Ann Univ Ferrara (2012) 58:359–369
DOI 10.1007/s11565-012-0155-7

Multiple positive solutions for fractional differential systems

Nemat Nyamoradi

Received: 14 February 2012 / Accepted: 26 March 2012 / Published online: 17 April 2012

Abstract In this paper, we study the existence of positive solution to boundary value problem for fractional differential system

$$\begin{cases} D_{0+}^{\alpha}u(t) + a_1(t)f_1(t, u(t), v(t)) = 0, & t \in (0, 1), \\ D_{0+}^{\alpha}v(t) + a_2(t)f_2(t, u(t), v(t)) = 0, & t \in (0, 1), \quad 2 < \alpha < 3, \\ u(0) = u'(0) = 0, \quad u'(1) - \mu_1 u'(\eta_1) = 0, \\ v(0) = v'(0) = 0, \quad v'(1) - \mu_2 v'(\eta_2) = 0, \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α. By using the Leggett-Williams fixed point theorem in a cone, the existence of three positive solutions for nonlinear singular boundary value problems is obtained.

Keywords Cone · Multi point boundary value problem · Fixed point theorem · Riemann-Liouville fractional derivative

Mathematics Subject Classification 47H10 · 26A33 · 34A08

1 Introduction

The purpose of this paper is to study the existence of positive solutions for the following boundary value problem for fractional differential system

N. Nyamoradi thanks Razi University for support.

N. Nyamoradi (✉)
Department of Mathematics, Faculty of Sciences,
Razi University, 67149 Kermanshah, Iran
e-mail: nyamoradi@razi.ac.ir

$$
\begin{cases}
D_{0+}^{\alpha}u(t)+a_1(t)f_1(t,u(t),v(t))=0, & t\in(0,1),\\
D_{0+}^{\alpha}v(t)+a_2(t)f_2(t,u(t),v(t))=0, & t\in(0,1),\ 2<\alpha<3,\\
u(0)=u'(0)=0, & u'(1)-\mu_1u'(\eta_2)=0,\\
v(0)=v'(0)=0, & v'(1)-\mu_2v'(\eta_2)=0,
\end{cases}
\tag{1}
$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α, $\eta_i\in(0,1)$, $\mu_i\in\left[0,\frac{1}{\eta_i^{\alpha-2}}\right)$ are two arbitrary constants, $a_i\in C((0,1);[0,+\infty))$, $f_i:[0,1]\times[0,+\infty)\to[0,+\infty)$ and $a_1(t)$, $f_1(t,0,0)$ or $a_2(t)$, $f_2(t,0,0)$ does not vanish identically on any subinterval of $(0,1)$, $i=1,2$.

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details, see [1–3] and the references therein.

The existence of solutions of initial value problems for fractional order differential equations have been studied in the literature [4–8] and the references therein. Saadi and Benbachir [9] considered the following boundary value problem

$$
\begin{cases}
D_{0+}^{\alpha}u(t)+a(t)f(u(t))=0, & t\in(0,1), \quad 2<\alpha<3,\\
u(0)=u'(0)=0, & u'(1)-\mu u'(\eta)=\lambda,
\end{cases}
\tag{2}
$$

where $\eta\in(0,1)$, $\mu\in\left[0,\frac{1}{\eta^{\alpha-2}}\right)$ are two arbitrary constants. They applied the Guo-Krasnosel'skii fixed point theorem and Schauder's fixed point theorem to establish some results on the existence, nonexistence and uniqueness of positive solutions (2).

Motivated by the work mentioned above, our purpose in this paper is to show the existence and multiplicity of positive solutions to the problem (1) by using the Leggett-Williams fixed point theorem.

The rest of the article is organized as follows: in Sect. 2, we present some preliminaries that will be used in Sect. 3. The main result and proof will be given in Sect. 3. Finally, in Sect. 4, an example is given to demonstrate the application of our main result.

2 Preliminaries

In this section, we present some notations and preliminary lemmas that will be used in the proofs of the main results.

