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ORIGINAL RESEARCH

Revisiting Reichenbach's logic

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Abstract
In this paper we show that, when analyzed with contemporary tools in logic—such as Dunn-style semantics, Reichenbach's three-valued logic exhibits many interesting features, and even new responses to some of the old objections to it can be attempted. Also, we establish some connections between Reichenbach's three-valued logic and some contra-classical logics.

Keywords Reichenbach's logic · Dunn semantics · Contra-classical logics · Quasi-implication · Cyclical negation

1 Introduction

Reichenbach's approach to the philosophy of quantum mechanics stems mainly from his views on language and meaning in science. He thought that the logical formulation of an adequate language for quantum mechanics would help us not only to cogently discuss the physical laws and phenomena involved in quantum mechanics, but should also invalidate any arguments leading to *causal anomalies* in quantum mechanics. Examples of these are illustrated by the Double-slit Experiment but include also action-at-a-distance (or non-local causality).

Though claims along these lines were not uncommon among logical empiricists as himself, Reichenbach's contribution to the discussion was the introduction of a non-classical logic which not only deviates from the logic endorsed by most Vienna Circle

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members (i.e. classical logic), but also, being a three-valued logic, deviates from the traditional quantum logic of Birkhoff and Von Neumann.

Nonetheless, Reichenbach's logic was never precisely popular, for two main reasons. The first one is that it has some uncommon features that made it too deviant by the lights of its time: for example, it is three-valued, with the usual problems in giving a precise meaning to the additional truth values; and it multiplies the connectives but the semantics seemingly does not justify that they are the intended connectives. The second reason is that the logic did not live up to its promises, and it was regarded as insufficient to address the representation and interpretation issues quantum mechanics is plagued with.

Our very modest aim in this paper is to show that, when seen through the glasses of more contemporary tools and discussions in logic, Reichenbach's logic exhibits many interesting features, and even new responses to some of the old objections to it can be attempted, or so we think when it comes not to those objections regarding its applications to quantum mechanics —which is an issue that lays out of the scope of this paper—, but to those objections regarding the apparent formal oddities of this logic.

The plan of the paper is as follows. In Sect. 2, we revisit Reichenbach's three-valued logic —we will call it '**R3V**' for short— and its underlying motivations. In Sect. 3, we give a Dunn-style semantics for **R3V**, that is, a semantics based on independent, relational truth and falsity conditions for the connectives. In Sect. 4, we reconstruct some of the objections raised against to the formal features of Reichenbach's logic, and show how the presentation of **R3V** in Sect. 3 helps in addressing some of these worries. Finally, in Sect. 5 we establish some connections between **R3V** and some contra-classical logics, a family of yet understudied logics which validate arguments that are invalid in classical logic.

2 Reichenbach's logic

Motivations. Following Carnap's idea of a physical language, Reichenbach considers an *observational language* and a *quantum mechanical language*, so as to make every question about the existence of physical entities a matter of the meaning of the propositions involved in such languages. Geiger counters, indicators of a dial, lines on photographic films, etc. are observational terms which belong to the observational language, these are directly related to measurement and experimentation, whereas the position q of an electron, its momentum p, etc. are terms of the quantum mechanical language, which describe a physical system, property or situation. The way we construct the quantum mechanical language is what Reichenbach calls an *interpretation* of quantum mechanics. Of course, the meaning of a proposition of the quantum mechanical language is determined in terms of the truth or falsity of the corresponding propositions of the observational language. More precisely, a proposition A of the quantum mechanical language has the same meaning as the set of observational propositions $a_1, \ldots, a_n$ which verify A (or make A highly probable) (Reichenbach 1935, p. 137).

Reichenbach is interested in showing that, while any *exhaustive* interpretation of quantum mechanics —i.e. any interpretation in which the values of unobserved entities are completely defined, such as p and q— leads to causal anomalies, this is not the case with a *restrictive* interpretation in which such values remain undefined. The restriction in question involves excluding statements about the combination of *noncommutative* properties or quantities, such as p and q; thus, for example, any statement about the simultaneous momentum and position of a given electron is meaningless.

An important feature of Reichenbach's physical language for quantum mechanics is the statistical completeness of the observational language, where an observational language is *statistically complete* when, for any possible situation defined in observational terms, the observational result of a measurement can be predicted with a determinate probability (Reichenbach 1935, p. 138). This sort of completeness is not about the values of unobserved physical entities or situations, which leads to causal anomalies, but about the predictive methods of quantum mechanics when expressed in observational terms. We may construct a restrictive interpretation of quantum mechanics by introducing a definition of *meaning* (so as to exclude meaningless statements) and this interpretation will also be statistically complete in observational terms.

The *Bohr-Heisenberg interpretation* is an example of interpretation by restricted meaning. According to Reichenbach, it employs the following definition for the values of measured entities: "the result of a measurement represents the value of the measured entity immediately after the measurement" and the following definition for meaning of statements: "in a physical state not preceded by a measurement of an entity u, any statement about the value of the entity u is meaningless". This interpretation does not assert nor deny that an entity is disturbed by the measurement but it clearly excludes simultaneous measurement of values such as p and q, as needed. This results in the fact that not every proposition like "the value of the entity is u" has meaning.

Some neat consequences of the Bohr-Heisenberg interpretation are (1) that a meaningless statement is not subject to propositional operations (if A is meaningful but B is meaningless, then 'A and B' and 'A or B' are meaningless, and not even 'B or not-B' holds) and (2) that when given two *complementary statements*, i.e. statements about simultaneous values of noncommutative entities, at most one of them is meaningful and the other is meaningless (Reichenbach 1935, pp. 142–143).[1]

If statements about values of unobserved entities are regarded as meaningless, then the language of physics includes meaningless statements. To avoid this, Reichenbach suggests an interpretation much in the line with that of Bohr and Heisenberg but which excludes statements like the aforementioned not from the domain of meaning but from the domain of *assertability*, which may be naturally captured in a three-valued logic, by introducing an intermediate truth value between truth and falsity, called the *indeterminate* truth value.[2]

The logic **R3V**. Consider a language L consisting of formulas built, in the usual way, from propositional variables with the connectives $\{\neg, \sim, -, \wedge, \vee, \supset, \rightarrow, \sqsupset\}$. We will use the first capital letters of the Latin alphabet, 'A', 'B', 'C'…as variables ranging over arbitrary formulas.

[1] Notice that it is not necessary that one of the two complementary statements is meaningful, since in a physical situation s determined by a wave function ψ which is not an eigen-value of one of the entities considered, both statements will be meaningless.

[2] This is different than just saying that the truth value is "unknown", since this we may apply to two-valued statements as well.

A *Reichenbach model* for L is a function i from propositional variables to the set of values $\{T, I, F\}$, understood as *truth*, *indeterminacy* and *falsity*, respectively. T is the only designated value, i.e. $D^+ = \{T\}$. The evaluation of formulas is extended and defined recursively according to the following tables.

Conjunction and disjunction

A	B	$A \wedge B$	$A \vee B$
T	T	T	T
T	I	I	T
T	F	F	T
I	T	I	T
I	I	I	I
I	F	F	I
F	T	F	T
F	I	F	I
F	F	F	F

Conjunction and disjunction are generalizations of the two-valued versions of these connectives.

Negations

	Cyclical	Diametrical	Complete
A	$\sim A$	$\neg A$	$\overline{A}$
T	I	F	I
I	F	I	T
F	T	T	T

Cyclical negation shifts a truth value to the next lower one, except for the case of the lowest, which is shifted to the highest value; *diametrical negation* reverses T and F, but leaves I unchanged; and *complete negation* shifts a truth value to the higher one of the other two.

Implications

		Standard	Alternative	Quasi-implication
A	B	$A \supset B$	$A \rightarrow B$	$A \sqsupset B$
T	T	T	T	T
T	I	I	F	I
T	F	F	F	F
I	T	T	T	I
I	I	T	T	I
I	F	I	T	I
F	T	T	T	I
F	I	T	T	I
F	F	T	T	I

Standard implication was already found in Łukasiewicz's three-valued logic $\mathbf{Ł}_3$, which is just a generalization of the two-valued version of this connective[3]; *alternative implication* takes any combination of the three truth values and returns either T or F; and *quasi-implication* returns the intermediate value for any combination except for the true-case $\langle T, T\rangle$ and the false-case $\langle T, F\rangle$ of a standard conditional.

