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Axiomatizing a Minimal Discussive Logic

Abstract. In the paper we analyse the problem of axiomatizing the minimal variant of discussive logic denoted as D_0. Our aim is to give its axiomatization that would correspond to a known axiomatization of the original discussive logic D_2. The considered system is minimal in a class of discussive logics. It is defined similarly, as Jaśkowski's logic D_2 but with the help of the deontic normal logic **D**. Although we focus on the smallest discussive logic and its correspondence to D_2, we analyse to some extent also its formal aspects, in particular its behaviour with respect to rules that hold for classical logic. In the paper we propose a deductive system for the above recalled discussive logic. While formulating this system, we apply a method of Newton da Costa and Lech Dubikajtis—a modified version of Jerzy Kotas's method used to axiomatize D_2. Basically the difference manifests in the result—in the case of da Costa and Dubikajtis, the resulting axiomatization is pure modus ponens-style. In the case of D_0, we have to use some rules, but they are mostly needed to express some aspects of positive logic. D_0 understood as a set of theses is contained in D_2. Additionally, any non-trivial discussive logic expressed by means of Jaśkowski's model of discussion, applied to any regular modal logic of discussion, contains D_0.

Keywords: Discussive logics, The smallest discussive logic, Discussive operators, Accessibility relations, Modal logic, da Costa and Dubikajtis, Embedding.

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1. Introduction

Stanisław Jaśkowski proposed a model presenting an analysis of inconsistent theories within some consistent framework. A certain model of discussion was used since it permits intuitively present theories containing inconsistent statements. Moreover, acting on such an inconsistent basis no one would

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conclude from the information collected during the conversation, that everything follows; in other words, debaters can formulate pairwise contradictory theses, but none of the observers would agree that everything follows from such a discussion.[1] The role of an external observer in Jaśkowski's intuitive model consists in rendering a discussion assertion, which relies on preceding each thesis of the system with a proviso: "according to the view of one of the participants in the discussion" or "with a certain acceptable meaning of the words used". In Jaśkowski's motivation, such a role could be played by an impartial mediator who might understand this way the theses of particular participants in the discussion. So, from an intuitive point of view, the external observer should rather not be a discussant simultaneously. The so-called discussive connectives of conjunction and implication represent some aspects of communication acts holding between participants.

We will assume the reader's familiarity with the basic notion connected with normal modal logics.

2. Variants of Discussive Logics

Jaśkowski considered a discussion in which every one of two debaters can respond to each other. It can be treated as a simulation of the full accessibility relation determining the modal logic **S5**.[2] While defining the discussive logic D_2 you do not need all theses of **S5** (see [5, 18–20, 24]). On the other hand it is not the case that using any normal modal logic one obtains the very same D_2.

From the intuitive point of view Jaśkowski's logic could be seen as 'democratic' in the sense that everyone is allowed to formulate a statement and everyone can respond to any statement made by anyone else. However, as we know from everyday experience, the situation may be different. One can observe that in some cases not everyone is in position to react to each statement. Such situations can be connected, for example, with some charisma of particular debaters. Hence, using the language of Kripke-style semantics, one can say that participants of a discussion are usually connected by some accessibility relation that determines which persons are in a position to react to the statements of a given debater. In this context, one can ask

[1] Further considerations on this matter can be found e.g. in [12].

[2] Although Kripke semantics was not discovered at that time yet, Wajsberg's result on the connection of **S5** with the first-order logic was known to Jaśkowski—he was referring to this connection in [7, 8].

whether each accessibility relation is suitable to represent this aspect of discussion within this more general variant of Jaśkowski's model. The answer is obvious—not every accessibility relation. As an extremum one can indicate the empty relation that would not lead to any reactions. In the context of Jaśkowski's way of describing the current stage of a discussion by means of the point of view of the external observer who treats voices of a discussion as possible, it is also obvious that members of a discussive group who have the empty set of alternatives, i.e. as one could say, people whose opinions are not taken into account by others while formulating their own statements, are not included in the outcome of the discussion. Moreover, since we are interested in stating at the end what follows logically, we should be able to vary the considered point of view, so everyone who is meant to be a debater in the given discussion should be connected to some other participant. Semantically this means that we have to consider discussive groups in which everyone is connected with at least one member (the case of self-connection is not excluded). So seriality is the most general and weakest stipulation for groups of debaters meant to intuitively represent models used to semantically express the considered smallest variant of discussive logic.

