



The Neglect of Epistemic Considerations in Logic: The Case of Epistemic Assumptions

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Abstract

The two different layers of logical theory—epistemological and ontological—are considered and explained. Special attention is given to epistemic assumptions of the kind that a judgement is granted as known, and their role in validating rules of inference, namely to aid the inferential preservation of epistemic matters from premise judgements to conclusion judgement, while ordinary Natural Deduction assumptions (that propositions are true) serve to establish the holding of consequence from antecedent propositions to succedent proposition.

Keywords Assumption · Consequence · Judgement · Proof-object · Demonstration · Analytic, immediate inference

1 Two Perspectives in Logic

Following Archbishop Whatley’s *Elements of Logic* from 1826 we say:

1.1 Logic may be Considered as the Science, and also as the Art, of Reasoning

When reasoning we carry out acts of passage, “inferences”, from granted premises to novel conclusions. Logic is Science because it investigates the principles that govern reasoning and Logic is Art because it provides practical rules that may be obtained from those principles. Reasoning is par excellence an epistemic matter, dependent on a judging agent. If the ultimate starting points for such a process of reasoning are items of knowledge, accordingly a chain of reasoning in the end brings us to novel knowledge.

In today’ logic, on the other hand, inferences are not primarily seen as acts, but as production-steps in the generation of derivations among metamathematical objects known as wff’s, that is, well-formed formulae. Furthermore, by the side of this metamathematical change regarding the status of inferences, an ontological approach has largely taken

over from the previous epistemological one. This ontological approach in logic began with another nineteenth century cleric, namely the Bohemian Bernard Bolzano and his *Wissenschaftslehre* (1837). As is by now well-known Bolzano avails himself of certain denizens in a Platonic “Third Realm” that are known as *Sätze an sich*, that is, *propositions-in-themselves*, precisely half of which, namely the truths-in-themselves, are true. This notion of truth (*-in-itself*), also considered as a Platonist in-itself notion, when applied to a proposition (*in-itself*), serves as the pivot for this novel rendering of logic.

In particular, Bolzano reduces the epistemic evaluative notions with respect to judgements and inferences, namely *correctness* and *validity*, to various matters of ontology pertaining to these propositions-in-themselves. Thus the judgement [A is true], in which truth is ascribed to the proposition (-in-itself) A that serves in the role of judgemental content, is deemed to be right, or correct (German *richtig*), if the proposition(-in-itself) in question really is a truth. Similarly the inference-scheme, or *figure*, I:

$$\frac{J_1 \dots J_k}{J}$$

where each judgment J_i is of the form *proposition A_i is true*, is deemed to be valid if, in modern terms, the relation of logical consequence, that is, preservation of truth “under all variations”, holds from the antecedent propositions A_1, A_2, \dots, A_k that serve as contents of the premise-judgements $J_1 \dots J_k$ of the inference I to the proposition C that serves as

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content of the conclusion. Another way of formulating the second Bolzano reduction may be found in Wittgenstein's *Tractatus* (5.11, 5.35.132, 5.133, 6.1201, 6.1221). The inference J is valid if the implication $A_1 \& A_2 \& \dots \& A_k \supset C$ is a logical truth, or, in the Tractarian terminology, a *tautology*. Both formulations of this Bolzano reduction are close enough to what Bolzano actually says; his particular cavils regarding the compatibility of the antecedent propositions, and his conjunctive, rather than the customary current *disjunctive* reading of consequences with multiple consequent propositions we may, at the present level of generality, disregard.¹

The epistemic conception of traditional logic is all-out Aristotelian and stems from the early sections of the *Posterior Analytics*. The Aristotelian conception of demonstrative science organizes a field of knowledge by using axioms that are self-evident in terms of primitive concepts and proceeds to gain novel insights by application of similarly self-evident rules of inference. Frege's great innovation in logic can be seen as refining this traditional Aristotelian axiomatic conception by joining it to his notion of a *formal language*, with its concomitant notion of logical inference. Frege's deployment of a novel form of judgement, namely *proposition* ("Thought") *A is true*, where the content A has function/argument structure P(a), allowed him to develop a much richer view of what follows from what, in particular when drawing upon quantification theory. He did not change anything, though, with respect to epistemic demonstration (*Beweis*), which remains Aristotelian through and through. Thus, both the Preface to the *Begriffsschrift* as well §3 of *Grundlagen der Arithmetik* bear strong resemblance to the well-known regress argument unto first principles, with which Aristotle opens the *Posterior Analytics*.

