Synthese (2022) 200:420
https://doi.org/10.1007/s11229-022-03856-y

ORIGINAL RESEARCH

Ante rem structuralism and the semantics of instantial terms

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Received: 25 January 2022 / Accepted: 10 August 2022 / Published online: 7 October 2022

Abstract

Ante rem structures were posited as the subject matter of mathematics in order to resolve a problem of referential indeterminacy within mathematical discourse. Nevertheless, *ante rem* structuralists are inevitably committed to the existence of indiscernible entities, and this commitment produces an exactly analogous problem. If it cannot be sorted out, then the postulation of *ante rem* structures is futile. In a recent paper, Stewart Shapiro argued that the problem may be solved by analysing some of the singular terms of mathematics not as genuinely referring expressions, but as instantial terms. In this paper, I discuss several competing accounts of the semantics of terms of this kind, and argue that they are all untenable for the *ante rem* structuralist. Shapiro, then, still owes us an account of the semantics of instantial terms that suits the *ante rem* structuralist project. Without it, the *ante rem* structuralist is still unable to determine the reference of the singular terms of mathematics.

Keywords Mathematical structuralism · Ante rem structuralism · Referential indeterminacy · Instantial terms · Instantial reasoning

1 Introduction

Ante rem structures are abstract entities akin to Platonic universals, each one of which is exemplified by all of the isomorphic models of a mathematical theory. *Ante rem* structuralism is the view that *ante rem* structures constitute the subject matter of mathematics. The central aim of this theory is to solve a well-known problem of referential indeterminacy described by Benacerraf in 1965.

Ante rem structuralists are inevitably committed to the truth of two contentious hypotheses: the first is that there are indiscernible entities; the second, that certain

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singular terms are used to denote individual indiscernibles. Black (1952) advanced a compelling case for the existence of indiscernible entities; but he also argued convincingly that an individual, if it is indiscernible from others, cannot be referred to. If Black is right, then *ante rem* structuralism faces a problem of referential indeterminacy. In his (2012), Shapiro defended *ante rem* structuralism from the threat of Black's argument. In this paper, I contend that the threat is still in force, and that it is more powerful than the *ante rem* structuralist has recognised thus far.

In § 2, I outline the *ante rem* structuralist's project: I present the rationale behind the postulation of *ante rem* structures as the subject matter of mathematics, and the basic tenets of *ante rem* structuralism. Afterwards, I relate how the *ante rem* structuralist became committed to indiscernible entities, and show that there is no obvious way for them to escape this commitment. In § 3, I describe how Black's insight poses a threat for the *ante rem* structuralist, and argue that this threat is more worrying than it has been recognised thus far. Then I explain how Shapiro proposes to dissolve this threat by advancing the surprising hypothesis that some syntactically proper names occurring in mathematical sentences are not *logically* proper names, but instantial terms. Finally, in § 4, I discuss several competing accounts of the semantics of terms of this kind, and contend that none of them is compatible with the project of *ante rem* structuralism as I described it in § 2. Shapiro, then, still owes us an account of the semantics of instantial terms that suits this project. Without it, the *ante rem* structuralist cannot claim to have fixed the reference of the singular terms of mathematics to *ante rem* structures.

2 Background: Benacerraf on set-theoretic foundationalism, *ante rem* structuralism, and the Burgess-Keränen objection

Set-theoretic foundationalism states that axiomatic set theory allows us to identify mathematical objects with sets. In his (1965), Benacerraf argued against this view: if numbers are sets, he noted, they must be *particular* sets—but which? The naturals can be identified with the finite von Neumann ordinals, the finite Zermelo ordinals, and infinitely many isomorphic systems. Similarly, the reals may be reduced to Dedekind cuts constructed set-theoretically, but they can also be accounted for in terms of a construction based on equivalence classes of Cauchy sequences on the rational numbers—as suggested, for instance, by Cantor. According to Benacerraf, "[t]here is no way connected with the reference of number words that will allow us to choose among [the candidate reductions of numbers to sets], *for the accounts differ at places where there is no connection whatever between features of the accounts and our uses of the words in question*" (1965, p. 62). Benacerraf observed, in summary, that there are several systems of set-theoretic objects that numbers could be identified with, and no reason for identifying the latter with any one of these systems rather than any other.

Benacerraf thus concluded that numbers cannot be identified with set-theoretic objects. In fact, he claimed, they cannot be identified with any objects whatsoever, for there are several, equally good candidates (1965, p. 69). He then suggested that numbers should be thought of as positions in structures instead: the naturals, for instance, are to be identified not with the finite von Neumann ordinals nor the finite Zermelo ordinals, but with the abstract entity instantiated by them and all isomorphic systems. Objects, Benacerraf thought, can be individuated in terms of intrinsic, non-relational properties; but numbers cannot: to be number 3, he said, "is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5, and so forth" (1965, p. 70). Numbers are best identified with positions in structures because positions in structures are like this as well: they lack intrinsic properties, and are characterised exhaustively by the relations that they bear to other positions within the structure that they belong to.

The details of Benacerraf's proposal were left open, but the idea that numbers are individuated exclusively by their relational properties has been generalised to a variety of other mathematical entities—e.g., groups, rings and fields—, and developed in two directions, by nominalist and realist structuralists respectively. The former (e.g., Hellman, 1989) believe that there is no specifically mathematical ontology that mathematical theories are about; that these theories describe all systems satisfying their axioms—regardless of whether their elements are abstract or concrete. Nominalist structuralists maintain, consequently, that mathematical sentences should not be interpreted at face value: numerals, as well as all the other seemingly singular terms that occur within mathematical statements, are in reality disguised bound variables ranging over the elements of systems.