In what follows we keep Jaśkowski's original understanding of discussive connectives of implication and conjunction. Jaśkowski's discussive implication $\rightarrow_{\mathrm{d}}$ is meant intuitively as saying: "if anyone states that p, then q" (see [7, p. 150, 1969], [8]), in the modal language: $\Diamond p \rightarrow q$. Nowadays, discussive conjunction added to discussive language later in 1949 ([9]) is usually treated as saying "p and someone said q" and in the modal language is translated as $p \wedge \Diamond q$. In the case of D_2, the possibility operator meant to give an interpretation of discussive connectives in the modal language, intuitively is used to express statements presented by some participant during the discussion. In our interpretation we would like to rely on the relation that 'connects' debaters. Possibility is used by Jaśkowski also to simulate an evaluation made by an 'impartial arbiter'. Hence, according to Jaśkowski, theses of the discussive system have to be preceded by the stipulation: "if a thesis is recorded in a discursive system, its intuitive sense ought to be interpreted so as if it were preceded by the symbol *Pos*" ([7, p. 149, 1969], [8]), where '*Pos*' was used to denote the possibility operator.

Preserving the described way of formulating discussive logic, one can define discussive logic by taking any normal modal logic or even any modal logic. Formulas of the discussive language $\mathrm{For}_{\mathrm{d}}$ become theses of some specific discussive system, if their translation into modal logic is a possible truth of the underlying modal system. In some cases, one can obtain even the empty set of theses, if a given modal logic does not have respective theses that could act as results of the translations used to define discussive logic. As an obvious example one can consider the logic **K** and try to define discussive logic on its basis. Since **K** has no thesis of the form $\Diamond A$, the resulting discussive logic would be the empty set. So, the necessary condition for a normal modal logic to be used to define a nontrivial discussive logic is to have at least one thesis of the form $\Diamond A$. But by monotonicity, such a logic contains the axiom (D): $\Diamond(p \to p)$. As one can easily see the same holds for regular modal logics. In particular, taking into account the standard semantics for regular logics, where frames with non-normal worlds are considered, one can observe that for any thesis $\Diamond A$ of the normal deontic logic **D**, $\Diamond A$ also belongs to **D2**[3]—the smallest regular logic containing the axiom (D). So, the intended discussive logic D_0 defined on the basis of the logic **D** (equivalently on the basis of **D2**) can be seen as the intersection of the family of all non-trivial discussive logics defined by means of normal or regular modal logics using Jaśkowski's translations. More precisely, D_0 is minimal in the following sense: consider any normal or regular modal logic $\boldsymbol{L}$ and define Jaśkowski's discussive logic obtained on the basis of $\boldsymbol{L}$ exactly in the same way as D_2 is defined on the basis of **S5** (of course, with some fixed translations for discussive connectives, in our case with right discussive conjunction and discussive implication). If one sticks only to non-empty resulting discussive logics (as we mentioned, starting with **K**, the resulting discussive logic would be just the trivial logic—the empty set), D_0 would be contained in every such discussive logic or in other words D_0 would be the intersection of all these discussive logics. In the language of D_0 we take $\neg, \vee, \wedge_{\mathrm{d}}, \to_{\mathrm{d}}, \leftrightarrow_{\mathrm{d}}$, however, if we would restrict the language, for example, to $\neg, \vee, \to_{\mathrm{d}}$, we would obtain a logic—let us call it—D_0^-. Thus D_0^- would be even a weaker than D_0 logic, but due to the reduction of the language.

From this follows that D_0 is really the smallest logic in the class of all non-empty discussive logics defined on the basis of regular logics (notice that the family of normal modal logics is contained in the family of all regular logics). But taking into account the fact that weaker modal logics can have the same modal theses as a given normal logic, one could suspect that the minimality of D_0 can be saved also for some bigger classes of modal logics. However, this question could be the matter of some further research. Summarising the introduction, we would like to stress that the postulated minimality refers to the very specific class of logics, obtainable by a natural generalisation of the model of discussion developed by Jaśkowski. By taking other possible

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[3]Notice that **D2** is a modal logic while D_2 is Jaśkowski's discussive logic.

explications of 'discussiveness' expressed possibly by a modified model of discussion, one could obtain other 'discussive-like' minimal logics.