Finally, let Γ be a set of formulas of **R3V**. A is a *logical consequence* of Γ in **R3V**, $\Gamma \models_{\mathbf{R3V}} A$, if and only if, for every interpretation i, $i(A) \in D^+$ if $i(B) \in D^+$ for every $B \in \Gamma$. A is *valid* or *holds* in **R3V** if and only if $\Gamma \models_{\mathbf{R3V}} A$ and $\Gamma = \varnothing$.

Reichenbach considered the following desirable properties of an implication connective © and noticed that the quasi-implication connective does not satisfy 3 and 4:

1. *Detachment I* if A is T and $A©B$ is T one may validly infer that B is T.
2. *Falsification* if A is T and B is F then $A©B$ is F.
3. *Detachment II* it is not the case that $A©B \dashv\vdash A \wedge B$. If the conditional $A©B$ behaves exactly like $A \wedge B$, then the separate occurrence of A's truth in *Detachment I* is completely irrelevant, since the truth of B may be deduced from the truth of $A \wedge B$ alone, without the need of the truth of A to detach the truth of B.
4. *Reflexivity* $A©A$ is always true.
5. *Non-symmetry* from the truth of $A©B$ one may not deduce the truth of $B©A$.

Quasi-implication does not satisfy Reflexivity and Detachment II, which are, for Reichenbach, desirable properties of an implication connective, and since Reflexivity is needed to get an equivalence relation, he only considers the following two connectives for equivalence, where $A \equiv B \overset{df}{=} (A \supset B) \wedge (B \supset A)$ and $A \leftrightarrow B \overset{df}{=} (A \rightarrow B) \wedge (B \rightarrow A)$:

Equivalences

		Standard	Alternative
A	B	$A \equiv B$	$A \leftrightarrow B$
T	T	T	T
T	I	I	F
T	F	F	F
I	T	I	F
I	I	T	T
I	F	I	F
F	T	F	F
F	I	I	F
F	F	T	T

[3] The influence of Łukasiewicz on Reichenbach's works on the possible applications of many-valued logics in physics is partly due to fruitful discussion with Zygmunt Zawirski, a Polish logician working on applications of $\mathbf{Ł}_3$ in quantum physics, the foundations of mathematics and other philosophical problems, who met Reichenbach in Paris for the 1935 International Congress for Scientific Philosophy; see Szumilewicz-Lachman et al. (2012), pp. 43–51. We thank an anonymous referee for pointing this fact to us. Actually, the 1935 Congress was momentous in several respects and attracted people with strikingly similar interests and ideas; de Finetti also gave a paper there where he presented a logic with quasi-implication.

Next, Reichenbach demands that the truth values are so defined that *only a statement having the truth value T can be asserted*. This means that when we wish to say that a statement has a truth value other than T, we may use $\sim\sim A$ to say that A is indeterminate and $\sim A$ or $\neg A$ to say that A is false. This enables us to eliminate statements in the metalanguage about truth values by relying only on statements in the object language; i.e. *what we wish to say is said in a true statement of the object language* (Reichenbach 1935, p. 153). Note that while Reichenbach correctly points out that when it is asserted (it is true) that $\sim A$ or that $\neg A$ it must be because A is false, he is not claiming that $\sim A \equiv \neg A$ or that $\sim A \leftrightarrow \neg A$ in general, only that a true occurrence of $\sim A$ and a true occurrence of $\neg A$ assert the same thing (to wit, A is false).

A formula A is called *tautological* if and only if $i(A) = T$, for all i; *contradictory* if and only if $i(A) = F$, for all i; and *synthetic* if and only if there are i and i' such that $i(A) = T$ and either $i'(A) = I$ or $i'(A) = F$. A formula A such that $i(A) \neq T$, for all i, but there is a i' such that $i'(A) = I$, is called *asynthetic*. Among synthetic formulas, those that receive all three values are called *fully synthetic statements*; those whose values are only T's and F's are called *true-false statements* (or *plain-synthetic statements*); and those whose values are only T's and I's are called *semi-synthetic statements*. Thus,

- The cyclical or the diametrical negation of a contradictory formula is a tautology;
- The complete negation of an asynthetic statement is a tautology;
- A synthetic statement cannot be made a tautology simply by the addition of a negation.

In the sense just defined, *all quantum mechanical statements are synthetic*, since they assert something about the physical world (Reichenbach 1935, p. 154).

The following are some tautological schemata from **R3V**:

1. *Identity*: $A \equiv A$
2. *Double Negation*: $A \equiv \neg\neg A$
3. *Triple Negation*: $A \equiv\sim\sim\sim A$
4. *Complete Double Negation*: $\overline{A} \equiv \overline{\overline{\overline{A}}}$
5. *Complete Negation*: $\overline{A} \equiv \sim A \vee \sim\sim A$
6. *Quartum Non Datur*: $A \vee \sim A \vee \sim\sim A$
7. *Pseudo-Tertium Non Datur*: $A \vee \overline{A}$ (from substitution of Complete Negation in Quartum Non Datur).
8. *Non-contradiction I*: $\overline{A \wedge \overline{\overline{A}}}$
9. *Non-contradiction II*: $\overline{A \wedge \sim A}$
10. *Non-contradiction III*: $\overline{A \wedge \neg A}$
11. *De Morgan I*: $\neg(A \wedge B) \equiv \neg A \vee \neg B$
12. *De Morgan II*: $\neg(A \vee B) \equiv \neg A \wedge \neg B$
13. *Distribution I*: $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$
14. *Distribution II*: $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$
15. *Contraposition I*: $\neg A \supset B \equiv \neg B \supset A$
16. *Contraposition II*: $\overline{A} \rightarrow B \equiv \overline{B} \rightarrow A$
17. *Dissolution of Equivalence*: $(A \equiv B) \equiv (A \leftrightarrow B) \wedge (\neg A \leftrightarrow \neg B)$
18. *Dissolution of Implication*: $A \rightarrow B \equiv\sim\neg(\overline{A} \vee B)$

19. *Reductio Ad Absurdum I*: $(A \supset \overline{A}) \supset \overline{A}$
20. *Reductio Ad Absurdum II*: $(A \rightarrow \overline{A}) \rightarrow \overline{A}$

Of special importance is the following principle which links **R3V** with quantum mechanical considerations:
Complementarity We call two statements A and B *complementary* if they satisfy the relation

$$(A \vee \sim A) \rightarrow \sim\!\sim B$$

that is, in Reichenbach's reading, if A is true or false, then B is indeterminate.

The *rule of complementarity of quantum mechanics* can be stated as

$$(U \vee \sim U) \rightarrow \sim\!\sim V$$

where U is the abbreviation of "the first entity has the value u", V stands for "the second entity has the value v", and u and v are noncommutative quantities. Furthermore, the condition of complementarity is *symmetrical*: if A is complementary to B, then B is complementary to A, and thus we may as well write the rule of complementarity of quantum mechanics as

$$(V \vee \sim V) \rightarrow \sim\!\sim U$$

Complementarity is not restricted to two entities. Consider the three components $\vec{x}$, $\vec{y}$ and $\vec{z}$ of the angular momentum, where each component is complementary to each of the other two; then we may express the complementarity between them as three formulas: $(X \vee \sim X) \rightarrow \sim\!\sim Y$, $(Y \vee \sim Y) \rightarrow \sim\!\sim Z$ and $(Z \vee \sim Z) \rightarrow \sim\!\sim X$.

Now, to show how **R3V** may avoid causal anomalies in quantum mechanics, Reichenbach considers the famous Double-slit Experiment.[4] His analysis depends on specifying some properties of disjunctions which are not available in two-valued logic (Reichenbach 1935, p. 161):

1. A disjunction of n disjuncts is *closed* if, in case $n-1$ disjuncts are false, the n-th disjunct must be true.
2. A disjunction is called *exclusive* if, in case one disjunct is true, all the others must be false.