Realist structuralists, by contrast, do countenance a realm of specifically mathematical entities that constitute the subject matter of mathematics: they maintain that mathematics is concerned with positions in structures. Accordingly, they claim that mathematical sentences are to be taken at face value: on their view, apparently singular terms are singular terms indeed, which refer genuinely to positions in structures. Resnik (1997) formulated a theory of this kind, but the most sophisticated and clearly articulated version of realist structuralism was put forward by Shapiro (1997). His theory he called '*ante rem* structuralism', and declared that its primary motivation is to allow for an anti-revisionary stance towards the semantics of mathematical statements—in particular, Shapiro insists, he wishes to account for the expressions that function as singular terms at the level of syntax as genuinely referential or, in other words, as logically proper names.

Shapiro's doctrine comprises two core tenets. The first one endows it with its name, and says that the structures that constitute the subject matter of mathematics exist independently of whether any non-mathematical systems instantiate them or not. The second tenet states that, just as Benacerraf supposed, the essence of mathematical entities is the relations that they bear to positions within the same structure (Shapiro, 1997, pp. 72–73). The latter of these tenets strongly suggests that Shapiro thinks that mathematical entities can be characterised uniquely in terms of their *structural properties*—the properties, that is to say, that mathematical entities possess, and can be defined in terms of the relations that exist among the positions of a given structure. If this is the case, then it seems that Shapiro is committed to a principle that individuates mathematical entities exclusively in terms of their structural properties. Witness:

IND. For any positions x, y in the same *ante rem* structure, x and y share all of their structural properties just in case $x=y$.

The supposition that Shapiro endorses (*IND*), however, has given rise to one of the most forceful and widely discussed objections against *ante rem* structuralism. The reason is that from (*IND*) follows a strong principle of identity of structurally indiscernible entities which delivers unacceptable results when applied to the positions of non-rigid structures. A structure is said to be rigid if it has only a trivial automorphism—i.e., the one based on the identity function. An instance of a non-rigid structure we can find in complex analysis: the function that takes a number $a+bi$ to its conjugate $a-bi$ is a non-trivial automorphism of the complex number structure. It follows that, for any formula Φx in the language of the theory with only x free, $\Phi(a+bi)$ just in case $\Phi(a-bi)$; and, in particular, $\Phi(i)$ just in case $\Phi(-i)$. Hence, i and $-i$ are indiscernible: they share all of their structural properties. But (*IND*) entails that, if this is the case, then i and $-i$ are identical—there is only one square root of -1. Another unacceptable conclusion follows from (*IND*) when it is applied to Euclidian geometry: any two points in Euclidean space can be connected with a rigid translation, which is an automorphism; so all the points in Euclidean space are structurally indiscernible. If (*IND*) is true, then there is only one such point. Shapiro, therefore, had better not commit to (*IND*).

Sadly, it is controversial whether or not rejecting (*IND*) suffices to vindicate *ante rem* structuralism. Burgess (1999, p. 288) complains that there seems to be nothing in the *ante rem* structuralist's picture which distinguishes i from $-i$, nor any one of the points in Euclidean space from the rest. He thus seems to invoke tacitly the Principle of Sufficient Reason in order to demand that Shapiro explicates what makes it the case that these mathematical entities are distinct from one another, if not qualitative differences. On his view, then, rejecting (*IND*) would not suffice: Shapiro would still have some explaining to do. Similarly, Keränen (2001, pp. 312–315) argues that, for the sake of the plausibility of his theory, Shapiro must identify the fact that makes each mathematical object the object that it is, distinct from the others, and to tell us under which circumstances an identity statement concerning *ante rem* positions is true. Thus Keränen requests that Shapiro not only rejects (*IND*), but produces a satisfactory substitute for it.

A number of authors advanced responses to the Burgess-Keränen objection. A particularly promising one was put forward by Ladyman (2005), and it consists of an attempt to fulfil Burgess's requirement that the *ante rem* structuralist finds something that distinguishes distinct positions: Ladyman proposes that, for any positions x and y, if $x \neq y$, then there is an irreflexive relation R such that Rxy. When applied to complex analysis and Euclidean geometry, this principle works: i and $-i$ hold the irreflexive relation of being non-zero and additive inverses to each other; a pair of distinct points in Euclidean space stand in the irreflexive relation of lying on exactly one line. Unfortunately, though, Ladyman's principle is not successful in general: it fails, for instance, in connection with simple graphs that have no relations. Given that there is no relation whatsoever holding between their nodes, there is, trivially, no relation holding amongst them that could satisfy Ladyman's criterion.

A wholly different strategy for responding to the Burgess-Keränen objection was employed by Ketland (2006), and by Shapiro himself (2008); and consists in rejecting the demand that a distinguishing or individuating principle for *ante rem* positions is provided by the *ante rem* structuralist. There are two claims that they adduce for justifying this rejection. The first one is that distinguishing principles like the one demanded by Burgess are circular. Consider, for instance, Ladyman's principle as applied to i and $-i$: according to it, the distinctness of i and $-i$ is grounded on the fact that they are non-zero and additive inverses of each other; but 'a is the additive inverse of b' is expressed by the formula $a+b=0$, and, as Ketland maintains, "one needn't be Sherlock Holmes to observe that this contains the identity predicate" (2006, p. 308). Evidently, Ketland suggests, a demand for identity criteria is misplaced if all plausible candidates are bound to presuppose the identity relation as primitive.

