



Sarmanov bivariate distribution: dependence structure—Fisher information in order statistics and their concomitants

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Abstract

The Sarmanov family of bivariate distributions is considered as the most flexible and efficient extended families of the traditional Farlie–Gumbel–Morgenstern family. The goal of this work is twofold. The first part focuses on revealing some novel aspects of the Sarmanov family's dependency structure. In the second part, we study the Fisher information (FI) related to order statistics (OSs) and their concomitants about the shape-parameter of the Sarmanov family. The FI helps finding information contained in singly or multiply censored bivariate samples from the Sarmanov family. In addition, the FI about the mean and shape parameter of exponential and power distributions in concomitants of OSs is evaluated, respectively. Finally, the cumulative residual FI in the concomitants of OSs based on the Sarmanov family is derived.

Keywords Fisher information · Concomitants · Order statistics · Sarmanov family · Cumulative residual Fisher information

Mathematics Subject Classification 62B10 · 62G30

1 Introduction

The Farlie-Gumbel-Morgenstern (FGM) family is an important and seminal class of bivariate distributions. For a given set of marginal distribution functions (DFs) $F_X(\cdot)$ and $F_Y(\cdot)$, the FGM family is defined as $G_{X,Y}(x, y) = F_X(x)F_Y(y)[1 + \lambda \bar{F}_X(x)\bar{F}_Y(y)]$, $-1 \leq \lambda \leq 1$, where $\bar{F}_X(x)$ and $\bar{F}_Y(y)$ are the survival functions of the random variables (RVs) X and Y , respectively. One important limitation of the FGM family is that the correlation coefficient between its marginals is restricted to a

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narrow range $[-\frac{1}{3}, \frac{1}{3}]$, where the maximum value of the correlation is attained for the FGM copula, i.e., with uniform marginals (cf. Schucany et al. [33]). Accordingly, this family can be used to model the bivariate data that exhibits low correlation. For this reason, since the inception of this family, several extensions of it have been introduced in the literature in an attempt to improve the correlation level. Among these extensions, we recall the one-shape-parameter Sarmanov family, which was suggested by Sarmanov [30] as a new mathematical model of hydrological processes. Its copula (with uniform marginals $F_X(x) = x$, $F_Y(y) = y$, $0 \leq x, y \leq 1$) is given by

$$F_{X,Y}(x, y; \alpha) = xy(1 + 3\alpha(1-x)(1-y) + 5\alpha^2(1-x)(1-y)\varphi(x)\varphi(y)), \quad |\alpha| \leq \frac{\sqrt{7}}{5}, \quad (1.1)$$

where $\varphi(x) = 2x - 1$. The corresponding PDF is given by

$$f_{X,Y}(x, y; \alpha) = 1 + 3\alpha\varphi(x)\varphi(y) + \frac{5}{4}\alpha^2(3\varphi^2(x) - 1)(3\varphi^2(y) - 1). \quad (1.2)$$

The correlation coefficient of the Sarmanov copula is α , which means that the copula's maximum correlation coefficient is 0.529 (cf. Balakrishnan and Lai, [11]; page 74). Clearly, the Sarmanov family is the most efficient one among many known extended families of the FGM family because on both the positive and negative sides, it delivers the best improvement in the correlation level. Barakat et al. [17] and Husseiny et al. [21] revisited the Sarmanov family with general marginals F_X and F_Y (denoted by $\text{SAR}(\alpha)$) and showed that this family is an extension of the FGM family and it belongs to a wider family suggested by Sarmanov [29], which has many recorded applications in the literature (see, e.g., Bairamov et al. [10], and Tank and Gebizlioglu, [37]). Moreover, they showed that the $\text{SAR}(\alpha)$ is the only one of the extended families of FGM with a radially symmetric copula about (0.5, 0.5). This property was used by them to reveal several prominent statistical properties for the concomitants of OSs and record values from this family and some of information measures, namely, the Shannon entropy, inaccuracy measure, extropy, cumulative entropy, and Fisher information number (FIN) were studied theoretically and numerically.

Due to its relevance in a wide range of scientific issues, concomitants of OSs have got a lot of attention in recent years. Suppose (X_i, Y_i) , $i = 1, 2, \dots, n$, is a random sample from a bivariate DF $F_{X,Y}$. If we order the sample by the X -variate, and obtain the OSs $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ for the X sample, then the Y -variate associated with the r th OS $X_{r:n}$ is the concomitant of the r th OS, and is denoted by $Y_{[r:n]}$, $r = 1, \dots, n$. The concept of concomitants of OSs was first introduced by David [20] and almost simultaneously under the name of induced OSs by Bhattacharya [19]. Biological selection problem is the most striking application of concomitants of OSs. Another application of concomitants of OSs is in reliability theory, see Barakat et al. [18]. Scaria and Mohan [31] investigated the concomitants of record values from the Cambanis family with logistic marginals. Moreover, Scaria and Mohan [32] investigated the concomitants of OSs based on the FGM and Cambanis families with exponentiated exponential marginals. Recently, Abd Elgawad et al. [4] studied the

concomitants of m -generalized OSs of the Cambanis family and some information measures. Furthermore, Alawady et al. [7] studied the concomitants of m -generalized OSs from the Cambanis family under general setting. For more recent works about this subject see Abd Elgawad and Alawady [1], Abd Elgawad et al. [2–5], Alawady et al. [7–9], Barakat et al. [14, 16], Jafari et al. [22], and Tahmasebi et al. [35, 36].

The FI is a way of determining how much information an observable RV has about an unknown parameter on which the probability of that RV depends. The FI is an important and fundamental criterion in statistical inference, physics, thermodynamics, information theory and some other fields. Traditionally, the FI has played a valuable role in statistical inference through the Cramér–Rao inequality and its association with the asymptotic properties of the maximum likelihood estimators. Knowing how much information a sample contains about an unknown parameter can assist determine bounds on the variance of a particular estimator of that parameter and approximate the sampling distribution of that estimator when the sample is large enough.

Consider a RV X , which has a PDF $f_X(x; \theta)$, where $\theta \in \Theta$ is an unknown parameter with a parameter space Θ . Under certain regularity conditions (see, Rao, [28], p. 329 and Abo-Eleneen and Nagaraja, [6]), FI about $\theta \in \Theta$, contained in the RV X , is $I_\theta(X; \theta) := -E\left(\frac{\partial^2 \ln f_X(x; \theta)}{\partial \theta^2}\right) = E\left(\frac{\partial \ln f_X(x; \theta)}{\partial \theta}\right)^2$. For some recent studies on the FI, one may refer to Barakat and Husseiny [13], Barakat et al. [15], and Kharazmi and Asadi [25]. Another important kind of the FI is the FIN, which is the second moment of the score function (the score function of a RV X with a PDF f_X is defined by $\rho(x) = \frac{\frac{\partial f_X(x; \theta)}{\partial x}}{f_X(x; \theta)} = \frac{\partial \ln f_X(x; \theta)}{\partial x}$), where the derivative is with respect to x in a given PDF $f_X(x; \theta)$ rather than the parameter θ . It is a FI for a location parameter. For some recent works about this measure, see Abd Elgawad et al. [2], Barakat and Husseiny [12], Barakat et al. [17], and Tahmasebi and Jafari [34].

In the definition of FI, one should be aware that, the FI is not robust to the presence of outliers. This is because the FI is an expectation with respect to X , it equally weights all X values, including large values that may be outliers. The cumulative residual Fisher information (abbreviated as CF) was recently presented by Kharazmi and Balakrishnan [26], which is naturally robust to the presence of outliers. This modified measure is defined as $CF_\theta(X; \theta) := \int_{-\infty}^{\infty} \left(\frac{\partial \ln \bar{F}_X(x; \theta)}{\partial \theta}\right)^2 \bar{F}_X(x; \theta) dx$, where $\bar{F}_X(x; \theta)$ is the survival function of X . It is worth mentioning that the CF can be used effectively for developing suitable goodness-of-fit tests for lifetime distributions by using empirical versions of this measure. Kharazmi and Balakrishnan [26] analogously defined the modified FIN via the survival function as $CF(X; \theta) := \int_{-\infty}^{\infty} \left(\frac{\partial \ln \bar{F}_X(x; \theta)}{\partial x}\right)^2 \bar{F}_X(x; \theta) dx$.

