Thermal convection with generalized friction

Brian Straughan[1]

Abstract

A model for thermal convection with generalized friction is investigated. It is shown that the linear instability threshold is the same as the global stability one. In addition, decay of the energy in the L^2 norm is shown for the perturbation velocity and temperature fields. However, due to the presence of the generalized friction we establish exponential decay in the $L^{\beta+1}$ norm for the perturbation temperature, where $\beta > 1$.

Keywords Thermal convection · Generalized friction · Nonlinear stability · Energy stability

1 Introduction

In a recent paper [1] have proposed a model for thermal convection in a linearly viscous fluid which is subject to a generalized friction term. [1] study the existence and regularity questions for their model.

The generalized friction consists of a term of form $-|\mathbf{v}|^{\beta-1} v_i$ for a constant $\beta > 1$, where v_i is the velocity. In porous medium flow such a term is known as a generalized Forchheimer term, cf. [2], chapter 1. The buoyancy effect which drives thermally convective motion is introduced via a Boussinesq approximation, see e.g. [3], pp. 16–21.

The goal of this paper is to study the stability of the steady conduction solution to the model of [1]. We demonstrate an optimum result in that we show the linear instability threshold for the Rayleigh number coincides with the global nonlinear stability one. The energy decay is in $L^2(V)$ for the velocity perturbation u_i, and the temperature perturbation θ, where V is a period cell for the solution. However, we here show that the generalized friction term allows one to demonstrate exponential decay of θ in the $L^{\beta+1}(V)$ norm, for $\beta > 1$.

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2 Thermal convection equations

The governing equations presented by [1] have form

$$
\begin{aligned}
v_{i,t} + v_j v_{i,j} &= -\frac{1}{\rho_0} p_{,i} + \nu \Delta v_i + \omega g k_i T - \hat{\alpha} |\mathbf{v}|^{\beta-1} v_i , \\
v_{i,i} &= 0, \\
T_{,t} + v_i T_{,i} &= \kappa \Delta T.
\end{aligned}
\tag{1}
$$

In these equations $v_i(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ and $T(\mathbf{x}, t)$ denote the velocity, pressure and temperature at position $\mathbf{x}$ and time t. Furthermore, ρ_0, ν, ω, g, $\hat{\alpha}$, β, κ denote the reference density, kinematic viscosity, thermal expansion coefficient, gravity, friction coefficient, friction index, and thermal diffusivity. Throughout this article we employ standard indicial notation in conjunction with the Einstein summation convention, so for example,

$$
\begin{aligned}
v_{i,i} \equiv \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} &= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \\
&\equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
\end{aligned}
$$

where $\mathbf{v} \equiv (u, v, w)$ and $\mathbf{x} \equiv (x, y, z)$. For a nonlinear example

$$
v_i T_{,i} \equiv \sum_{i=1}^{3} v_i \frac{\partial T}{\partial x_i} \equiv u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} .
$$

The term Δ denotes the Laplacian in $\mathbb{R}^3$.

Equation (1) are defined on the horizontal layer $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, d)\}$ for $t > 0$. The boundary conditions are that

$$
v_i = 0, \ z = 0, d; \qquad T = T_L , \ z = 0; \qquad T = T_U , \ z = d; \tag{2}
$$

$0 < T_U < T_L$, with T_L, T_U constants.

The steady conduction solution to Eqs. (1), (2) in whose stability we are interested has form

$$
\bar{v}_i \equiv 0, \qquad \bar{T} = T_L - \gamma z, \tag{3}
$$

where γ is the temperature gradient,

$$
\gamma = \frac{T_L - T_U}{d} > 0.
$$

The steady pressure is a quadratic function of z determined from $(1)_1$.

