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On the Existence of Solutions to Nonlinear Equations Involving Singular Mappings with Non-zero *p*-Kernel

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Abstract The paper presents the continuation of the previous results devoted to the problem of solutions existence to nonlinear equations in singular case where a linear part of considered mapping determining the equation may be degenerate at the corresponding initial point. We study the case when the p-kernel of the mapping is non trivial. Such type of problems appears in various mathematical models and applications. The p-regularity theory is used in our analysis and some concepts and technics of set-valued approach.

Keywords p-regularity · Singularity · Nonlinear mapping · Multimapping · Contraction · Factor operator · p-kernel

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1 Introduction

In this paper we continue consideration of the solution existence problem to the nonlinear equation which was studied in [15].

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Let us consider the following nonlinear equation:

$$F(x) = 0, \tag{1}$$

where $F : X \to Y$ and X, Y are Banach spaces. There are many results on solution existence of (1) in non-degenerate case, i.e. when in some point x_0 we have $\operatorname{Im} F'(x_0) = Y$ (see [9]). But in degenerate case such results either don't exist or at least there are not unified approach for analysis of such type problems.

In present paper in the main result (Theorem 2) we give sufficient conditions for existence of local solutions to some equations in non-regular (singular) case when p-kernel of pth order derivative of F at the initial point x_0 is nonempty. Earlier we have considered the trivial p-kernel case (see [15]). This result can be applied for instance to the following nonlinear boundary value problem:

$$F(u) = \Delta u - (\varepsilon + \bar{\varepsilon})g(u) = 0 \quad \text{in } \Omega \tag{2}$$

with the boundary conditions

$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded region in $\mathbb{R}^n$ with smooth boundary, Δ is Laplacian, u belongs to a suitable Banach space (let say H_0^{s+2}, such that $F : H_0^{s+2} \times \mathbb{R} \to H^s$), $\bar{\varepsilon}$ is an eigenvalue of Δ, $g(0) = 0$, $g'(0) = 1$, ε is sufficiently small number and H^s is Hilbert space of s-times continuously differentiable functions with traditional scalar product.

In this case at the initial point $u_0 = 0$ operator $F'(u_0)$ is singular and (as we will show later) 2-kernel of $F''(u_0)$ is nonempty.

Another example is the following integral equation

$$F(x) = x(t) - \frac{1}{\sqrt{\pi}}\varepsilon \cos t - \frac{1}{\pi}\int_0^{2\pi} \cos(t-\tau)x(\tau)d\tau$$
$$- \frac{1}{\sqrt{\pi}}\sin t \int_0^{2\pi} \sin\tau \cdot \cos\tau\,(x(\tau))^2 d\tau, \quad t \in [0, 2\pi] \tag{3}$$

where $\varepsilon \in \mathbb{R}$ is a small parameter, $F : X \to X$, $X = L_2[0, 1]$.

The problem is the solution existence to the above equation for $\varepsilon \in \mathbb{R}$ sufficiently small. In this case at the initial point $x_0 = 0$, $F'(x_0)$ is singular and 2-kernel of $F''(x_0)$ is nonempty similarly like in Eq. 2. We consider this example in the final part of our paper as well.

Let us recall some definitions and denotations.

The problem (1) is called regular at x_0 if $\operatorname{Im} F'(x_0) = Y$. Otherwise, the problem (1) is called nonregular (singular or degenerate) at x_0.

We use the p-regularity theory (see e.g. [2–7, 11, 13, 16–19]) for description and investigation of solutions existence in the degenerate case.

Let p be a natural number and let $B : X \times \ldots \times X \to Y$ be a continuous p-multilinear mapping. A p-form associated to B is the map $B[\cdot]^p : X \to Y$ defined by

$$B[x]^p = B(\underbrace{x, \ldots, x}_{p})$$

for $x \in X$. Alternatively, we may simply view $B[\cdot]^p$ as homogeneous polynomial map $B : X \to Y$ of degree p, i.e. $B(\alpha x) = \alpha^p \cdot B(x)$.

Throughout this paper we assume that the mapping $F : X \to Y$ is continuously p-times Fréchet differentiable on X and write $F \in \mathcal{C}^p(X)$. Its p-th order derivative at $x \in X$ we denote as usual by $F^{(p)}(x)$ (a symmetric multilinear map of p copies of X to Y) and the associated p-form is

$$F^{(p)}(x)[h]^p = F^{(p)}(x)[\underbrace{h, \ldots, h}_{p}].$$

Furthermore, we use the following notation for the p-kernel of the mapping $F^{(p)}(x)$ (zero locus of $F^{(p)}(x)$)

$$\mathrm{Ker}^p F^{(p)}(x) = \left\{h \in X : F^{(p)}(x)[h]^p = 0\right\}.$$

Denote also by $\mathcal{L}(X, Y)$ a space of all continuous linear operators from X to Y. The set

$$M(x^*) = \left\{x \in U : F(x) = F(x^*) = 0\right\},$$

is called the *solution set* for the mapping F in neighborhood $U \subseteq X$.

We call *h a tangent vector* to a set $M \subseteq X$ at $x^* \in M$ if there exist $\varepsilon > 0$ sufficiently small and a map $r : [0, \varepsilon] \to X$ with the property that for $t \in [0, \varepsilon]$, we have $x^* + th + r(t) \in M$ and $\|r(t)\| = o(t)$. The collection of all tangent vectors at x^* is called *the tangent cone* to M at x^* and it is denoted by $T_1 M(x^*)$ (see e.g. [1]).

By the mapping $\Phi : X \to 2^Y$ we mean multimapping (or a multivalued mapping) from X to the collection of all subsets of Y.

For a linear operator $\Lambda : X \to Y$ we denote by Λ^{-1} its right inverse, that is $\Lambda^{-1} : Y \to 2^X$ which maps any element $y \in Y$ on its complete inverse image of the mapping Λ, $\Lambda^{-1}y = \{x \in X : \Lambda x = y\}$, and of course $\Lambda\Lambda^{-1} = I_Y$.

By the "norm" of such right inverse operator we mean the number

$$\|\Lambda^{-1}\| = \sup_{\|y\|=1} \inf\{\|x\| : \Lambda x = y, \ x \in X\}. \tag{4}$$

Note that if Λ is one-to-one, then $\|\Lambda^{-1}\|$ can be considered as the usual norm of the element Λ^{-1} in the space $\mathcal{L}(Y, X)$.

In our further considerations, by Λ^{-1} we shall mean just right inverse operator (multivalued) with the norm defined by (4).

2 Elements of p-Regularity Theory

Let $F : X \to Y$ is p-times Fréchet differentiable mapping. If $F^{(i)}(x_0) = 0$, where $i = 1, \ldots, p-1$, then we say that F is completely degenerate at x_0 up to the order p.

In this paper we consider the case when the regularity condition does not hold, i.e. $\mathrm{Im} F'(x_0) \neq Y$, but the mapping F is p-regular. First of all, let us remind the definition of p-regularity and construction of p-factor operator.