The second idea supporting the rejection of the distinguishing demand has it that, in fact, the whole enterprise of mathematics too requires that the identity relation is taken as primitive:

> To characterize the (rigid) structure of the natural numbers, one invokes a non-logical successor function. In calling it a 'function', we presuppose that, in any interpretation, each number has a unique successor. That is, if b and c are successors of a, then $b=c$. Similarly, one of the axioms of arithmetic is that the successor function is one-to-one: for any natural numbers a, b, if $sa=sb$, then $a=b$. There is simply no way to say that the successor relation is a function, or that it is one-to-one, without invoking identity, or something else that presupposes identity (Shapiro, 2008, p. 293).

After supporting his point with several similar illustrations, Shapiro concludes that the *ante rem* structuralist presupposes no more than is presupposed in ordinary mathematical practice by countenancing the identity relation as primitive. Ordinary mathematicians can define the cardinal-two structure with the following, categorical axiom:

$\exists x \exists y (x \neq y \wedge \forall z (z=x \vee z=y))$.

This axiom is not different in kind from any other—except because it contains no non-logical terminology, Shapiro observes, and perhaps because it is trivial. The complex number structure also has a standard axiomatisation, which entails that there are two square roots of -1; and, just as the ordinary mathematician needs nothing more than the aforestated axiom in order to establish that there are two members of the cardinal-two structure, the *ante rem* structuralist needs nothing more than the abovementioned standard axiomatisation of the complex number structure for establishing that there are two square roots of -1 (2008, p. 294).

On the grounds of this argument, and of Ketland's remarks, Shapiro, in his (2012, p. 381) asks his readers to assume that the *ante rem* structuralist "has won the metaphysical war" against the Burgess-Keränen objection; that there need not be anything distinguishing certain mathematical entities from each other. The rationale behind this request, Shapiro says, is that he wishes to discuss an interesting logical problem that arises from the supposition that indiscernible mathematical entities exist. The following section will be devoted to discussing this problem, and Shapiro's proposed solution to it.

3 Indiscernibility and reference

Like Shapiro, Black believes that indiscernibles are ontologically admissible: he argued for this contention in his (1952).[1] However, he also argued that an entity, if it is indiscernible from others, cannot be referred to (1952, pp. 156–157). In many of our ordinary mathematical assertions, we use syntactically singular terms in connection with the entities that Shapiro judges to be indiscernible. If Black is right, this is problematic. Consider, for instance, the term '*i*', which behaves syntactically like a name. If it really is a name, it must refer to a specific entity. Evidently, the only plausible candidate referents are the two square roots of −1; but which one does '*i*' refer to? Given that they are indiscernible, there is no reason to prefer one over the other; and picking either randomly as the referent of '*i*' would be impossible: every single description that we could ever formulate for trying to single out one of them will be satisfied by both. Relatedly, the locution 'the square root of −1' is often used felicitously; but if its denotation is not unique, there are, it seems, no grounds for its felicitousness. If the two square roots of −1 are indiscernible, moreover, then then we cannot suppose that the locution in question is elliptical for a description satisfied by only one of them.

The difficulty of accounting for the use of 'the square root of −1' and '*i*' has not created a stir like the Burgess-Keränen objection; but I think that it too constitutes a significant threat to the tenability of *ante rem* structuralism. If Shapiro cannot explain why the sentences where the aforementioned definite description ordinarily occurs are felicitous, then his theory could hardly be regarded as fostering a satisfactory anti-revisionary account of the semantics of mathematical discourse. Additionally, as I explained in § 2, Shapiro's motivation for postulating *ante rem* structures is the desire to account for the expressions that function as singular terms at the level of syntax as logically proper names. If Black is right, though, and proper names cannot be used in connection with indiscernibles, then Shapiro's project is not feasible.

Importantly, Black's insight makes the project of *ante rem* structuralism appear not merely unfeasible, but also futile. As I explained earlier, Shapiro's theory was formulated as an attempt to solve the problem that it is impossible to choose from among the many candidate reductions of numbers to sets; and, hence, to assign referents to numerals. In a way, the issue of reference to indiscernible entities leaves the *ante rem* structuralist precisely before an exactly analogous difficulty: before a variety of entities, none of which can be singled out as the referent of the singular terms of mathematics.[2]

Fortunately, Shapiro has a response to this difficulty. It is twofold, and its first part is founded on Roberts's (2003) work in linguistics on the semantics and prag-

[1] The entities whose existence Black defends are, in reality, not utterly indiscernible like unlabelled edgeless graphs, but *weakly discernible*—which equates to saying that they are discernible from one another by Ladyman's criterion mentioned above. However, as Assadian (2019, pp. 2557–2559) compellingly demonstrated, there is no reason for rejecting utterly indiscernible entities, like simple graphs with no relations, if merely weakly discernible entities have been accepted.

[2] This result—that *ante rem* structuralism suffers from a problem of referential indeterminacy—may also be reached by entirely different means through the so-called permutation argument. See, e.g., Shapiro (2016, pp. 131–144) and Assadian (2018).

matics of definite noun phrases—viz. definite descriptions, and singular pronouns and demonstratives. On the basis of a wealth of illustrations, Roberts argues that the existence and the uniqueness of the denotations of phrases of this kind are generally registered at the level not of semantics, but of pragmatics. In what concerns definite descriptions, of course, this hypothesis diametrically opposes Russell's, which states, famously, that the logical analysis of definite descriptions contains both an existence clause and a uniqueness clause. Surprisingly, Roberts claims, further, that in some cases no assumption about the existence or the uniqueness of a denotation is required at any level for the felicitous use of definite noun phrases, and adduces the following pieces of evidence:

1. If a strange man and a curious woman live here, the strange man will scare my cat, and the curious woman will make friends with it.
2. Remember that chess set that came with an extra pawn? I could have used an extra king, but I never needed the extra pawn.

