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• ADDENDA •

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Addendum to: Exotic structures arising from fake projective planes (Sci China Math, 2013, 56: 43–54)

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The goal of this addendum is to generalize the argument of the original article [5] to handle the case of Cartwright-Steger surface, and to correct misprints in the tables in [5].

It is known that 3 is the smallest Euler number achievable by a smooth surface of general type. Moreover, smooth surfaces of general type with $c_2 = 3$ are complex ball quotients $B^2_{\mathbb{C}}/\Pi$ consisting of 100 fake projective planes and 2 Cartwright-Steger surfaces. The list corresponds to 51 choices of Π, each of which gives rise to two non-biholomorphic complex structures. We refer the readers to [3, 4, 6] for the above results. Natural examples of exotic

$$pP^2_{\mathbb{C}}\#q\overline{P^2_{\mathbb{C}}}$$

with some relatively small p and q are obtained from the list of fake projective planes with non-trivial automorphisms and are tabulated in [5]. Our first goal in this addendum is to complete the picture by showing that the Cartwright-Steger surface gives rise to some exotic manifold as well, albeit different from the examples in [5].

Theorem 1. *Let M be a Cartwright-Steger surface. Let N be the quotient of M by its automorphism group. Let Y be the minimal resolution of singularities of N. Then Y gives rise to an exotic*

$$3P^2_{\mathbb{C}}\#17\overline{P^2_{\mathbb{C}}}.$$

Here a manifold A is said to be an exotic B if A is homeomorphic but not diffeomorphic to B. We need the following lemma. From this point on, we denote by M the Cartwright-Steger surface.

Lemma 1. *The Cartwright-Steger surface M has the following properties*:

(a) *The automorphism group of M,*

$$H := \mathrm{Aut}(M) = \mathbb{Z}_3,$$

the cyclic group of order 3.

(b) *The fixed point set of H consists of* 3 *fixed points of type $\frac{1}{3}(1,1)$ and* 6 *fixed points of type $\frac{1}{3}(1,2)$.*

(c) *The quotient $N := M/H$ is a simply-connected orbifold.*

Proof. (a) follows from the work of [3], see [2, Theorem 2]. (b) follows from [2, Proposition 12], and is also known to Igor Dolgachev and Tim Steger. (c) is a consequence of the presentation of π given in the references above. In fact,

$$N = B^2_{\mathbb{C}}/\mathcal{N},$$

where $\mathcal{N}$ is the normalizer of Π in the maximal arithmetic group $\bar{\Gamma}$ in the commensurable class of Π. In the notation of [2, Theorem 2], $\mathcal{N}$ is generated by an element j of order 4, and Π, the latter is generated by three elements a_1, a_2, a_3 which are of finite order. Hence $\mathcal{N}$ is generated by elements of finite order and the result follows from [1]. The author is indebted to Donald Cartwright for the presentation here. □

Proof of Theorem 1. Recall that N as a quotient of M is an orbifold with isolated singularities. Let $\pi : Y \to N$ be a minimal resolution of singularities of N. The strategy is to compute the Euler number and index of Y so that the fundamental results in geometric topology can be applied to conclude the proof as in [5].

Since H acts with isolated singularities, the canonical line bundle K_N on N is Cartier with

$$K_N^2 = \frac{1}{3}K_M^2 = 3.$$

As M has Euler number 3 and there are 9 fixed points of order 3 under the action of H, the Euler number of N is

$$e(N) = \frac{1}{3}(3-9)+9 = 7.$$

A singularity of type $\frac{1}{3}(1,1)$ is resolved to a (-3) curve on Y, and a singularity of type $\frac{1}{3}(1,2)$ gives rise to a chain of two (-2) curves on Y. Hence we have three (-3) curves $E_i, i = 1,2,3$ and three separate two chains of (-2) curves (F_{i1}, F_{i2}), $i = 1,\dots,6$, on Y. It follows that

$$K_Y = \pi^* K_N + \sum_{i=1}^{3} a_i E_i + \sum_{j=1}^{6}(b_{j1}F_{j1} + b_{j2}F_{j2}).$$

As E_i and F_{jk} are rational curves of self-intersection (-3) and (-2) respectively, we obtain the following from taking intersection of K_Y with each curve F_{jk}:

$$0 = K_Y \cdot F_{j1} = b_{j1}(-2) + b_{j2},$$
$$0 = K_Y \cdot F_{j2} = b_{j1} + b_{j2}(-2).$$

It follows that $b_{jk} = 0$ for all j and k. Similarly from intersection with E_i and adjunction formula, we get

$$1 = K_Y \cdot E_i = a_i(-3).$$

Hence,

$$a_i = -\frac{1}{3}$$

and

$$K_Y = \pi^* K_N - \frac{1}{3}\sum_{i=1}^{3} E_i.$$

It follows that

$$c_1(Y)^2 = K_Y \cdot K_Y = 3 + \frac{1}{9}(3)(-3) = 2.$$

Moreover, from Hurwitz formula,

$$c_2(Y) = e(N) + 3 + 2\cdot 6 = 22.$$

Hence the index is given by

$$\sigma(Y) = \frac{1}{3}(c_1^2 - 2c_2) = -14.$$

As E_1 has self-intersection -3, the quadratic form Q_Y on $H^2(Y,\mathbb{Z})$ is odd. Hence Freedman's result as stated in [5, Theorem 2.1] implies that Y is homeomorphic to $pP^2_{\mathbb{C}}\#q\overline{P^2_{\mathbb{C}}}$ for some integers p and q. Since N and hence Y is simply connected from Lemma 1,

$$b_1(Y) = b_3(Y) = 0$$

and we obtain

$$\begin{aligned} p+q &= c_2(Y) - 2 = 20, \\ p-q &= \sigma(Y) = -14. \end{aligned}$$