[4] Recall that, in this experiment, a source shoots individual electrons (which we may think as corpuscles) to a screen which helps us detect where they landed, but the path between the source and the screen is blocked by a barrier with two slits which we may open or close at will. If slit 1 is open but slit 2 is closed, the electrons leave a mark in the screen's area closest to the open slit, showing that they landed there after coming through slit 1 but got blocked (did not land on the screen) by slit 2; similarly if slit 1 is closed but slit 2 is open. No electron gets to the screen if both slits are closed, but if they are both open, then what is observed in the screen is an interference pattern—typical of waves, not corpuscles—instead of a distribution of electrons that only landed on the screen's area closest to each opened slit. This experiment is typically thought to show a "causal anomaly" because the expected result of the experiment when the two slits are open is radically different from what we observe (which suggests that individually fired electrons may go through *both* slits as a wave would); it is also an experiment considered to establish the wave-particle duality in quantum mechanics.

3. A disjunction is called *complete* if one of its disjuncts must be true; or, equivalently, if the disjunction is true.

In classical two-valued logic, a closed disjunction is also complete, and vice versa, so we may derive that the disjunction $B_1 \vee B_2 \vee \cdots \vee B_n$ is true if the disjunction is both closed and exclusive, which is given by the following relations

$$\begin{aligned} B_1 &\equiv \neg B_2 \wedge \neg B_3 \wedge \cdots \wedge \neg B_n \\ B_2 &\equiv \neg B_1 \wedge \neg B_3 \wedge \cdots \wedge \neg B_n \\ &\vdots \\ B_n &\equiv \neg B_1 \wedge \neg B_2 \wedge \cdots \wedge \neg B_{n-1} \end{aligned}$$

where '$\neg$' is the two-valued diametrical negation, i.e. the classical negation for two-valued logic, and similarly for '$\wedge$' and '$\equiv$'.

In a three-valued logic like **R3V**, since the disjunction $B_1 \vee B_2 \vee \cdots \vee B_n$ may be indeterminate if some of the B_i are indeterminate and the others are false, the truth of the disjunction $B_1 \vee B_2 \vee \cdots \vee B_n$ does not follow from the fact that the disjunction is closed and exclusive, which is given by the following relations

$$\begin{aligned} B_1 &\leftrightarrow \neg B_2 \wedge \neg B_3 \wedge \cdots \wedge \neg B_n \\ B_2 &\leftrightarrow \neg B_1 \wedge \neg B_3 \wedge \cdots \wedge \neg B_n \\ &\vdots \\ B_n &\leftrightarrow \neg B_1 \wedge \neg B_2 \wedge \cdots \wedge \neg B_{n-1} \end{aligned}$$

where '$\neg$' is the three-valued diametrical negation, '$\wedge$' is the three-valued conjunction and '$\leftrightarrow$' is the alternative equivalence.

These distinctions are important because in two-valued logic a statement C is proved when

$$(B \vee \neg B) \supset C$$

has been proved. In three-valued logic, its analogue

$$(B \vee \overline{B}) \supset C$$

which is equivalent to

$$(B \vee \sim B \vee \sim\sim B) \supset C$$

requires a proof of C when B is true, another one when B is indeterminate, and another one when B is false. So while the disjunction $B \vee \neg B$ is closed and exclusive, in **R3V** the relation $(B \vee \neg B) \supset C$ is not a proof of C; more generally, if the disjunction $B_1 \vee B_2 \vee \cdots \vee B_n$ is closed and exclusive, then the relation $(B_1 \vee B_2 \vee \cdots \vee B_n) \supset C$

does not represent a proof of C since the disjunction may be indeterminate. Only a complete disjunction in the antecedent of that implication would lead to a proof of C.

Now, as to the Double-slit Experiment, let us consider a generalization in which n slits $S_1, \ldots, S_n$ are used. Reichenbach lets B_i be the statement "The particle passes through slit S_i" and says that after we observe that an electron landed on the screen we immediately know that the disjunction $B_1 \vee B_2 \vee \cdots \vee B_n$ is closed and exclusive, i.e. we know that:

- if the particle did not go through $n-1$ of the slits, it went through the n-th slit
- if the particle went through one of the slits, it did not go through the others

However, recall that a closed and exclusive disjunction in **R3V** is not necessarily also a complete disjunction, i.e. it may not be a true disjunction since it may be indeterminate instead. All we can say about the disjunction $B_1 \vee B_2 \vee \cdots \vee B_n$ after an electron hits the screen is that the disjunction *is not false*.

To illustrate this, consider the case when $n = 2$. The conditions for a closed and exclusive disjunction $B_1 \vee B_2$ are

$$B_1 \leftrightarrow \neg B_2$$
$$B_2 \leftrightarrow \neg B_1$$

which is equivalent to $B_1 \equiv \neg B_2$, a relation which makes the disjunction $B_1 \vee B_2$ what Reichanbach calls a *diametrical disjunction*, which is a three-valued version of the two-valued exclusive "or". Accordingly, we know that:

- B_1 is true if B_2 is false
- B_1 is false if B_2 is true
- B_1 is indeterminate if B_2 is indeterminate

hence, not all B_i can be false simultaneously, so $B_1 \vee B_2$ may be true or may be indeterminate and thus it need not be a complete disjunction as well.

This, for Reichenbach, represents the physical situation at hand; indeed, if an observation at one slit is made, then the statement about the passage of the particle at the other slit is no longer indeterminate, for if the observation was positive then the proposition stating that the particle went through the other slit must be false, and it must be true if the observation was negative. Moreover, if no observation of the particle at one of the slits has been made, the disjunction $B_1 \vee B_2$ is indeterminate; this will also be the case if an observation is made at the n-th slit with the result that the particle did not go through this slit.

The latter shows that causal anomalies expressed as C may not be derived from $B_1 \vee B_2 \supset C$ since the diametrical disjunction $B_1 \vee B_2$ cannot be proved as true—or, from another perspective, $B_1 \vee B_2$ is not a tautological but a semi-synthetic formula. Reichenbach puts C for "The probability holding for the particle has the value $P(A \wedge (B_1 \vee B_2 \vee \cdots \vee B_n) \mid L)$", where

$$P(A \wedge (B_1 \vee B_2 \vee \cdots \vee B_n) \mid L) = \frac{\sum_{i=1}^{n} P(A \mid B_i) P(A \wedge B_i \mid L)}{\sum_{i=1}^{n} P(A \mid B_i)}$$

and where A stands for "The particle leaved the source F_0", L stands for "The particle is observed to arrive at location U of the screen", and the bar '|' in probability functions as $P(X \mid Y)$ indicates the conditional probability of X when given Y.

The reason for Reichenbach's choice of C above is that the latter equation, the principle of corpuscular superposition, can only be applied when A and $B_1 \vee B_2 \vee \cdots \vee B_n$ are true; since we have shown that $B_1 \vee B_2 \vee \cdots \vee B_n$ may not be true, the equation is inapplicable. The principle of corpuscular superposition states that the statistical pattern occurring on the screen, when all slits are open simultaneously, is a superposition of the individual patterns resulting when only one slit is open; but this, we know, is not the case when the experiment is performed. Thus, a causal anomaly is logically avoided.