2 Two Views on Logical Language

Aristotle's detailed account of consequence from the *Prior Analytics*, on the other hand, was of course superseded by Frege's introduction of the formal ideography that comprises also quantification theory. Frege's conception of a formal language, though, was different from our modern notion of a formal language (or perhaps better today: *formal system*) that distinguishes between syntax and semantics and deploys two turnstiles: one "syntactic" \vdash that really is a metamathematical theorem-predicate with respect to wff's, and indicates the existence of a suitable formal derivation,

and one semantic \models that indicates "satisfaction" in a suitable model. Both turnstiles furthermore are relativized by including also assumptions in the guise of antecedent-formulae to the left of the respective turnstile, thereby making matters even more complex. The second, model-theoretic notion plays no role in Frege, and his uses of the "syntactic" turnstile is radically different from the modern one: Frege's sign serves as a pragmatic assertion indicator, whereas the modern one is a predicate—a propositional function if you want—that is defined on well-formed formulae. This difference is symptomatic of the difference in use between Frege's formal language, i.e. his ideography (*Begriffsschrift*), on the one hand, and modern formal languages that, as a rule, are construed meta-mathematically, on the other hand.² The latter can only be *talked about*; they are objects of study *only*, but are not intended for use. For instance, in Solomon Feferman's authoritative treatment of Gödel's two Incompleteness Theorems one finds no "*object language*"; instead Feferman (1960) proceeds directly to the Gödel numbers. Since the object "language" in question is never used for saying anything—its "metamathematical expressions" are not real expressions and *do not express*, but instead are *expressed* as the referents of real expressions—there is no need to display such an object language: it is only talked about, but in contradistinction to other languages, it is not a vehicle for the expression of thoughts.³

Frege's ideography, on the other hand, is an interpreted formal language, and he spent a tremendous effort on meaning explanations, for instance, in the early sections of *Begriffsschrift*, for the predicate logic version of the ideography from 1879, and in the opening sections §§1–32 of *Grundgesetze der Arithmetik*, Vol I, from 1893, especially the §§27–31. It should be noted that this *Grundgesetze* version of the Fregean ideography is *not* a predicate logic, but a *term logic*, which sometimes serves to make matters hard to understand when viewed from the prevalent standard of today, where theories are routinely formulated in predicate logic. In Frege's late piece of writing, the *Nachlass* fragment *Logische Allgemeinheit* that was left uncompleted at the time of his death, we find a distinction between a *Hilfssprache* and a *Darlegungssprache*. The Editors of Frege's "*Posthumous Writings*" deliberately point to Tarski and translate *Hilfssprache* as object-language and *Darlegungssprache* as meta-language. This translation, however, is not felicitous. The term *Hilfssprache* is the

¹ Detailed attributions in the *Wissenschaftslehre* for the claims regarding Bolzano can be found in my 2009, § 3 'Revolution: Bolzano's Annus Mirabilis'.

² Barnes (2002) convincingly argues the use of the term *ideography* as a translation of German *Begriffsschrift*.

³ Sundholm (2002) treats in some detail of the distinction between expressions and metamathematical expressions, whereas the advantages of interpreted formal languages are argued in Sundholm (2001, 2003).