(1) is felicitous; and, evidently, it does not entail, implicate, nor even presuppose, that there exist a strange man and a curious woman living here. The definite descriptions occurring in (1), then, can be used felicitously without there being an assumption of any kind concerning the existence of denotations. Similarly, (2) illustrates that uniqueness need not be assumed for the felicitous use of definite noun phrases: supposing that the chess set that is mentioned in this sentence came with nine white paws packed inside the box, there is no unique denotation for 'the extra pawn', since nothing allows us to identify any one of these pieces as a spare.

On Roberts's view, the felicitousness of (1) and (2) is to be explained not in terms of assumptions concerning existence and uniqueness, but of discourse referents—i.e., conversational files, so to speak, which do not require, nor presume, the existence or uniqueness of entities denoted by the relevant phrases. In the first half of (2), for instance, the locution 'an extra pawn' is used to express the idea that the chess set came with nine, equally coloured pawns. There is no piece that could be identified as a spare; but the locution in question introduces a discourse referent that later on legitimises the felicitous use of the definite description that occurs in the second half of the sentence.

According to Shapiro (2012, p. 399), 'the square root of -1' in our ordinary mathematical assertions is very much akin to 'the extra pawn' in (1). Mathematicians, Shapiro supposes, realised that, in the complex numbers, or in any algebraic closure of the reals, there is an x such that $x^2=-1$; and this realisation created a discourse referent that licensed the felicitous use of the expression 'the square root of -1'. So, just like 'the extra pawn' in (1), the felicitousness of 'the square root of -1' is not grounded on an assumption of uniqueness, but on the existence of a discourse referent.

Moreover, Shapiro hypothesises, at some point mathematicians decided to baptise the square root of -1, as it were; they decided, in other words, to introduce a singular term to denote the discourse referent introduced by their realisation that there is an x such that $x^2=-1$. Consider:

3. Remember that chess set that came with an extra pawn? Let us call it 'Max'. I could have used an extra king, but I never used Max.

In (2), we have seen, the locution 'an extra pawn', creates a discourse referent that sanctions the use of the definite description 'the extra pawn'. In (3) too the above-mentioned locution creates a discourse referent; but this time what it sanctions is the use of the singular term 'Max'. Similarly, Shapiro believes, the mathematicians' realisation that there is an x such that $x^2 = -1$ created a discourse referent which grounded not only the felicitous use of 'the square root of -1', but also of the singular term 'i' as anaphoric to it (2012, p. 399).

Nevertheless, 'Max' in (3) is not really a proper name, but an instantial term—also known as a parameter or a dummy name. In typical deduction systems, instantial terms are introduced, for example, by means of applications of the rule of existential elimination. Consider a reasoner who reaches a conclusion conforming to the following schema:

$\exists x \Phi x$.

She may then assume 'Φb', in accordance with the rule of existential elimination; and, if she then derives a formula 'Ψ' in which 'b' does not occur, she is entitled to discharge this assumption, and have 'Ψ' rest on the premises or assumptions that '$\exists x \Phi x$' rests on. Within this deductive process, 'b' works as an instantial term; and, according to Shapiro,

> [i]n some ways, parameters function as constants; in others they function as variables. In the case of existential elimination, we have it that some object in the domain satisfies Φ. The semantic role of the term b is to denote one such object. So in that sense, the parameter is like a constant. But it is crucial that we do not specify which such object, even if we could. The rules of engagement require the reasoner to avoid saying anything about b that does not hold of any object that satisfies Φ. In that sense, b functions more like a variable, ranging over the Φ's (2012, p. 403).

The core of Shapiro's proposal concerning 'i' is that, just as 'Max' in (3), this singular term is not a name, but an instantial term; its semantic role, therefore, is not to refer to either one of the square roots of -1, but to denote *any one* of the square roots of -1; none in particular.

In order to endow this claim with even minimal plausibility, though, Shapiro must advance a rather contentious hypothesis. As I said earlier, instantial terms are often introduced via applications of the rule of existential elimination. After reaching '$\exists x \Phi x$', the rule of existential elimination allows a reasoner to proceed by assuming 'Φb', provided that 'b' does not occur earlier in the proof. This assumption may be discharged whenever the reasoner reaches a formula where 'b' does not occur. Shapiro, though, must deny that instantial terms can only be introduced via existential elimination: if this were the case, then they could only occur within subproofs; but, clearly, we do not use the term 'i' only when doing subproofs. Shapiro contends, accordingly, that instantial terms are best understood as singular terms that are introduced by means not of existential elimination, but existential instantiation. According to the rule of existential instantiation, after reaching a conclusion of the form '$\exists x \Phi x$', one can *infer*, rather than assume, 'Φb', provided that 'b' has not occurred earlier in the proof. This inference is valid as long as 'b' does not occur in the conclusion of the proof. In support of his contention, Shapiro appeals to the language of informal reasoning. Suppose, he says, that, in doing a derivation, someone reaches '$\exists x \Phi x$'. A natural move to perform at that point would be to make the following stipulation: 'Let b be a Φ'. Saying 'Assume that Φb', by contrast, would be unnatural (2012, p. 404).

This argument is quite flimsy, however. It seems perfectly adequate to follow '$\exists x \Phi x$' with, for instance, 'Suppose that b is a Φ'; so the language of informal reasoning does not definitely tip the balance on the side of Shapiro's contention that instantial terms are introduced via inferences, rather than assumptions. Shapiro is aware of this issue (2012, p. 404), but proceeds as supposing that the linguistic evidence that he provided is enough to make his contention sufficiently plausible. He then affirms that we can uncontentiously adopt his hypothesis that 'i' is an instantial term.