3 Dunn semantics for R3V

Omori and Sano (2015) propose a method to obtain the truth and falsity conditions of the connectives in several many-valued logics in terms of *Dunn conditions*. Dunn conditions are defined as follows (taken almost verbatim from Omori and Sano 2015, p. 889):

Consider four distinct truth values a, b, c, d partially ordered as follows: $x < a$, for any $x \in \{b, c, d\}$; $d < x$, for any $x \in \{a, b, c\}$; and $b \not< c$ and $c \not< b$. Represent those values by the four subsets of the set $\{0, 1\}$ of classical values, i.e. $\{1\}$, $\{1, 0\}$, $\varnothing$, and $\{0\}$, respectively, and let $E = \{\{1\}, \{1, 0\}, \varnothing, \{0\}\}$. Then

- $x_i (1 \leq i \in \omega)$ is a variable (at the meta-logical level) which runs over E. We use X and X_k for $\{x_i | 1 \leq i \in \omega\}$ and $\{x_1, \ldots, x_k\}$, respectively.
- The expressions '$1 \in x_i$' and '$0 \in x_i$' $(1 \leq i \in \omega)$ are called *Dunn atoms*. Furthermore, we write the set of all Dunn atoms whose variables are from X_k as $\mathrm{Datom}(X_k)$.
- Let $f : E^k \longrightarrow E (1 \leq k \in \omega)$ be a finitary mapping. *Positive* and *negative Dunn condition for f* are conditions of the following forms respectively:

$$i \in f(x_1, \ldots, x_k) \text{ iff } \mathrm{B}_i(\mathrm{Datom}(X_k))(i \in \{1, 0\})$$

  where $\mathrm{B}_i(\mathrm{Datom}(X_k))$ are Boolean combinations (at the meta-logical level) constructed from $\mathrm{Datom}(X_k)$. We refer to $\mathrm{B}_1(\mathrm{Datom}(X_k))$ and $\mathrm{B}_0(\mathrm{Datom}(X_k))$ as *positive and negative clauses for f* respectively.
- *Dunn conditions for a finitary mapping f: $E^k \longrightarrow E$* is a pair of positive Dunn condition for f and negative Dunn condition for f".

Said briefly, Dunn conditions are essentially all the combinations of 1's and 0's in disjunctive normal form that show the cases in which a given connective is either true or false. Notice that this construction defines non-standard assignments (like "both" and "neither") in terms of the standard values 1 (true) and 0 (false) and the set-theoretic relation $\in$, thus eliminating the philosophical difficulty of answering about the nature and interpretation of additional values, since to talk about formulas with non-standard assignments is, in a Dunn semantics, exactly the same as to talk about the positive and negative Dunn conditions for that formula being satisfied in a particular manner.

Given any n-valued truth table, with $n \leq 4$, one can obtain its Dunn conditions following just four steps (taken almost verbatim from Omori and Sano (2015), pp. 891–892):

(i) Suppose that we have a truth table for an n-ary connective f written in terms of some of $\{1\}$, $\{1, 0\}$, $\varnothing$ and $\{0\}$, respectively.

(ii) Separate the truth tables into two parts, one having only 1's in the truth table, and the other having 0's in the truth table.

(iii) Let $E^n_{(f,i)}$ be the set $\{(\epsilon_1, \ldots, \epsilon_n) \in E^n | i \in f(\epsilon_1, \ldots, \epsilon_n)\}$ $(i \in \{0, 1\})$. Then, we have the following two conditions through the two separated truth tables obtained in the previous step (where '$\mathrm{OR}_{(\epsilon_1,\ldots,\epsilon_n)}$' represents the combinations of 1's and 0's in disjunctive normal form):

- $1 \in f(x_1, \ldots, x_n)$ iff $\mathrm{OR}_{(\epsilon_1,\ldots,\epsilon_n)\in E^n_{(f,1)}}$ $(x_1 = \epsilon_1$ and… and $x_n = \epsilon_n)$
- $0 \in f(x_1, \ldots, x_n)$ iff $\mathrm{OR}_{(\epsilon_1,\ldots,\epsilon_n)\in E^n_{(f,0)}}$ $(x_1 = \epsilon_1$ and… and $x_n = \epsilon_n)$

If $E^n_{(f,i)} = \varnothing$, then we set $\mathrm{OR}_{(\epsilon_1,\ldots,\epsilon_n)\in E^n_{(f,1)}}$ $(x_1 = \epsilon_1$ and… and $x_n = \epsilon_n)$ to be $x_1 = \{1\}$ and $x_1 = \{0\}$.

(iv) Finally, apply the following rules to rewrite the conditions above:

$$x = \{1\} \text{ iff } 1 \in x \text{ and } 0 \notin x, \qquad x = \{0, 1\} \text{ iff } 1 \in x \text{ and } 0 \in x$$
$$x = \varnothing \text{ iff } 1 \notin x \text{ and } 0 \notin x, \qquad x = \{0\} \text{ iff } 1 \notin x \text{ and } 0 \in x.$$

Let us apply this to **R3V**. The three-valued presentation of Reichenbach's logic, along with the number of elements in D^+, motivate the representation of Reichenbach's T, I and F as three subsets of the set of classical values $\{1, 0\}$, namely $\{1\}$, $\varnothing$ and $\{0\}$, respectively, leaving the remaining subset $\{1, 0\}$ aside. Thus, a *Reichenbach model* for the language L of Sect. 1 is a mapping i from L to $\{\{1\}, \varnothing, \{0\}\}$. Consider the truth table for conjunction:

Conjunction

A	B	$A \wedge B$
T	T	T
T	I	I
T	F	F
I	T	I
I	I	I
I	F	F
F	T	F
F	I	F
F	F	F

We then replace T, I and F by $\{1\}$, $\varnothing$ and $\{0\}$ respectively, as in the table below:

A	B	$A \wedge B$
$\{1\}$	$\{1\}$	$\{1\}$
$\{1\}$	$\varnothing$	$\varnothing$
$\{1\}$	$\{0\}$	$\{0\}$
$\varnothing$	$\{1\}$	$\varnothing$
$\varnothing$	$\varnothing$	$\varnothing$
$\varnothing$	$\{0\}$	$\{0\}$
$\{0\}$	$\{1\}$	$\{0\}$
$\{0\}$	$\varnothing$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$

Afterwards, we separate truth tables in two tables, one having just 1's, and the other having just 0's, as in the tables below:

A	B	$A \wedge B$	A	B	$A \wedge B$
$\{1\}$	$\{1\}$	1	$\{1\}$	$\{1\}$	
$\{1\}$	$\varnothing$		$\{1\}$	$\varnothing$	
$\{1\}$	$\{0\}$		$\{1\}$	$\{0\}$	0
$\varnothing$	$\{1\}$		$\varnothing$	$\{1\}$	
$\varnothing$	$\varnothing$		$\varnothing$	$\varnothing$	
$\varnothing$	$\{0\}$		$\varnothing$	$\{0\}$	0
$\{0\}$	$\{1\}$		$\{0\}$	$\{1\}$	0
$\{0\}$	$\varnothing$		$\{0\}$	$\varnothing$	0
$\{0\}$	$\{0\}$		$\{0\}$	$\{0\}$	0

We obtain thus the following two conditions:

$$\begin{aligned}
(A \wedge B) = \{1\} \text{ iff } & A = \{1\} \text{ and } B = \{1\} \\
(A \wedge B) = \{0\} \text{ iff } & A = \{1\} \text{ and } B = \{0\} \text{ or} \\
& A = \varnothing \;\; \text{ and } B = \{0\} \text{ or} \\
& A = \{0\} \text{ and } B = \{1\} \text{ or} \\
& A = \{0\} \text{ and } B = \varnothing \text{ or} \\
& A = \{0\} \text{ and } B = \{0\}
\end{aligned}$$

Rewriting both conditions in terms of Dunn atoms, we obtain the following presentation of those conditions:

$$\begin{aligned}
1 \in (A \wedge B) \text{ iff } & 1 \in A \text{ and } 0 \notin A \text{ and } 1 \in B \text{ and } 0 \notin B \\
0 \in (A \wedge B) \text{ iff } & 1 \in A \text{ and } 0 \notin A \text{ and } 1 \notin B \text{ and } 0 \in B \text{ or} \\
& 1 \notin A \text{ and } 0 \notin A \text{ and } 1 \notin B \text{ and } 0 \in B \text{ or}
\end{aligned}$$

$$1 \notin A \text{ and } 0 \in A \text{ and } 1 \in B \text{ and } 0 \notin B \text{ or}$$
$$1 \notin A \text{ and } 0 \in A \text{ and } 1 \notin B \text{ and } 0 \notin B$$
$$1 \notin A \text{ and } 0 \in A \text{ and } 1 \notin B \text{ and } 0 \in B$$

which in turn can be simplified as follows:

$$1 \in (A \wedge B) \text{ iff } 1 \in A \text{ and } 1 \in B$$
$$0 \in (A \wedge B) \text{ iff } 0 \in A \text{ or } 0 \in B$$

(Omori and Sano use classical logic on making simplifications like the previous one. Note also that we allowed ourselves a little abuse of notation by ommiting the interpretations i and writing expressions like '$1 \in ©(A_1, \ldots A_n)$', '$0 \in ©(A_1, \ldots A_n)$', '$©(A_1, \ldots A_n) = \{1\}$' or '$©(A_1, \ldots A_n) = \{0\}$' to give the evaluation conditions for any n-nary connective ©.)