To proceed we define perturbation variables u_i, θ, π by

$$v_i = \bar{v}_i + u_i, \qquad T = \bar{T} + \theta, \qquad p = \bar{p} + \pi.$$

These expressions are now substituted into Eq. (1) and are non-dimensionalized with the scalings

$$x_i = x_i^* d, \qquad t = \mathcal{T} t^*, \qquad \mathcal{T} = \frac{d^2}{\nu},$$

$$U = \frac{\nu}{d}, \qquad Pr = \frac{\nu}{\kappa}, \qquad \alpha = \frac{d^2 \hat{\alpha}}{\nu} U^{\beta-1},$$

and the Rayleigh number Ra is introduced as

$$Ra \equiv R^2 = \frac{\omega g \gamma d^4}{\nu \kappa}.$$

In this way we arrive at the non-dimensional perturbation equations, where the *s are dropped,

$$
\begin{aligned}
&u_{i,t} + u_j u_{i,j} = -\pi_{,i} + \Delta u_i + R k_i \theta - \alpha |\mathbf{u}|^{\beta-1} u_i, \\
&u_{i,i} = 0, \\
&Pr(\theta_{,t} + u_i \theta_{,i}) = Rw + \Delta \theta,
\end{aligned}
\tag{4}
$$

where (4) hold on $\mathbb{R}^2 \times (0,1) \times \{t > 0\}$, with $w \equiv u_3$. The non-dimensional perturbation boundary conditions are

$$u_i = 0, \quad z = 0, 1; \qquad \theta = 0, \quad z = 0, 1; \tag{5}$$

together with horizontal periodicity of the solution, cf. [4], p. 51. The period cell of the solution is denoted by V.

3 Global nonlinear stability

Before we commence a global nonlinear stability analysis for Eqs. (4), (5), we observe that an analogous problem for the Navier–Stokes equations, i.e. the system

$$
\begin{aligned}
&u_{i,t} + u_j u_{i,j} = -\pi_{,i} + \Delta u_i - \alpha |\mathbf{u}|^{\beta-1} u_i, \\
&u_{i,i} = 0,
\end{aligned}
\tag{6}
$$

has been analysed by other writers. [5] establish extensive existence, regularity and evolutionary behaviour results for a solution to (6).

For system (4), (5), the linear operator is symmetric and thus the linear instability threshold coincides with the nonlinear stability one, cf. [6], [4], chapter 4. For clarification we multiply Eq. (4)₁ by u_i, we multiply equation (4)₃ by θ, and integrate each

resulting equation over V. Add the resulting equations and after some integration by parts and use of the boundary conditions one may deduce the energy equation.

$$\frac{dE}{dt} = RI - D - \alpha \int_V |\mathbf{u}|^{\beta+1} dx, \tag{7}$$

where the energy function is given by

$$E = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{Pr}{2}\|\theta\|^2,$$

the dissipation is

$$D = \|\nabla \mathbf{u}\|^2 + \|\nabla \theta\|^2,$$

and the production term has form

$$I = 2(\theta, w),$$

where $(\cdot, \cdot)$ and $\|\cdot\|$ denote the inner product and norm on $L^2(V)$.

From (7) we discard the $\|\mathbf{u}\|_{\beta+1}^{\beta+1}$ term and then derive

$$\frac{dE}{dt} \le -D\left(1 - \frac{R}{R_E}\right), \tag{8}$$

where

$$\frac{1}{R_E} = \max_H \frac{I}{D}, \tag{9}$$

H being the space of admissible solutions. One may now show that the Euler-Lagrange equations which arise from (9) have exactly the same form as the linear instability equations which arise from (4). Thus, we obtain the optimal result that the linear instability boundary is the same as the nonlinear stability one, and provided $R < R_E$ the stability is global. In addition, one shows $\|\mathbf{u}\|$ and $\|\theta\|$ decay at least exponentially.