The purpose of this work is twofold. The first part (Sect. 2) is to reveal some new salient features pertaining to the dependence structure of SAR(α), where some bounds and relationships are shown for the correlation coefficient. In the second part (Sects. 3, 4, and 5), we theoretically and numerically study in Sect. 3 the FI, $I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha)$, for a single pair $(X_{r:n}, Y_{[r:n]})$. Moreover, in Sect. 4, we theoretically and numerically discuss the FI about the mean of the exponential distribution marginals and the

shape parameter of the power function distribution marginals for SAR(α). In Sect. 5, $CF(Y_{[r:n]}; \alpha)$ is theoretically and numerically investigated.

2 Dependence structure

In this section, we first obtain the maximum bounds for the correlation coefficient, between two arbitrary continuous RVs with finite nonzero variances, which are arising from the SAR(α), whose copula is defined by (1.1). Secondly, we deduce the sufficient conditions under which the marginal components become uncorrelated, among these conditions, we have a direct condition that $\alpha = 0$. The upper bound for the correlation coefficient, given in the following theorem, is so large as to confirm the usefulness and distinction of the Sarmanov family.

Theorem 2.1 *Let $F_X(x)$ and $F_Y(y)$ be the DFs of some RVs X and Y , respectively. If the RVs X and Y are bivariate SAR(α) with finite nonzero variances and correlation coefficient, $\rho(X, Y)$, where $F_X = F_Y$, then $\rho(X, Y) \leq 0.8091502$.*

Proof First, we assume that $F_X(x)$ and $F_Y(y)$ are different. Then in view of (1.1) and (1.2) and by using the result of Schucany et al. [33], we get

$$\begin{aligned}
 COV(X, Y) &= 3\alpha\delta_X\delta_Y + \frac{5}{4}\alpha^2 \int_{-\infty}^{\infty} x(3\varphi^2(F_X(x)) - 1)dF_X(x) \\
 &\quad \times \int_{-\infty}^{\infty} y(3\varphi^2(F_Y(y)) - 1)dF_Y(y), \tag{2.1}
 \end{aligned}$$

where $\delta_j = \int_{-\infty}^{\infty} x dF_j^2(x) - \int_{-\infty}^{\infty} x dF_j(x) = \int_{-\infty}^{\infty} x\varphi^2(F_j(x))dF_j(x)$, $j = X, Y$. Clearly, $\delta_j > 0$, $j = X, Y$, since δ_j is the difference between the mean of the larger of two independent RVs from F_j and the mean of F_j (cf. Johnson and Kotz, [24], Schucany et al. [33]). On the other hand, we can easily verify that

$$\begin{aligned}
 W_j &:= \int_{-\infty}^{\infty} x(3\varphi^2(F_j(x)) - 1)dF_j(x) \\
 &= 4 \int_{-\infty}^{\infty} x dF_j^3(x) - 6 \int_{-\infty}^{\infty} x dF_j^2(x) + 2 \int_{-\infty}^{\infty} x dF_j(x) \\
 &= 4 \left(\int_{-\infty}^{\infty} x dF_j^3(x) - \int_{-\infty}^{\infty} x dF_j^2(x) \right) - 2 \left(\int_{-\infty}^{\infty} x dF_j^2(x) - \int_{-\infty}^{\infty} x dF_j(x) \right) \\
 &:= 4W_{j1} - 2W_{j2}. \tag{2.2}
 \end{aligned}$$

Moreover, we can show that

$$W_j = 0, \text{ if } F_j(x) = x, \text{ i.e., for the uniform marginal case.} \tag{2.3}$$

Note that, W_{j1} is the difference between the mean of the larger of three independent RVs from F_j and the mean of two independent RVs from F_j . Furthermore, W_{j2} is the

difference between the mean of the larger of two independent RVs from F_j and the mean of F_j . Then we get $W_{ji} > 0, i = 1, 2$. Upon combining (2.1) and (2.2), we get

$$COV(X, Y) = 3\alpha\delta_X\delta_Y + \frac{5}{4}\alpha^2W_XW_Y, \tag{2.4}$$

where for the uniform marginal DFs F_U and F_V , (2.3) entails that $COV(U, V) = 3\alpha\delta_U\delta_V = \frac{\alpha}{12}$, which implies, as it is known, $\rho(U, V) = \alpha$. Now, let $F := F_X(x) = F_Y(y)$, where F has finite nonzero variance, σ^2 , and finite mean, μ . Furthermore, let $W := W_j, j = X, Y$. Then, we get

$$W^2 = \left(\int_{-\infty}^{\infty} x(3\varphi^2(F(x)) - 1)dF(x) \right)^2 = \left(\int_{-\infty}^{\infty} (x - \mu)(3\varphi^2(F(x)) - 1)dF(x) \right)^2,$$

since $\int_{-\infty}^{\infty} (3\varphi^2(F(x)) - 1)dF(x) = \int_0^1 (3\varphi^2(u) - 1)du = 0$. Thus, by applying Cauchy–Schwartz inequality, we get

$$\begin{aligned} W^2 &\leq \sigma^2 \int_{-\infty}^{\infty} (3\varphi^2(F(x)) - 1)^2 dF(x) = \sigma^2 \int_0^1 (3\varphi^2(u) - 1)^2 du \\ &= \frac{\sigma^2}{2} \int_{-1}^1 (3v^2 - 1)^2 dv = 0.8\sigma^2. \end{aligned} \tag{2.5}$$

In view of the inequality $\alpha \leq \frac{\sqrt{7}}{5}$ and the result of Schucany et al. [33], a combination between (2.5) and (2.4), thus yields $\rho(X, Y) \leq 0.8091502$. This completes the proof. □

Corollary 2.1 *Under the conditions of Theorem 2.1, possibly except the condition $F_X = F_Y$, we have $0.8091502 \leq \max \rho(X, Y) \leq 1$.*

Proof The proof follows from the general relation $\max\{x : x \in A\} \leq \max\{x : x \in B\}$, if $A \subseteq B$, and the result of Theorem 2.1. □

Corollary 2.2 *Let the conditions of Theorem 2.1, possibly except the condition $F_X = F_Y$, be satisfied. Assume that $W_X, W_Y \neq 0, \alpha \neq 0$, and $|\frac{\delta_X\delta_Y}{W_XW_Y}| \leq \frac{\sqrt{7}}{12}$. Then, we get the following interesting result $\rho(X, Y) = 0$, while X and Y are dependent, if $\alpha = \frac{-12}{5} \frac{\delta_X\delta_Y}{W_XW_Y}$.*

Proof The proof follows directly from (2.4). □

3 FI in $(X_{r:n}, Y_{[r:n]})$ about α based on the copula of SAR(α)

Let X and Y be uniformly distributed RVs over $(0, 1)$, written $X, Y \sim U(0, 1)$, and let them be jointly distributed as the Sarmanov copula (1.1). This copula is free of any

unknown parameters except the parameter α . The joint PDF of $(X_{r:n}, Y_{[r:n]})$ is given by (cf. Abo-Eleneen and Nagaraja [6], Barakat et al. [15])

$$f_{[r:n]}(x, y; \alpha) = C_{r,n} f_{X,Y}(x, y; \alpha) F_X(x)^{r-1} (1 - F_X(x))^{n-r}, \tag{3.1}$$

where $C_{r,n} = \frac{1}{\beta(r, n-r+1)}$ and $\beta(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} du$, $a, b > 0$, is the beta function.