Again, note that the Dunn semantics does not amount to the formally trivial replacement of F by $\{0\}$, I by $\varnothing$, and T by $\{1\}$; the predication of such values for formulas has been modified as follows: instead of the requisites of *truth-functionality and three-values*—i.e. to each formula there is one and only one of the three truth values assigned to it—we have *relational assignments of just two truth values* given in terms of $\in$ in the positive and negative Dunn conditions, which allows for formulas which have neither the value true nor the value false assigned to them.

Thus, under the Dunn semantics, **R3V** is not three-valued; there are only two truth values, those in the set $\{1, 0\}$. What one gets are three admissible valuations on those two truth values:

- A formula may be assigned just 1: $1 \in i(A)$ and $0 \notin i(A)$, represented by '$\{1\}$';
- A formula may be assigned just 0: $0 \in i(A)$ and $1 \notin i(A)$, represented by '$\{0\}$';
- A formula may be assigned neither 1 nor 0: $1 \notin i(A)$ and $0 \notin i(A)$, represented by '$\varnothing$'.

Finally, the truth tables rewritten would look as follows:

Conjunction and disjunction

A	B	$A \wedge B$	$A \vee B$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{1\}$	$\varnothing$	$\varnothing$	$\{1\}$
$\{1\}$	$\{0\}$	$\{0\}$	$\{1\}$
$\varnothing$	$\{1\}$	$\varnothing$	$\{1\}$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$\varnothing$	$\{0\}$	$\{0\}$	$\varnothing$
$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$
$\{0\}$	$\varnothing$	$\{0\}$	$\varnothing$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

Negations

	Cyclical	Diametrical	Complete
A	$\sim A$	$\neg A$	$\overline{A}$
{1}	∅	{0}	∅
∅	{0}	∅	{1}
{0}	{1}	{1}	{1}

Implications

		Standard	Alternative	Quasi-implication
A	B	$A \supset B$	$A \to B$	$A \sqsupset B$
{1}	{1}	{1}	{1}	{1}
{1}	∅	∅	{0}	∅
{1}	{0}	{0}	{0}	{0}
∅	{1}	{1}	{1}	∅
∅	∅	{1}	{1}	∅
∅	{0}	∅	{1}	∅
{0}	{1}	{1}	{1}	∅
{0}	∅	{1}	{1}	∅
{0}	{0}	{1}	{1}	∅

Equivalences

		Standard	Alternative
A	B	$A \equiv B$	$A \leftrightarrow B$
{1}	{1}	{1}	{1}
{1}	∅	∅	{0}
{1}	{0}	{0}	{0}
∅	{1}	∅	{0}
∅	∅	{1}	{1}
∅	{0}	∅	{0}
{0}	{1}	{0}	{0}
{0}	∅	∅	{0}
{0}	{0}	{1}	{1}

Thus, note that Reichenbach's **R3V** can be presented as (Strong) Kleene $\mathbf{K}_3$ with a two-valued relational semantics—consider the fragment $\{\neg, \wedge, \vee\}$—expanded with five new connectives, two negations and three implications[5], whose truth and falsity conditions are as follows:

$1 \in i(\sim A)$ iff $0 \in i(A)$
$0 \in i(\sim A)$ iff $1 \notin i(A)$ and $0 \notin i(A)$
$1 \in i(\overline{A})$ iff $1 \notin i(A)$
$0 \in i(\overline{A})$ iff $1 \in i(A)$ and $0 \in i(A)$
$1 \in i(A \supset B)$ iff $0 \in i(A)$ or $0 \notin i(B)$
$0 \in i(A \supset B)$ iff $1 \in i(A)$ and $1 \notin i(B)$
$1 \in i(A \to B)$ iff $1 \notin i(A)$ or $1 \in i(B)$

[5] Or as an expansion of Ł_3, if standard implication is included among the primitive connectives, with two more negations and two more implications.

$0 \in i(A \rightarrow B)$ iff $1 \in i(A)$ and $1 \notin i(B)$
$1 \in i(A \sqsupset B)$ iff $1 \in i(A)$ and $1 \in i(B)$
$0 \in i(A \sqsupset B)$ iff $1 \in i(A)$ and $0 \in i(B)$
(As we have seen, the equivalences are defined with the vocabulary already available.)

In the next section we explore whether this new presentation helps in addressing some of the old misgivings about the formal features of **R3V**.

4 A new glance at the objections against R3V

Philosophers of science raised several objections against Reichenbach's logic **R3V**. These may be summarized as follows:

O_1) **Feyerabend's**: Reichenbach's move to render statements of causal anomalies unassertable in the object language of quantum mechanics is a move that makes refuting instances of the theory of quantum mechanics unassertable (Nilson 1979, p. 441; Feyerabend 2012).

O_2) **Suppes'**: it seems that **R3V** has little to do with the underlying logic required for quantum mechanical probability spaces; moreover, what he calls the logic for quantum mechanic events is not truth-functional, while Reichenbach's logic is so (Patrick 2012b, a).

O_3) **Strauss and Gardner's**: some causal anomalies are avoided, but others are not (e.g. Schrödinger's cat and the Einstein-Podolsky-Rosen paradox) (Gardner 1972; Strauss 1971).

This kind of objections questions the suitability of **R3V** as a logic to work in quantum mechanics. Addressing them would imply to take a stance in a variety of issues that go beyond the scope of the paper, and we leave them for another occasion.[6]

There are other criticisms, directed not towards the applicability of **R3V** in its intended domain, but rather towards its formal features. For example, according to Hempel and Nagel, Reichenbach has not given sufficient indication of how we are to understand the various truth values in **R3V**—see for example (Hempel 1945; Nagel 1946; Nilson 1979, pp. 442–443). In particular, they claim that

HN1. "Truth" in a two-valued language does not mean the same as "truth" in a three-valued language.

HN2. Conditions under which a statement is true, false or indeterminate are left unspecified.

HN3. "Indeterminate" means the same as "meaningless".

HN4. Certain statements in **R3V** claimed by Reichenbach to make the same assertion, in fact do not do so. Hempel argues that if two sentences state the same fact, they must have identical truth tables, but the truth tables for $\sim A$ and $\neg A$ are different and yet Reichenbach tells us both of them state "A is false".

HN5. Reichenbach's attempt to move certain statements ordinarily made meta-linguistically into the object language of quantum mechanics does not work. Besides the case for $\sim A$ and $\neg A$ not stating the same, Hempel is worried about

[6] For some replies to the objections above, sympathetic to Reichenbach's project, see Nilson (1979).

the non-standard nature of the "if then" relation used in $U \vee \sim U \rightarrow \sim\sim V$ and thinks that the Rule of Complementarity should be stated in the meta-language.