Definition 2.1 Let X be a real Banach space. A non-empty closed set $P\subset X$ is called a cone of X if it satisfies the following conditions:

(1) $x\in P$, $\mu\geq 0$ implies $\mu x\in P$,
(2) $x\in P$, $-x\in P$ implies $x=0$.

Definition 2.2 The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3 The Riemann-Liouville fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in \mathbb{N}$ is defined as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n-1)$.

Lemma 1 ([10]). *The equality* $D_{0+}^{\gamma} I_{0+}^{\gamma} f(t) = f(t), \gamma > 0$ *holds for* $f \in L(0,1)$.

Lemma 2 ([10]). *Let* $\alpha > 0$. *Then the differential equation*

$$D_{0+}^{\alpha} u = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, c_i \in \mathbb{R}, i = 1, \ldots, n,$ *where* $n-1 < \alpha \leq n$.

Lemma 3 ([10]). *Let* $\alpha > 0$. *Then the following equality holds for* $u \in L(0,1)$, $D_{0+}^{\alpha} u \in L(0,1)$;

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

$c_i \in \mathbb{R}, i = 1, \ldots, n,$ *where* $n-1 < \alpha \leq n$.

In the following, we present the Green function of fractional differential equation boundary value problem.

Lemma 4 *Let* $y(t) \in C[0,1]$, *for* $i = 1$ *or* $i = 2$, *then the boundary value problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, & u'(1) - \mu_i u'(\eta_i) = 0, \end{cases} \tag{3}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\mu_i t^{\alpha-1}}{(1-\mu_i \eta_i^{\alpha-2})} \int_0^1 G_{1i}(\eta_i, s)y(s)ds, \tag{4}$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1 \end{cases} \tag{5}$$

and

$$G_{1i}(\eta_i, s) = \begin{cases} \frac{\eta_i^{\alpha-2}(1-s)^{\alpha-2}-(\eta_i-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le s \le \eta_i \le 1, \\ \frac{\eta_i^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le \eta_i \le s \le 1. \end{cases} \tag{6}$$

Proof The proof is similar to that of Lemma 5 in [9], so we omit it here. □

Lemma 5 ([9]). *For all* $(t, s) \in [0, 1] \times [0, 1]$, *we have*

(i) $0 \le G_{1i}(\eta_i, s) \le \frac{1}{\Gamma(\alpha)}\eta_i^{\alpha-2}(1-s)^{\alpha-2}$, $G(t, s) \ge 0$;
(ii) $\gamma G(1, s) \le G(t, s) \le G(1, s)$, $(t, s) \in [\tau, 1] \times [0, 1]$,

where $G(1, s) = \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-2}$, $\gamma = \tau^{\alpha-1}$, *and* τ *satisfies*

$$\int_0^1 s(1-s)^{\alpha-2}a_i(s)ds > 0, \tag{7}$$

for $i = 1, 2$.

Now, we consider the system (1). Obviously, $(u, v) \in C^2(0, 1) \times C^2(0, 1)$ is a solution of the system (1) if and only if $(u, v) \in C[0, 1] \times C[0, 1]$ is a solution of the following nonlinear integral system:

$$\begin{cases} u(t) = \int_0^1 G(t, s)a_1(s) f_1(s, u(s), v(s))ds \\ \qquad\quad + \frac{\mu_1 t^{\alpha-1}}{(1-\mu_1\eta_1^{\alpha-2})} \int_0^1 G_{11}(\eta_1, s)a_1(s) f_1(s, u(s), v(s))ds \\ v(t) = \int_0^1 G(t, s)a_2(s) f_2(s, u(s), v(s))ds \\ \qquad\quad + \frac{\mu_2 t^{\alpha-1}}{(1-\mu_2\eta_2^{\alpha-2})} \int_0^1 G_{12}(\eta_2, s)a_2(s) f_2(s, u(s), v(s))ds. \end{cases} \tag{8}$$

To establish the existence three positive solutions of system (1), we will employ the following Leggett-Williams fixed point theorem.

For the convenience of the reader, we present here the Leggett-Williams fixed point theorem [11].