Consider a sufficiently smooth nonlinear mapping $F : X \to Y$. We construct the p-factor operator under the assumption that Y is decomposed into a direct sum

$$Y = Y_1 \oplus \ldots \oplus Y_p, \tag{5}$$

where $Y_1 = \overline{\mathrm{Im} F'(x_0)}$ (the closure of the image of the first derivative of F evaluated at x_0), and the remaining spaces are defined as follows. Let $Z_1 = Y$, Z_2 be closed complementary subspace to Y_1 (we are assuming that such closed complement exists), and let $P_{Z_2} : Y \to Z_2$ be the projection operator onto Z_2 along Y_1. Let Y_2 be the closed linear span of the image of the quadratic map $P_{Z_2} F^{(2)}(x_0)[\cdot]^2$. More generally, define inductively,

$$Y_i = \overline{\text{span}}\, \text{Im} P_{Z_i} F^{(i)}(x_0)[\cdot]^i \subseteq Z_i, \quad i = 2, \dots, p-1,$$

where Z_i is a choice of closed complementary subspace for $(Y_1 \oplus \dots \oplus Y_{i-1})$ with respect to Y, $i = 2, \dots, p$ and $P_{Z_i} : Y \to Z_i$ is the projection operator onto Z_i along $(Y_1 \oplus \dots \oplus Y_{i-1})$ with respect to Y, $i = 2, \dots, p$. Finally, $Y_p = Z_p$. The order p is chosen as the minimum number for which (4) holds. Now, define the following mappings (see [11, 15, 19]), $f_i : U \to Y_i$, $f_i(x) = P_{Y_i} F(x)$, $i = 1, \dots, p$, where $P_{Y_i} : Y \to Y_i$ is the projection operator onto Y_i along $(Y_1 \oplus \dots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \dots \oplus Y_p)$ with respect to Y, $i = 1, \dots, p$.

Definition 1 The linear operator $\Lambda_h \in \mathcal{L}(X, Y_1 \oplus \dots \oplus Y_p)$ is defined for some $h \in X$ by

$$\Lambda_h(x) = f_1'(x_0)[x] + f_2''(x_0)[h, x] + \dots + \frac{1}{(p-1)!} f_p^{(p)}(x_0)[h, \dots, h, x], \quad x \in X$$

and is called the p-factor operator.

We will also use more exact denotation $\Lambda_h = (\Lambda_{h,1} + \Lambda_{h,2} + \dots + \Lambda_{h,p})$, where $\Lambda_{h,1} = \frac{1}{(k-1)!} f_k^{(k)}(x_0)[h]^{k-1}$.

Sometimes it is convenient to use the following equivalent definition of p-factor operator $\widetilde{\Lambda}_h \in \mathcal{L}(X, Y_1 \times \dots \times Y_p)$ for some $h \in X$,

$$\widetilde{\Lambda}_h(x) = \left(f_1'(x_0)[x],\ f_2''(x_0)[h, x], \dots, \frac{1}{(p-1)!} f_p^{(p)}(x_0)[h, \dots, h, x] \right), \quad x \in X.$$

Note that in *completely degenerate* case the p-factor operator has the form $\frac{1}{(p-1)!} F^{(p)}(x_0)[h]^{p-1}$.

In other words, we construct a decomposition of "non-regular part" of the mapping F on partial mappings f_i in such a way that all of these mappings are completely degenerate up to the order $i-1$ where $i = 2, \dots, p$.

For our further considerations we need the following generalization of the notion of regular mapping.

Definition 2 We say that the mapping F is p-regular at x_0 along h if

$$\text{Im}\Lambda_h = Y.$$

Let us introduce corresponding nonlinear operator

$$\Psi[x]^p = f_1'(x_0)[x] + f_2''(x_0)[x]^2 + \dots + f_p^{(p)}(x_0)[x]^p$$

and

$$\text{Ker}^p \Psi[x]^p = \left\{ x : f_1'(x_0)[x] + f_2''(x_0)[x]^2 + \dots + f_p^{(p)}(x_0)[x]^p = 0 \right\}.$$

It is easy to see that $\Psi[h]^p = \Lambda_{h,1}(h) + \Lambda_{h,2}(h) + \dots + (p-1)!\Lambda_{h,p}(h)$.

Definition 3 We say that the mapping F is p-regular at x_0 if either it is p-regular along every h belonging to the set

$$\mathrm{Ker}^p \Psi_p[x]^p \setminus \{0\}$$

or $\mathrm{Ker}^p \Psi_p[x]^p = \{0\}$.

Remark 1 Sets $\mathrm{Ker}^p \Psi_p[x]^p$, and $\bigcap_{k=1}^p \mathrm{Ker}^k f_k^{(k)}(x_0)$ coincide. It follows from the definitions of $\mathrm{Ker}^p \Psi[x]^p$ and $f_k(x)$, $k = 1, \ldots, p$.

Let us consider some examples that illustrate the above introduced construction of p-regularity.

Example 1 Consider a type of (2) equation

$$F(u, \varepsilon) = \Delta u - (\varepsilon - 10)g(u) = 0 \tag{6}$$

on $\Omega = [0, \pi] \times [0, \pi]$ in $\mathbb{R}^2$ with $u = 0$ on $\partial\Omega$, $\bar{\varepsilon} = -10$. Assume that $g(0) = 0$, $g'(0) = 1$, $g''(0) = 1$.

We would like to find out whether the mapping $F(u, \varepsilon)$ is p-regular ($p = 2$) at the point $x_0 = (u_0, \varepsilon_0) = (0, 0)$ along some $h = (h_u, h_\varepsilon) \in \mathrm{Ker} F'(0, 0) \cap \mathrm{Ker}^2 P_2 F''(0, 0)$. To answer this question let us formulate the following result that gives sufficient conditions for p-regularity of mapping $F(u, \varepsilon)$.

Consider a general equation (2) for $p = 2$, i.e.

$$F(u, \varepsilon) = \Delta u - (\varepsilon + \bar{\varepsilon})g(u),$$

$F : H_0^{s+2} \times \mathbb{R} \to H^s$, $g \in \mathcal{C}^{p+1}$, $u = u(y)$, $y \in \mathbb{R}^m$ and $\bar{\varepsilon}$ is an eigenvalue of Δ with multiplicity $l > 1$. Here $g(0) = 0$, $g'(0) = 1$, $g''(0) = 1$, $p = 2$. Let us give some auxiliary assumptions and denotations that will be necessary for proving p-regularity property of mapping $F(x)$.

Let $F'_u(0, 0)$ be a Fredholm operator and $\mathrm{Ker} F'(0, 0) = \mathrm{Ker}(\Delta - \bar{\varepsilon} I)$ be spanned by the orthogonal functions $u_1, \ldots, u_l$ from $\mathcal{L}_2$. Consider an element

$$u = z_1 u_1 + \ldots + z_l u_l, \quad z_i \in \mathbb{R},\ i = 1, \ldots, l.$$

Since $g(0)=0$, we obtain $Y_1 = \mathrm{Im} F'_x(0) = \mathrm{Im} F'_u(0)$, $Y_2 = (\mathrm{Im} F'_x(0))^\perp = (\mathrm{Im} F'_u(0))^\perp$, and $H^s = Y_1 \oplus Y_2$. For Fredholm operator $F'_u(0, 0)$ we have $\mathrm{Ker}(\Delta - \bar{\varepsilon} I) = (\mathrm{Im} F'_u(0))^\perp$, $F''(0)[u, \varepsilon]^2 = -\bar{\varepsilon} g''(0) u^2 - 2u\varepsilon = -\bar{\varepsilon} u^2 - 2u\varepsilon$.