German rendering of the French *langue auxiliaire*, which term stands for the artificial languages that were considered in the artificial languages movement, of which Frege's correspondents Couturat and Peano were prominent members.⁴ Examples that spring to mind are *Volapük*, *Bolak*, *Esperanto*, and today also *Klingon*, and on the scientific side *Interlingua*, *Latine sine flexione* in which Peano wrote a famous paper on differential equations. Frege's *Begriffsschrift* is precisely such an artificial auxiliary language—a *Hilfssprache*—and the difference between it and other auxiliary languages is that it is a formal one. Nevertheless, just as Esperanto and Volapük, it was intended for expressing meaning, and accordingly one needs a “language of display” in order to set it out properly. All the languages in the Russell–Tarski tower of “meta-languages” (over the first object-language) are also *object-languages*, and are ultimately only spoken *about*.⁵ The real meta-language is Curry's “U language”—U for *use*—and it needs a vantage point *outside* the Russell–Tarski hierarchy in question.⁶ Frege's *Darlegungssprache* matches Curry's U language and his *Hilfssprache* is an auxiliary language like *Volapük*, *Bolak*, and *Esperanto* championed by Couturat and Peano (*Interlingua*, *Latine sine flexione*).

Of course, the two different versions of Frege's ideography in *Begriffsschrift* and *Grundgesetze* are *Hilfssprachen* and must be explained, that is, *dargelegt*, or spelled out. The editors of the *Nachlass* compliment Frege for having here anticipated the precise object-language/meta-language distinction that was put firmly onto the philosophical firmament a decade later by Carnap (1934) in *Logische Syntax der Sprache* and by Tarski in *Der Wahrheitsbegriff in den formalisierten Sprachen*. However, as we saw Frege's *Hilfssprache* is not an artefact void of meaning, that is, it is not an uninterpreted, “object-language”: on the contrary, it is an *auxiliary language* in the terminology of the artificial language movement.

Up to \pm 1930 every logician of note followed Frege's lead when constructing formal calculi, marrying their formal languages to the Aristotelian conception of Science: Whitehead and Russell, Ramsey, Lesniewski, early Carnap (*Aufbau* and *Abriss*), Curry, Church, early Heyting⁷ Their systems were interpreted calculi intended as epistemological tools. The mathematical study of mathematical language was naturally begun by Hilbert as part of his ideological programme

of applying positivistic verificationism to mathematics. Here equations between finitistically computable terms serve as analogues of positivist observation sentences. Such formulae [$s = t$] are even known as “*verifiable* propositions” in the magisterial Hilbert and Bernays (1934, 1939).⁸

In the Warsaw seminar of Lukasiewicz and Tarski during the second half of the 1920s, the study of formal languages and formal systems—Many-valued Logics!—was liberated from the Göttingen finitist ideological shackles of Hilbert. From hence on ordinary mathematical means were allowed in the meta-mathematical study of formal systems, much in the same way that naïve set theory was used in the development of set theoretic topology and cardinal arithmetic at which Polish mathematicians then excelled. With this liberating move, yet a further radical shift of perspective occurs. The formal systems no longer serve any epistemological role *per se*. Instead, strictly speaking, the “well-formed formulae” lack meaning, and do not as such express. They are mathematical objects on par with other mathematical objects; in fact, formally speaking, the meta-mathematical expressions are elements of freely generated semi-groups of strings. With this shift in the role of the “languages” of logic, epistemic matters are driven even further into the background. The logical calculi are not used for epistemological purposes anymore. One only proves theorems *about* them.

During the 1920s the *Grundlagenstreit* came to the fore and sharp epistemological problems were raised. After Brouwer's criticism of the unlimited use of the *Law of Excluded Middle*, there appear to be only two viable options with respect to logic. We may keep Platonistic impredicativity and **LEM** as freely used in classical analysis after the fashion of Weierstrass, or we may jettison them. We have already seen the other dichotomy of options, namely to consider formal systems based on languages with meaning, on the one hand, and based on uninterpreted formal calculi, on the other. After Gödel's work, attempts to resuscitate Fregean logicism, for instance by Carnap, no longer seemed viable and were abandoned: retaining classical logic as well as impredicativity, while insisting on explicit meaning-explanations that render axioms and rules of inference self-evident, simply seems to be asking too much. Thus we may jettison either meaning for the full formal language, while retaining classical logic and impredicativity, which is the option chosen by Hilbert's formalism. Only his “real” sentences, that is, the “*verifiable*” equations between finitist terms, and which serve as the analogue to the observation sentences of positivism, have meaning, whereas other sentences, the “*ideal*” ones, strictly speaking, are not given meaning-explanations.