3. Syntax

As usually for the case of discussive logic, to formally express the logic under consideration one can make a translation from discussive language to the modal one. Modal formulas are formed in the standard way from propositional letters: 'p', 'q', 'r', 's', 't', 'p_0', 'p_1', 'p_2', ...; truth-value operators: '$\neg$', '$\vee$', '$\wedge$', '$\rightarrow$', and '$\leftrightarrow$'; modal operators: the necessity and possibility operators '$\Box$' and '$\Diamond$'; and the brackets. Let $\mathrm{For_m}$ denote the set of all modal formulas and Greek letters φ, ψ, etc. range over $\mathrm{For_m}$.

The object language of discussive logic is built out of propositional letters, truth-value operators '$\neg$' and '$\vee$', discussive implication ($\rightarrow_\mathrm{d}$) and discussive conjunction ($\wedge_\mathrm{d}$) $\mathrm{For_d}$ denotes the set of all discussive formulas, while letters A, B, C, etc. range over $\mathrm{For_d}$.

Let us recall that in Jaśkowski's intuitive model of $\mathsf{D_2}$ every two debaters are in connection, while the antecedent of discussive implication is interpreted as saying: 'if *anyone* states that p'. The similar modal operation is applied to the whole formula, i.e. the possibility functor before the whole formula is meant as a kind of tool used by an external observer who acts as a judge. This external observer adjudicates the validity of a formula by referring to a given discussive group. Of course, since we are interested in setting the logical truth, we have to consider any discussive group treated as a model. Thus, formally, Jaśkowski's discussive logic $\mathsf{D_2}$ is definable by means of **S5** as follows:

$$\mathsf{D_2} := \{\, A \in \mathrm{For_d} : \Diamond \mathtt{i}_1(A) \in \mathbf{S5} \,\},$$

where $\mathtt{i}_1$ is a translation of the discussive language to the modal one, in particular $\mathtt{i}_1$ is a function from $\mathrm{For_d}$ to $\mathrm{For_m}$, where we stipulate:

1 $\mathtt{i}_1(a) = a$, for any propositional letter a,

2 for any $A, B \in \mathrm{For_d}$:

(a) $\mathtt{i}_1(\neg A) = \neg \mathtt{i}_1(A)$,
(b) $\mathtt{i}_1(A \vee B) = \mathtt{i}_1(A) \vee \mathtt{i}_1(B)$,
(c) $\mathtt{i}_1(A \wedge_\mathrm{d} B) = \mathtt{i}_1(A) \wedge \Diamond \mathtt{i}_1(B)$,
(d) $\mathtt{i}_1(A \rightarrow_\mathrm{d} B) = \Diamond \mathtt{i}_1(A) \rightarrow \mathtt{i}_1(B)$.

We continue an investigation on the system $\mathsf{D_0}$ given in [13] by the definition:

$$\mathsf{D_0} := \{\, A \in \mathrm{For_d} : \Diamond \mathtt{i}_1(A) \in \mathbf{D} \,\} \qquad (\mathrm{def}_{\mathsf{D_0}})$$

The set $\mathsf{D_0}$ is a logic:

FACT 1. ([13]) *The set* $\mathsf{D_0}$ *is closed under substitution and modus ponens with respect to* $\to_{\mathrm{d}}$.

4. Axiomatization of the Smallest Discussive Logic

As it was observed in [13], various classical theses fail to belong to $\mathsf{D_0}$. Also standard—for classical logic—inferences are not saved in the case of $\mathsf{D_0}$:

$$\frac{A \to_{\mathrm{d}} B \qquad B \to_{\mathrm{d}} C}{A \to_{\mathrm{d}} C} \qquad (\mathrm{Syl})$$

To see this it is enough to take $A := p$, $B := (p \vee \neg p) \wedge_{\mathrm{d}} p$ and $C := (p \vee \neg p) \wedge_{\mathrm{d}} ((p \vee \neg p) \wedge_{\mathrm{d}} p)$. These circumstances can be given as an explanation for the specific form of the given axiom system.