Shapiro does not discuss the consequences of his hypothesis for his anti-revisionary project; but, clearly, if his aim, as I explained in § 2, is to account for the totality of the syntactically singular terms of mathematics as logically proper names, he has failed. In absence of any declaration on his part concerning these affairs, though, we may assume that he would be satisfied with the accomplishment of having accounted for the felicitous use of the definite description 'the square root of -1', and the singular term 'i'. In the following section, however, I will argue that, in fact, Shapiro has not yet achieved this either.

4 The semantics of instantial terms

As Shapiro himself recognises, his hypothesis concerning 'i' begs for an account of the semantics of instantial terms. This matter is a much-debated one; but, in his (2012), Shapiro advances an original proposal (2012, pp. 405–408), and considers a number of others (2012, pp. 410–412), so as to contend that there are several plausible candidates. His proposal, briefly stated, is to think of instantial terms as functioning very much like pronouns without an explicit or implicit antecedent (2012, p. 407). If someone said 'He is cold' without having in any way supplied an antecedent for the pronoun, it would be impossible to judge whether she has spoken truly or not. The sentence 'He is cold', then, does not have a truth-value by itself; and, similarly, Shapiro maintains, sentences containing instantial terms do not have truth-values by themselves. In more technical jargon, what Shapiro proposes is to conceive of instantial terms along the lines of free variables: that, if 'b' is an instantial term, then 'Φb' is as devoid of a truth-value as the open formula 'Φx'—even under an interpretation.

Sentences where instantial terms occur, Shapiro continues, acquire truth-values only if an assignment of referents for their instantial terms is provided; and, when elaborating our model-theoretic semantics, we can introduce one such assignment—we can, that is to say, introduce a function mapping every instantial term to an individual of the relevant sort (2012, pp. 406–407). If, for instance, the sentence 'Φb' is the result of an application of the rule of existential instantiation, and 'b' is, consequently, an instantial term, then we can introduce a function mapping '*b*' onto one of the Φs. Only then, Shapiro maintains, will the sentence 'Φ*b*' earn a truth-value.

Shapiro, moreover, declares that all of this can be done even if the Φs are indiscernible (2012, p. 408); and this move is a crucial one. The rationale behind Shapiro's *ante rem* structuralism, recall, is semantic non-revisionism; and no theory could be more revisionary than those that deny the truth of intuitively true sentences. Shapiro, then, cannot fail to confer a truth-value to the sentences where '*i*' occurs. But, by his own lights, these sentences have a truth-value only if a referent is assigned to '*i*'; so Shapiro must allow for the existence of a reference-assigning function mapping '*i*' onto one of the indiscernible square roots of − 1.

Nevertheless, it is clear that, for each one of the Φs, there is a function mapping '*b*' onto it; and, if the Φs are indiscernible, then it is impossible to specify which one of these functions is the one that assigns a referent to '*b*'. We know that, if the Φs cannot be discerned, then every single description that we could come up with for trying to select one of them will be satisfied by all. Hence, they cannot be individually denoted, nor referred to. But then the functions that we are presently concerned with are indiscernible as well: they are all functions that can only be described as mapping '*b*' to one of the Φs; they cannot be distinguished by a specification of the particular Φ that they map '*b*' onto. All we can do, therefore, whilst building our model-theoretic semantics, is to assert that there is a function *f* that assigns one of the Φs to '*b*' as a referent; but, if all such functions are indiscernible, then the singular term '*f*' cannot be a genuinely referring name—it cannot be anything but an instantial term.

Shapiro recognises this as a result of his proposal, and thus diagnoses it as being circular: in order to provide a semantics for instantial terms, it invokes instantial terms. He then declares that this circularity is unavoidable and, more importantly, unproblematic (2012, p. 408). To my mind, however, Shapiro's proposal is not innocuously circular, but prone to generate a pernicious infinite regress. The problem with the use of the instantial term '*f*' in the formulation of our model-theoretic semantics is not that it pertains to the very kind of items for which we are trying to provide a semantics: the problem is that, according to Shapiro's own account of the semantics of instantial terms, '*f*' is empty until we assign a referent to it, and we cannot tolerate that it remains this way: if '*f*' is empty, then no referent has been assigned to '*b*' and, consequently, sentences where '*b*' occurs do not yet have a truth-value. If we want to assign a referent to '*b*', we must decide which one of the functions that map it onto one of the Φs is the one doing the assigning; and, in order to do this, we need the instantial term '*f*' to be non-empty: we need it to point to one of these functions in particular. Hence, we must give it a referent.

If we were to use Shapiro's strategy, we would do so by asserting that there is a higher-order function *g* that assigns a referent to '*f*' from among the members of the set of indiscernible functions that we are considering—call it *F*. But, of course, for each one of the members of *F*, there is a higher-order function mapping '*f*' onto it; and, if the members of *F* cannot be discerned, then there is no way to specify which one of these higher-order functions is the one that assigns a referent to '*f*'. If the members of *F* are indiscernible, then they cannot be individually denoted, nor referred to; but then the higher-order functions that map '*f*' onto each one of them are themselves indiscernible: they are all higher-order functions that can only be described as mapping *'f'* onto one of the members of *F*; they cannot be distinguished by a specification of the particular member of *F* that they map '*f*' onto. All we can do, therefore, is to assert that there is a higher-order function *g* that maps '*f*' onto one of the members of *F*. But, if all such members are indiscernible, then the singular term *'g'* cannot be genuinely referring—and must be, consequently, an instantial term. On Shapiro's account, then, '*g*' is, by itself, empty; but we cannot tolerate that it remains this way, and thus we must assign it a referent.