This may be improved upon by considering the $L^{\beta+1}$ norm for θ. To see this note that from (4)$_3$, cf. [7],

$$\frac{d}{dt}\frac{Pr}{(\beta+1)}\int_V |\theta|^{\beta+1} dx = Pr \int_V \theta_{,t}|\theta|^\beta (\text{sign}\,\theta) dx$$

$$= R\int_V w|\theta|^\beta(\text{sign}\,\theta)dx - \beta\int_V |\theta|^{\beta-1}\theta_{,i}\theta_{,i}dx.$$

We next employ Poincaré's inequality on the second term on the right and employ Young's inequality on the first term on the right to derive

$$\frac{d}{dt}\frac{Pr}{(\beta+1)}\int_V |\theta|^{\beta+1}dx$$
$$\leq \frac{\beta}{(\beta+1)}\left[\frac{R}{a^{(\beta+1)/\beta}} - \frac{4\lambda_1}{(\beta+1)}\right]\|\theta\|_{\beta+1}^{\beta+1} + \frac{R\,a^{\beta+1}}{\beta+1}\|w\|_{\beta+1}^{\beta+1}, \tag{10}$$

where a is a constant at our disposal, and $\lambda_1 = \pi^2$.

Next, form the combination (7)+ξ(10), for a constant $\xi > 0$ to be selected. We now pick a so that

$$a^{\beta+1} = \frac{\alpha(\beta+1)}{R\xi}$$

and then we require

$$\xi < \frac{(4\lambda_1)^\beta \alpha}{R^{\beta+1}(\beta+1)^{\beta-1}}.$$

This allows us to define ϵ so that

$$\frac{\beta}{(\beta+1)}\left[\frac{4\lambda_1\xi}{(\beta+1)} - \frac{(R\xi)^{1+1/\beta}}{[\alpha(\beta+1)]^{1/\beta}}\right] = \epsilon > 0.$$

In this manner we derive an energy inequality of form

$$\frac{d}{dt}\left(E + \frac{\xi Pr}{(\beta+1)}\|\theta\|_{\beta+1}^{\beta+1}\right) \leq -D\left(1 - \frac{R}{R_E}\right) - \epsilon\|\theta\|_{\beta+1}^{\beta+1}. \tag{11}$$

If now $R < R_E$ we may employ Poincaré's inequality on D to show that

$$\frac{d}{dt}\left(E + \frac{\xi Pr}{(\beta+1)}\|\theta\|_{\beta+1}^{\beta+1}\right)$$
$$\leq -2\lambda_1 c\left(\frac{1}{2}\|\mathbf{u}\|^2 + \frac{1}{Pr}\frac{Pr}{2}\|\theta\|^2\right) - \frac{\epsilon(\beta+1)}{\xi Pr}\left(\frac{\xi Pr}{\beta+1}\|\theta\|_{\beta+1}^{\beta+1}\right), \tag{12}$$

where $c = 1 - R/R_E > 0$. When $Pr \geq 1$, as it usually is, we let

$$k = \min\left\{2\left(1 - \frac{R}{R_E}\right)\lambda_1, \frac{\epsilon(\beta+1)}{\xi Pr}\right\}$$

and then from (12) we may obtain

$$\frac{d\mathcal{E}}{dt} \leq -k\mathcal{E}, \tag{13}$$

where

$$\mathcal{E} = E + \frac{\xi Pr}{(\beta+1)}\|\theta\|_{\beta+1}^{\beta+1}.$$

From Eq. (13) we see that when $R < R_E$, $\|\theta\|_{\beta+1}^{\beta+1}$ also decays exponentially and the stronger decay effect induced by the generalized friction is demonstrated.

Remark We have chosen here $Pr \geq 1$. The same proof works when $Pr < 1$, *mutatis matuandis.*

4 Conclusions

We have addressed the global nonlinear stability problem for thermal convection with generalized friction in a model recently proposed by [1]. It is shown that the linear instability theory correctly captures the onset of thermal convection by showing that the Rayleigh number threshold for global nonlinear stability employing the energy method is exactly the same as the one obtained by a linear analysis. When the Rayleigh number is below the instability threshold it is shown that the solution decays exponentially in time in L^2 norm for the velocity, and in $L^{\beta+1}$ norm for the temperature, $\beta > 1$.

Declarations

Conflict of interest There are no conflicts of interest.

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