3.1 Theoretical result

Before formulating Theorem 3.1 about the FI in $(X_{r:n}, Y_{[r:n]})$, we consider the set $\Omega = \{\alpha : | 3\alpha\varphi(x)\varphi(y) + \frac{5}{4}\alpha^2(3\varphi^2(x) - 1)(3\varphi^2(y) - 1) | < 1, \forall 0 \leq x, y \leq 1\}$. From now on, we deal only with $\alpha \in \Omega \cap \Upsilon$, where $\Upsilon = \{\alpha : |\alpha| \leq \frac{\sqrt{7}}{5}\}$. Since $\alpha = 0 \in \Omega \cap \Upsilon$, then the set $\Omega \cap \Upsilon$ is not empty. On the other hand, the set $\{\alpha : | 3\alpha\varphi(x)\varphi(y) + \frac{5}{4}\alpha^2(3\varphi^2(x) - 1)(3\varphi^2(y) - 1) | > 1, \forall 0 \leq x, y \leq 1\} = \emptyset$ (empty set). Therefore, $\Omega \cup \tilde{\Omega} = \mathcal{U}$ (while $\Omega \cap \tilde{\Omega} \neq \emptyset$), where \mathcal{U} is the universal set and $\tilde{\Omega} = \{\alpha : | 3\alpha\varphi(x)\varphi(y) + \frac{5}{4}\alpha^2(3\varphi^2(x) - 1)(3\varphi^2(y) - 1) | < 1, 0 \leq x \leq x_0, 0 \leq y \leq y_0, \text{ and } | 3\alpha\varphi(x)\varphi(y) + \frac{5}{4}\alpha^2(3\varphi^2(x) - 1)(3\varphi^2(y) - 1) | \geq 1, x > x_0, y > y_0, \text{ for some } 0 < x_0, y_0 < 1\}$. In order to check $\alpha \in \Omega$, for any $\alpha \in \Upsilon$, draw the function $\mathcal{F}(x, y; \alpha) = | 3\alpha\varphi(x)\varphi(y) + \frac{5}{4}\alpha^2(3\varphi^2(x) - 1)(3\varphi^2(y) - 1) |, 0 \leq x, y \leq 1$, as 3D diagram (x, y, \mathcal{F}) , by using Mathematica 12. If the curve of \mathcal{F} falls entirely within the cube $\mathcal{C} = \{(x, y, z) : -1 \leq x, y, z \leq +1\}$, then $\alpha \in \Omega$, otherwise $\alpha \notin \Omega$. The fact that $\Omega \cup \tilde{\Omega} = \mathcal{U}$ means that we have only two possibilities. The first possibility is that the curve of \mathcal{F} falls entirely within the cube \mathcal{C} , represented by the set Ω . The second possibility is that a portion of that curve falls within \mathcal{C} and the other portion is outside the cube \mathcal{C} , represented by the set $\tilde{\Omega}$. Parts a,b,c, and d of Fig. 1 show how we can achieve this check for some values of $\alpha \in \Omega \cap \Upsilon$.

Theorem 3.1 *Suppose that X and $Y \sim U(0, 1)$ with joint PDF (1.2), then for any $1 \leq r \leq n$, and $\alpha \in \Omega \cap \Upsilon$, the FI in $(X_{r:n}, Y_{[r:n]})$ about α is given by*

$$I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha) = C_{r,n} \sum_{i=0}^\infty \sum_{j=0}^i \binom{i}{j} (-1)^j (\alpha)^i \left(\frac{1}{2}\right)^j [I_1 + I_2 + I_3], \tag{3.2}$$

where

$$I_1 = \sum_{k=0}^{i-j+2} \sum_{l=0}^j \sum_{m=0}^{2l} \sum_{s=0}^{i-j+2} \sum_{t=0}^j \sum_{p=0}^{2t} (-2)^{s+p+k+m} (-3)^{l+t} (3)^{i-j+2} \left(\frac{5\alpha}{2}\right)^j \times \binom{i-j+2}{k} \binom{j}{l} \binom{2l}{m} \binom{i-j+2}{s} \binom{j}{t} \binom{2t}{p} \frac{\beta(k+m+r, n-r+1)}{s+p+1}, \tag{3.3}$$

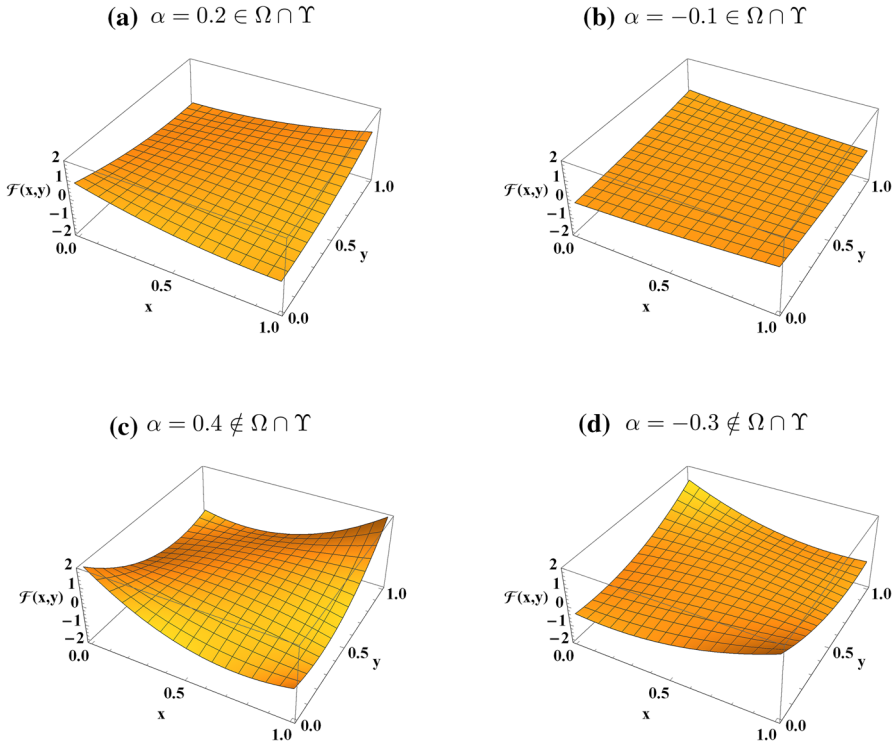


Fig. 1 3D Diagrams for checking the belonging relationship $\alpha \in \Omega \cap \Upsilon$

$$\begin{aligned}
 I_2 = & \sum_{u=0}^{i-j} \sum_{v=0}^{j+2} \sum_{w=0}^{2v} \sum_{z=0}^{i-j} \sum_{c=0}^{j+2} \sum_{b=0}^{2c} (-2)^{u+w+z+b} (-3)^{v+c} (3)^{i-j} \left(\frac{5\alpha}{2}\right)^{j+2} \\
 & \times \binom{i-j}{u} \binom{j+2}{v} \binom{2v}{w} \binom{i-j}{z} \binom{j+2}{c} \binom{2c}{b} \frac{\beta(u+w+r, n-r+1)}{z+b+1},
 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
 I_3 = & \sum_{a=0}^{i-j+1} \sum_{d=0}^{j+1} \sum_{e=0}^{2d} \sum_{f=0}^{i-j+1} \sum_{g=0}^{j+1} \sum_{h=0}^{2g} (-2)^{a+e+f+h} (-3)^{d+g} (3)^{i-j+1} (5\alpha)^{j+1} \left(\frac{1}{2}\right)^j \\
 & \times \binom{i-j+1}{a} \binom{j+1}{d} \binom{2d}{e} \binom{i-j+1}{f} \binom{j+1}{g} \binom{2g}{h} \frac{\beta(a+e+r, n-r+1)}{f+h+1}.
 \end{aligned} \tag{3.5}$$

Proof From (1.2) and (3.1), we get

$$\begin{aligned}
 \ln f_{[r:n]}(x, y; \alpha) = & \ln C_{r,n} + \ln \left(1 + 3\alpha\varphi(x)\varphi(y) + \frac{5}{4}\alpha^2(3\varphi^2(x) - 1)(3\varphi^2(y) - 1) \right) \\
 & + (r-1)\ln x + (n-r)\ln(1-x),
 \end{aligned}$$

Thus,

$$\frac{\partial^2 \ln f_{[r:n]}(x, y; \alpha)}{\partial \alpha^2} = \frac{\frac{5}{2}(3\varphi^2(x) - 1)(3\varphi^2(y) - 1)}{f_{X,Y}(x, y; \alpha)} - \frac{\left(3\varphi(x)\varphi(y) + \frac{5}{2}\alpha(3\varphi^2(x) - 1)(3\varphi^2(y) - 1)\right)^2}{f_{X,Y}^2(x, y; \alpha)}. \tag{3.6}$$