This regress continues *ad infinitum*. And, if this is the case, then the point where the instantial terms in question are finally assigned referents is endlessly deferred, and never reached. According to Shapiro's own theory of the semantics of instantial terms, then, they will remain empty forever; but then this theory has failed in explaining to us how we manage to talk about one of the square roots of -1 by using '*i*'—and, even worse: we can now conclude that from this theory follows that '*i*' is meaningless and, thus, that the sentences where it occurs do not have a truth-value.

At this point, we might wonder whether Shapiro could confer a truth-value to sentences where instantial terms occur, even if he has failed in using referent-assigning functions to select referents for them. [3] Supervaluation would do the trick. Suppose that a reasoner reaches '$\exists x\Phi x$', and infers 'Φb' by existential instantiation. Then, for any formula 'Ψx', with only x free, 'Ψb' is super-true just in case Ψ is true of all of the Φs, and super-false just in case Ψ is false of all of the Φs. However, as Shapiro himself notes (2012, p. 408), super-truth is not compositional; and, importantly, the instantial term '*b*' cannot be construed as a genuinely referring singular term if 'Φb' is supervaluated in this way, for '*b*' will not be assigned a particular Φ as a referent. Consequently, adopting the supervaluation strategy would be heavily detrimental for Shapiro's project of fostering a face-value semantics of mathematical discourse.

Fortunately, there are several accounts of the semantics of instantial terms which differ significantly from Shapiro's, and could deliver better results for the *ante rem* structuralist. The first of them was put forward by Fine (1985), and states that, besides the totality of particular objects, there exist *arbitrary objects* to which instantial terms refer. Besides particular men, for instance, there are arbitrary men; besides particular natural numbers, there are arbitrary natural numbers. Each one of these objects, Fine claims, has a "value- range" (1985, p. 55): a set of objects whose shared properties determine which properties can be correctly ascribed to the corresponding arbitrary object. Roughly, the attributes that an arbitrary object may be said to bear are, on Fine's view, those that are shared by all of the objects pertaining to its value-range. Shapiro (2012, pp. 410–411) imagines that the use of the instantial term '*i*' can be easily accounted for from Fine's framework: after concluding that there is an x such that $x^2 = -1$, mathematicians introduced *'i'* by means of an application of the rule of existential instantiation. Then it acquired its permanent fixture within the language of complex analysis. From the very moment when it was introduced, though, it referred not to either of the square roots of -1, but to the corresponding arbitrary object—the arbitrary object, in other words, that has exactly these two square roots in its value-range. If this is the case, then '*i*' is genuinely referential, and, thus, Shapiro's anti-revisionary project has been vindicated.

[3] I am grateful to an anonymous reviewer for pointing this out.

I think, however, that Fine's theory is problematic for Shapiro for two mutually independent reasons. Suppose that a reasoner reaches '$\exists x(x \leq 3)$', and that 'x' ranges over the positive integers. Suppose, moreover, that, by applying the rule of existential instantiation, the reasoner derives '$n \leq 3$'. The singular term 'n' here is an instantial term, and, as such, it refers to an arbitrary object—more specifically, to the arbitrary object whose value-range is constituted by the three positive integers that are less than, or equal to 3. Since all of the particular objects within the value-range of n share the properties of being positive integers, and less than, or equal to 3, n possesses them as well; but then it follows that there are *four* positive integers that are less than, or equal to 3.

This result is obviously unacceptable; and, in order to elude it, Fine must reply that, when we claim that there are three positive integers that are less than, or equal to 3, we speak elliptically rather than literally; that what we really assert is that there are three *particular* positive integers that are less than, or equal to three. Importantly, though, this idea is frankly at odds with the felicity of the following natural language introduction of 'n', which explicitly states that n is one of the three numbers that we speak of when we say that there are three positive integers that are less than, or equal to 3:

4. There are three positive integers that are less than, or equal to 3. Let n be one of them.

Breckenridge & Magidor (2012, p. 389) adduced a similar remark against the theory of arbitrary objects; but I believe that Fine need not be too worried about it—he might simply deny that the natural language formulation of the introduction of instantial terms is relevant in the slightest for an analysis of their semantics. Shapiro, by contrast, cannot do this without falling into a methodological inconsistency: his crucial contention that instantial terms do not only occur within subproofs, recall, was motivated by a reminder that, in natural language, it is felicitous to introduce an instantial term via a stipulation, and not necessarily via an assumption.

Additionally, Shapiro's adherence to the theory of arbitrary objects is inadequate because, even if it can provide a fairly satisfactory account of the semantics of instantial terms, it is itself subject to the problem of reference to indiscernibles. Suppose that a reasoner soundly reaches a conclusion conforming to the following schema:

$\exists x \exists y(\Phi x \wedge \Phi y \wedge x \neq y)$.

According to Shapiro's own contention, by two applications of the rule of existential instantiation, she is allowed to infer the following, as long as neither 'c' nor 'e' occur earlier in the proof:

$\Phi c \wedge \Phi e \wedge c \neq e$.

If this inference is sound, 'c' and 'e' refer to distinct referents. We must, then, countenance two arbitrary objects associated with the very same value-range—namely, the set of the Φs. Since arbitrary objects are characterised exhaustively by the properties that are shared by the totality of the members of their value-ranges, the two arbitrary Φs are indiscernible. But then it is a mistake to maintain that 'c' and 'e' refer to them: if they cannot be discerned, it is impossible to pick one out in order to fix the reference of 'c', or of 'e'. The reference of these names, then, cannot be fixed at all. Ultimately, if Shapiro agreed with Fine's proposal that instantial terms refer to arbitrary objects, the question that he was trying to answer when he proposed that '*i*' is a parameter would simply reappear: how can we refer to an object if it is indiscernible from others?