For the second option on the other hand we may jettison classical logic and Platonist impredicativity, but then offer

⁴ I owe my awareness of these origins of Frege's *Hilfssprache* to the scholarship of Wolfgang Künne, cf. Künne (2010), Chap. 5, §5, pp. 725–738.

⁵ Russell's Introduction to Wittgenstein (1922) and Tarski (1936).

⁶ Curry (1963), Chap 2, §§1 and 2, is the *locus classicus* for the U language.

⁷ Sundholm (2001).

⁸ ((1934, §6) section c, third part: *Verifizierbare Formeln*).

meaning explanations for constructivist language after the now familiar fashion of Heyting.⁹

	Classical logic / \	
Language with content	Accept	Reject
Yes	<i>Logicism</i>	<i>Intuitionism</i>
No	<i>Formalism</i>	?

The hope of Carnap and others for meaning-explanations for the full language of say, second order analysis that render evident classical logic and impredicativity appears to be forlorn. We may then follow Hilbert confining meaning only to a “real” fragment, while the “ideal sentences” of full language remain uninterpreted, or we may jettison classical logic and impredicativity, and follow Heyting’s by now well-known way of giving constructive meaning-explanations with respect to the full language.

3 Constructive Meaning-Explanations and the Two Layers of Logic

With his *Constructive Type Theory* Per Martin-Löf has given streamlined form to Heyting’s “Proof Explanation of the intuitionistic logical constants”: a proposition A is explained by laying down how its canonical proofs may be put together out of parts (and when two such canonical proofs are equal canonical proofs of the proposition A).¹⁰ Accordingly, for each proposition A, we have a “type” Proof(A) and define a notion of truth for propositions by means of an application of the truthmaker analysis: A is true = Proof(A) exists.¹¹

Here the relevant notion of existence cannot be, on pain of an infinite regress, that of the existential quantifier. Classically, we may choose it to be Platonist set-theoretic existence and drawing upon classical reasoning one readily checks that the semantics verifies the Law of Excluded Middle. Thus, if we are prepared to reason Platonistically when justifying the rules of inference and axioms, casting the semantics in

terms of the Heyting proof-explanation does not force us to abandon classical logic. This, however, yields no epistemic benefits, and so I prefer to use the Brouwer–Weyl constructive notion of existence with respect to types α .¹² When α is a type (general concept),

α exists
is a judgement and its assertion condition is given by a rule of instantiation

$\frac{a \text{ is an } \alpha}{\alpha \text{ exists}}$

We note that propositions are given by truth-conditions that are defined in terms of (canonical) proofs, and (epistemic) judgements are explained in terms of assertion conditions. Thus we get an ensuing bifurcation of notions at both the ontological level of propositions, their truth, and their proofs (that is, their truthmakers), and on the epistemic level of judgements and their demonstrations.¹³

In the table below the epistemological and ontological two sides of logic are spelled out for a fairly large number of notions, and in other writings I have dealt with most of the lines. In the sequel of the present paper I intend to deal with the line contrasting an assumption that a proposition is true with an epistemic assumption that a judgement is known, with as a special case an assumption that a proposition is *known* to be true.

Epistemic notion	Ontological (“Alethic”) notion
Judgement (assertion)	Proposition
Demonstration	Proof (-object), truthmaker
Truth of judgement	Truth of proposition
Demonstrability	Existence of proof
Self-evident/mediated	Direct/indirect
Axiomatic/derived	Canonical/non-canonical
Intuitive/discursive	Simple/composite
Inference	Consequence
Validity	Holding
Assumption that a judgement is known	Assumption that a proposition is true
Hypothetical demonstration	Dependent proof-object
Hypothetical judgement	Implicational proposition
Definitional (criterial) equality	Propositional identity
(Function) Generality	Quantifier

⁹ The various options regarding retention of classical reasoning and meaning explanations are spelled out in some details in my 1998a.

¹⁰ Martin-Löf (1984).

¹¹ A fairly comprehensive introduction to Martin-Löf’s CTT can be found in my (1977). See also the paper by Ansten Klev in the present issue of TOPOI. That Heyting’s explanation of truth as existence of a proof (-object) is a kind of truth-maker analysis was first suggested in my (1994a).