Define $z = (z_1, \ldots, z_l)$, $h = (h_u, h_\varepsilon) = (z_1 u_1 + \ldots + z_l u_l, \varepsilon)$ and

$$\begin{aligned} Q[h]^2 &= P_2 F''(0)[h]^2 = \left(\left\langle F''(0)[h]^2, u_1\right\rangle, \ldots, \left\langle F''(0)[h]^2, u_l\right\rangle\right) \\ &= \left(Q_1[h]^2, \ldots, Q_l[h]^2\right), \end{aligned}$$

where

$$\begin{aligned} Q_i[h]^2 &= Q_i[z, \varepsilon]^2 = \left\langle F''(0)[z, \varepsilon]^2, u_i\right\rangle \\ &= 2\varepsilon z_i + \bar{\varepsilon} \int_\Omega u_i (z_1 u_1 + \ldots + z_l u_l)^2 dy \\ &= 2\varepsilon z_i + \sum_{j,k} C_i^{jk} z_j z_k, \end{aligned} \tag{7}$$

$i = 1, \ldots, l$ where C_i^{jk} are so called branching coefficients such that $C_i^{jk} = \int_\Omega u_j u_k u_i dy,\ j, k, i = 1, \ldots, l.$

Let us introduce $M(z) = M(z_1, \ldots, z_l)$, the $l \times l$ symmetric matrix, where (ij) entry is

$$\sum_{k=1}^{l} \left(\int_\Omega u_i u_j u_k dy \right) z_k$$

and let

$$\det(M(z) - \varepsilon I) = D_1(\varepsilon) = \varepsilon^l + A_1 \varepsilon^{l-1} + \ldots + A_l,$$

$$\det(M(z) - 2\varepsilon I) = D_2(\varepsilon) = 2^l \varepsilon^l + 2^{l-1} A_1 \varepsilon^{l-1} + \ldots + A_l,$$

where A_i is an homogeneous polynomial in the variables $z_1, \ldots, z_l$ of degree i. Let $R(D_1, D_2)$ be a resultant of $D_1(\varepsilon)$ and $D_2(\varepsilon)$ and

$$R(D_1, D_2) = \det \begin{pmatrix} 1 & A_1 & A_2 & \cdots & A_l & 0 & \cdots\ 0 \\ 0 & 1 & A_1 & \cdots & A_{l-1} & A_l & \cdots\ 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots \\ 0 & \cdots & \cdots & \cdots & 1 & A_1 & \cdots\ A_l \\ 2^l & 2^{l-1}A_1 & 2^{l-2}A_2 & \cdots & A_l & 0 & \cdots\ 0 \\ 0 & 2^l & 2^{l-1}A_1 & 2^{l-2}A_2 & \cdots & A_l & \cdots\ 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots \\ 0 & \cdots & \cdots & 0 & 2^l & 2^{l-1}A_1 & \cdots\ A_l \end{pmatrix} \tag{8}$$

Denote $\bar{D}_2(\varepsilon) = (2^l - 1)\varepsilon^{l-1} + (2^{l-1} - 1)A_1 \varepsilon^{l-2} + \ldots + A_{l-1}.$

We formulate sufficient conditions of 2-regularity of the mapping $F(x)$.

Theorem 1 *Let* $(z, \varepsilon) \in \operatorname{Ker}^2 P_2 F''(0)$ *(equivalently* $(z, \varepsilon) \in \operatorname{Ker}^2 Q$*) and* $(z, \varepsilon) \neq 0$. *Under the above assumptions and notations for general equation* (2) *we have:*

If $\varepsilon \neq 0$ *and* $R(D_1, D_2) \neq 0$, *then* $F(x)$ *is 2-regular along* (z, ε) *at* $(0, 0)$.

If $\varepsilon = 0$ *and* $Rank M(z) = l - 1$ *then* $F(x)$ *is 2-regular along* (z, ε) *at* $(0, 0)$.

Proof Consider the case $\varepsilon \neq 0$. Property of 2-regularity along (z, ε) means that $l \times (l + 1)$ matrix $(M(z) + 2\varepsilon I, 2Iz)$ must be nonsingular for

$$h = (z, \varepsilon) \in \operatorname{Ker}^2 P_2 F''(0) = \operatorname{Ker}^2 Q.$$

It means that ε can not be an eigenvalue of $M(z)$.

Element (z, ε) is in $\operatorname{Ker}^2 Q$ if and only if either z is an eigenvector of $M(z)$ corresponding to the eigenvalue 2ε or $z = (0, \ldots, 0)$. At $(0, \ldots, 0, \varepsilon)$, $\varepsilon \neq 0$, Q is always nonsingular (and of course, $R(D_1, \bar{D}_2) \neq 0$). At any $(z, \varepsilon) \in \operatorname{Ker}^2 Q$, Q is 2-regular along (z, ε) if ε is not an eigenvalue of $M(z)$. It follows that a sufficient conditions for 2-regularity along (z, ε) is that neither ε, nor 2ε are eigenvalue of the matrix $M(z)$.

It means that polynomial

$$\varepsilon^l + A_1\varepsilon^{l-1} + \ldots + A_l$$

$$(2^{l-1} - 1)\varepsilon^{l-1} + (2^{l-2} - 1)A_1\varepsilon^{l-2} + \ldots + A_{l-1}$$

does not have a common roots. This condition is equivalent to non-vanishing of the determinant of (8), which is equivalent to $R(D_1, \bar{D}_2) \neq 0$ for $\varepsilon \neq 0$.

The case $\varepsilon = 0$ is simpler. For 2-regularity along $(z, 0)$ it is sufficient that the matrix $(M(z), Iz)$ is not singular for $(z, 0)$ such that $(M(z), Iz)(z, 0)^{\perp} = 0$.

It means that either $\dim \mathrm{Ker}(M(z), Iz) = 1$ and we obtain Rank $M(z) = l - 1$, or $\dim \mathrm{Ker}(M(z), Iz) = 0$ and $(z, 0) \notin \mathrm{Ker}^2 Q$. This contradiction completes the proof. □

Let us return to (6). We take eigenfunctions $u_1 = \sin 3y_1 \sin y_2$, $u_2 = \sin y_1 \sin 3y_2$ corresponding to eigenvalue $\bar{\varepsilon} = -10$ and $a = \int_\Omega u_1^3 dy = \int_\Omega u_2^3 dy = \frac{16}{27}$, $b = \int_\Omega u_1^2 u_2 dy = \int_\Omega u_1 u_2^2 dy = \frac{68}{75}$. Then

$$P_2 F''(0,0)[z,\varepsilon]^2 = 10(az_1^2 + 2z_1z_2b + z_2^2 b + 2\varepsilon z_1, bz_1^2 + 2z_1z_2b + az_2^2 + 2\varepsilon z_2),$$

$$M(z) = \begin{pmatrix} az_1 + bz_2 & bz_2 + bz_1 \\ bz_2 + bz_1 & bz_1 + az_2 \end{pmatrix}.$$

From calculation of resultant (8) we have the following expression

$$-2((a+b)z_1 + (a+b)z_2)^2 + 9\left((az_1 + bz_2)(bz_1 + az_2) - (bz_1 + az_2)^2\right)$$

which can be rewritten as $Az_1^2 + 2Bz_1z_2 + Cz_2^2$, where $AC - B^2 > 0$, $AC = (2a - 5ab + 11b^2)^2$, $B^2 = \frac{1}{4}(13b^2 - 5a^2 + 8ab)^2$ for $a = 16/27$, $b = 68/75$. It means that $R(D_1, \bar{D}_2) \neq 0$ and in accordance to Theorem 1, the mapping F is 2-regular along $(z(\varepsilon), \varepsilon) \in \mathrm{Ker}^2 Q$, $\varepsilon \neq 0$.