Therefore, by using (3.1), (3.6) and the binomial expansion under the condition $\alpha \in \Omega \cap \Upsilon$, the FI about the shape parameter α is given as

$$\begin{aligned} &I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha) \\ &= -E\left(\frac{\partial^2 \ln f_{[r:n]}(X_{r:n}, Y_{[r:n]}; \alpha)}{\partial \alpha^2}\right) \\ &= C_{r,n} \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^i \binom{i}{j} (\alpha)^i \left(\frac{1}{2}\right)^j \\ &\quad \times \int_0^1 \int_0^1 \left(3\varphi(x)\varphi(y) + \frac{5}{2}\alpha(3\varphi^2(x) - 1)(3\varphi^2(y) - 1)\right)^2 (3\varphi(x)\varphi(y))^{i-j} \\ &\quad \times \left(\frac{5}{2}\alpha(3\varphi^2(x) - 1)(3\varphi^2(y) - 1)\right)^j x^{r-1}(1-x)^{n-r} dx dy \\ &= C_{r,n} \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^i \binom{i}{j} (\alpha)^i \left(\frac{1}{2}\right)^j [I_1 + I_2 + I_3], \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} I_1 &= \int_0^1 \int_0^1 (3\varphi(x)\varphi(y))^{i-j+2} \left(\frac{5}{2}\alpha(3\varphi^2(x) - 1)(3\varphi^2(y) - 1)\right)^j x^{r-1}(1-x)^{n-r} dx dy \\ &= \int_0^1 (3\varphi(x))^{i-j+2} \left(\frac{5}{2}\alpha(3\varphi^2(x) - 1)\right)^j x^{r-1}(1-x)^{n-r} dx \\ &\quad \int_0^1 (\varphi(y))^{i-j+2} (3\varphi^2(y) - 1)^j dy \\ &= \sum_{k=0}^{i-j+2} \sum_{l=0}^j \sum_{m=0}^{2l} \sum_{s=0}^{i-j+2} \sum_{t=0}^j \sum_{p=0}^{2t} (-2)^{s+p+k+m} (-3)^{l+t} (3)^{i-j+2} \left(\frac{5\alpha}{2}\right)^j \\ &\quad \times \binom{i-j+2}{k} \binom{j}{l} \binom{2l}{m} \binom{i-j+2}{s} \binom{j}{t} \binom{2t}{p} \frac{\beta(k+m+r, n-r+1)}{s+p+1}, \end{aligned} \tag{3.8}$$

$$\begin{aligned}
 I_2 &= \int_0^1 \int_0^1 (3\varphi(x)\varphi(y))^{i-j} \left(\frac{5}{2}\alpha(3\varphi^2(x) - 1)\right. \\
 &\quad \left.(3\varphi^2(y) - 1)\right)^{j+2} x^{r-1} (1-x)^{n-r} dx dy \\
 &= \int_0^1 (3\varphi(x))^{i-j} \left(\frac{5}{2}\alpha(3\varphi^2(x) - 1)\right)^{j+2} x^{r-1} (1-x)^{n-r} dx \\
 &\quad \int_0^1 (\varphi(y))^{i-j} (3\varphi^2(y) - 1)^{j+2} dy \\
 &= \sum_{u=0}^{i-j} \sum_{v=0}^{j+2} \sum_{w=0}^{2v} \sum_{z=0}^{i-j} \sum_{c=0}^{j+2} \sum_{b=0}^{2c} (-2)^{u+w+z+b} (-3)^{v+c} (3)^{i-j} \left(\frac{5\alpha}{2}\right)^{j+2} \\
 &\quad \times \binom{i-j}{u} \binom{j+2}{v} \binom{2v}{w} \binom{i-j}{z} \binom{j+2}{c} \binom{2c}{b} \frac{\beta(u+w+r, n-r+1)}{z+b+1}, \tag{3.9}
 \end{aligned}$$

and

$$\begin{aligned}
 I_3 &= \int_0^1 \int_0^1 (3\varphi(x)\varphi(y))^{i-j+1} (5\alpha(3\varphi^2(x) - 1)(3\varphi^2(y) - 1))^{j+1} \\
 &\quad \left(\frac{1}{2}\right)^j x^{r-1} (1-x)^{n-r} dx dy \\
 &= \int_0^1 (3\varphi(x))^{i-j+1} (5\alpha(3\varphi^2(x) - 1))^{j+1} \left(\frac{1}{2}\right)^j x^{r-1} (1-x)^{n-r} dx \\
 &\quad \times \int_0^1 (\varphi(y))^{i-j+1} ((3\varphi^2(y) - 1))^{j+1} dy \\
 &= \sum_{a=0}^{i-j+1} \sum_{d=0}^{j+1} \sum_{e=0}^{2d} \sum_{f=0}^{i-j+1} \sum_{g=0}^{j+1} \sum_{h=0}^{2g} (-2)^{a+e+f+h} (-3)^{d+g} (3)^{i-j+1} (5\alpha)^{j+1} \left(\frac{1}{2}\right)^j \\
 &\quad \times \binom{i-j+1}{a} \binom{j+1}{d} \binom{2d}{e} \binom{i-j+1}{f} \binom{j+1}{g} \binom{2g}{h} \frac{\beta(a+e+r, n-r+1)}{f+h+1}. \tag{3.10}
 \end{aligned}$$

Combining (3.8), (3.9), and (3.10), we get $I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha)$. The theorem is proved. □

3.2 Computing $I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha)$ with discussion

Table 1 displays the FI $I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha)$ as a function of $n, r \leq \frac{n+1}{2}$, and α , for $n = 1, 3, 5, 15, \alpha = -0.2, -0.15, -0.1, 0.1, 0.15, 0.2$, where $\alpha \in \Omega \cap \Upsilon$. The entries are computed using the relations (3.2)–(3.5). We only compute 9 terms from the infinite series included in (3.2), as this procedure gives us satisfactory results. The first row of Table 1 represents the FI $I_\alpha(X, Y; \alpha)$. Based on the fact that the FI $I_\alpha(X, Y; \alpha)$ in a random sample of size n is $nI_\alpha(X, Y; \alpha)$, Table 1 allows us to compute the proportion of the sample FI $I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha)$ contained in a single pair. For example, when $n = 5$, the FI $I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha)$ about α in the extreme pair ranges from 26% to 35.5% of the total information, as α ranges from -0.2 to 0.2 . In addition,

Table 1 FI for $(X_{r:n}, Y_{[r:n]})$ about the parameter α

Sample size n	Rank r	$\alpha = 0.1$	$\alpha = -0.1$	$\alpha = 0.15$	$\alpha = -0.15$	$\alpha = 0.2$	$\alpha = -0.2$
1	1	1.03326	1.03326	1.07828	1.08118	1.31893	0.965084
3	1	1.23267	1.23267	1.27645	1.27657	1.3496	1.36565
3	2	0.634429	0.634428	0.679603	0.67801	0.906024	0.486046
5	1	1.60332	1.60332	1.64701	1.647	1.71117	1.71155
5	2	0.747139	0.747139	0.789877	0.789897	0.860308	0.853786
5	3	0.465364	0.465364	0.513183	0.513331	0.853771	0.602498
15	1	2.37775	2.37775	2.43437	2.43437	2.51868	2.51868
15	2	1.79123	1.79123	1.82747	1.82747	1.88187	1.88188
15	3	1.30306	1.30306	1.33433	1.33433	1.38107	1.38098
15	4	0.90896	0.90896	0.943696	0.943696	0.995571	0.995437
15	5	0.605641	0.605641	0.647302	0.647302	0.709143	0.708444
15	6	0.390654	0.390654	0.439363	0.439339	0.510503	0.509824
15	7	0.262317	0.262317	0.316036	0.315932	0.38843	0.383647
15	8	0.219646	0.219646	0.27528	0.275038	0.366608	0.322666

the FI $I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha)$ in the central pair ranges from 9% to 13% of what is available in the complete sample in all cases. Another useful application of Table 1 is that it may be used to quickly extract the FI contained in singly or multiply censored bivariate data sample from the SAR(α). Simply sum up the FI in each pair that constitutes the censored sample. For example, when $n = 5$, the FI about α in the type-II censored sample consisting of the bottom (or the top) three pairs ranges from 52% to 65% as α ranges from -0.2 to 0.2 . The following interesting features can be extracted from Table 1:

- $I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha)$ increases when the difference between the rank r and the sample size n , increases for $r \leq \frac{n+1}{2}$.
- For fixed n and r , the value of $I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha)$ is close to $I_\alpha(X_{r:n}, Y_{[r:n]}; -\alpha)$. Moreover, $I_\alpha(X_{r:n}, Y_{[r:n]}; \alpha) = I_\alpha(X_{r:n}, Y_{[r:n]}; -\alpha)$ in some cases.