¹² As is well known, Tarski’s definition of truth does not *on its own* yield the Law of Excluded Middle for the notion of truth thus defined. Classical reasoning in the meta-theory is required for that. In my (2004) I carry out the pendant reasoning and show that, when classical meta-theory is allowed, it is very easy to validity LEM, also under the Heyting semantics.

¹³ In my (1997), (2000), and (2012) the demonstration versus proof distinction is given more substance.

4 Four Different Notions of Consequence

Apart from the two changes already indicated—the meta-mathematical shift and the Bolzano reduction of inferential validity to logical truth (or logical consequence) in “all variations”—we then have occasion to consider another major invention of the early 1930s, namely Gentzen’s Natural Deduction derivations and his Sequent Calculi.

Within the interpreted perspective of an interpreted formal language, with respect to two propositions A and B , there are at least four relevant notions of consequence here.

- (1) the *implication* proposition $A \supset B$, which may be **true** (or even *logically true* “in all variations”);
- (2) the *conditional* [if A is true then B is true], or, in other words,

B is **true**, **oncondition** that A is true
underhypothesis that A is true
under assumption that A is true
- (3) the *consequence* [$A \Rightarrow B$] may **hold**;
- (4) the *inference* [A is true. Therefore: B is true] may be **valid**.¹⁴

Fact 1 “implies” takes that-clauses, whereas “if-then” takes complete declaratives. Ergo:*implication* and *conditional* are not the same. The conditional (2) is a hypothetical judgement in which **hypothetical truth** is ascribed to the proposition B . Its verification-object is a dependent proof-object $b:\text{Proof}(B)$ [$x:\text{Proof}(A)$], that is, b is a proof of B under the assumption (hypothesis, supposition) that x is a proof of A .

The *consequence* (3) is a Gentzen *sequent* (German *Sequenz*). (Why, we may ask, did Gentzen drop the prefix *Kon* here?)

The *judgement*

[$A \Rightarrow B$] **holds**

is a generalization of [A is **true**] and demands for its verification a mapping (higher-level function) $f: \text{Proof}(A) \rightarrow \text{Proof}(B)$. Since implication and conditional are different, this is *not* the proof-object demanded for the truth of an implication: these have the canonical form $\lambda (A, B, [x]b)$, or if you prefer the logical formulation, rather than the set-theoretical one:

$\supset I(A, B, [x]b)$,

where b is a dependent proof of B , under the assumption that x is a proof(A), and have a special application function $ap(y,x)$, whereas application in the case of f is primitive:

when $a : \text{Proof}(A)$, then $f(a) : \text{Proof}(B)$.

Fact 2 The judgement (1)–(3) have different meaning-explanations—their assertion conditions are not the same—and accordingly do not mean the same, are not synonymous, while (4) indicates acts of passage. The first three notions, however, are *equi-assertible*. Given a verification-object for one of the three, verification-objects for the other two are readily found in a couple of trivial steps. Furthermore, all four relations are refuted by the same counter-example, namely a situation in which A is known to be true and B known to be false. This might serve to explain why the four notions have sometimes been hard to keep apart, especially from the classical point of view.¹⁵

Fact 3 Bolzano deals ably with consequence, whereas his account of inference is inadequate and quite psychologistic in terms of *Gewissmachungen*.¹⁶ Frege, on the other hand, deals ably with inference, but (logical) consequence has no place in his system. Only with Gentzen’s 1936 sequential formulation of Natural Deduction, where the derivable objects are sequents, that is consequences, and where the principal introduction and elimination inferences all take place to the right of the sequent-arrow, do we get a system that can cope both with inference and consequence.¹⁷

Fact 4 Consequence, not *logical* consequence, is the primary notion. Gentzen’s system deals with *arithmetic*; his rules of inference that take us from premise-sequent(s) to conclusion-sequent are obviously valid, but they do not hold logically in all variations. They are only “arithmetically valid”.