Given a cone K in a real Banach space E, a map α is said to be a nonnegative continuous concave (resp. convex) functional on K provided that $\alpha : K \to [0. +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y),$$
$$(resp.\alpha(tx + (1-t)y) \le t\alpha(x) + (1-t)\alpha(y)),$$

for all $x, y \in K$ and $t \in [0, 1]$.

Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha, a, b)$ by

$$P_r = \{x \in K \mid \|x\| < r\},$$

and

$$P(\alpha, a, b) = \{x \in K | a \leq \alpha(x), \|x\| \leq b\}.$$

Theorem 1 (Leggett-Williams fixed point theorem). *Let $A : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator and let α be a nonnegative continuous concave functional on K such that $\alpha(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < a < b < d \leq c$ such that*
(A1) $\{x \in P(\alpha, b, d) | \alpha(x) > b\} \neq \emptyset$, *and* $\alpha(Ax) > b$ *for* $x \in P(\alpha, b, d)$,
(A2) $\|Ax\| < a$ *for* $\|x\| \leq a$, *and*
(A3) $\alpha(Ax) > b$ *for* $x \in P(\alpha, b, c)$ *with* $\|Ax\| > d$.
Then A has at least three fixed points x_1, x_2, and x_3 and such that $\|x_1\| < a$, $b < \alpha(x_2)$ and $\|x_3\| > a$, with $\alpha(x_3) < b$.

3 Main result

For convenience, we introduce the following notations. Let

$$M_i = \max_{0 \leq t \leq 1} \Big[\int_0^1 G(t,s) a_i(s) ds + \frac{\mu_i t^{\alpha-1}}{(1 - \mu_i \eta_i^{\alpha-2})} \int_0^1 G_{1i}(\eta_i, s) a_i(s) ds \Big],$$

$$m_i = \min_{\tau \leq t \leq 1} \Big[\int_\tau^1 G(t,s) a_i(s) ds + \frac{\mu_i \tau^{\alpha-1}}{(1 - \mu_i \eta_i^{\alpha-2})} \int_\tau^1 G_{1i}(\eta_i, s) a_i(s) ds \Big], \quad i = 1, 2.$$

Then $0 < m_i < M_i$, $i = 1, 2$.

The basic space used in this paper is a real Banach space $E = C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ with the norm $||(u,v)|| := ||u|| + ||v||$, where $\|u\| = \max_{t \in [0,1]} |u(t)|$.

Then, choose a cone $K \subset E$, by

$$K = \{(u,v) \in E \mid u(t) \geq 0,\ v(t) \geq 0,\ \min_{\tau \leq t \leq 1} (u(t) + v(t)) \geq \gamma \|(u,v)\|\}.$$

It is obvious that K is a cone.
Define an operator T by

$$T(u,v)(t) = (A(u,v)(t), B(u,v)(t)), \quad \forall\, t \in (0,1), \tag{9}$$

where

$$\begin{cases} A(u,v)(t) = \int_0^1 G(t,s) a_1(s) f_1(s, u(s), v(s)) ds \\ \qquad\qquad + \frac{\mu_1 t^{\alpha-1}}{(1-\mu_1 \eta_1^{\alpha-2})} \int_0^1 G_{11}(\eta_1, s) a_1(s) f_1(s, u(s), v(s)) ds \\ B(u,v)(t) = \int_0^1 G(t,s) a_2(s) f_2(s, u(s), v(s)) ds \\ \qquad\qquad + \frac{\mu_2 t^{\alpha-1}}{(1-\mu_2 \eta_2^{\alpha-2})} \int_0^1 G_{12}(\eta_2, s) a_2(s) f_2(s, u(s), v(s)) ds. \end{cases} \tag{10}$$

Lemma 6 *The operator defined in* (9) *is completely continuous and* $T : K \to K$.