Example 2 Let us consider some modification of Eq. 2

$$F(u) = \Delta u - \bar{\varepsilon} g(u) - \varepsilon\varphi(y) = 0, \tag{9}$$

where $\varphi \in H^s$, $\Omega = [0,\pi] \times [0,\pi]$, $m = 2$, $g(0) = 0$, $g'(0) = 1$, $g''(0) = 1$, $\bar{\varepsilon}$ is an eigenvalue of Δ, $\varepsilon \in \mathbb{R}_+$ a sufficiently small parameter and $\|\varphi(y)\| = 1$. Similarly to how it was shown in previous example the mapping $F(u)$ is 2-regular at $u_0 = 0$ along some $z \in \mathrm{Ker}^2 P_2 F_u''(0)$, where $z = \bar{z}_1 u_1 + \bar{z}_2 u_2$,

$$P_2 F_u''(0)[z]^2 = 10\left(2z_1^2 + 2z_1z_2 + z_2, z_1^2 + 2z_1z_2 + z_2^2\right)$$

if matrix $M(z)$ has $Rank M(z) = l$ and z is a solution of

$$P_2 F_u''(0)[z]^2 = 0.$$

Example 3 Let us consider Eq. 3 where $\varepsilon > 0$ is sufficiently small parameter, $F : X \to X$, $X = L_2[0, 1]$, $x_0 = 0$. Mapping F is infinitely differentiable on X and moreover [12]

$$(F'(x_0)[\xi])(t) = \xi(t) - \frac{1}{\pi}\int_0^{2\pi} \cos(t-\tau)\xi(\tau)d\tau,$$

$$(F''(x_0)[\xi_1, \xi_2])(t) = -\frac{2}{\pi}\sin t \int_0^{2\pi} \sin\tau\cos\tau\,\xi_1(\tau)\xi_2(\tau)d\tau.$$

Operator $F'(x_0)$ is Fredholm of second kind with continuous kernel and $\mathrm{Ker}F'(x_0) = \mathrm{span}\,\{u_1(t), u_2(t)\}$ where $u_1(t) = \frac{1}{\pi}\sin t$, $u_2(t) = \frac{1}{\pi}\cos t$. Then

$$\mathrm{Im}F'(x_0) = \left(\mathrm{Ker}F'(x_0)\right)^{\perp} \neq X$$

since $\cos t \notin \mathrm{Im}F'(x_0)$, $\langle u_2(t), \cos t\rangle = -1 \neq 0$, i.e. F' is non-surjective at $x_0 = 0$ (singular). Following the construction in Example 1 we obtain

$$C^{jk} = c_1^{jk} = \langle u_1, F''(x_0)[u_j, u_k]\rangle, \quad j, k = 1, 2,$$

$C^{11} = C^{22} = 0$, $C^{12} = C^{21} = -\frac{1}{2}$ and corresponding quadratic form

$$Q[z]^2 = \sum_{j,k=1,2} C^{jk} z_j z_k = -z_1 z_2$$

is 2-regular along $h_1 = (1, 0)$, $h_2 = (0, 1)$ (which belongs to $\mathrm{Ker}^2 Q[\cdot]^2$) since the matrices $M(z) = (-z_2, -z_1)$ are non-singular for h_1, $M(h_1) = (0, -1)$ and for h_2,$M(h_2) = (-1, 0)$.

In next two examples we present mappings for which regularity conditions fails.

Example 4 Let us consider the following boundary value problem

$$x''(t) + x(t) + x^2(t) = 0, \quad x(0) = x(2\pi) = 0, \tag{10}$$

where $x \in \mathcal{C}^2[0, 2\pi]$ and verify existence of the solution to this problem. In order to apply Theorem 2 introduce the mapping $F : X \to Y$, $F(x) = x'' + x + x^2$, where $X = \{x \in \mathcal{C}^2[0, 2\pi] :\ x(0) = x(2\pi) = 0\}$, $Y = \mathcal{C}[0, 2\pi]$. Let $x_0 = 0$. Then

$$Y_1 = \mathrm{Im}F'(x_0) = \left\{ y \in Y : \int_0^{2\pi} y(\tau)\sin\tau d\tau = 0 \right\}.$$

Pay attention that $Y_1 \neq Y$, $\mathrm{Ker}F'(x_0) = \mathrm{span}\,(\sin t) = Y_2$,

$$P_2 y = 2\sin t \int_0^{2\pi} \sin\tau\, y(\tau)d\tau, \quad y \in Y.$$

Here $\mathrm{Ker}^2 P_2 F''(x_0) = \left\{ x \in X :\ \int_0^{2\pi} \sin\tau\, x^2(\tau)d\tau = 0 \right\}$,
$\mathrm{Ker}F'(x_0) \cap \mathrm{Ker}^2 P_2 F''(x_0) = \mathrm{span}\,\{\sin t\}$.

The 2-regularity condition of F along $h(t)=\sin t$ is equivalent to $\mathrm{Im}\, P_2 F''(0)h(t)\big|_{\mathrm{Ker} F'(0)} = Y_2$. But

$$P_2 F''(0)h(t)\mathrm{Ker} F'(0) = \text{span } \sin t \int_0^{2\pi} \sin^3 t dt = 0 \neq Y_2.$$

It means that the mapping F is not 2-regular along $h(t) = \sin t$ at the point $x_0 = 0$.

Example 5 Let us consider similar equation to (10), where its righthand side is equal to $\varepsilon \sin^2 t$, namely:

$$x''(t) + x(t) + x^2(t) = \varepsilon \sin^2 t, \quad x(0) = x(2\pi) = 0, \tag{11}$$

Here $F(x) = x''(t) + x(t) + x^2(t) - \varepsilon \sin^2 t$, $x_0 = 0$ and analogously to the above example the condition of 2-regularity of the mapping F along $h(t) = \sin t$ at $x_0 = 0$ fails.

Below we give an example of 3-regular mapping, which is useful in applications.

Example 6 Let us consider the equation of beam deformation which appears often in applications

$$F(x) = x''(t) + x(t) + \mu x^3(t) + \varepsilon \sin t = 0, \quad x(0) = x(2\pi) = 0, \tag{12}$$

where $\mu \neq 0$. For $x_0 = 0$ the operator $F'(x_0)$ is singular i.e. the equation

$$x''(t) + x(t) - \varepsilon \sin t = 0, \quad x(0) = x(2\pi) = 0,$$

has no solutions. We have $\mathrm{Ker} F'(x_0) \cap \mathrm{Ker}^2 P_2 F''(x_0) = \text{span } (\sin t)$. Since $\mathrm{Ker}^2 P_2 F''(x_0) = X = \{x \in \mathcal{C}^2[0, 2\pi] :\ x(0) = x(2\pi) = 0\}$, we omit $\mathrm{Ker}^2 P_2 F''(x_0)$ in the sequel. But for $h(t) = \sin t$, 3-factor operator

$$\left\{F'(x_0) + P_3 F'''(x_0)h^2(t)\right\}(\cdot) = (\cdot)'' + (\cdot) + 3\mu \sin t \int_0^{2\pi} \sin\tau \cdot h^2(\tau)(\cdot)d\tau$$

is surjective, i.e. the mapping $F(\cdot)$ is 3-regular along $h(t) = \sin t$ at the point $x_0 = 0$.