Fact 5 A completeness theorem for an interpreted formal language would state: all truths (and in the case of Gentzen’s system: all sequents that hold) are derivable by means of these rules. For Gödelian reasons, interesting systems with theorems of the form [A is true] are not complete.¹⁸

When we now consider how one would establish that (1) to (4) obtain, we see that for (1)–(3) ordinary natural deduction derivations are involved in one way or another. In

¹⁴ My (1998) and (2012) explain the inter-relations of notions (1)–(4) in considerable detail.

¹⁵ The afterword to my (2012) gives more details concerning the kinds of function—Euler-Frege functions, Dedekind mappings, and courses-of-value—that serve as verification witnesses for, respectively, conditionals, closed consequences (“sequents”), and implicational propositions.

¹⁶ Volume III of the *Wissenschaftslehre* contains Bolzano’s account of *Gewissmachungen*.

¹⁷ In (2006), at p. 632, and (2009), at p. 298, the links between Frege and Gentzen are explored further.

¹⁸ I explore these Gödel phenomena in (2004a, §8).

all three cases one needs a hypothetical proof $b:\text{Proof}(B)$ [$x:\text{Proof}(A)$].

The implication $A \supset B$ is established by forming the course-of-value $\lambda(A, B, [x]b)$, whereas the conditional is already established by the hypothetical, dependent proof-object in question. Finally, forming the function $[x] b:\text{Proof}(A) \rightarrow \text{Proof}(B)$ by means of “lambda” abstraction $[\]$ (Curry’s notation!) on the hypothetical proof establishes that the closed consequence (“sequent”) holds.

5 Blind Judgement and Inference

Under the Bolzano reduction, when the proofs (“verification objects”) work also in all variations, then *classically* one says that the inference (4) is valid. However, the Bolzano reduction validates what we may, in the excellent terminology of Brentano, call *blind* judgement and inference.¹⁹ The epistemic link to the judging reasoner has here been severed, whereas I am concerned to preserve this link.

Consequence preserves truth from antecedent propositions to consequent proposition, and logical consequence does so “under all variations”. The demonstration of the Prime Number Theorem (PNT) by De la Vallée-Poussin and Hadamard in 1896 certainly could be formalized within NBG, the set theory of Von Neumann, Bernays and Gödel.²⁰ Since this theory is finitely axiomatized, we may conjoin its axioms into one proposition VNBG and then consider the inference

$$(*) \frac{\text{VNBG is true}}{\text{PNT is true}}$$

The inference (*), certainly, is truth-preserving, in the light of the formalized demonstration offered and the Soundness Theorem for the Predicate Calculus: every time an NBG axiom is used in the predicate logic derivation we replace it by the proposition VNBG and then apply conjunction elimination. Hence we get a formal derivation of PNT from VNBG, whence the Soundness Theorem guarantees truth-preservation. So under the Bolzano reduction this is a valid inference, because truth-preserving under all variations, but it provides no epistemic insight at all.

6 Epistemic Assumptions

Instead, validity of inference, rather than (logical) holding of consequence, involves preservation, or transmission, of *epistemic* matters from premises to conclusion and it is here that epistemic assumptions that judgements are known (or *granted*) become helpful. In order to validate the inference I one makes the assumption that one knows the premise-judgments, or that they are being given as evident, and under this epistemic assumption one has to make clear that also the conclusion can be made evident.²¹

The difference between the two types of assumptions is especially clear when we consider Gentzen derivations in Natural Deduction. An ordinary assumption A of Natural Deduction corresponds to an alethic, ontological assumption that proposition A is true. From such an assumption we may, for instance, obtain a conclusion that B is true, when we have already established the conditional judgement, $(\$)$ B is true, on hypothesis that A is true,

Furthermore, if we wish to do so, from this we readily obtain also the outright assertion that the implication $A \supset B$ is true by implication introduction, or, for that matter, if we so wish, but now with the aid of functional abstraction on the dependent proof-object that warrants $(\$)$, we also may conclude that the sequent $[A \rightarrow B]$ holds.

An epistemic assumption that a judgement $[A \text{ is true}]$ is known, or perhaps better granted, corresponds for Natural Deduction derivations to the hypothesis that we have been provided with a closed derivation of the proposition A . This is patently a different kind of assumption from the ordinary Natural Deduction assumption of the wff A .