4 FI in concomitant $Y_{[r:n]}$ based on some distributions

Barakat et al. [17] studied the concomitants of OSs based on the Sarmanov family with marginal generalized exponential DFs, where the DF of the generalized exponential distribution is defined as $F_X(x) = (1 - e^{-\theta x})^a$, $x, a, \theta > 0$, and is denoted by $GE(\theta; a)$. Barakat et al. [17] expressed the marginal PDF of the concomitant $Y_{[r:n]}$ as

$$f_{[r:n]}(y; \alpha, \theta) = \left(1 - 3\Delta_{1,r:n}^{(\alpha)} + \frac{5}{2}\Delta_{2,r:n}^{(\alpha)}\right) f_Y(y) + \left(3\Delta_{1,r:n}^{(\alpha)} - \frac{15}{2}\Delta_{2,r:n}^{(\alpha)}\right) f_{V_1}(y) + 5\Delta_{2,r:n}^{(\alpha)} f_{V_2}(y), \quad (4.1)$$

where $V_1 \sim \text{GE}(\theta; 2a)$, $V_2 \sim \text{GE}(\theta; 3a)$, $\Delta_{1,r:n}^{(\alpha)} = \frac{\alpha(2r-n-1)}{n+1}$ and $\Delta_{2,r:n}^{(\alpha)} = 2\alpha^2 \left[1 - 6\frac{r(n-r+1)}{(n+1)(n+2)}\right]$.

Proposition 1 *Let the FI associated with concomitants of OSs based on SAR(α) with marginal DFs GE(θ ; a) about unknown parameter θ be denoted by $I_\theta(Y_{[r:n]}; \alpha, \theta)$. Then,*

1. $I_\theta(Y_{[r:n]}; -\alpha, \theta) = I_\theta(Y_{[n-r+1:n]}; \alpha, \theta)$.
2. $I_\theta(Y_{[\frac{n+1}{2}:n]}; \alpha, \theta) = I_\theta(Y_{[\frac{n+1}{2}:n]}; -\alpha, \theta)$.

Proof From (4.1), we get

$$I_\theta(Y_{[r:n]}; -\alpha, \theta) = E \left[\frac{\partial^2 \ln f_{[r:n]}(Y_{[r:n]}; -\alpha, \theta)}{\partial \theta^2} \right]^2.$$

By applying the easy-check relations

$$\Delta_{1,r:n}^{(\alpha)} = \frac{\alpha(2r-n-1)}{n+1} = \Delta_{1,n-r+1:n}^{(-\alpha)},$$

$$\Delta_{2,r:n}^{(\alpha)} = 2\alpha^2 \left[1 - 6\frac{r(n-r+1)}{(n+1)(n+2)}\right] = \Delta_{2,n-r+1:n}^{(\alpha)}, \text{ and } \Delta_{2,r:n}^{(\alpha)} = \Delta_{2,r:n}^{(-\alpha)}$$

we immediately get the relation $f_{[r:n]}(y; -\alpha, \theta) = f_{[n-r+1:n]}(y; \alpha, \theta)$. The first part of the proposition is thus proved. Also, for the second part we have $\Delta_{1, \frac{n+1}{2}:n}^{(\alpha)} = 2\Delta_{1, \frac{n+1}{2}:n}^{(-\alpha)} = 0$. This completes the proof. □

4.1 FI in $Y_{[r:n]}$ about $E(Y)$ of exponential distribution marginal

By putting $a = 1$ in (4.1), we get the marginal PDF of the concomitant $Y_{[r:n]}$ based on the exponential distribution as

$$f_{[r:n]}(y; \alpha, \theta) = \frac{1}{\theta} \exp\left(\frac{-y}{\theta}\right) + 3\Delta_{1,r:n}^{(\alpha)} \left(\frac{2}{\theta} \exp\left(\frac{-y}{\theta}\right) \left(1 - \exp\left(\frac{-y}{\theta}\right)\right) - \frac{1}{\theta} \exp\left(\frac{-y}{\theta}\right)\right) + \frac{5}{2}\Delta_{2,r:n}^{(\alpha)} \left(\frac{1}{\theta} \exp\left(\frac{-y}{\theta}\right) - \frac{6}{\theta} \exp\left(\frac{-y}{\theta}\right) \left(1 - \exp\left(\frac{-y}{\theta}\right)\right) + \frac{6}{\theta} \exp\left(\frac{-y}{\theta}\right) \left(1 - \exp\left(\frac{-y}{\theta}\right)\right)^2\right).$$

This expression, after some algebra, can be written as

$$f_{[r:n]}(y; \alpha, \theta) = \frac{1}{\theta} \exp\left(\frac{-y}{\theta}\right) \left(A_1 + A_2 \exp\left(\frac{-y}{\theta}\right) + A_3 \exp\left(\frac{-2y}{\theta}\right) \right),$$

where $A_1 = \left(1 + 3\Delta_{1,r:n}^{(\alpha)} + \frac{5}{2}\Delta_{2,r:n}^{(\alpha)}\right)$, $A_2 = -\left(6\Delta_{1,r:n}^{(\alpha)} + 15\Delta_{2,r:n}^{(\alpha)}\right)$, and $A_3 = \left(15\Delta_{2,r:n}^{(\alpha)}\right)$. Therefore,

$$\begin{aligned} & \frac{\partial \ln f_{[r:n]}(y; \alpha, \theta)}{\partial \theta} \\ &= \frac{1}{\theta} \left(\frac{y}{\theta} - 1 + \frac{y}{\theta} \frac{A_2 \exp\left(\frac{-y}{\theta}\right) + 2A_3 \exp\left(\frac{-2y}{\theta}\right)}{A_1 + A_2 \exp\left(\frac{-y}{\theta}\right) + A_3 \exp\left(\frac{-2y}{\theta}\right)} \right) \\ &= \frac{1}{\theta} \left(\frac{2y}{\theta} - 1 - \frac{A_1 \frac{y}{\theta}}{A_1 + A_2 \exp\left(\frac{-y}{\theta}\right) + A_3 \exp\left(\frac{-2y}{\theta}\right)} + \frac{A_3 \frac{y}{\theta} \exp\left(\frac{-2y}{\theta}\right)}{A_1 + A_2 \exp\left(\frac{-y}{\theta}\right) + A_3 \exp\left(\frac{-2y}{\theta}\right)} \right). \end{aligned}$$

Thus,

$$\begin{aligned} \left(\frac{\partial \ln f_{[r:n]}(y; \alpha, \theta)}{\partial \theta} \right)^2 &= \frac{1}{\theta^2} \left(4w^2 + 1 + \frac{A_1^2 w^2}{(A_1 + A_2 e^{-w} + A_3 e^{-2w})^2} \right. \\ &\quad + \frac{A_3^2 w^2 e^{-4w}}{(A_1 + A_2 e^{-w} + A_3 e^{-2w})^2} - 4w - \frac{4A_1 w^2}{A_1 + A_2 e^{-w} + A_3 e^{-2w}} \\ &\quad + \frac{4A_3 w^2 e^{-2w}}{A_1 + A_2 e^{-w} + A_3 e^{-2w}} + \frac{2A_1 w}{A_1 + A_2 e^{-w} + A_3 e^{-2w}} \\ &\quad \left. - \frac{2A_3 w e^{-2w}}{A_1 + A_2 e^{-w} + A_3 e^{-2w}} - \frac{2A_1 A_3 w^2 e^{-2w}}{(A_1 + A_2 e^{-w} + A_3 e^{-2w})^2} \right), \end{aligned} \tag{4.2}$$

where $w = \frac{y}{\theta}$. On the other hand, the PDF of the RV $W = \frac{Y_{[r:n]}}$ is $f_W(w) = e^{-w} (A_1 + A_2 e^{-w} + A_3 e^{-2w})$. Therefore, the relation (4.2) yields

$$I_{\theta}(Y_{[r:n]}; \alpha, \theta) = \int_0^{\infty} \left(\frac{\partial \ln f_W(w)}{\partial \theta} \right)^2 f_W(w) dw = \frac{1}{\theta^2} \sum_{i=1}^{10} T_i,$$

where

$$T_1 = 4 \int_0^{\infty} w^2 e^{-w} (A_1 + A_2 e^{-w} + A_3 e^{-2w}) dw = 4 \left(2A_1 + \frac{A_2}{4} + \frac{2A_3}{27} \right),$$

$$T_2 = \int_0^{\infty} e^{-w} (A_1 + A_2 e^{-w} + A_3 e^{-2w}) dw = 1,$$

$$T_3 = A_1^2 \int_0^{\infty} \frac{w^2 e^{-w}}{A_1 + A_2 e^{-w} + A_3 e^{-2w}} dw, \quad T_4 = A_3^2 \int_0^{\infty} \frac{w^2 e^{-5w}}{A_1 + A_2 e^{-w} + A_3 e^{-2w}} dw,$$

$$\begin{aligned}
 T_5 &= -4 \int_0^\infty w e^{-w} (A_1 + A_2 e^{-w} + A_3 e^{-2w}) dw = -4 \left(A_1 + \frac{A_2}{4} + \frac{A_3}{9} \right), \\
 T_6 &= -4A_1 \int_0^\infty w^2 e^{-w} dw = -8A_1, \quad T_7 = 4A_3 \int_0^\infty w^2 e^{-3w} dw = \frac{8A_3}{27}, \\
 T_8 &= 2A_1 \int_0^\infty w e^{-w} dw = 2A_1, \quad T_9 = -2A_3 \int_0^\infty w e^{-3w} dw = \frac{-2A_3}{9},
 \end{aligned}$$

and

$$T_{10} = -2A_1 A_3 \int_0^\infty \frac{w^2 e^{-3w}}{A_1 + A_2 e^{-w} + A_3 e^{-2w}} dw.$$