The following theorem gives a description of a solution set in singular (degenerate) case.

Theorem (Generalized Lyusternik Theorem) [5, 11] *Let X and Y be Banach spaces and U be a neighborhood of $x_0 \in X$. Assume that $F : X \to Y$, $F \in C^p(U)$ is p-regular at x_0. Then*

$$T_1 M(x_0) = \mathrm{Ker}^p \Psi_p[x]^p.$$

We shall give two auxiliary lemmas. The first of these lemmas is a "multivalued" generalization of the contraction mapping principle and is independent interest. By $\mathrm{dist}_{\mathrm{H}}(A_1, A_2)$ we mean the Hausdorff distance between sets A_1 and A_2.

Lemma 1 (Contraction Multimapping Principle) [1, 8, 10, 14] *Let Z be a complete metric space with distance ρ. Assume that we are given a multimapping*

$$\Phi : U_\nu(z_0) \to 2^Z,$$

on a ball $U_\nu(z_0) = \{z : \rho(z, z_0) < \nu\}$ $(\nu > 0)$ *where sets* $\Phi(z)$ *are non-empty and closed for any* $z \in U_\nu(z_0)$. *Further, assume that there exists a number* θ, $0 < \theta < 1$ *such that*

1. $\mathrm{dist}_{\mathrm{H}}(\Phi(z_1), \Phi(z_2)) \le \theta\rho(z_1, z_2)$ *for any* $z_1, z_2 \in U_\nu(z_0)$
2. $\rho(z_0, \Phi(z_0)) < (1-\theta)\nu$.

Then, for every number ε_1 *which satisfies the inequality*

$$\rho(z_0, \Phi(z_0)) < \varepsilon_1 < (1-\theta)\nu,$$

there exists $z \in B_{\varepsilon_1/(1-\theta)}(z_0) = \{\omega : \rho(\omega, z_0) \le \varepsilon_1/(1-\theta)\}$ *such that*

$$z \in \Phi(z) \tag{13}$$

Lemma 2 [1, 10] *Let* $\Lambda \in \mathcal{L}(X, Y)$. *We set*

$$C(\Lambda) = \sup_{\|y\|=1} \inf\{\|x\| : x \in X,\ \Lambda x = y\}.$$

If $\mathrm{Im}\Lambda = Y$, *then* $C(\Lambda) < \infty$.

Lemma 3 (Mean Value Theorem) [1, 10] *Let* U *be an open subset of the Banach space* X *such that* $[a, b] \in U$. *If* $f : U \to Y$ *and* $f \in \mathcal{C}^1(U)$ *then*

$$\|f(b) - f(a) - \Lambda(b-a)\| \le \sup_{\xi\in[a,b]} \|f'(\xi) - \Lambda\| \cdot \|a - b\|,$$

for any $\Lambda \in \mathcal{L}(X, Y)$.

For better understanding of the main result of this paper we consider more simple analogous result in non-degenerate (regular) case.

3 Regular Case

We quote one modification of the theorem on existence of solutions to the equations with non-degenerate mappings (see in [15]). The essential idea of the proof in the singular case is based on the similar construction to the one in the regular case.

Let us consider the mapping $F : X \to Y$. We are interested in the existence of such a point x^* that $F(x^*) = 0$. Throughout this section we assume that an initial point x_0 is given such that $F(x)$ is regular at x_0, i.e. $F'(x_0)X = Y$ and that $\left\{F'(x_0)\right\}^{-1}$ is a multivalued mapping. Moreover, let $\mathrm{Ker} F'(x_0) \neq \{0\}$ and $U_\varepsilon(x_0) = \{x \in X : \|x - x_0\| \le \theta\}$ where $0 < \theta < 1$.

Theorem 2 *Let* $F \in \mathcal{C}^2(X)$, $\|F(x_0)\| = \delta$, $F'(x_0) : X \xrightarrow{on} Y$, $\left\|\{F'(x_0)\}^{-1}\right\| = \eta$ *and* $\sup_{x\in U_\nu(x_0)} \|F''(x)\| = c < \infty$, $h \in \mathrm{Ker} F'(x_0)$, $\|h\| = 1$. *If there exists* ω, $0 < \omega < \frac{1}{2}\nu$ *such that the following inequalities*

1. $\delta \cdot \eta \le \frac{\omega}{8}$,
2. $\omega \cdot c \cdot \eta \le \frac{1}{6}$

hold then the equation $F(x) = 0$ *has a solution* $x^* = x_0 + \omega h + \bar{x}(\omega) \in U_\nu(x_0)$ *and* $\|\bar{x}(\omega)\| \le \frac{1}{2}\omega$.

Proof Define a multivalued mapping

$$\Phi : U_\nu(0) \to 2^X, \quad U_\nu(0) \subset X,$$

$$\Phi(x) = x - \{F'(x_0)\}^{-1}(F(x_0 + \omega h + x)), \quad x \in U_\omega(0).$$

The sets $\Phi(x)$ are nonempty and closed for any $x \in U_\omega(0)$. It follows from the assumption that $F'(x_0)$ is a surjection and that the sets $\{F'(x_0)\}^{-1}(y)$ for $y \in Y$, are linear manifolds parallel to $\mathrm{Ker} F'(x_0)$.

We shall show that

$$\mathrm{dist_H}(\Phi(s_1), \Phi(s_2)) \le \frac{1}{2}\|s_1 - s_2\|, \text{ for any } s_1, s_2 \in U_\omega(0). \tag{14}$$

Indeed, by Lemmas 2 and 3 we have

$$\mathrm{dist_H}(\Phi(s_1), \Phi(s_2)) \le 3\eta \cdot c \cdot \omega \|s_1 - s_2\|$$

and (14) is proved.

Moreover we have

$$\begin{aligned} \varrho(\Phi(0), 0) &= \inf\left\{\|z\| : F'(x_0)z = -F(x_0 + \omega h)\right\} \le \delta\|F(x_0 + \omega h)\| \\ &\le \delta\|F(x_0\| + \frac{1}{2}\eta \cdot c \cdot \omega^2 < \frac{1}{8}\omega + \frac{1}{12}\omega < \frac{1}{4}\omega. \end{aligned} \tag{15}$$

It means that all the assumptions of Lemma 1 are fulfilled and there exists $\bar{x}(\omega) \in \Phi(\bar{x}(\omega))$. Consequently, $\bar{x}(\omega) \in \bar{x}(\omega) + \{F'(x_0)\}^{-1}(-F(x_0 + \omega h + \bar{x}(\omega)))$. Hence $0 \in \{F'(x_0)\}^{-1}(-F(x_0 + \omega h + \bar{x}(\omega))$ and $F(x_0 + \omega h + \bar{x}(\omega)) = 0$ or $F(x^*) = 0$. From (15), $\|\bar{x}(\omega)\| \le \frac{1}{2}\omega$ or $\|\omega h + \bar{x}(\omega)\| \le \frac{1}{2}\nu + \frac{1}{4}\nu < \nu$, i.e. $x^* \in U_\nu(x_0)$. □

Remark 2 It is easy to see, that if in the condition 1 of Theorem 2 we assume that $\delta \cdot \eta \le \omega^{1+\sigma}$, $\sigma > 0$, then we obtain $\|\bar{x}(\omega)\| = o(\omega)$.

