to\infty} (\lambda_i^2 (\mathbf{x}_i^\mathsf{T} P\mathbf{x}_i + 2\mathbf{p}^\mathsf{T}\mathbf{x}_i + p_0)) \\
&= \lim_{i\to\infty} ((\lambda_i \mathbf{x}_i)^\mathsf{T} P(\lambda_i \mathbf{x}_i) + 2\lambda_i \mathbf{p}^\mathsf{T}(\lambda_i \mathbf{x}_i) + \lambda_i^2 p_0)) \\
&= \mathbf{d}^\mathsf{T} P\mathbf{d}.
\end{aligned}
$$

2. We consider an arbitrary $\mathbf{d} \in \mathbb{R}^n$ such that $\mathbf{d}^\mathsf{T} P\mathbf{d} < 0$. There exists a $\hat{\omega} > 0$ such that for all $\omega \geq \hat{\omega}$ we have $\omega^2 \mathbf{d}^\mathsf{T} P\mathbf{d} + 2\omega \mathbf{p}^\mathsf{T}\mathbf{d} + p_0 \leq 0$. We now define the sequence $\{(\lambda_i, \mathbf{x}_i) \mid i \in \mathbb{N}\} \subseteq \mathbb{R}_+ \times \mathcal{K}$ such that $\lambda_i = (\hat{\omega} + i)^{-1}$ and

$\mathbf{x}_i = (\hat{\omega} + i)\mathbf{d}$. Using this sequence, we can immediately see that $\mathbf{d} \in \mathcal{K}_\infty$, and thus $\mathcal{K}_\infty \supseteq \{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{d}^\mathsf{T} P\mathbf{d} < 0\}$.

We now suppose that $P \notin \mathcal{S}_+$ and consider an arbitrary $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}^\mathsf{T} P\mathbf{x} = 0$. There exists $\mathbf{y}$ such that $\mathbf{y}^\mathsf{T} P\mathbf{y} < 0$. As $(-\mathbf{y})^\mathsf{T} P(-\mathbf{y}) = \mathbf{y}^\mathsf{T} P\mathbf{y} < 0$, we may assume w.l.o.g. that $\mathbf{x}^\mathsf{T} Q\mathbf{y} \leq 0$. Then for all $\varepsilon > 0$ we have $(\mathbf{x} + \varepsilon\mathbf{y})^\mathsf{T} P(\mathbf{x} + \varepsilon\mathbf{y}) < 0$. Therefore, by the previous paragraph, $\mathbf{x} + \varepsilon\mathbf{y} \in \mathcal{K}_\infty$ for all $\varepsilon > 0$. Hence, using the fact that $\mathcal{K}_\infty$ is a closed cone, we have $\mathbf{x} \in \mathrm{cl}\{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{d}^\mathsf{T} P\mathbf{d} < 0\} \subseteq \mathrm{cl}\,\mathcal{K}_\infty = \mathcal{K}_\infty$.

3. From now on we suppose that $P \in \mathcal{S}_+$ and $\mathcal{K} \neq \emptyset$.

We first consider an arbitrary $\mathbf{d} \in \mathcal{K}_\infty$. From the first part, and $P \in \mathcal{S}_+$, we have $\mathbf{d}^\mathsf{T} P\mathbf{d} = 0$, and there exists a sequence $\{(\lambda_i, \mathbf{x}_i) \mid i \in \mathbb{N}\} \subseteq \mathbb{R}_+ \times \mathcal{K}$ such that $\lim_{i \to \infty} \lambda_i = 0$ and $\lim_{i \to \infty} \lambda_i \mathbf{x}_i = \mathbf{d}$. We then have that

$$
\begin{aligned}
2\mathbf{p}^\mathsf{T}\mathbf{d} &= \lim_{i \to \infty} (\lambda_i 2\mathbf{p}^\mathsf{T}\mathbf{x}_i) \\
&\leq \lim_{i \to \infty} (\lambda_i(-\mathbf{x}_i^\mathsf{T} P\mathbf{x}_i - p_0)) \\
&\leq \lim_{i \to \infty} (-\lambda_i p_0) \\
&= 0.
\end{aligned}
$$

Therefore $\mathcal{K}_\infty \subseteq \{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{d}^\mathsf{T} P\mathbf{d} = 0,\ \mathbf{p}^\mathsf{T}\mathbf{d} \leq 0\}$.