Another alternative that Shapiro could resort to is Breckenridge's and Magidor's theory of arbitrary reference (2012). On their view, if '*b*' is introduced as an instantial term by means of an application of the rule of existential instantiation on '$\exists x\Phi x$', '*b*' refers arbitrarily to one of the Φs: even if the reasoner has not selected any specific Φ as its referent, or cannot do so, the instantial term '*b*' refers to one of them. In fact, Breckenridge and Magidor insist, *nothing* whatsoever determines which one of the Φs '*b*' refers to; and, because of this, which Φ is the referent of '*b*' can never be known. Shapiro, as well as Breckenridge and Magidor, believe that the theory of arbitrary reference allows for a simple and convincing explanation of the semantic functioning of '*i*': it refers arbitrarily, they maintain, to one of the square roots of -1, even if nobody was ever able to single out one of them in order to fix its reference.

Clearly, Breckenridge's and Magidor's theory has the advantage over Fine's of doing justice to the felicitousness of (4); but it is problematic for Shapiro nevertheless. As I explained in § 2, there are several candidate reductions of numbers to sets, and no justification for choosing one over the others. Shapiro posited *ante rem* structures in the face of this fact in order to supply referents for the singular terms of mathematics. If there is arbitrary reference, though, we may hypothesise that the singular terms of mathematics refer arbitrarily; that '2', for instance, arbitrarily refers to {Ø, {Ø}}, {{Ø}}, or any one of the other set-theoretical objects that it could be reduced to. Then we could conjecture that the reference of '3' is fixed by means of the following definite description: 'the successor of 2'. If the reference-fixing process continues in the obvious manner, then the referents of all numerals will pertain to the same system of set-theoretical objects; the naturals, that is to say, will be identical to the finite von Neumann ordinals, the finite Zermelo ordinals, or some other isomorphic system.

For Shapiro, this result is undesirable: it proves that, if the theory of arbitrary reference is right, then the project of analysing the singular terms of mathematics as genuinely referring expressions can be carried out without positing *ante rem* structures. It follows that, if the theory of arbitrary reference is correct, then the conviction that mathematical singular terms must be accounted for as genuinely referring expressions cannot motivate the postulation of *ante rem* structures as the subject matter of mathematics. Adopting the theory of arbitrary reference, therefore, would force the *ante rem* structuralist to find other motivations for their theory. This task may not be unsurmountable: perhaps, for instance, Shapiro could find compelling reasons to suppose that mathematical theories require a *sui generis* subject matter.[4] However, I believe that there is a weightier reason why the *ante rem* structuralist should not take the theory of arbitrary reference to constitute a satisfactory account of the semantics of instantial terms. Briefly stated, I think that, under the assumption that *ante rem* structures are the subject matter of mathematics, there are certain instances of instantial reasoning in mathematics that the theory of arbitrary reference

[4] I am grateful to an anonymous reviewer for pointing this out.

does not support. Suppose, again, that a reasoner reaches a conclusion conforming to the following schema:

$\exists x \exists y(\Phi x \wedge \Phi y \wedge x \neq y)$.

Suppose, moreover, that the variables of the sentence above range over a certain domain of mathematical objects and, consequently, over *ante rem* positions. According to Shapiro, our reasoner may validly derive.

$\exists y(\Phi c \wedge \Phi y \wedge c \neq y)$

and, subsequently,

$\Phi c \wedge \Phi e \wedge c \neq e$,

provided that neither '*c*' nor '*e*' occur earlier in the proof. If the inference is sound, then '*c*' and '*e*' have distinct referents. According to the theory of arbitrary reference, when introducing '*c*' via the first existential instantiation, our hypothetical reasoner intended it to refer to one of Φs, and so it did. However, she cannot have intended '*e*' to refer merely to one of the Φs, for that would allow for the possibility that both '*c*' and '*e*' ended up referring to the same individual. If our reasoner was to make sure that '*c*' and '*e*' refer to distinct Φs, then we must suppose that, once the reference of '*c*' had been fixed, '*e*' was stipulated to refer not merely to one of the Φs, but to a Φ that was *distinct* from *c*. Unfortunately, this move clashes with the *ante rem* structuralist doctrine. *Ante rem* structuralists, we have seen, maintain that *ante rem* structures bear only structural properties; properties, that is to say, that can be defined in terms of the relations that exist amongst the positions of a given structure. If *c* is an *ante rem* position, then the property of being identical to *c* can be defined in terms of the relations that hold amongst the positions of the structure that *c* belongs to: it can be defined in terms of the identity relation that holds between *c* and *c* itself. It seems, consequently, that being identical with *c* is a structural property, and our reasoner intended '*e*' to refer to a Φ that did *not* bear it. Unfortunately, the property of being identical with *c* lacks the hallmark of structural properties.

As we saw in § 2, *ante rem* structuralism states that *ante rem* structures are types instantiated by systems of objects. The *ante rem* structure of the natural numbers, for instance, is instantiated by the finite von Neumann ordinals, the finite Zermelo ordinals, and all other isomorphic systems. The properties that are borne by the *ante rem* structure of the natural numbers, therefore, are those that are shared by the finite von Neumann ordinals, the finite Zermelo ordinals, etc. These properties are the structural properties that are supposed to characterise *ante rem* structures exhaustively.[5] In more precise terms, a property is borne by an *ante rem* structure just in case it is borne by any one of the systems that instantiate it. This means, however, that the structural properties that exhaustively characterise an *ante rem* structure must be not only definable in terms of the relations that hold amongst its positions, but definable by formulae containing no individual constants denoting any one of these positions (see, e.g., Keränen 2001, p. 316). Let us see why. Suppose, for example, that ***S*** is an *ante rem* structure comprising two positions, ***a*** and ***b***, and a relation *R* such that *R**ab***.