Proof For any $(u, v) \in K$, then from properties of $G(t, s)$, $G_{11}(t, s)$ and $G_{12}(t, s)$, $A(u, v)(t) \geq 0$, $B(u, v)(t) \geq 0$, $t \in [0, 1]$, and it follows from (10) that

$$\begin{aligned}
||A(u, v)|| &= \int_0^1 G(1, s)a_1(s) f_1(s, u(s), v(s))ds \\
&\quad + \frac{\mu_1}{(1 - \mu_1 \eta_1^{\alpha-2})} \int_0^1 G_{11}(\eta_1, s)a_1(s) f_1(s, u(s), v(s))ds \\
||B(u, v)|| &= \int_0^1 G(1, s)a_2(s) f_2(s, u(s), v(s))ds \\
&\quad + \frac{\mu_2}{(1 - \mu_2 \eta_2^{\alpha-2})} \int_0^1 G_{12}(\eta_2, s)a_2(s) f_2(s, u(s), v(s))ds. \qquad (11)
\end{aligned}$$

Thus, for any $(u, v) \in K$, it follows from Lemma 5 and (11) that

$$\begin{aligned}
\min_{\tau \leq t \leq 1} A(u, v)(t) &= \min_{\tau \leq t \leq 1} \left[\int_0^1 G(t, s)a_1(s) f_1(s, u(s), v(s))ds \right. \\
&\quad \left. + \frac{\mu_1 t^{\alpha-1}}{(1 - \mu_1 \eta_1^{\alpha-2})} \int_0^1 G_{11}(\eta_1, s)a_1(s) f_1(s, u(s), v(s))ds \right] \\
&\geq \gamma \int_0^1 G(1, s)a_1(s) f_1(s, u(s), v(s))ds \\
&\quad + \frac{\mu_1 \tau^{\alpha-1}}{(1 - \mu_1 \eta_1^{\alpha-2})} \int_0^1 G_{11}(\eta_1, s)a_1(s) f_1(s, u(s), v(s))ds \\
&\geq \gamma ||A(u, v)||.
\end{aligned}$$

In the same way, for any $(u, v) \in K$, we have

$$\min_{\tau \leq t \leq 1} B(u, v)(t) \geq \gamma ||B(u, v)||.$$

Therefore

$$\begin{aligned}\min_{\tau\le t\le 1}(A(u,v)(t)+B(u,v)(t)) &\ge \gamma\|A(u,v)\|+\gamma\|B(u,v)\|\\ &=\gamma\|(A(u,v),B(u,v))\|.\end{aligned}$$

From the above, we conclude that $T(u,v)(t)=(A(u,v)(t),B(u,v)(t))\in K$, that is, $T(K)\subset K$. This completes the proof. □

It is clear that the existence of a positive solution for the system (1) is equivalent to the existence of a nontrivial fixed point of T in K. Finally, we define the nonnegative continuous concave functional on K by

$$\alpha(u,v)=\min_{\tau\le t\le 1}(u(t)+v(t)).$$

It is obvious that, for each $(u,v)\in K, \alpha(u,v)\le\|(u,v)\|$.

Throughout this section, we assume that $p_i, i=1,2$, are two positive numbers satisfying $\frac{1}{p_1}+\frac{1}{p_2}\le 1$.

To state our main result, we will assume that the following conditions are satisfied: (H1) $a_i(t)$ do not vanish identically on any subinterval of $(0,1)$, and there exists $t_0\in(0,1)$ such that $a_i(t_0)>0$ and $0<\int_0^1 a_i(s)G(t,s)ds<+\infty, 0<\int_0^1 a_i(s)$ $G_{1i}(t,s)ds<+\infty, i=1,2$.

Now, we can state our main result.

Theorem 2 *Assume that (H1) holds. In addition, assume there exist nonnegative numbers* a,b,c *such that* $0<a<b\le\min\{\tau,\frac{m_1}{p_1M_1},\frac{m_2}{p_2M_2}\}c$, *and* $f_i(t,u,v)$ *satisfy the following conditions:*

(H2) $f_i(t,u,v)<\frac{1}{p_i}\cdot\frac{c}{M_i}, \forall t\in[0,1], u+v\in[0,c], i=1,2,$

(H3) $f_i(t,u,v)<\frac{1}{p_i}\cdot\frac{a}{M_i}, \forall t\in[0,1], u+v\in[0,a], i=1,2,$

(H4) *(i)* $f_1(t,u,v)>\frac{b}{m_1}\forall t\in[\tau,1], u+v\in[b,\frac{b}{\gamma}]$, *or*

(ii) $f_2(t,u,v)>\frac{b}{m_2}\forall t\in[\tau,1], u+v\in[b,\frac{b}{\gamma}]$.