Let us recall a result of the adaption of Kotas' method [11] (used by him to determine an axiomatization of $\mathsf{D_2}$) that was used to axiomatize $\mathsf{D_0}$ (see [13]).

Let Φ be a set of modal formulas. The result of 'removing' $\Diamond$ from elements of Φ will be denoted by $\Diamond$-Φ, while let $\Box\Phi$[4] denote the set of all formulas resulting from adding $\Box$ before every element of Φ. Thus, we use the following notation:

$$\Diamond\text{-}\Phi := \{A \in \mathrm{For_m} : \Diamond A \in \Phi\} \qquad (1)$$

$$\Box\Phi := \{\Box A : A \in \Phi\} \qquad (2)$$

$$\Diamond\Phi := \{\Diamond A : A \in \Phi\} \qquad (3)$$

By definitions for any normal logic $\mathbf{S} \supseteq \mathbf{D}$:

$$\Box\mathbf{S} \subsetneq \mathbf{S} \subseteq \Diamond\text{-}\mathbf{S}$$

It is known that (see [24, p. 68]):

FACT 2.

$$\Diamond\text{-}\mathbf{D} = \mathbf{D} \qquad (4)$$

[4] The respective similar symbols **M-S5** and **LS5** for the case of **S5** were used by J. Kotas [11].

For the sake of legibility let us denote for any $A, B \in \mathrm{For_d}$ (resp. $\varphi, \psi \in \mathrm{For_m}$), the formula $\ulcorner \neg A \vee B \urcorner$ as $\ulcorner A \rightarrow_{\mathrm{c}} B \urcorner$ (resp. $\ulcorner \neg\varphi \vee \psi \urcorner$ as $\ulcorner \varphi \rightarrow_{\mathrm{c}} \psi \urcorner$).

Consider the following Frege-Łukasiewicz-Hilbert axiomatization of classical propositional logic:

$$p \rightarrow (q \rightarrow p) \tag{A1}$$
$$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)) \tag{A2}$$
$$p \wedge q \rightarrow p \tag{A3}$$
$$p \wedge q \rightarrow q \tag{A4}$$
$$p \rightarrow (q \rightarrow p \wedge q) \tag{A5}$$
$$p \rightarrow p \vee q \tag{A6}$$
$$q \rightarrow p \vee q \tag{A7}$$
$$(p \rightarrow q) \rightarrow ((r \rightarrow q) \rightarrow (p \vee r \rightarrow q)) \tag{A8}$$
$$(p \leftrightarrow q) \rightarrow (p \rightarrow q) \tag{A9}$$
$$(p \leftrightarrow q) \rightarrow (q \rightarrow p) \tag{A10}$$
$$(p \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow (p \leftrightarrow q)) \tag{A11}$$
$$(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p) \tag{A12}$$

As in the case of sets of formulas, we use a similar custom for the case of names of formulas resulting from preceding a given formula with $\Box$. In this way, for example, for formulas:

$$\Diamond p \leftrightarrow \neg\Box\neg p \tag{$\mathtt{df}\,\Diamond$}$$
$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \tag{$\mathtt{K}$}$$
$$\Box p \rightarrow \Diamond p \tag{$\mathtt{D}$}$$

we have respectively:

$$\Box(\Diamond p \leftrightarrow \neg\Box\neg p) \tag{$\Box\mathtt{df}\Diamond$}$$
$$\Box(\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)) \tag{$\Box\mathtt{K}$}$$
$$\Box(\Box p \rightarrow \Diamond p) \tag{$\Box\mathtt{D}$}$$

We put $\Omega := \{(\Box\mathrm{A}i) : 1 \leqslant i \leqslant 12\} \cup \{(\Box\mathtt{df}\Diamond), (\Box\mathtt{K}), (\Box\mathtt{D})\}$.