Brouwer did not accept hypothetical proofs—I hesitate to call them proof-objects in his case. His proofs are all epistemic demonstrations: an assumption that a proposition is true amounts to an assumption that the assumed proposition is known to be true, for instance in his demonstration of the Bar Theorem.²²

¹⁹ Brentano (1889, Anm. 27, pp. 64–72) and Brentano (1930), where, in particular, the fragments in part IV are important.

²⁰ Mendelson (1964, Chap. 4) contains a rich exposition of NBG.

²¹ Martin-Löf (1984), for instance at p. 41, avails himself of epistemic assumptions: “Assuming that we *know* the premisses ...” (my emphasis). He does not, however, then formulate the explicit notion, which, or so it appears, was introduced in my (1997, p. 210).

²² Brouwer’s Demonstration of the Bar Theorem, with its particular use of an epistemic assumption, is discussed in detail by Sundholm and Van Atten (2008).

7 Gentzen’s Two Frameworks for Natural Deduction Ans Epistemic Assumptions

Over the past decades I have had a discussion with Dag Prawitz about the status of the proofs in the BKH explanation: I have claimed that they are not demonstrations with epistemic power, but that they are mathematical witnesses, corresponding to truthmakers in currently popular theories of grounding. Prawitz, on the other hand, has held that they are epistemically binding.²³ With my present terminology I can formulate my principal objection thus: the distinction between epistemic and alethic assumptions collapses if proofs are held to be epistemically binding. There will be no difference between assuming that proposition A is true and assuming that one knows that A is true.

In type theory the difference between the two kinds of assumption comes out in different treatments of proof-objects. An ordinary assumption has the form $x:\text{Proof}(A)$:assume that x is a proof for A

An epistemic assumption with respect to the same proposition takes a *closed* proof-object as given:assume that I am given a closed proof $a:\text{Proof}(A)$

Against the background of these distinctions we can now explain the difference between the two Gentzen frameworks for Natural Deduction.

The 1932 format from the dissertation is the usual one with assumption formulae as top nodes in derivations

$$\Pi : \begin{array}{c} A_1 \dots A_k \\ \cdot \quad \cdot \\ C \end{array}$$

1936 format, on the other hand, is an axiomatic calculus for deriving consequences of the form, where the assumption formulae are listed

$$A_1 \dots A_k \rightarrow C$$

1936 derivations are best seen as **demonstrations of judgements** of the form:

$$[A_1 \dots A_k \rightarrow C] \text{hold}$$

Derivations in the 1932 format, on the other hand, are to my mind best seen, not as epistemic demonstrations, but as *dependent proof-objects* Π of the form

$$\Pi : C(x_1 A_1 \dots x_k : A_k),$$

that is, Π is a proof of C under the assumptions that $x_1 \dots x_k$ are proofs of $A_1 \dots A_k$, respectively.²⁴

²³ For an early instalment in this debate, see my 2000, with a reply by Prawitz in the same issue of *Theoria*.

²⁴ My (2006) is devoted to spelling out the differences, *with respect to an interpreted calculus*, between Gentzen’s 1932 and 1936 ways of setting out his derivations.

8 Epistemic Assumptions and Analytic Validation of Inferences

In recent work, Per Martin-Löf has given an interesting dialogical twist to epistemic assumptions.²⁵ Already in his first 1946 paper on performatives, etc., John Austin wrote:

If I say “S is P” when I don’t even believe it, I am lying: if I say it when I believe it but am not sure of it, I may be misleading but I am not exactly lying.
When I say “I know”, I *give others my word*: I *give others my authority for saying* that “S is P”.²⁶

Assertions contain implicit, first-person knowledge claims (recall G. E. Moore and asserting that it is raining, but that one does not believe it!), so assertions grant authority.

When I first read Austin in 2009 I was led to formulate an **Inference Criterion** of the same kind:

When I say “Therefore” I give others my authority for asserting the conclusion, given theirs for asserting the premisses.

Martin-Löf has now noted that one does not need to *know* that the premisses are evident for the validation of an inference: what one must be prepared to undertake is to make the conclusion known or evident under the assumption that *someone else* grants the premisses as evident.