Therefore, we get

$$I_\theta(Y_{[r:n]}; \alpha, \theta) = \frac{1}{\theta^2} \left(1 - 2A_1 - \frac{2A_3}{27} + T_3 + T_4 + T_{10} \right). \tag{4.3}$$

We use Mathematica 12 to compute the values of the infinite integrals $T_3, T_4,$ and T_{10} , consequently, the FI $I_\theta(Y_{[r:n]}; \alpha, \theta)$ can be computed by using (4.3). Table 2 provides the values of $I_\theta(Y_{[r:n]}; \alpha, \theta)$, for $n = 5, 15, \theta = 1$. From Table 2, the following properties can be extracted:

- In the vast majority of the cases $I_\theta(Y_{[r:n]}; \alpha, 1)$ increases when the difference $n - r$ decreases. In contrast, almost $I_\theta(Y_{[r:n]}; -\alpha, 1)$ increases when $n - r$ increases. Moreover, Table 2 reveals that the greatest values of FI are obtained almost at the maximum OSs.
- Generally, we have $I_\theta(Y_{[\frac{n+1}{2};n]}; \alpha, 1) = I_\theta(Y_{[\frac{n+1}{2};n]}; -\alpha, 1)$.
- $I_\theta(Y_{[r:n]}; \alpha, 1) = I_\theta(Y_{[n-r+1;n]}; -\alpha, 1)$, which endorse Proposition 2.

4.2 FI in $Y_{[r:n]}$ about the shape parameter of power distribution marginal

Let $f_Y(y) = cy^{c-1}, c > 0, 0 \leq y \leq 1$. By using (4.1) we get the marginal PDF of the concomitant $Y_{[r:n]}$ based on the power function distribution as:

$$\begin{aligned}
 f_{[r:n]}(y; \alpha, c) &= cy^{c-1} + 3\Delta_{1,r:n}^{(\alpha)} (2cy^{2c-1} - cy^{c-1}) \\
 &\quad + \frac{5}{2}\Delta_{2,r:n}^{(\alpha)} (cy^{c-1} - 6cy^{2c-1} + 6cy^{3c-1}).
 \end{aligned}$$

This expression, after some algebra, can be written as $f_{[r:n]}(y; \alpha, c) = cy^{c-1} (B_1 + B_2 y^c + B_3 y^{2c})$, where $B_1 = \left(1 - 3\Delta_{1,r:n}^{(\alpha)} + \frac{5}{2}\Delta_{2,r:n}^{(\alpha)} \right), B_2 = \left(6\Delta_{1,r:n}^{(\alpha)} - 15\Delta_{2,r:n}^{(\alpha)} \right)$ and $B_3 = \left(15\Delta_{2,r:n}^{(\alpha)} \right)$. Therefore,

$$\frac{\partial \ln f_{[r:n]}(y; \alpha, c)}{\partial c} = \frac{1}{c} + \ln y + \frac{B_2 y^c \ln y + 2B_3 y^{2c} \ln y}{B_1 + B_2 y^c + B_3 y^{2c}},$$

Table 2 FI in $Y_{[r;n]}$ for $\theta = 1$

n	r	$\alpha = 0.2$	$\alpha = -0.2$	$\alpha = 0.3$	$\alpha = -0.3$	$\alpha = 0.4$	$\alpha = -0.4$	$\alpha = 0.52$	$\alpha = -0.52$
5	1	0.82882	1.23558	0.763119	1.38588	0.711398	1.56495	0.67454	1.82972
5	2	0.922576	1.11853	0.905656	1.19139	0.910542	1.27439	0.960892	1.39079
5	3	1.01742	1.01742	1.04359	1.04359	1.0887	1.0887	1.18118	1.18118
5	4	1.11853	0.922576	1.19139	0.905656	1.27439	0.910542	1.39079	0.960892
5	5	1.23558	0.82882	1.38588	0.763119	1.56495	0.711398	1.82972	0.67454
15	1	0.773173	1.3233	0.686806	1.55626	0.643062	1.87458	0.767559	2.46482
15	2	0.809611	1.27026	0.738228	1.44891	0.685666	1.66892	0.665784	2.01052
15	3	0.846409	1.22231	0.795982	1.36136	0.76549	1.52163	0.761468	1.74688
15	4	0.883039	1.17839	0.854839	1.28825	0.855837	1.41278	0.920566	1.58601
15	5	0.919256	1.13757	0.911963	1.22542	0.942169	1.32908	1.07137	1.48366
15	6	0.955041	1.09903	0.966278	1.16954	1.01866	1.26109	1.1832	1.41415
15	7	0.990552	1.06208	1.01794	1.11782	1.08517	1.2016	1.25921	1.36165
15	8	1.02609	1.02609	1.06794	1.06794	1.14468	1.14468	1.31404	1.31404
15	9	1.06208	0.990552	1.11782	1.01794	1.2016	1.08517	1.36165	1.25921
15	10	1.09903	0.955041	1.16954	0.966278	1.26109	1.01866	1.41415	1.1832
15	11	1.13757	0.919256	1.22542	0.911963	1.32908	0.942169	1.48366	1.07137
15	12	1.17839	0.883039	1.28825	0.854839	1.41278	0.855837	1.58601	0.920566
15	13	1.22231	0.846409	1.36136	0.795982	1.52163	0.76549	1.74688	0.761468
15	14	1.27026	0.809611	1.44891	0.738228	1.66892	0.685666	2.01052	0.665784
15	15	1.3233	0.773173	1.55626	0.686806	1.87458	0.643062	2.46482	0.767559

which implies

$$\begin{aligned} \left(\frac{\partial \ln f_{[r:n]}(y; \alpha, c)}{\partial c}\right)^2 &= \frac{1}{c^2} + (\ln y)^2 + \left(\frac{B_2 y^c \ln y + 2B_3 y^{2c} \ln y}{B_1 + B_2 y^c + B_3 y^{2c}}\right)^2 + \frac{2 \ln y}{c} \\ &\quad + \frac{2}{c} \left(\frac{B_2 y^c \ln y + 2B_3 y^{2c} \ln y}{B_1 + B_2 y^c + B_3 y^{2c}}\right) + 2 \left(\frac{B_2 y^c \ln y + 2B_3 y^{2c} \ln y}{B_1 + B_2 y^c + B_3 y^{2c}}\right) \ln y. \end{aligned} \tag{4.4}$$

Thus, (4.4) yields

$$I_c(Y_{[r:n]}; \alpha, c) = \int_0^1 \left(\frac{\partial \ln f_{Y_{[r:n]}}(y; \alpha, c)}{\partial c}\right)^2 f_{[r:n]}(y; \alpha, c) dy = \sum_{i=1}^6 k_i,$$

where

$$\begin{aligned} k_1 &= \frac{1}{c^2} \int_0^1 c y^{c-1} (B_1 + B_2 y^c + B_3 y^{2c}) dy = \frac{1}{c^2}, \\ k_2 &= c \int_0^1 y^{c-1} (B_1 + B_2 y^c + B_3 y^{2c}) (\ln y)^2 dy = \frac{27(8B_1 + B_2) + 8B_3}{108c^2}, \\ k_3 &= c \int_0^1 y^{c-1} \frac{(B_2 y^c \ln y + 2B_3 y^{2c} \ln y)^2}{B_1 + B_2 y^c + B_3 y^{2c}} dy, \\ k_4 &= 2 \int_0^1 y^{c-1} (B_1 + B_2 y^c + B_3 y^{2c}) \ln y dy = -\frac{36B_1 + 9B_2 + 4B_3}{18c^2}, \\ k_5 &= 2 \int_0^1 y^{c-1} (B_2 y^c \ln y + 2B_3 y^{2c} \ln y) dy = -\frac{9B_2 + 8B_3}{18c^2}, \end{aligned}$$

and

$$k_6 = 2c \int_0^1 y^{c-1} (B_2 y^c \ln y + 2B_3 y^{2c} \ln y) \ln y dy = \frac{27B_2 + 16B_3}{54c^2}.$$