As it is known, every normal logic has ($\mathtt{K}^{\Diamond}$) as theses:

$$\Box(p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q) \tag{$\mathtt{K}^{\Diamond}$}$$

Let us recall the logic $\mathbf{D}^{\Vdash}$ ([13]) being a set of theses with respect to the consequence relation $\Vdash$ determined by the set Ω as axioms, the substitution rule and the following ones:

$$\frac{\Box\varphi}{\Box\Box\varphi} \tag{$\Box$nec}$$

$$\frac{\Box\varphi, \quad \Box(\varphi \to \psi)}{\Box\psi} \tag{$\Box$mp}$$

$$\frac{\varphi, \quad \Box(\varphi \to \psi)}{\psi} \tag{$\Box$mp$_{-}$}$$

$$\frac{\Diamond\varphi}{\varphi} \tag{pos$_{\Leftarrow}$}$$

We know that:

Lemma 3 ([13]) $\mathbf{D} = \mathbf{D}^{\Vdash}$.

We recall an axiomatization of D_2 given in [21]. It is indirectly an adaptation of an axiomatization given in [4] (which is further modified in [1]) and directly a correction of an axiomatization proposed in [3]. The axiomatization given in [4] refers to a variant of discussive logic with left discussive conjunction. Non-adequacy of this axiomatization with respect to the original D_2 was observed by Ciuciura [3].[5]

We recall the final axiomatization of D_2 given in [21].

(D_1) $A \to_{\mathrm{d}} (B \to_{\mathrm{d}} A)$

(D_2) $(A \to_{\mathrm{d}} (B \to_{\mathrm{d}} C)) \to_{\mathrm{d}} ((A \to_{\mathrm{d}} B) \to_{\mathrm{d}} (A \to_{\mathrm{d}} C))$

(D_3) $((A \to_{\mathrm{d}} B) \to_{\mathrm{d}} A) \to_{\mathrm{d}} A$

(D_4) $A \wedge_{\mathrm{d}} B \to_{\mathrm{d}} A$

(D_5) $A \wedge_{\mathrm{d}} B \to_{\mathrm{d}} B$

(D_6) $A \to_{\mathrm{d}} (B \to_{\mathrm{d}} (A \wedge_{\mathrm{d}} B))$

(D_7) $A \to_{\mathrm{d}} A \vee B$

(D_8) $B \to_{\mathrm{d}} A \vee B$

(D_9) $(A \to_{\mathrm{d}} C) \to_{\mathrm{d}} ((B \to_{\mathrm{d}} C) \to_{\mathrm{d}} (A \vee B \to_{\mathrm{d}} C))$

(D_{10}) $A \to_{\mathrm{d}} \neg\neg A$

(D_{11}) $\neg\neg A \to_{\mathrm{d}} A$

[5] The problem was that at some point Achtelik and others' axiomatization was treated as an axiomatization of D_2 with right discussive conjunction.

(D_{12}) $\neg(A \vee \neg A) \to_{\mathrm{d}} B$

(D_{13}) $\neg(A \vee B) \to_{\mathrm{d}} \neg(B \vee A)$

(D_{14}) $\neg(A \vee B) \to_{\mathrm{d}} (\neg A \wedge_{\mathrm{d}} \neg B)$

(D_{15}) $\neg(\neg\neg A \vee B) \to_{\mathrm{d}} \neg(A \vee B)$

(D_{16}) $(\neg(A \vee B) \to_{\mathrm{d}} C) \to_{\mathrm{d}} ((\neg A \to_{\mathrm{d}} B) \vee C)$

(D_{17}) $\neg((A \vee B) \vee C) \to_{\mathrm{d}} \neg(A \vee (B \vee C))$

(D_{18}) $\neg((A \to_{\mathrm{d}} B) \vee C) \to_{\mathrm{d}} (A \wedge_{\mathrm{d}} \neg(B \vee C))$

(D_{19}) $\neg((A \wedge_{\mathrm{d}} B) \vee C) \to_{\mathrm{d}} (B \to_{\mathrm{d}} \neg(A \vee C))$

(D_{20}) $\neg(\neg(A \vee B) \vee C) \to_{\mathrm{d}} (\neg(\neg A \vee C) \vee \neg(\neg B \vee C))$

(D_{21}) $\neg(\neg(A \to_{\mathrm{d}} B) \vee C) \to_{\mathrm{d}} (A \to_{\mathrm{d}} \neg(\neg B \vee C))$