Remark 3 (Sequential Regularity) The assumptions of Theorem 2 may be weakened if, instead of the mapping $\Phi(x)$, we consider the following sequence $x_0 = 0$, $x_{n+1} \in x_n - \{F'(x_0)\}^{-1}F(x_0 + \omega h + x_n)$, $n = 1, 2, \ldots$. And consequently, instead of surjection of the mapping $F'(x_0)$ we can assume that $\mathrm{Im} F'(x_0) = W \subset Y$, where W is some closed subspace of Y and $F(x_0 + \omega h + x_n) \subset W$, $n = 0, 1, \ldots$.

4 *p*-Regular (Singular) Case

In the previous paper [15] we have considered the case of the trivial p-kernel of p-order derivatives of the mapping $F(x)$ at the initial point x_0 (for completely degenerate case). For these purposes we required much stronger assumptions on the mapping $F(x)$ such as invertibility of nonlinear operator $F^{(p)}(x_0)[\cdot]^p$, uniform boundedness of $\left\{F^{(p)}(x_0)[\cdot]^p\right\}^{-1}$ and so on. In nontrivial p-kernel case, $\mathrm{Ker}^p F^{(p)}(x_0) \neq \{0\}$ these assumptions we omit and remain only assumptions of p-regularity of the mapping

$F(x)$ along some element belonging to p-kernel of $F^{(p)}(x_0)$ and some quantitative assumptions like in the case of trivial p-kernel of mapping $F^{(p)}(x_0)$.

Let us introduce the following notations and assumptions.

$$\delta = \|F(x_0)\|, \tag{16}$$

$$h \in \bigcap_{k=1}^{p} \mathrm{Ker}^k f_k^{(k)}(x_0) \tag{17}$$

$$\eta = \left\|\Lambda_h^{-1}\right\| < \infty, \quad \|h\| = 1, \tag{18}$$

$$c = \max_{k=1,\dots,p} \sup_{x \in U_\nu(x_0)} \left\| f_k^{(k+1)}(x) \right\|, \tag{19}$$

$$d = 4 \max_{k=1,\dots,p} \frac{1}{(k-1)!} \left\| f_k^{(k)}(x_0) \right\|, \tag{20}$$

$$\alpha = \min \left\{ \frac{3}{4^{p+2}\eta}, \min_{k=1,\dots,p} \frac{\left\| f_k^{(k)}(x_0) \right\|}{(k-1)!} \right\}. \tag{21}$$

Now we can state the main result.

Theorem 3 *Let $F: X \to Y$, and X, Y – Banach spaces, $F \in \mathcal{C}^{p+1}(X)$. Assume that there exists $h \in \bigcap_{k=1}^{p} \mathrm{Ker}^k f_k^{(k)}(x_0)$, $\|h\| = 1$ such that F is p-regular mapping at x_0 along h.*

If there exists ω, $0 < \omega < \frac{1}{2}\nu$, $\nu \in (0,1)$, such that the inequalities

1. $\eta\delta \leq \alpha \frac{\omega^p}{2pd}$,
2. $\frac{4^{p+2}}{3} c\omega\eta \leq \frac{1}{2}$,

hold, then the equation $F(x) = 0$ has a solution $x^* = x_0 + \omega h + \bar{x}(\omega) \in U_\nu(x_0)$ and $\|\bar{x}(\omega)\| \leq \frac{1}{2}\omega$.

Remark 4 The following conditions are equivalent:

1. F satisfies p-regularity condition at x_0 along h.
2. Operator $\Lambda_{\omega h}$ is surjective for any $\omega \neq 0$.
3. For h condition (18) holds.
4. $\|\Lambda_{\omega h}^{-1} y\| \leq \eta(1 + \frac{1}{\omega} + \frac{1}{w^2} + \ldots + \frac{1}{\omega^{p-1}})$ for $\omega \neq 0$.

Proof of Theorem 3 Similarly like in the regular case let us consider a multivalued mapping

$$\Phi_{\omega h}(x) : U_\nu(0) \to Y$$

such that

$$\Phi_{\omega h}(x) = x - \Lambda_{\omega h}^{-1} \left\{ f_1(x_0 + \omega h + x), \ldots, f_p(x_0 + \omega h + x) \right\},$$

where $x \in U_\nu(0)$.

Like in regular case, the assumptions of contraction multimapping principle hold for $\Phi_{\omega h}(x)$. Indeed, the sets $\Phi_{\omega h}(x)$ are non-empty because $\Lambda_{\omega h}$ is a surjection for any $x \in U_\nu(0)$.

Moreover, for any $y \in Y_1 \times \ldots \times Y_p$ the sets $\Lambda_{\omega h}^{-1}(y)$ are linear manifolds parallel to $\mathrm{Ker}\Lambda_{\omega h}$, and hence the sets $\Phi_{\omega h}(x)$ are closed for any $x \in U_\nu(0)$.

We prove that

$$\mathrm{dist}_{\mathrm{H}}\left(\Phi_{\omega h}(x_1), \Phi_{\omega h}(x_2)\right) \leq \frac{1}{2}\|x_1 - x_2\|, \tag{22}$$

for $x_1, x_2 \in U_\nu(0)$ such that $\|x_i\| \leq \frac{\omega}{R}$, $i = 1, 2$, where $R = \max_{k=1,\ldots,p} R_k$, and $R_k = \frac{\left\| f_k^{(k)}(x_0)\right\|}{\alpha(k-1)!}$.

Let us point out that $R_k \geq 1$, $\forall k = 1, \ldots, p$ and it implies that $R \geq 1$ and $\|x_i\| \leq \frac{\omega}{R} \leq \omega$. Let $s_1 = x_0 + th + x_1$, $s_2 = x_0 + th + x_2$. Then

$$\begin{aligned}
&\mathrm{dist}_{\mathrm{H}}\left(\Phi_{\omega h}(x_1), \Phi_{\omega h}(x_2)\right) = \inf\left\{\|z_1 - z_2\| : z_j \in \Phi_{\omega h}(x_j),\ \ j = 1, 2\right\} \\
&\quad = \inf\left\{\|z_1 - z_2\| : \Lambda_{\omega h}(z_j) = \Lambda_{\omega h}(x_j) - \left(f_1(s_j), \ldots, f_p(s_j)\right),\ j = 1, 2\right\} \\
&\quad \leq \inf\left\{\|z\| : \Lambda_{\omega h}(z) = \Lambda_{\omega h}(x_1 - x_2) - \left(f_1(s_1) - f_1(s_2), \ldots, f_p(s_1) - f_p(s_2)\right)\right\} \\
&\quad = \inf\Bigg\{\|z\| : \Lambda_{\omega h}(z) = \Bigg(\Lambda_{\omega h,1}(x_1 - x_2) - f_1(s_1) + f_1(s_2), \ldots \\
&\qquad\qquad \ldots, \frac{1}{\omega^{p-1}}\left(\Lambda_{\omega h,p}(x_1 - x_2) - f_p(s_1) + f_p(s_2)\right)\Bigg)\Bigg\} \\
&\quad \leq \inf\Bigg\{\|z\| : z = \Lambda_{\omega h}^{-1}\Bigg(\Lambda_{\omega h,1}(x_1 - x_2) - f_1(s_1) + f_1(s_2), \ldots \\
&\qquad\qquad \ldots, \frac{1}{\omega^{p-1}}\left(\Lambda_{\omega h,p}(x_1 - x_2) - f_p(s_1) + f_p(s_2)\right)\Bigg)\Bigg\} \\
&\quad \leq \eta \cdot \sum_{k=1}^{p} \frac{1}{\omega^{k-1}}\left\| f_k(s_1) - f_k(s_2) - \Lambda_{\omega h,k}(x_1 - x_2)\right\|.
\end{aligned}$$