Therefore, we get $I_c(Y_{[r:n]}; \alpha, c) = \frac{1}{c^2} (1 - \frac{1}{4} B_2 - \frac{8}{27} B_3) + k_3$. Table 3 provides the values of $I_c(Y_{[r:n]}; \alpha, c)$, for $n = 5, 15, c = 2$. From Table 3, the following properties can be extracted:

- Generally, we have that $I_c(Y_{[r:n]}; \alpha, 2)$ increases when the difference $n - r$ increases. In contrast, $I_c(Y_{[r:n]}; \alpha, 2)$ increases when the difference $n - r$ decreases. Moreover, Table 3 reveals that the greatest values of FI are obtained at the maximum OSs.
- Generally, we have that $I_c(Y_{[\frac{n+1}{2}:n]}; \alpha, 2) = I_c(Y_{[\frac{n+1}{2}:n]}; -\alpha, 2)$.
- $I_c(Y_{[r:n]}; \alpha, 2) = I_c(Y_{[n-r+1:n]}; -\alpha, 2)$.

Table 3 FI in $Y_{[r;m]}$, for $c = 2$

n	r	$\alpha = 0.2$	$\alpha = -0.2$	$\alpha = 0.3$	$\alpha = -0.3$	$\alpha = 0.4$	$\alpha = -0.4$	$\alpha = 0.52$	$\alpha = -0.52$
5	1	0.308896	0.207205	0.34647	0.19078	0.391238	0.177849	0.457429	0.168635
5	2	0.279633	0.230644	0.297849	0.226414	0.318599	0.227635	0.347699	0.240223
5	3	0.254354	0.254354	0.260898	0.260898	0.272176	0.272176	0.295295	0.295295
5	4	0.230644	0.279633	0.226414	0.297849	0.227635	0.318599	0.240223	0.347699
5	5	0.207205	0.308896	0.19078	0.34647	0.177849	0.391238	0.168635	0.457429
15	1	0.330825	0.193293	0.389065	0.171701	0.468645	0.160765	0.616204	0.19189
15	2	0.317564	0.202403	0.362228	0.184557	0.417231	0.171416	0.50263	0.166446
15	3	0.305578	0.211602	0.340339	0.198996	0.380408	0.191372	0.436721	0.190367
15	4	0.294598	0.22076	0.322062	0.21371	0.353196	0.213959	0.396504	0.230141
15	5	0.284392	0.229814	0.306356	0.227991	0.332271	0.235542	0.370916	0.267842
15	6	0.274758	0.23876	0.292384	0.24157	0.315273	0.254664	0.353537	0.295801
15	7	0.26552	0.247638	0.279456	0.254486	0.3004	0.271293	0.340413	0.314803
15	8	0.256523	0.256523	0.266986	0.266986	0.286171	0.286171	0.328511	0.328511
15	9	0.247638	0.26552	0.254486	0.279456	0.271293	0.3004	0.314803	0.340413
15	10	0.23876	0.274758	0.24157	0.292384	0.254664	0.315273	0.295801	0.353537
15	11	0.229814	0.284392	0.227991	0.306356	0.235542	0.332271	0.267842	0.370916
15	12	0.22076	0.294598	0.21371	0.322062	0.213959	0.353196	0.230141	0.396504
15	13	0.211602	0.305578	0.198996	0.340339	0.191372	0.380408	0.190367	0.436721
15	14	0.202403	0.317564	0.184557	0.362228	0.171416	0.417231	0.166446	0.50263
15	15	0.193293	0.330825	0.171701	0.389065	0.160765	0.468645	0.19189	0.616204

5 Cumulative residual FI in $Y_{[r:n]}$

Theorem 5.1 Let $F_{[r:n]}(y; \alpha)$ be the DF of the concomitant of $Y_{[r:n]}$ based on SAR(α). Then, the CF for location parameter based on $Y_{[r:n]}$, is given by

$$\begin{aligned} CF(Y_{[r:n]}; \alpha) &= \int_{-\infty}^{\infty} \left(\frac{\partial \ln \bar{F}_{[r:n]}(y; \alpha)}{\partial y} \right)^2 \bar{F}_{[r:n]}(y; \alpha) dy \\ &= CF(Y) + \zeta(\alpha) + 2\eta(\alpha) + \xi(\alpha), \end{aligned}$$

where

$$\zeta(\alpha) = \int_{-\infty}^{\infty} \left(\frac{\partial \ln \bar{F}_Y(y)}{\partial y} \right)^2 \left(3\Delta_{1,r:n}^{(\alpha)} F_Y(y) + \frac{5}{2} \Delta_{2,r:n}^{(\alpha)} F_Y(y) \varphi(F_Y(y)) \right) \bar{F}_Y(y) dy,$$

$$\eta(\alpha) = - \int_{-\infty}^{\infty} \left(3\Delta_{1,r:n}^{(\alpha)} - \frac{5}{2} \Delta_{2,r:n}^{(\alpha)} + 10\Delta_{2,r:n}^{(\alpha)} F_Y(y) \right) f_Y^2(y) dy,$$

and

$$\xi(\alpha) = \int_{-\infty}^{\infty} \frac{\left[f_Y(y) \left(3\Delta_{1,r:n}^{(\alpha)} - \frac{5}{2} \Delta_{2,r:n}^{(\alpha)} + 10\Delta_{2,r:n}^{(\alpha)} F_Y(y) \right) \right]^2}{1 + (3\Delta_{1,r:n}^{(\alpha)} - \frac{5}{2} \Delta_{2,r:n}^{(\alpha)}) F_Y(y) + 5\Delta_{2,r:n}^{(\alpha)} F_Y^2(y)} \bar{F}_Y(y) dy.$$

Proof By using (4.1), the CF of $Y_{[r:n]}$ is given by

$$\begin{aligned} CF(Y_{[r:n]}; \alpha) &= \int_{-\infty}^{\infty} \left(\frac{\partial \ln \bar{F}_Y(y)}{\partial y} + \frac{\partial \ln(1 + 3\Delta_{1,r:n}^{(\alpha)} F_Y(y) + \frac{5}{2} \Delta_{2,r:n}^{(\alpha)} F_Y(y) \varphi(F_Y(y)))}{\partial y} \right)^2 \\ &\quad \times \left(\bar{F}_Y(y) \left[1 + 3\Delta_{1,r:n}^{(\alpha)} F_Y(y) + \frac{5}{2} \Delta_{2,r:n}^{(\alpha)} F_Y(y) \varphi(F_Y(y)) \right] \right) dy \\ &= \int_{-\infty}^{\infty} \left(\frac{\partial \ln \bar{F}_Y(y)}{\partial y} \right)^2 \bar{F}_Y(y) dy + \int_{-\infty}^{\infty} \left(\frac{\partial \ln \bar{F}_Y(y)}{\partial y} \right)^2 \\ &\quad \times \left(3\Delta_{1,r:n}^{(\alpha)} F_Y(y) + \frac{5}{2} \Delta_{2,r:n}^{(\alpha)} F_Y(y) \varphi(F_Y(y)) \right) \bar{F}_Y(y) dy \\ &\quad + \int_{-\infty}^{\infty} \left(\frac{\partial \ln(1 + 3\Delta_{1,r:n}^{(\alpha)} F_Y(y) + \frac{5}{2} \Delta_{2,r:n}^{(\alpha)} F_Y(y) \varphi(F_Y(y)))}{\partial y} \right)^2 \\ &\quad \times \left(1 + 3\Delta_{1,r:n}^{(\alpha)} F_Y(y) + \frac{5}{2} \Delta_{2,r:n}^{(\alpha)} F_Y(y) \varphi(F_Y(y)) \right) \bar{F}_Y(y) dy \\ &\quad + 2 \int_{-\infty}^{\infty} \left(\frac{\partial \ln \bar{F}_Y(y)}{\partial y} \right) \times \left(\frac{\partial \ln(1 + 3\Delta_{1,r:n}^{(\alpha)} F_Y(y) + \frac{5}{2} \Delta_{2,r:n}^{(\alpha)} F_Y(y) \varphi(F_Y(y)))}{\partial y} \right) \end{aligned}$$

$$\times \left(1 + 3\Delta_{1,r:n}^{(\alpha)} F_Y(y) + \frac{5}{2}\Delta_{2,r:n}^{(\alpha)} F_Y(y)\varphi(F_Y(y)) \right) \bar{F}_Y(y)dy.$$