(D_{22}) $\neg(\neg(A \wedge_{\mathrm{d}} B) \vee C) \to_{\mathrm{d}} (\neg(\neg A \vee C) \wedge_{\mathrm{d}} B)$

where the only rule of inference is modus ponens ($\mathrm{MP}^{\to_{\mathrm{d}}}$) for $\to_{\mathrm{d}}$. First observe that:

FACT 4. *(D_{11}) and (D_{15}) are dependent on the rest of the above axiomatization.*

PROOF. Indeed by (D_{16}), we have $(\neg(\neg A \vee A) \to_{\mathrm{d}} \neg(A \vee \neg A)) \to_{\mathrm{d}} ((\neg\neg A \to_{\mathrm{d}} A) \vee \neg(A \vee \neg A))$, hence by ($D_{13}$) and ($\mathrm{MP}^{\to_{\mathrm{d}}}$) we infer $((\neg\neg A \to_{\mathrm{d}} A) \vee \neg(A \vee \neg A))$. However, standardly by positive logic ((D_1), (D_2) and (D_9)) and (D_{13}) we have $B \vee \neg(A \vee \neg A) \to_{\mathrm{d}} B$. Using $\neg\neg A \to_{\mathrm{d}} A$ as B and again applying ($\mathrm{MP}^{\to_{\mathrm{d}}}$) we get ($D_{11}$).

For the case of (D_{15}), again, by (D_{16}), we have $(\neg((\neg\neg A \vee B) \vee \neg(A \vee B)) \to_{\mathrm{d}} \neg(A \vee \neg A)) \to_{\mathrm{d}} ((\neg(\neg\neg A \vee B) \to_{\mathrm{d}} \neg(A \vee B)) \vee \neg(A \vee \neg A))$. While by ($D_{13}$), ($D_{20}$) and transitivity of $\to_{\mathrm{d}}$,

$$\neg((\neg\neg A \vee B) \vee \neg(A \vee B)) \to_{\mathrm{d}} \neg(\neg A \vee (\neg\neg A \vee B)) \vee \neg(\neg B \vee (\neg\neg A \vee B)) \quad (5)$$

Standardly, as is the case of the usual associativity, using (D_{13}), (D_{17}) and transitivity of $\to_{\mathrm{d}}$ we have $\neg(\neg A \vee (\neg\neg A \vee B)) \to_{\mathrm{d}} \neg((\neg A \vee \neg\neg A) \vee B)$, hence by ($D_{14}$), ($D_4$), ($D_{12}$) and transitivity of $\to_{\mathrm{d}}$ we receive $\neg(\neg A \vee (\neg\neg A \vee B)) \to_{\mathrm{d}} C$. Quite similarly, using (D_{13}), (D_{17}), (D_{14}), (D_5), (D_{12}) and transitivity of $\to_{\mathrm{d}}$ we receive $\neg(\neg B \vee (\neg\neg A \vee B)) \to_{\mathrm{d}} C$. Thus, using the last two formulas, (D_9), (5), transitivity of $\to_{\mathrm{d}}$ and substitution for C, we get $\neg((\neg\neg A \vee B) \vee \neg(A \vee B)) \to_{\mathrm{d}} \neg(A \vee \neg A)$, hence using the initial formula of this proof and ($\mathrm{MP}^{\to_{\mathrm{d}}}$) we obtain $(\neg(\neg\neg A \vee B) \to_{\mathrm{d}} \neg(A \vee B)) \vee \neg(A \vee \neg A)$. Next, similarly as in the case of the proof of (D_{11}) we get the announced result. ∎

The proofs given above are minorly adapted versions of the proofs given in [1]. We will use the idea of the first proof also in the case of $\mathsf{D_0}$ for ($\mathrm{D}_0^{\vdash}3$) on page 11.

For any formula A we use the following shortcuts:

$$\top_A := A \rightarrow_{\mathrm{c}} A \quad (6)$$

$$\bot_A := \neg(A \rightarrow_{\mathrm{c}} A) \quad (7)$$

In particular, $\top_p$ and $\bot_p$ denote $p \rightarrow_{\mathrm{c}} p$ and $\neg(p \rightarrow_{\mathrm{c}} p)$, respectively.