Then, the system (1) has at least three positive solutions $(u_1,v_1),(u_2,v_2),(u_3,v_3)$ *such that* $\|(u_1,v_1)\|<a, b<\min_{\tau\le t\le 1}(u_2(t)+v_2(t))$, *and* $\|(u_3,v_3)\|>a$, *with* $\min_{\tau\le t\le 1}(u_3(t)+v_3(t))<b$.

Proof First, we show that $T:\overline{P_c}\to\overline{P_c}$ is a completely continuous operator. If $(u,v)\in\overline{P_c}$, then by condition (H2), we have

$$\begin{aligned}\|T(u,y)\| &= \max_{0\le t\le 1}|A(u,v)(t)|+\max_{0\le t\le 1}|B(u,v)(t)|\\ &=\max_{0\le t\le 1}\left\{\int_0^1 G(t,s)a_1(s)f_1(s,u(s),v(s))ds+\frac{\mu_1 t^{\alpha-1}}{(1-\mu_1\eta_1^{\alpha-2})}\right.\end{aligned}$$

$$
\begin{aligned}
&\quad \times \int_0^1 G_{11}(\eta_1, s)a_1(s)f_1(s, u(s), v(s))ds \Bigg\} \\
&\quad + \max_{0\le t\le 1} \Bigg\{ \int_0^1 G(t,s)a_2(s)f_2(s,u(s),v(s))ds + \frac{\mu_2 t^{\alpha-1}}{(1-\mu_2\eta_2^{\alpha-2})} \\
&\quad \times \int_0^1 G_{12}(\eta_2, s)a_2(s)f_2(s, u(s), v(s))ds \Bigg\} \\
&\le \frac{1}{p_1}\cdot\frac{c}{M_1}\max_{0\le t\le 1} \Bigg\{ \int_0^1 G(t,s)a_1(s)ds + \frac{\mu_1 t^{\alpha-1}}{(1-\mu_1\eta_1^{\alpha-2})} \\
&\quad \times \int_0^1 G_{11}(\eta_1, s)a_1(s)ds \Bigg\} \\
&\quad + \frac{1}{p_2}\cdot\frac{c}{M_2}\max_{0\le t\le 1} \Bigg\{ \int_0^1 G(t,s)a_2(s)ds + \frac{\mu_2 t^{\alpha-1}}{(1-\mu_2\eta_2^{\alpha-2})} \\
&\quad \times \int_0^1 G_{12}(\eta_2, s)a_2(s)ds \Bigg\} \\
&\le \frac{1}{p_1}\cdot c + \frac{1}{p_2}\cdot c + \frac{1}{q_1}\cdot c + \frac{1}{q_2}\cdot c \le c.
\end{aligned}
$$

Therefore, $\|T(u, y)\| \le c$, that is, $T : \overline{P_c} \to \overline{P_c}$. The operator T is completely continuous by an application of the Ascoli-Arzela theorem.

In the same way, the condition (H3) implies that the condition (A2) of Theorem 1 is satisfied. We now show that condition (A1) of Theorem 1 is satisfied. Clearly, $\{(u, v) \in P(\alpha, b, \frac{b}{\gamma})|\alpha(u, v) > b\} \neq \emptyset$. If $(u, v) \in P(\alpha, b, \frac{b}{\gamma})$, then $b \le u(s) + v(s) \le \frac{b}{\gamma}$, $s \in [\tau, 1]$.