□

Proposition 2 Let $\bar{F}_{[r:n]}(y; \alpha)$ and $R_{[r:n]}(y; \alpha)$ be the survival and hazard functions of concomitant $Y_{[r:n]}$ from SAR(α), respectively, where $R_{[r:n]}(y; \alpha) = \frac{f_{[r:n]}(y; \alpha)}{\bar{F}_{[r:n]}(y; \alpha)}$. Then,

$$CF(Y_{[r:n]}; \alpha) = E [R_{[r:n]}(Y_{[r:n]}; \alpha)] \geq \frac{1}{E(Y_{[r:n]})}.$$

Proof From the definition of $CF(Y_{[r:n]}; \alpha)$ and by using the result of Nanda [27], that $E[R_{F_X}(X)] \geq \frac{1}{E(X)}$, we get

$$\begin{aligned} CF(Y_{[r:n]}; \alpha) &= \int_{-\infty}^{\infty} \left(\frac{\partial \ln \bar{F}_{[r:n]}(y; \alpha)}{\partial y} \right)^2 \bar{F}_{[r:n]}(y; \alpha)dy \\ &= \int_{-\infty}^{\infty} \left(-\frac{f_{[r:n]}(y; \alpha)}{\bar{F}_{[r:n]}(y; \alpha)} \right)^2 \bar{F}_{[r:n]}(y; \alpha)dy \\ &= \int_{-\infty}^{\infty} R_{[r:n]}(y; \alpha) f_{[r:n]}(y; \alpha)dy \geq \frac{1}{E(Y_{[r:n]})}. \end{aligned}$$

The proof is completed. □

Example 5.1 Let X and Y have exponential distributions with means $\frac{1}{\theta^*}$ and $\frac{1}{\theta}$, respectively. Then, $CF(Y_{[r:n]}; \alpha) = \theta$, $\zeta(\alpha) = \frac{\theta}{2}(3\Delta_{1,r:n}^{(\alpha)} - \frac{5}{2}\Delta_{2,r:n}^{(\alpha)}) + \frac{5\theta}{3}\Delta_{2,r:n}^{(\alpha)}$, and $\eta(\alpha) = \frac{\theta}{2}(-3\Delta_{1,r:n}^{(\alpha)} + \frac{5}{2}\Delta_{2,r:n}^{(\alpha)}) - \frac{5\theta}{3}\Delta_{2,r:n}^{(\alpha)}$. Thus, the CF of $Y_{[r:n]}$ is given by

$$\begin{aligned} CF(Y_{[r:n]}; \alpha) &= (1 - 3\Delta_{1,r:n}^{(\alpha)} + \frac{5}{2}\Delta_{2,r:n}^{(\alpha)})\theta + (3\Delta_{1,r:n}^{(\alpha)} - \frac{5}{2}\Delta_{2,r:n}^{(\alpha)})\frac{\theta}{2} - \frac{5\theta}{3}\Delta_{2,r:n}^{(\alpha)} + \theta^2 J(\theta), \end{aligned}$$

where

$$J(\theta) = \int_0^{\infty} \frac{\left(3\Delta_{1,r:n}^{(\alpha)} - \frac{5}{2}\Delta_{2,r:n}^{(\alpha)} + 10\Delta_{2,r:n}^{(\alpha)}(1 - e^{-\theta y}) \right)^2 e^{-3\theta y}}{1 + (3\Delta_{1,r:n}^{(\alpha)} - \frac{5}{2}\Delta_{2,r:n}^{(\alpha)})(1 - e^{-\theta y}) + 5\Delta_{2,r:n}^{(\alpha)}(1 - e^{-\theta y})^2} dy.$$

Table 4 provides $CF(Y_{[r:n]}; \alpha)$ values for $n = 5, 15, \theta = 1$. From Table 4 following properties can be extracted:

- Generally, we have $CF(Y_{[\frac{n+1}{2};n]}; \alpha) = CF(Y_{[\frac{n+1}{2};n]}; -\alpha)$. Moreover, $CF(Y_{[r:n]}; \alpha) = CF(Y_{[n-r+1;n]}; -\alpha)$.
- with fixed n and $\alpha > 0$, the value of $CF(Y_{[r:n]}; \alpha)$ slowly decreases as r increases. In contrast, the value of $CF(Y_{[r:n]}; \alpha)$ slowly increases as r increases for $\alpha < 0$.

Table 4 CF in $Y_{[r:n]}$ for SAR(α) at $\theta = 1$

n	r	$\alpha = 0.2$	$\alpha = -0.2$	$\alpha = 0.3$	$\alpha = -0.3$	$\alpha = 0.4$	$\alpha = -0.4$	$\alpha = 0.52$	$\alpha = -0.52$
5	1	1.26042	0.83349	1.46297	0.767355	1.75298	0.708855	2.29742	0.648172
5	2	1.11734	0.919724	1.18917	0.896785	1.2715	0.888018	1.38963	0.899691
5	3	1.01127	1.01127	1.0303	1.0303	1.06628	1.06628	1.14575	1.14575
5	4	0.919724	1.11734	0.896785	1.18917	0.888018	1.2715	0.899691	1.38963
5	5	0.83349	1.26042	0.767355	1.46297	0.708855	1.75298	0.648172	2.29742
15	1	1.38543	0.783103	1.77591	0.696888	2.49056	0.62807	4.3142	0.584758
15	2	1.3053	0.815152	1.56231	0.742408	1.95846	0.67891	2.77907	0.616316
15	3	1.23912	0.847936	1.41136	0.792379	1.63732	0.746464	2.01561	0.701314
15	4	1.18339	0.881135	1.30256	0.844461	1.44385	0.822728	1.65181	0.816421
15	5	1.13531	0.914582	1.22106	0.896903	1.32365	0.901309	1.48505	0.942764
15	6	1.09269	0.948274	1.15622	0.94861	1.24164	0.977503	1.39997	1.06566
15	7	1.0538	0.982397	1.10065	0.99924	1.17597	1.04851	1.33589	1.17471
15	8	1.01736	1.01736	1.04936	1.04936	1.11383	1.11383	1.26445	1.26445
15	9	0.982397	1.0538	0.99924	1.10065	1.04851	1.17597	1.17471	1.33589
15	10	0.948274	1.09269	0.94861	1.15622	0.977503	1.24164	1.06566	1.39997
15	11	0.914582	1.13531	0.896903	1.22106	0.901309	1.32365	0.942764	1.48505
15	12	0.881135	1.18339	0.844461	1.30256	0.822728	1.44385	0.816421	1.65181
15	13	0.847936	1.23912	0.792379	1.41136	0.746464	1.63732	0.701314	2.01561
15	14	0.815152	1.3053	0.742408	1.56231	0.67891	1.95846	0.616316	2.77907
15	15	0.783103	1.38543	0.696888	1.77591	0.62807	2.49056	0.584758	4.3142

6 Concluding remarks

Among all the known families that generalize the FGM family, we exclusively found that the Sarmanov family shares two properties with the FGM family of theoretical and practical importance. The first property is that they both have one shape parameter, which makes it easy to work with them for bivariate data modeling. The second property is the radial symmetry, which gives many of the measures of information associated with them, mathematical flexibility in terms of simplifying the mathematical relationships that describe those measures. On the other hand, the Sarmanov family surpassed all known families in terms of the high coefficient of correlation between its components.

If we transfer to the issue of modelling multivariate data (with more than two variables), we find the multivariate FGM, which was proposed by Johnson and Kotz [23]. The multivariate FGM family, which has a direct structure and can describe the interrelationships of two or more variables, is useful as an alternative to a multivariate normal distribution and it has been applied to statistical modeling in various research fields. The formulation of the Sarmanov family as a multivariate distribution, which benefits from the radial symmetry property and offers high partial correlations between its variables, will be a very useful contribution in the field of multivariate data modelling and represents a potential future development for the current work.

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Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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