Van Fraassen has also objected that Reichenbach's Rule of Complementarity is satisfied by certain parameters which are not incompatible (Van Fraassen 2012), so the object language formula $U \vee \sim U \rightarrow \sim\sim V$ is not enough, in his view, to capture the phenomena of complementarity; moreover, this rule may be expressed through a single connective $\mathbin{\text{ó}}$ (Van Fraassen's "apple" connective) such that $A \mathbin{\text{ó}} B$ if and only if $A \vee \sim A \rightarrow \sim\sim B$, where

$A \mathbin{\text{ó}} B$	$\{1\}$	$\varnothing$	$\{0\}$
$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
$\varnothing$	$\{1\}$	$\{1\}$	$\{1\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{0\}$

which makes Van Fraassen's criticism evident.[7] Van Fraassen shows that there are models where the formula $U \vee \sim U \rightarrow \sim\sim V$ (or $U \mathbin{\text{ó}} V$) is true, but where their corresponding quantities u and v commute, i.e. $[u, v] = 0$. To see this, let Q be the operator for the x-coordinate position of a system, Q' be the y-coordinate position operator, and P be the x-coordinate momentum operator. Suppose $Qx = rx$ and $Q'x = r'x$, i.e. the x-coordinate position operator applied to a state vector x results in the eigenvalue r (similarly for Q' and r'). Then the proposition $U(q', r')$, i.e. "the y-coordinate position magnitude has eigenvalue r'" is true and $V(p, r'')$, i.e. "the x-coordinate momentum magnitude has eigenvalue r''" is indeterminate, whence $U \vee \sim U \rightarrow \sim\sim V$ (or $U \mathbin{\text{ó}} V$) is true. However, $[q', p] = 0$ even though $[q, p] \neq 0$, i.e. while by Heisenberg's Uncertainty Principle, the x-coordinate position magnitude q does not commute with the x-coordinate momentum magnitude p, still the y-coordinate position magnitude q' commutes with the x-coordinate momentum magnitude p.

The latter argument is used to suggest that Reichenbach's Rule of Complementarity will not work when stated as an object language formula, and so that the only hope is to resort to other formulation in the meta-language. We will return to this a bit later, but notice that Van Fraassen's argument only proves that

$$U \vee \sim U \rightarrow \sim\sim V \nVdash [u, v] \neq 0$$

Let us start with Hempel's and Nagel's objections. Regarding HN1, we can say that "truth" does mean the same in classical logic and in **R3V**: the set of truth values in each case is the same, $\{1, 0\}$, and truth is the same element in both. There is nonetheless a difference in how we treat the predicate *is true* ("truth" as a predicate of formulas) in Reichenbach's logic, which our Dunn-style semantics makes clear. A functional semantics like the one Hempel and Nagel had in mind predicates truth of a formula

[7] The truth and falsity conditions for the apple connective are as follows:
$1 \in i(A \mathbin{\text{ó}} B)$ iff $1 \notin i(A)$ and $0 \notin i(A)$, or $1 \notin i(B)$ and $0 \notin i(B)$
$0 \in i(A \mathbin{\text{ó}} B)$ iff $1 \in i(A)$ or $0 \in i(A)$, and $1 \in i(B)$ or $0 \in i(B)$

A exactly when there is a function i such that $i(A) = 1$, i.e. predication of truth happens via the identity relation with the element 1 of $\{1, 0\}$. However, in a relational semantics like the Dunn-style semantics we give here, predication of truth happens via the membership relation, i.e. $1 \in i(A)$. Thus, even when the semantics for two logics have the same set of truth values, $\{1, 0\}$, a difference in the predication of truth values would allow for truth-value gluts—the assignation of both truth and falsity—or gaps—the assignation of neither truth nor falsity. This is what happens in **R3V**.

HN2 is difficult to assess. Reichenbach thought that his truth tables specified every truth, indeterminacy and falsity conditions for propositions in general—see Nilson (1979), p. 443, and we think he is right. If the demand was to give truth, indeterminacy and falsity conditions for atomic propositions, in the sense that they say when an atomic proposition get each of the valuations, instead of simply stating that it can get one of them, the demand is simply too high to meet not only for **R3V**, but for any formal semantics.

Regarding HN3, the claim is not true. We have seen that **R3V** is an expansion of $\mathbf{K}_3$ and the additional admissible valuation is not to be understood as "meaningless"; it is rather "neither true nor false". Besides the Dunn reconstruction above, there is one more consideration for us to entertain to show why we should not think of the additional assignment as meaninglessness. It is customary to think of meaninglessness as *infectious*: an assignment m is infectious if and only if $i(k(A_1, \ldots, A_n)) = m$, for every A_i and n-ary connective k, whenever $i(A_j) = m$, for some A_j, with $1 \leq i \leq j \leq n$. ("One bad apple spoils the whole barrel", as the saying goes.) This is not the case in **R3V**. A quick inspection on the tables reveals that there are connectives whose evaluation is not $\varnothing$ when some of its components gets $\varnothing$.[8]

The fourth objection is a bit more pressing, but still not compelling. What Reichenbach seems to have in mind when he says that both $\sim A$ and $\neg A$ state that A is false is the following:

- A is false iff $\sim A$ (is true)
- A is false iff $\neg A$ (is true)

Moreover, "Both $\sim A$ and $\neg A$ state that A is false", Reichenbach's claim, is not the same as "$\sim A$ and $\neg A$ state the same fact", which is Hempel's reconstruction of Reichenbach's claim. To see this more clearly, suppose that

(a) A and B state the same fact
 means
(b) A and B have exactly the same evaluations
 as it is suggested by a reviewer. Nonetheless, (b) in general is not equivalent to
(c) A and B have exactly the same truth conditions
 because in non-classical contexts the truth conditions do not determine by themselves the falsity conditions too. Therefore, without further argument, there is no need to accept Hempel's (stronger) claim as equivalent to Reichenbach's.[9]

[8] For more on meaninglessness—or "nonsense"—and infectious logics, see Ferguson (2017), Ch. 2.

[9] Note that $\overline{A}$ has different truth conditions: if $\overline{A}$ is true, A is untrue, although not necessarily false.

Hardegree (1979), pp. 501–503 has come up with a solution to the objection by adding to **R3V** the unary connectives **T**, **I** and **F** for "…is true", "…is indeterminate" and "…is false", respectively, with the following truth tables:

A	$\mathbf{T}A$	$\mathbf{I}A$	$\mathbf{F}A$
$\{1\}$	$\{1\}$	$\{0\}$	$\{0\}$
$\varnothing$	$\{0\}$	$\{1\}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{1\}$

Nonetheless, in our opinion, this move makes no improvement with respect to Reichenbach's negations[10], because its truth condition

- A is false iff $\mathbf{F}A$ (is true)

is the same that of $\sim A$ and $\neg A$. There is nonetheless one sense in which $\mathbf{F}A$ would be a negation more acceptable to Hempel: its falsity condition is *Boolean*, in the sense that the falsity of $\mathbf{F}A$ is determined not by the truth of A but by its non-falsity.[11]

HN5 and Van Fraassen's objection are very similar, so we will address them jointly. Van Fraassen (2012), p. 586 observes that the formulas $U \vee \sim U \rightarrow \sim\sim V$ and U ⫰ V belong in the object language but fail to capture the incompatibility relation of complementarity, and suggests, very much in line with Hempel's own remarks, a reformulation of the rule in terms of (semantic) entailment thus: whenever U and V are incompatible,

$$U \vee \sim U \Vdash \sim\sim V$$

Van Fraassen also thinks that **R3V** must rely on special rules governing the incompatibility relation, such as the following generalized form of Disjunctive Syllogism (or Gamma Rule): where C is incompatible with B,

$$A \vee B,\ C \Vdash A$$

However, we believe that the latter formulation of the Rule of Complementarity may be stated more carefully. Nilson (1979), p. 446 shows that Reichenbach's formulation of the rule (when carefully stated, i.e. "**if** U and V are non-commutative entities, **then** $U \vee \sim U \rightarrow \sim\sim V$") strictly speaking belongs to the meta-language and involves both the alternative implication $\rightarrow$ and the standard (i.e. two-valued) meta-linguistic "if then" relation sought by Hempel. A rule like

$$[u, v] \neq 0 \Vdash (U \vee \sim U) \rightarrow \sim\sim V$$

[10] Although it produces some nice properties: The set of connectives of **R3V**'s plus the Hardegree connectives is functionally complete, as a reviewer correctly notes.