Taking into account Lemma 3 we have

$$\begin{aligned}
\left\| f_k(s_1) - f_k(s_2) - \Lambda_{\omega h,k}(x_1 - x_2)\right\| \leq &\sup_{\theta \in [0,1]}\left\| f_k'(s_k + \theta(x_1 - x_2)) - \Lambda_{\omega h,k}\right\| \\
&\cdot \|x_1 - x_2\|.
\end{aligned} \tag{23}$$

As $f_k(x)$ is completely degenerate up to the order k we obtain the following Taylor expansion

$$\begin{aligned}
f_k'(s_2 + \theta(x_1 - x_2)) &= f_k'(x_0) + \ldots + \frac{f_k^{(k)}(x_0)[s_2 - x_0 + \theta(x_1 - x_2)]^{k-1}}{(k-1)!} \\
&\quad + \omega_k(\omega h, x_1, x_2, \theta) \\
&= \frac{f_k^{(k)}(x_0)[s_2 - x_0 + \theta(x_1 - x_2)]^{k-1}}{(k-1)!} \\
&\quad + \omega_k(\omega h, x_1, x_2, \theta),
\end{aligned} \tag{24}$$

where $\|\omega_k(\omega h, x_1, x_2, \theta)\| \le \sup_{x\in U_\nu(x_0)} \left\| \frac{1}{k!} f_k^{(k+1)}(x)[\omega h + x_2 + \theta(x_1 - x_2)]^k \right\|$. Then from the assumption (21) we have $\|\omega h + x_2 + \theta(x_1 - x_2)\| \le 4\omega$. This and previous formulae imply

$$\|\omega_k(\omega h, x_1, x_2, \theta)\| \le \frac{1}{k!} c\|\omega h + x_2 + \theta(x_1 - x_2)\|^k \le \frac{1}{k!} 4^k \omega^k c \tag{25}$$

and

$$\begin{aligned} & f_k^{(k)}(x_0)[s\omega h + x_2 + \theta(x_1 - x_2)]^{k-1} \\ &= \sum_{n=0}^{k-1} \binom{k-1}{n} f_k^{(k)}(x_0)[\omega h]^{k-1-n}[x_2 + \theta(x_1 - x_2)]^n \\ &= f_k^{(k)}(x_0)[\omega h]^{k-1} + \sum_{n=1}^{k-1} \binom{k-1}{n} f_k^{(k)}(x_0)[\omega h]^{k-1-n}[x_2 + \theta(x_1 - x_2)]^n. \end{aligned} \tag{26}$$

Moreover

$$\|x_2 + \theta(x_1 - x_2)\| \le 3\frac{\omega}{R} \le \frac{3\omega}{R_k}. \tag{27}$$

Taking into account the definition of R_k, we get

$$\begin{aligned} & \left\| \sum_{n=1}^{k-1} \binom{k-1}{n} f_k^{(k)}(x_0)[\omega h]^{k-1-n}[x_2 + \theta(x_1 - x_2)]^n \right\| \\ & \le \left\| f_k^{(k)}(x_0) \right\| \cdot \sum_{n=1}^{k-1} \binom{k-1}{n} \omega^{k-1-n}(3\omega)^n / R_k^n \\ & \le \left\| f_k^{(k)}(x_0) \right\| \cdot \omega^{k-1} \cdot 4^{k-1}/R_k \le 4^k (k-1)! \omega^{k-1} \alpha. \end{aligned} \tag{28}$$

Now, inserting (24)–(28) into (23) we obtain

$$\| f_k(s_1) - f_k(s_2) - \Lambda_{\omega h,k}(x_1 - x_2) \| \le 4^k \omega^{k-1} \left(\alpha + \frac{c\omega}{k!} \right) \cdot \|x_1 - x_2\|.$$

Hence

$$\begin{aligned} \mathrm{dist}_{\mathrm{H}}(\Phi_{\omega h}(x_1), \Phi_{\omega h}(x_2)) &\le \eta \cdot \sum_{k=1}^{p} \frac{1}{\omega^{k-1}} 4^k \omega^{k-1} (\alpha + c\omega)\|x_1 - x_2\| \\ &= \eta \sum_{k=1}^{p} 4^k (\alpha + c\omega)\|x_1 - x_2\| \\ &\le \frac{\eta \cdot 4^{p+1}}{3} (\alpha + c\omega)\|x_1 - x_2\| \le \frac{1}{2}\|x_1 - x_2\| \end{aligned}$$

(by virtue of (21) and assumption 2), which proves (22).

Now let us estimate $\|\Phi_{\omega h}(0)\|$. We have

$$\|\Phi_{\omega h}(0)\| = \inf\{\|z\| : z \in \Phi_{\omega h}(0)\} = \eta \sum_{k=1}^{p} \frac{1}{\omega^{k-1}} \| f_k(x_0) - \omega_k(x_0, \omega h)\|$$

$$\leq \eta \sum_{k=1}^{p} \left(\frac{1}{\omega^{k-1}} \| f_k(x_0)\| + \frac{1}{\omega_{k-1}} \frac{c\omega^{k+1}}{(k-1)!} \right)$$

$$\leq \eta \sum_{k=1}^{p} \left(\frac{1}{\omega^{k-1}} \| f_k(x_0)\| + \frac{c\omega^2}{(k-1)!} \right)$$

$$\leq \eta \cdot p \frac{\delta}{\omega^{p-1}} + p \cdot \eta \cdot c \cdot \omega^2. \tag{29}$$

It follows from assumption 1, that $\frac{\eta p \delta}{\omega^{p-1}} \leq \frac{\omega}{8R}$, since $\frac{\alpha}{d} \leq \frac{1}{4}$. From assumption 2) we obtain $p\eta c \cdot \omega^2 \leq \frac{\omega}{8R}$, and finally $\|\Phi_{\omega h}(0)\| \leq \frac{\omega}{4R}$.