We next consider an arbitrary $\mathbf{d} \in \mathbb{R}^n$ such that $\mathbf{d}^\mathsf{T} P\mathbf{d} = 0$ and $\mathbf{p}^\mathsf{T}\mathbf{d} \leq 0$. As $P \in \mathcal{S}_+$, we have $P\mathbf{d} = \mathbf{0}$. Now, for any $\mathbf{x} \in \mathcal{K}$ and $\omega \geq 0$, we observe that

$$
\begin{aligned}
(\mathbf{x} + \omega\mathbf{d})^\mathsf{T} P(\mathbf{x} + \omega\mathbf{d}) &+ 2\mathbf{p}^\mathsf{T}(\mathbf{x} + \omega\mathbf{d}) + p_0 \\
&= \underbrace{\mathbf{x}^\mathsf{T} P\mathbf{x} + 2\mathbf{p}^\mathsf{T}\mathbf{x} + p_0}_{\leq 0} + \underbrace{2\omega\mathbf{p}^\mathsf{T}\mathbf{d}}_{\leq 0} \leq 0,
\end{aligned}
$$

and thus $\mathbf{x} + \omega\mathbf{d} \in \mathcal{K}$, from which it can immediately be seen that $\mathbf{d} \in \mathrm{recc}(\mathcal{K}) \subseteq \mathcal{K}_\infty$.
□

We now get the following theorem connecting the quadratic case to one of our earlier assumptions.

Theorem 19 *For $\mathcal{K}$ as given at the start of this section, we have that Assumption 7 holds.*

Proof We consider an arbitrary $\mathbf{d} \in \mathcal{K}_\infty$, $\mathbf{x} \in \mathcal{K}$, $\widetilde{\lambda} \in \mathbb{R}_+$, and let $\lambda \in \mathbb{R} \setminus (-\widetilde{\lambda}, \widetilde{\lambda})$ such that

$$
\lambda = \begin{cases} -\widetilde{\lambda} & \text{if } \mathbf{d}^\mathsf{T}(2P\mathbf{x} + 2\mathbf{p}) \geq 0 \\ +\widetilde{\lambda} & \text{if } \mathbf{d}^\mathsf{T}(2P\mathbf{x} + 2\mathbf{p}) < 0. \end{cases}
$$

From the previous theorem we have that $\mathbf{d}^\mathsf{T} P \mathbf{d} \leq 0$, and thus

$$(\mathbf{x} + \lambda \mathbf{d})^\mathsf{T} P (\mathbf{x} + \lambda \mathbf{d}) + 2\mathbf{p}^\mathsf{T} (\mathbf{x} + \lambda \mathbf{d}) + p_0$$
$$= \underbrace{\mathbf{x}^\mathsf{T} P \mathbf{x} + 2\mathbf{p}^\mathsf{T} \mathbf{x} + p_0}_{\leq 0} + \underbrace{\lambda \mathbf{d}^\mathsf{T} (2P\mathbf{x} + 2\mathbf{p})}_{\leq 0} + \underbrace{\lambda^2 \mathbf{d}^\mathsf{T} P \mathbf{d}}_{\leq 0} \leq 0.$$

Therefore $\mathbf{x} + \lambda \mathbf{d} \in \mathcal{K}$, and so Assumption 7 holds. □

Corollary 20 *Corollary* 14 *from* [4] *is correct.*

5 Conclusion

Due to precise observation of P.J.C. Dickinson, it has been found that the proof of Lemma 9 in [4] by Eichfelder and Povh has a nontrivial gap. All three authors of this erratum paper worked jointly to provide new proofs to show that although Theorem 10 from [4] is not true in general, it is possible to correct this by adding some commonly used assumptions, as presented in Theorem 17. Using this, the three authors of this erratum were then able to show that Corollary 14 from [4] is fortunately still correct.

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