We conclude that

$$p = 3 \quad \text{and} \quad q = 17.$$

Hence Y is homeomorphic to $3P^2_{\mathbb{C}}\#17\overline{P^2_{\mathbb{C}}}$. On the other hand, from the result of Donaldson as stated in [5, Theorem 2.3], we conclude that any fourfold M diffeomorphic to $3P^2_{\mathbb{C}}\#17\overline{P^2_{\mathbb{C}}}$ does not carry any complex structure. We conclude that Y is an exotic $3P^2_{\mathbb{C}}\#17\overline{P^2_{\mathbb{C}}}$. □

Here we correct some clerical errors from data entry in the two tables in [5]. No change is needed for the arguments there. We also tabulate the result for the Cartwright-Steger surface obtained in this addendum in Table 3.

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Table 1 corrections: Fake projective planes with $k=\mathbb{Q}$

$(k,\ell,\mathcal{T})$	Class	M	$\|H\|$	M/H	$\pi_1(M/H)$	Exotic
$(\mathbb{Q},\mathbb{Q}(\sqrt{-1}),\{5\})$	$(a=1,p=5,\{2\})$	$(a=1,p=5,\{2\},D_3)$	3	$(a=1,p=5,\{2\})$	$\mathbb{Z}_4$	$7P^2_{\mathbb{C}}\#27\overline{P}^2_{\mathbb{C}}$
$(\mathbb{Q},\mathbb{Q}(\sqrt{-2}),\{3\})$	$(a=2,p=3,\{2\})$	$(a=2,p=3,\{2\},D_3)$	3	$(a=2,p=3,\{2\})$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$7P^2_{\mathbb{C}}\#27\overline{P}^2_{\mathbb{C}}$
$(\mathbb{Q},\mathbb{Q}(\sqrt{-7}),\{2\})$	$(a=7,p=2,\emptyset)$	$(a=7,p=2,\emptyset,7_{21})$	N	N	N	N
		$(a=7,p=2,\emptyset,D_3X_7)$	3	$(a=7,p=2,X_7)$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$
	$(a=7,p=2,\{7\})$	$(a=7,p=2,\{7\},D_32_7)$	3	$(a=7,p=2,\{7\},2_7)$	$\mathbb{Z}_2$	$3P^2_{\mathbb{C}}\#13\overline{P}^2_{\mathbb{C}}$
			21	$(a=7,p=2,\{7\})$	$\{1\}$	$P^2_{\mathbb{C}}\#9\overline{P}^2_{\mathbb{C}}$
$(\mathbb{Q},\mathbb{Q}(\sqrt{-7}),\{2,3\})$	$(a=7,p=2,\{3\})$	$(a=7,p=2,\{3\},3_3)$	N	N	N	N
	$(a=7,p=2,\{3,7\})$	$(a=7,p=2,\{3,7\},3_3)$	N	N	N	N
$(\mathbb{Q},\mathbb{Q}(\sqrt{-15}),\{2\})$	$(a=15,p=2,\{3\})$	$(a=15,p=2,\{3\},(D3)_3)$	3	$(a=15,p=2,\{3\})$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$

Table 2 corrections: Fake projective planes with $\deg_{\mathbb{Q}} k = 2$

$(k,\ell,\mathcal{T})$	Class	M	$\|H\|$	M/H	$\pi_1(M/H)$	Exotic
$(\mathcal{C}_{10},\{v_2\})$	$(\mathcal{C}_{10},p=2,\{17-\})$	$(\mathcal{C}_{10},p=2,\{17-\},D_3)$	3	$(\mathcal{C}_{10},p=2,\{17-\})$	$\{1\}$	$P^2_{\mathbb{C}}\#6\overline{P}^2_{\mathbb{C}}$
$(\mathcal{C}_{18},\{v_3\})$	$(\mathcal{C}_{18},p=3,\emptyset)$	$(\mathcal{C}_{18},p=3,\emptyset,d_3D_3)$	9	$(\mathcal{C}_{18},p=3,\emptyset)$	$\{1\}$	$P^2_{\mathbb{C}}\#8\overline{P}^2_{\mathbb{C}}$

Table 3 Cartwright-Steger surface

(k,ℓ)	$\|H\|$	$\pi_1(M/H)$	Exotic
$(\mathbb{Q}(\sqrt{3}),\mathbb{Q}(\sqrt{3},\sqrt{-1}))$	3	$\{1\}$	$3P^2_{\mathbb{C}}\#17\overline{P^2_{\mathbb{C}}}$

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