From the above and from (22) we obtain that for the mapping $\Phi_{\omega h}(x)$ all the assumptions of contraction multimapping principle hold with $\theta = \frac{1}{2}$, $\varepsilon_1 = \frac{\omega}{4R}$ and hence there exist an element $\bar{x}(\omega) \in \Phi_{\omega h}(\bar{x}(\omega))$ or $F(x_0 + \omega h + \bar{x}(\omega)) = 0$ such that $\|\bar{x}(\omega)\| \leq \frac{\omega}{2R} \leq \frac{\omega}{2}$. It follows that $x_0 + \omega h + \bar{x}(\omega)$ is the solution to the Eq. 1. □

Remark 5 As in the regular case if we assign $\alpha = \omega^{\sigma}$, $\sigma > 0$ then we obtain

$$\|\bar{x}(\omega)\| = o(\omega).$$

Remark 6 (Sequential p-Regularity) Similarly, like in the regular case the assumptions of Theorem 3 can be weakened if we consider the sequence: $x_0 = 0,\ x_{n+1} \in x_n - \Lambda_{\omega h}^{-1}\left(f_1(x_0 + \omega h + x_n), \ldots, f_p(x_0 + \omega h + x_n)\right),\ n = 1, 2, \ldots,$ instead of the mapping $\Phi_{\omega h}(x)$. Moreover, instead of p-regularity condition of the mapping F at the point x_0 we can assume p-regularity of F at $\{x_n\}$ i.e. instead of operator Λ_h surjectivity we can require that $\mathrm{Im}\Lambda_h = W \subset Y$, where W is some closed subspace of Y and $F(x_0 + \omega h + x_n) \in W$, $n = 0, 1, \ldots$.

We conclude the discussion with the examples from the beginning of the hereby paper, which serve now to illustrate the basic idea of Theorem 3.

5 Applications and Examples

We continue consideration of Examples 1–5 and we verify for them assumptions of Theorem 3.

Example 1 As it was shown earlier the main assumption of Theorem 3, that is p-regularity condition of the mapping $F(u, \varepsilon)$, is fulfilled at the point $u_0 = 0$, $\varepsilon_0 = 0$. We calculate $\delta = \|F(0, 0)\| = 0$. It means that assumption 1 of Theorem 3 holds. Second assumption is true for $\nu > 0$ sufficiently small if

$$\omega \leq \frac{3}{2 \cdot 4^{p+2} \cdot c \cdot \eta} \quad \text{and } \omega < \frac{1}{2}\nu.$$

It means that there exists a solution of Eq. 6 of the following form

$$u^*(\omega) = u^*(\varepsilon) = \varepsilon(\bar{z}_1 u_1 + \bar{z}_2 u_2) + \bar{u}(\varepsilon),$$

where $\|\bar{u}(\varepsilon)\| = o(\varepsilon)$.

Example 2 In Section 2 it was shown that the mapping $F(u)$ is p-regular at the point $u_0 = 0$ if

$$\text{Rank } P_2 F''(0)[z] = \text{Rank } M(z) = l$$

for $z = z_1 u_1 + \ldots + z_l u_l \in \text{Ker}^2 P_2 F''_u(0)$. Here $l = 2$. Assumption 1 of Theorem 3 holds, because we have $\delta = \|F(0)\| = \varepsilon$, $\omega \sim \sqrt{\varepsilon}$ and ε is sufficiently small.

Assumption 2 we obtain taking $\omega < \min\left\{\frac{3}{2\cdot 4^{p+2}\cdot c\cdot\eta}, \frac{1}{2}\nu\right\}$. Hence there exists a solution to (9) of the form

$$u(y, \varepsilon) = \sqrt{\varepsilon}(\bar{z}_1 u_1 + \bar{z}_2 u_2) + \bar{u}(y, \varepsilon),$$

where $\|\bar{u}(y, \varepsilon)\| = o(\sqrt{\varepsilon})$.

Example 3 It was verified in Section 2 that the mapping $F(x)$ is 2-regular at the point $x_0 = 0$ along $h_1 = (1, 0)$, $h_2 = (0, 1)$. Moreover $\delta = \|F(0)\| = \frac{\varepsilon}{\sqrt{\pi}}$. Hence, for

$$\sqrt{\frac{4\varepsilon \cdot \eta \cdot d}{\sqrt{\pi} \cdot \alpha}} \leq \omega < \frac{1}{2}\nu$$

and for sufficiently small $\varepsilon > 0$ assumptions 1 and 2 of Theorem 3 hold. It means that there exist solutions $x(t, \varepsilon)$ to Eq. 3 of the following form

$$x_1(t, \varepsilon) = \frac{\sqrt{\varepsilon}}{\sqrt{\pi}} \cos t + \bar{x}_1(t, \varepsilon)$$

$$x_2(t, \varepsilon) = \frac{\sqrt{\varepsilon}}{\sqrt{\pi}} \sin t + \bar{x}_2(t, \varepsilon),$$

where $\|\bar{x}_{1,2}(t, \varepsilon)\| = o(\sqrt{\varepsilon})$.

Example 4 We have shown already that $F(x)$ is not p-regular mapping at the point x_0 along $h(t) = \sin t \in \text{Ker} F'(x_0) \cap \text{Ker}^2 P_2 F''(x_0)$.

Although assumption of Theorem 3 fails, the assumptions of Remark 6 are fulfilled because considered mapping is sequentially 2-regular. Indeed, let $W = \text{Im}\Lambda_{h,1}$, where $\Lambda_{h,1} = \ddot{(\cdot)} + 1(\cdot)$. Then $F(x_0 + \omega h + x_n) \in \text{Im} W$ for $x_1 = 0$, $x_{n+1} = x_n - \Lambda_{h,1}^{-1} F(x_0 + \omega h + x_n)$. Let us consider for instance x_2. We have

$$\ddot{x}_2 + x_2 + F(x_0 + \omega h) = \ddot{x}_2 + x_2 + \omega^2 \sin^2 t = 0.$$

Since $\int_0^{2\pi} \omega^2 \sin^3 \tau d\tau = 0$. This equation has a solution $x_2(t)$ such that $x_2(0) = x_2(2\pi) = 0$. Similarly we can calculate elements x_n, for $n > 2$.

Both assumptions of Theorem 3 are fulfilled trivially for $0 < \omega < 1/2\nu, \omega \leq \frac{3}{2\cdot 4^4\cdot c\cdot\eta}$, because $\delta = \|F(0)\| = 0$ and $\eta = \left\|\Lambda_{h,1}^{-1}\right\| > 0$. It means that if we substitute ω^σ, $\sigma > 0$ (see Remark 5) such that $\omega > 0$ for α in assumption 1 of Theorem 3, then there exists a solution to (10) of the following form

$$x^*(t) = \omega \cdot \sin t + \bar{x}(\omega, t), \qquad \text{where} \qquad \|\bar{x}(\omega, t)\| = o(\omega).$$

Example 5 Similarly like in the previous example, mapping $F(x)$ from (11) is sequentially 2-regular at the point $x_0 = 0$ along $h(t) = \sin t$. Assumptions 1 and 2 of Theorem 3 are satisfied because $\delta = \|F(0)\| = \varepsilon > 0$. If we take into account $\alpha = \omega^\sigma$, $\sigma > 0$, then for such $\delta = \varepsilon$ that $\eta\varepsilon \leq \frac{\omega^{2+\sigma}}{4d}$ we obtain that there exists a solution $x^*(t, \varepsilon)$ to the Eq. 11 of the following form

$$x^*(t, \varepsilon) = \sqrt{\varepsilon} \sin t + \bar{x}(t, \varepsilon),$$

where $\|\bar{x}(t, \varepsilon)\| = o(\sqrt{\varepsilon})$. In fact $x^*(t, \varepsilon) = \sqrt{\varepsilon} \sin t$ is a solution of (11).

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