[11] Note however that a full Boolean negation, i.e. a connective **N** with the following truth and falsity conditions

$1 \in i(\mathbf{N}A)$ iff $1 \notin i(A)$

$0 \in i(\mathbf{N}A)$ iff $0 \notin i(A)$

is not expressible in **R3V**, even if expanded with the Hardegree connectives: if A is neither true nor false, $\mathbf{N}A$ would be both true and false, which is not an admissible evaluation in this context.

where u and v are non-commuting quantities, i.e. $[u, v] \neq 0$, might be the correct statement of the Rule of Complementarity, for it would state that every model where u and v are non-commuting quantities is a model where the object language implication $U \vee \sim U \rightarrow \sim\sim V$ holds. This strategy accepts Van Fraassen's criticism of the sole implication $U \vee \sim U \rightarrow \sim\sim V$ in the object language (or the formula U ♁ V for that matter) not being able to capture appropriately those models (and only those) which involve non-commuting quantities, but recasts Van Fraassen's meta-language formulation of the rule in a way that does not deviate too much from Reichenbach's ideas.

Although the issue is complicated and probably the best route to handle it is moving Complementarity to the meta-language as suggested by Hempel, Van Fraassen and Nilson, we just want to mention that working in the object language is not a complete dead end. One can add to the language a new complementarity connective that captures the intuition that, for A and B to be complementary, one of them (but not both) has to be either true or false:

A ♁* B	$\{1\}$	$\varnothing$	$\{0\}$
$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
$\varnothing$	$\{1\}$	$\{0\}$	$\{1\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{0\}$

Moreover, there are other connectives that can be considered to express certain facts. Adding the Hardegree unary connectives or the just mentioned complementarity connective would enrich the expressivity of **R3V**. Moreover, there are a number of other conditionals that can be taken into account. For example, the *extensional* conditional, with the following evaluation conditions

$1 \in i(A \rightarrow_e B)$ iff $0 \in i(A)$ or $1 \in i(B)$

$0 \in i(A \rightarrow_e B)$ iff $1 \in i(A)$ and $0 \in i(B)$

and which is already definable in the language of **R3V**—in fact, already in that of $\mathbf{K}_3$—as $\neg A \vee B$. A more systematic treatment of the conditionals that can be added to, or defined in, **R3V**, is left for further work.

5 R3V and contra-classical logics

Until very recently, most of the more well-known non-classical logics—constructive logics, relevance logics, paraconsistent logics, and so on—were subclassical, i.e. all their valid arguments are classically valid, but not all classically valid arguments are valid in them. More generally, and formally, let $\mathcal{L}$ be the base language of a logic **L**—thought of as a collection of valid arguments, and $Con(\mathcal{L})$ the collection of its connectives. Then, a logic **L*** is *sub*-**L** if

- $\mathcal{L}^* \subseteq \mathcal{L}$
- $\mathbf{L^*} \subset \mathbf{L}$

- for any © $\in Con(\mathcal{L})$, the interpretation of © in **L**, $i_{\mathbf{L}}(©(A_1, \ldots, A_n))$, entails (in **L**) the interpretation for © in **L***

But some non-classical logics are *contra-classical*. This means that, in them there are valid arguments that are invalid in classical logic over the same kind of underlying language. Again more generally and formally, a logic **L*** is *contra*-**L** if

- $\mathcal{L}^* \subseteq \mathcal{L}$
- **L*** $\not\subseteq$ **L**
- for any © $\in Con(\mathcal{L})$, the interpretation of © in **L**, $i_{\mathbf{L}}(©(A_1, \ldots, A_n))$, entails (in **L**) either the truth or the falsity condition for © in **L***

Parenthetical remark. Humberstone (2000) requires *profoundness*, i.e. that a contra-classical logic is non-translatable into a fragment of classical logic, to avoid cases were an intended connective —say, an implication— is really another connective in disguise —say, a biconditional. We take that as an excessive demand, though. Our third condition is there to ensure some commonality of meaning suitable to our purposes between the connectives of classical logic and those of a contra-classical logic, but not as our final word on the topic on the meaning of connectives in contra-classical logics. They bring even more perplexities than the more common non-classical logics, but discussing them would lead us too far from our actual topic. (For defenses of ideas akin to our assumption regarding connectives, see Hjortland (2014)) and Estrada-González (2020).

Given that, in classical logic, for any A, $1 \in i(A)$ iff $0 \notin i(A)$ and $0 \in i(A)$ iff $1 \notin i(A)$, it is straightforward to verify that the evaluation conditions for negation in classical logic imply the truth conditions for $\neg A$ and $\overline{A}$ in **R3V**, and that those of the conditional imply both truth an falsity conditions for $A \supset B$ and $A \rightarrow B$, and the falsity condition for $A \sqsupset B$. (Conjunction and disjunction are even easier cases.) **End of remark.**

An example of contra-classical logic is Aristotelian logic. Aristotelian logic and its medieval successors have two central parts: one of them is the theory of oppositions and the other is syllogistic.[12] Nowadays, the most natural translation of the syllogistic forms into classical logic is as follows:

Medieval mnemonics	English-with-variables	First-order language
SaP	All Ss are Ps	$\forall x(Sx \rightarrow Px)$
SeP	No Ss are Ps	$\sim \exists x(Sx \wedge Px)$
SiP	Some Ss are Ps	$\exists x(Sx \wedge Px)$
SoP	Some Ss are not Ps	$\exists x(Sx \wedge \sim Px)$

Given this translation, syllogistic gives verdicts concerning the validity of some syllogisms that are inconsistent with classical logic. Consider the argument forms called by the medievals *Darapti* and *Camestros*, which are, respectively:

All Bs are Cs	All Cs are Bs
All Bs are As	No As are Bs
Hence some As are Cs	Hence some As are not Cs

[12] This is not the place to even start giving the basics of Aristotelian and medieval logics. Here we will assume the reader's familiarity with the theory of oppositions and syllogistic. Since there are several good starting points on the topic, we recommend George (2004) almost randomly.

Both of these are valid syllogisms. Both are invalid in classical logic. Hence, syllogistic is contra-classical.[13]

We mentioned syllogistic only to show a well-known example of a contra-classical logic, that cannot be dismissed as an eccentricity derived from a too mathematical approach to logic. Having said that, we will restrict ourselves to zero-order logics; here are two examples of contra-classical logics developed during the 20th and 21st centuries that will be useful in what follows:

Connexive logics (McCall 2012; Wansing 2020; Wansing et al. 2016) logics such that

$\sim(A \to \sim A)$	Aristotle's Thesis
$\sim(\sim A \to A)$	Variant of Aristotle's Thesis
$(A \to B) \to \sim(A \to \sim B)$	Boethius' Thesis
$(A \to \sim B) \to \sim(A \to B)$	Variant of Boethius' Thesis

are logically valid, and, furthermore,

$(A \to B) \to (B \to A)$	Symmetry of implication

is not valid.

Actually, connexive implication is motivated in McCall (1966) by reproducing in a first-order language all valid moods of Aristotle's syllogistic.

The name 'connexive logic' suggests that systems of connexive logic are motivated by some ideas about coherence or connection between premises and conclusions of valid inferences or between formulas of a certain shape, and in that general, intuitive sense, they are closely related to relevance logics; see for example Routley (1978). Other motivations for connexive logic include subjunctive and causal conditionals, because it seems that 'If an object is dropped, it will not hit the floor' contradicts 'If an object is dropped, it will hit the floor'. On this, see McCall (2014).

Abelian logic (Meyer and Slaney 1989, 2002): logics containing the following axiom schema:

$((A \to B) \to B) \to A$	Axiom of Relativity

In spite of being a classical contingency, the Axiom of Relativity is closely related to the double negation schema. Remember that negation can usually be defined as follows:

$$\neg A =_{def} A \to \bot$$

where the nullary connective $\bot$ stands, as usual, for an arbitrary falsehood. The double negation schema can thereby be reformulated in the following way:

(*Double negation schema*) $((A \to \bot) \to \bot) \to A$

Meyer and Slaney point out that negation can be generalized if an arbitrary proposition, and not only a falsehood, is put in the consequent of the *definiens*. To this effect, Meyer and Slaney say that "[l]ogic should be in the business of telling us what follows from what, and not of advising us as to which propositions are despised and rejected. Is there a proposition so true that it logically cannot be taken as f [the value false] for some purposes? Our Modest Proposal is that there is not" (Meyer and Slaney 1989, p.