In order to undertake that responsibility it is enough if I possess a chain of immediately evidence-preserving steps (in terms of meaning-explanations) that link premisses to conclusion.²⁷ Here the introduction rules of Gentzen may be seen as immediate *and* meaning explanatory, whereas the elimination rules are immediate, but not meaning explanatory. In Kantian terms, both the introduction and elimination rules are *analytically valid*, but only the introduction rules are explicitly analytic, or “identical”, whereas the analyticity of the elimination rules is implicit, and might need to be made explicit in terms of the meaning explanations offered by the introduction rules, in analogy with:

All rational animals are rational

is an explicitly analytic (identical) judgement, whereas

²⁵ In lectures at SND, Paris 2015, and at Marseille 2016, at the meeting that provides the source for the present issue of TOPOI.

²⁶ Austen (1946, p. 171).

²⁷ I suggested this treatment of inferential validity in an invited lecture at LOGICA 1996, and published it the next year in the LOGICA Yearbook; it is now readily available in my (2012), p. 950. It is also dealt with in (2004a), pp. 454–455.

All humans are rational

is also an analytic judgement, but only implicitly so, and one resolution-step, replacing the term *human* by its definition *rational animal*, is needed to bring this judgement to explicitly analytic form.²⁸

In order to complete the comparison, we consider the question:

Why is $\&$ -elimination rule valid?

We are then, in an epistemic assumption, given as evident the premise-judgement

- (i) c :Proof(A&B)
for an application of $\&$ -elimination.
Under this epistemic assumption we have to make evident the conclusion
- (ii) $p(c)$: Proof(A).
Since c is a proof of A&B, it executes, (evaluates, is definitionally equal) to a *canonical* proof of A&B that accordingly has the form
- (iii) $\langle a, b \rangle$: Proof(A&B) and $c = \langle a, b \rangle$: Proof(A&B),
where we know that
- (iv) a :Proof(A) and b :Proof(B).
But granted this, it is a *meaning stipulation* for the ordered-pair- and projection-operators that
- (v) $p(\langle a, b \rangle) = a$:Proof(A)
but, since $c = \langle a, b \rangle$: Proof(A&B), we also get $p(c) = p(\langle a, b \rangle) = a$:Proof(A), whence we are done.

Note here these deliberations are all pursuant to the relevant meaning explanations for the notions Proof, $\&$, $\langle \rangle$, and p . The step from (i) to (iii) and (iv) matches the resolution-step that replaces *human* by *rational animal*.

9 Axiom and Lemma from an Epistemic Point of View

Finally, what does this mean for axioms in the traditional sense? Such axioms were *self-evident* judgements, and known as such. The work of Pasch and Hilbert in geometry initiated a change that led to a hypothetical-deductive conception, which replaced the epistemic notion of inference from self-evident axioms with the model-theoretic notion of logical consequence “under all variations” or “in all

models”. Natural Deduction added one more feature here to the dethroning of axioms: they now become ordinary assumptions among other ordinary assumptions, but as such they are privileged, because they need never be discharged, and may be discounted, when standing in antecedent position in consequences. Nevertheless, contrary to axioms in the old-fashioned sense, they are not known, nor are they asserted whenever they occur. An axiom in the old sense was not an assumption: it was asserted, whereas now that epistemic status is gone, and instead axioms are unasserted assumptions among other assumptions, with the privilege of not carrying the onus of discharge on them.

In conclusion then let me just note that epistemic assumptions are well known in mathematical practice when one draws upon a *lemma*, the demonstration of which is left out until the main demonstration has been completed. Nevertheless, within the main demonstration, the lemma does not work as an additional *assumption*, but avails itself of assertoric force, even though proper grounding by means of a demonstration is as yet absent. A very clear case here is the so-called *Zorn’s Lemma*, whose epistemic status is highly debatable from the point of view of constructivism, but classically is granted axiomatic status.

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Compliance with Ethical Standards

Conflict of interest The author declared that he has no conflict of interest.

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²⁸ Leibniz considered patently analytic judgements under the nomen “identical” and preferred Latin *resolution* to Greek *analysis* used by Kant. The Kantian Jäsche *Logik* §37 *Tautologische Sätze* is helpful here.

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