[5] The doctrine that *ante rem* positions bear only structural properties has been considered problematic, for it seems that *ante rem* positions bear properties that are not structural. The objects occupying *ante rem* positions, for instance, do not bear the property of possessing only structural properties; so the property of possessing only structural properties is not structural. This issue, however, has been discussed elsewhere (see, e.g., Shapiro 2006; Assadian, 2022), and I will not discuss it here.

Clearly, any system *S* instantiating ***S*** comprises two elements *a* and *b*, and is such that *Rab*. Let us now say that *P* is a property borne by all and only those individuals satisfying '*Rx**b***'. It seems that ***a*** satisfies this formula and, therefore, bears *P*. We know, though, that the properties that are borne by *ante rem* positions are also borne by the objects that occupy them. It follows that *a*, the object occupying ***a***, bears *P* as well: *a* in other words, satisfies '*Rx**b***'. But this is wrong: if *S* instantiates ***S***, then *a* must hold relation *R* to *b*, not to ***b***. Hence, the structural properties that are possessed by *ante rem* structures must be definable by formulae not comprising any constants referring to *ante rem* positions.

Let us now go back to the hypothetical reasoner that we were considering earlier. She has fixed the reference of '*c*' to a certain Φ, and now needs '*e*' to refer to a Φ not satisfying '$x=c$'. *c*, recall, is an *ante rem* position; so the formula '$x=c$' contains a constant denoting an *ante rem* position and, thus, it cannot be taken to define a structural property. It follows that, in order to fix the reference of '*e*' correctly, our hypothetical reasoner must attribute to an *ante rem* position a non-structural property; and, consequently, a property that, according to the doctrine of *ante rem* structuralism, *ante rem* positions cannot bear. In fact, it makes a lot of sense to exclude the property defined by '$x=c$' from the set of structural properties that are borne by the *ante rem* position in question. Since the objects occupying the position of an *ante rem* structure bear all of the properties that are borne by this position, then any object occupying *c* would have to bear the property defined by '$x=c$'. But then the only object that could occupy *c* is *c* itself; and, hence, the only *ante rem* structure that could instantiate the *ante rem* structure that *c* belongs to is that structure itself. By assumption, though, the structure that we were considering is any *ante rem* structure; and *ante rem* structures are generally not such that they can only be instantiated by themselves.[6]

If my argument is correct, then, under the assumption that there are *ante rem* structures, there are cases of instantial reasoning that cannot be accounted for by the theory of arbitrary reference. Shapiro, consequently, should not suppose that this theory may be unproblematically endorsed by the *ante rem* structuralist.

There are two other proposals concerning the semantics of instantial terms, but both are outright at odds with Shapiro's purposes. The first is King's quantificational theory (1991), according to which instantial terms are not singular terms, but disguised variables bound to elided quantifiers. Clearly, adopting this theory would be immensely detrimental to Shapiro's project of advocating for an anti-revisionary account of the semantics of mathematical discourse. The second is the instrumental account of instantial terms, which states that they are meaningless inferential devices that pertain to a conservative extension of deductive systems with semantically standard languages. On this view, if '*i*' is a parameter, then every mathematical sentence where it occurs is incomplete; and, obviously, Shapiro cannot accept this.

[6] It is important to note that what I have said does not imply that *ante rem* structuralism does not support the seemingly platitudinous hypothesis that *ante rem* positions are self-identical. Clearly, the property of being self-identical can be defined by a formula not comprising any constants denoting *ante rem* positions, namely '$x=x$'. The property of being self-identical, therefore, can be safely considered structural.

5 Conclusion

By way of conclusion, I will present a brief summary of what has been accomplished. There are several, equally plausible reductions of numbers to sets. According to Benacerraf, this means that numbers cannot be conceived of as objects of any kind. If he is right, then there are no particular objects that could be determinately identified as the referents of arithmetical singular terms. On the face of Benacerraf's compelling arguments, *ante rem* structures were postulated as the subject matter of mathematics in order to sanction an anti-revisionary semantics of mathematical discourse.

Unfortunately, the problem of the indeterminacy of reference in mathematics survived the postulation of *ante rem* structures. In § 2, I showed that the *ante rem* structuralist is inevitably committed to the existence of indiscernibles. It follows that there are, again, multiple entities, none of which can be identified as the referents of the singular terms of mathematics. In § 3, I contended that this result poses a pressing threat of futility for the postulation of *ante rem* structures.

Shapiro (2012) tried to resolve the second occurrence of the problem of referential indeterminacy by making a remarkably clever claim: some of the singular terms that occur within mathematical discourse are not genuinely referential expressions, but instantial terms. In § 4, however, I argued that the available accounts of the semantics of instantial terms are all untenable for the *ante rem* structuralist. Shapiro's own take on the matter produces a vicious infinite regress; Fine's theory of arbitrary objects generates yet a further version of the problem of referential indeterminacy; the theory of arbitrary reference makes *ante rem* structuralism futile, and cannot account for certain cases of instantial reasoning in mathematics; King's quantificational theory precludes the *ante rem* structuralist from carrying out her anti-revisionary project; and so does the instrumental account of the semantics of instantial terms. If my arguments are right, then Shapiro still owes us an account of the semantics of instantial terms that suits the *ante rem* structuralist project. Without it, the *ante rem* structuralist remains unable to determine the reference of some of the singular terms of mathematics.

Conflict of interest no conflicts of interest.

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