By condition (H4)(i), we get

$$
\begin{aligned}
\alpha(T(u,v)(t)) &= \min_{\tau\le t\le 1}(A(u,v)(t) + B(u,v)(t)) \\
&\ge \min_{\tau\le t\le 1} \Bigg\{ \int_\tau^1 G(t,s)a_1(s)f_1(s,u(s),v(s))ds \\
&\qquad + \frac{\mu_1\tau^{\alpha-1}}{(1-\mu_1\eta_1^{\alpha-2})}\int_0^1 G_{11}(\eta_1,s)a_1(s)f_1(s,u(s),v(s))ds \Bigg\}
\end{aligned}
$$

$$+\min_{\tau\le t\le 1}\left\{\int_{\tau}^{1} G(t,s)a_2(s)f_2(s,u(s),v(s))ds\right.$$

$$\left.+\frac{\mu_2\tau^{\alpha-1}}{(1-\mu_2\eta_2^{\alpha-2})}\int_0^1 G_{12}(\eta_2,s)a_2(s)f_2(s,u(s),v(s))ds\right\}$$

$$>\frac{b}{m_1}\min_{\tau\le t\le 1}\left\{\int_{\tau}^{1} G(t,s)a_1(s)ds+\frac{\mu_1\tau^{\alpha-1}}{(1-\mu_1\eta_1^{\alpha-2})}\right.$$

$$\left.\times\int_0^1 G_{11}(\eta_1,s)a_1(s)ds\right\}=\frac{b}{m_1}\cdot m_1=b.$$

Similarly, by (H4)(ii), we get

$$\alpha(T(u,v)(t))>\frac{b}{m_2}\min_{\tau\le t\le 1}\left\{\int_{\tau}^{1} G(t,s)a_2(s)ds+\frac{\mu_2\tau^{\alpha-1}}{(1-\mu_2\eta_2^{\alpha-2})}\right.$$
$$\left.\times\int_0^1 G_{12}(\eta_1,s)a_2(s)ds\right\}=\frac{b}{m_2}\cdot m_2=b.$$

Therefore, condition $A1$ of Theorem 1 is satisfied. Finally, we show that the condition $A3$ of Theorem 1 is also satisfied. If $(u,v)\in P(\alpha,b,c)$, and $\|T(u,v)\|>\frac{b}{\gamma}$, then

$$\alpha(T(u,v)(t))=\min_{\tau\le t\le 1}(A(u,v)(t)+B(u,v)(t))\ge\gamma\|T(u,v)\|>b.$$

Therefore, the condition $A3$ of Theorem 1 is also satisfied. By Theorem 1, there exist three positive solutions (u_1,v_1), (u_2,v_2), (u_3,v_3) such that $\|(u_1,v_1)\|<a$, $b<\min_{\tau\le t\le 1}(u_2(t)+v_2(t))$, and $\|(u_3,v_3)\|>a$, with $\min_{\tau\le t\le 1}(u_3(t)+v_3(t))<b$. we have the conclusion. □

4 Application

Example 1. Consider the following singular boundary value problems

$$\begin{cases} D_{0+}^{\frac{5}{2}}u(t)+a_1(t)f_1(t,u(t),v(t))=0, & t\in(0,1),\\ D_{0+}^{\frac{5}{2}}v(t)+a_2(t)f_2(t,u(t),v(t))=0, & t\in(0,1),\\ u(0)=u'(0)=0, & u'(1)-\frac{1}{\sqrt{2}}u'(\frac{1}{2})=0,\\ v(0)=v'(0)=0, & v'(1)-\frac{1}{2\sqrt{2}}v'(\frac{1}{2})=0, \end{cases}\tag{12}$$

where $a_1(t) = t, a_2(t) = 1, \mu_1 = \frac{1}{\sqrt{2}}, \mu_2 = \frac{1}{2\sqrt{2}}, \eta_1 = \eta_2 = \frac{1}{2}$ and