[13] Whether Aristotle held that *Camestros* is valid is a moot point, but it certainly was regarded as valid in medieval syllogistic.

252). Then, the motivation to define a relative negation in this fashion is precisely that, in principle, any proposition can play the logical role of a falsehood.[14] Thus, putting B uniformly instead of $\perp$ in the Double negation schema, one gets Relativity.

No contra-classical logics are subclassical, even if they can be built upon some subclassical logics. As we have seen, **R3V** is built upon $\mathbf{K}_3$ by adding some further negations and conditionals. In what follows, we will briefly show how the fragments $\{\sim\}$, $\{\sqsupset\}$, $\{\sqsupset, -\}$ and $\{\sqsupset, \rightarrow\}$ of **R3V** give rise to contra-classicality. However, before we proceed, an important note is in order. In their standard formulations reproduced above, the contra-classical schemas do not contain different kinds of negations or conditionals: we will allow different occurrences of negations or conditionals to be replaced by actual different negations and conditionals, though.[15]

Cyclical negation Typically, a triple negation is inter-derivable with a single negation, but in **R3V** a triple cyclical negation of A, $\sim\sim\sim A$, is inter-derivable with the non-negated A.

Given this result, one may wonder whether cyclical negation is a negation at all. This question has already been raised in Omori and Wansing (2018) for demi-negation, a connective δ such that $\delta\delta A$ is inter-derivable with NA, with N an already acceptable negation, like diametrical negation.

In Omori and Wansing (2018), the authors show that there is at least one sense of 'negation' in which demi-negation—in the particular logic they study, Kamide's **CP**—is a negation, namely, because it satisfies some minimal requirements for any negation $\ominus A$ in a logic **L**, such as

- There are formulas A and B such that $A \nVdash_{\mathbf{L}} \ominus A$ and $\ominus B \nVdash_{\mathbf{L}} B$;
- $\ominus$ expresses a semantic opposition between truth and falsity, such as "$\ominus A$ is true iff A is false", or "$\ominus A$ is false iff A is true";

It can be easily verified that cyclical negation satisfies these requirements as well. This does not mean that the issue is settled, but merely that the negational character of cyclical negation is not as hopeless as it seemed at first sight because of its contra-classicality.

Quasi-implication Both $(A \sqsupset B) \Vdash A$ and $(A \sqsupset B) \Vdash B$ are valid in **R3V**, which may cast doubt on the conditionality of Reichenbach's quasi-implication. Recall the evaluation conditions for $A \sqsupset B$:

$1 \in i(A \sqsupset B)$ if and only if $1 \in i(A)$ and $1 \in i(B)$

$0 \in i(A \sqsupset B)$ if and only if $1 \in i(A)$ and $0 \in i(B)$

that is, it has the typical truth condition of a conjunction and the typical falsity condition of a conditional. Then one may wonder whether quasi-implication is not rather a conjunction, and actually there would be some reasons to think this is the case. Consider a conjunction with its usual truth and falsity conditions:

[14] At an algebraic level, Meyer and Slaney's main concern was that the relevance logic **R** seemed suitable to be interpreted with groups, but that was not the case. With the *Relativity axiom* on top of **RW**, it is possible to understand negation as a complement and this allows to interpret the resulting logic **A** using Abelian groups.

[15] This is not so uncommon, at least in connexive logic, where the main conditional in Boethius' Theses has been replaced by the material conditional, whereas the inner ones are distinctive connexive conditionals. For a brief but illuminating discussion on the topic of contra-classical theses with different conditionals, see Kapsner and Omori (2020). See also Nicolás-Francisco (2020).

$1 \in i(A \wedge B)$ if and only if $1 \in i(A)$ and $1 \in i(B)$
$0 \in i(A \wedge B)$ if and only if $0 \in i(A)$ or $0 \in i(B)$

If, moreover, logical consequence is defined as

(*T-consequence*) $\Gamma \Vdash A$ if and only if $1 \in i(A)$ whenever for every B such that $B \in \Gamma$, $1 \in i(B)$

then $A \sqsupset B \dashv\Vdash A \wedge B$, and this seems evidence enough to hold that quasi-implication is a conjunction and not a conditional. Moreover, as we have seen, both $(A \sqsupset B) \Vdash A$ and $(A \sqsupset B) \Vdash B$, on the one hand, and $(A \wedge B) \Vdash A$ and $(A \wedge B) \Vdash B$ hold; but $(A \sqsupset B) \sqsupset A$ and $(A \sqsupset B) \sqsupset B$, and $(A \wedge B) \sqsupset A$ and $(A \wedge B) \sqsupset B$ do not hold. This strongly suggests that quasi-implication is not a conditional at all, and the alleged failure of Simplification is simply due to the fact that the quasi-implication is just another conjunction.

In fact, Egré et al. (2021) have shown that quasi-implication belongs to a family of intended conditionals $>$ such that, with T-consequence will fail at least one of the following:

- Detachment, $A, A > B \Vdash B$
- Self-identity, $\Vdash A > A$
- Non-symmetry ($A > B \nVdash B > A$) or Non-entailment of conjunction ($A > B \nVdash A \wedge B$).

Nonetheless, they have also shown that quasi-implication plus *TT-consequence*, that is

(*TT-consequence*) $\Gamma \Vdash A$ if and only if $0 \notin i(A)$ whenever for every B such that $B \in \Gamma$, $0 \notin i(B)$

satisfies Self-identity, Non-symmetry and Non-entailment of conjunction, although it still fails to validate Detachment. Let us grant for the sake of the argument that Detachment is so central to conditionality that if a connective satisfies a bunch of other conditional-ish properties but it fails to validate Detachment, it is not really a conditional.

This result for quasi-implication resembles the situation of the conditional in González-Asenjo's/Priest's **LP**—which is like $\mathbf{K}_3$, with the exception that the admissible valuations are $i(A) = \{1\}$, $i(A) = \{1, 0\}$ and $i(A) = \{0\}$. Beall has stressed several times (see for example Beall 2011, 2015) that even if Detachment is invalid for the conditional in **LP**, it is *default valid* in the sense that $A, A \rightarrow B \Vdash B \vee (A \wedge \neg A)$ holds, that is, either Detachment holds or the antecedent is a formula with the value $\{1, 0\}$, which arguably is not the case in most situations. The second disjunct internalizes in the conclusion, so to speak, the structure of valuations into the object language. In the case of quasi-implication, the admissible valuations are given by $\mathcal{P}(V) \setminus \{1, 0\}$, and one actually has that $A, A \sqsupset B \Vdash B \vee \neg(A \vee \neg A)$ holds, that is, either Detachment holds or the antecedent has the value $\varnothing$.

Quasi-implication and negations The fragments $\{\sqsupset, \sim\}$ and $\{\sqsupset, -\}$ are contra-classical as well. To wit, consider that the following version of Aristotle's Thesis, $\overline{(A \sqsupset NA)}$, with $N \in \{\sim, -\}$, is a valid schema. Moreover, $N \sim (A \sqsupset\sim A)$, again with $N \in \{\sim, -\}$ is a valid schema. Variations of Boethius' Theses are left to the reader.

Implications Finally, the fragment $\{\sqsupset, \rightarrow\}$ is contra-classical too. Within it, one can get the following version of Relativity: $((A \sqsupset B) \sqsupset B) \rightarrow A$.

6 Conclusions

In this paper we tried to show that, in relating it to certain contemporary tools and discussions in logic, Reichenbach's logic exhibits many interesting features, and even new responses to some of the old objections can be attempted. After revisiting Reichenbach's three-valued logic (**R3V**) and its underlying motivations, we gave a Dunn-style semantics for it, that is, a semantics based on independent, relational truth and falsity conditions for the connectives. Then we reconstructed some of the objections raised against the formal features of Reichenbach's logic, and showed how the presentation of **R3V** in terms of Dunn semantics helps in addressing some of these worries. Finally, we established some connections between **R3V** and some contra-classical logics. We hope that this contributes to renew the interest in a logic that is not so arcane as it can seem at first sight.

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