$$f_1(t,u,v) = \begin{cases} \frac{\sqrt{1-t^2}}{100} + \frac{1}{200}(u+v)^2, \quad t \in [0,1], \quad 0 \le u+v \le 1, \\ \frac{\sqrt{1-t^2}}{100} + 10[(u+v)^2 - (u+v)] + \frac{1}{200}, \quad t \in [0,1], \\ 1 < u+v < 2, \\ \frac{\sqrt{1-t^2}}{100} + 6[\log_2(u+v) + 2(u+v)] + \frac{1}{200}, \quad t \in [0,1], \\ 2 \le u+v \le 4 \\ \frac{\sqrt{1-t^2}}{100} + \frac{\sqrt{u+v}}{2} + 59 + \frac{1}{200}, \quad t \in [0,1], \quad 4 < u+v < +\infty, \end{cases}$$

and

$$f_2(t,u,v) = \begin{cases} \frac{\sqrt{1-t^2}}{1000} + \frac{1}{400}(u+v)^2, \quad t \in [0,1], \quad 0 \le u+v \le 1, \\ \frac{\sqrt{1-t^2}}{1000} + 20[(u+v)^2 - (u+v)] + \frac{1}{400}, \quad t \in [0,1], \\ 1 < u+v < 2, \\ \frac{\sqrt{1-t^2}}{1000} + 8[\log_2(u+v) + 2(u+v)] + \frac{1}{400}, \quad t \in [0,1], \\ 2 \le u+v \le 4 \\ \frac{\sqrt{1-t^2}}{1000} + \frac{\sqrt{u+v}}{2} + 79 + \frac{1}{400}, \quad t \in [0,1], \quad 4 < u+v < +\infty, \end{cases}$$

It is easy to check (H1) holds. Choose $\tau = \frac{1}{4}, p_1 = 20, p_2 = 2$. Then by direct calculations, we can obtain that

$$\begin{aligned} M_1 &= \frac{1}{\Gamma(\frac{5}{2})} 0.44761904, \quad M_2 = \frac{1}{\Gamma(\frac{5}{2})} 0.4888888, \\ m_1 &= \frac{1}{\Gamma(\frac{5}{2})} 0.0465565, \quad m_2 = \frac{1}{\Gamma(\frac{5}{2})} 0.0429012. \end{aligned}$$

So, we choose $a = \frac{1}{2}, b = 2, c = 800$. It is easy to check that f satisfy the conditions (H2)–(H4). Thus, system (12) has at least three positive solutions $(u_1, v_1), (u_2, v_2), (u_3, v_3)$ such that $\|(u_1, v_1)\| < \frac{1}{2}, 2 < \min_{\frac{1}{4} \le t \le 1}(u_2(t) + v_2(t))$, and $\|(u_3, v_3)\| > \frac{1}{2}$, with $\min_{\frac{1}{4} \le t \le 1}(u_3(t) + v_3(t)) < 2$.

References

1. El-Sayed, A.M.A.: Nonlinear functional differential equations of arbitrary orders. Nonlinear Anal. **33**, 181–186 (1998)
2. Kilbas, A.A., Trujillo, J.J.: Differential equations of fractional order: Methods, results and problems I. Appl. Anal. **78**, 153–192 (2001)
3. Kilbas, A.A., Trujillo, J.J.: Differential equations of fractional order: Methods, results and problems II. Appl. Anal. **81**, 435–493 (2002)
4. Agarwal, R.P., Benchohra, M., Hamani, S.: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. **109**, 973–1033 (2010)
5. Lakshmikantham, V., Leela, S., Devi, J.V.: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
6. Lakshmikantham, V., Vatsala, A.S.: Basic theory of fractional differential equations. Nonlinear Anal. TMA **69**(8), 2677–2682 (2008)
7. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
8. Samko, G., Kilbas, A., Marichev, O.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Amsterdam (1993)
9. Saadi, A., Benbachir, M.: Positive solutions for three-point nonlinear fractional boundary value problems. Electron. J. Qual. Theory Differ. Equ. **2**, 1–19 (2011)
10. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
11. Leggett, R.W., Williams, L.R.: Multiple positive fixed points of nonlinear operators on ordered Banach spaces. Indiana Univ. Math. J. **28**, 673–688 (1979)