Definition 5. Let $\mathsf{D}_0^{\Vdash}$ be the set of theses with respect to an inference system $\Vdash$ determined by the set of axiom schemes:

($\mathrm{D_0}1$) $A \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (B \rightarrow_{\mathrm{d}} A)$

($\mathrm{D_0}2$) $A \wedge_{\mathrm{d}} B \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} A$

($\mathrm{D_0}3$) $A \wedge_{\mathrm{d}} B \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} B)$

($\mathrm{D_0}4$) $A \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \vee B)$

($\mathrm{D_0}5$) $B \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \vee B)$

($\mathrm{D_0}6$) $\neg(A \vee B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(B \vee A)$

($\mathrm{D_0}7$) $\neg((\neg(A \vee B) \rightarrow_{\mathrm{d}} C) \vee D) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg(B \vee A) \rightarrow_{\mathrm{d}} C) \vee D)$

($\mathrm{D_0}8$) $\neg(A \vee B) \rightarrow_{\mathrm{d}} ((\top_p \wedge_{\mathrm{d}} \neg A) \wedge_{\mathrm{d}} \neg B)$

($\mathrm{D_0}9$) $(A \rightarrow_{\mathrm{d}} B) \wedge_{\mathrm{d}} A \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B$

($\mathrm{D_0}10$) $\neg(\neg(A \rightarrow_{\mathrm{d}} C) \vee \neg(B \rightarrow_{\mathrm{d}} C)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \vee B \rightarrow_{\mathrm{d}} C)$

($\mathrm{D_0}11$) $A \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg\neg A$

($\mathrm{D_0}12$) $A \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} A$

($\mathrm{D_0}13$) $\neg(A \vee \neg A) \rightarrow_{\mathrm{d}} B$

($\mathrm{D_0}14$) $\neg(\neg\neg A \vee B) \rightarrow_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg(A \vee B))$

($\mathrm{D_0}15$) $\neg(\neg(\neg\neg A \rightarrow_{\mathrm{d}} B) \vee C) \rightarrow_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg(\neg(A \rightarrow_{\mathrm{d}} B) \vee C))$

($\mathrm{D_0}16$) $\neg(\neg(A \rightarrow_{\mathrm{d}} B) \vee C) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(\neg\neg A \rightarrow_{\mathrm{d}} B) \vee C)$

($\mathrm{D_0}17$) $\neg((A \rightarrow_{\mathrm{d}} B) \vee C) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg\neg A \rightarrow_{\mathrm{d}} B) \vee C)$

($\mathrm{D_0}18$) $\neg(((\neg\bot_p \wedge_{\mathrm{d}} A) \vee (\neg B \rightarrow_{\mathrm{d}} C)) \vee D) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg(A \vee B) \rightarrow_{\mathrm{d}} C) \vee D)$

($\mathrm{D_0}19$) $(\neg(A \vee B) \rightarrow_{\mathrm{d}} C) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((\neg A \rightarrow_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} B)) \vee C)$

($\mathrm{D_0}20$) $\neg((A \vee B) \vee C) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee (B \vee C))$

($\mathrm{D_0}21$) $\neg((A \rightarrow_{\mathrm{d}} B) \vee C) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(B \vee C) \wedge_{\mathrm{d}} A)$

(D$_0$22) $\neg(\neg(A \vee B) \vee C) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\neg A \vee C) \vee \neg(\neg B \vee C))$

(D$_0$23) $\neg(\neg(A \rightarrow_{\mathrm{d}} B) \vee C) \rightarrow_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg((\neg B \wedge_{\mathrm{d}} A) \vee C))$

(D$_0$24) $\neg(\neg(A \wedge_{\mathrm{d}} B) \vee C) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\neg A \vee C) \wedge_{\mathrm{d}} B)$

(D$_0$25) $(A \vee \neg A \rightarrow_{\mathrm{d}} \bot_p) \rightarrow_{\mathrm{d}} \bot_p$

(D$_0$26) $(A \rightarrow_{\mathrm{d}} (B \vee C)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (B \vee (A \rightarrow_{\mathrm{d}} C))$

(D$_0$27) $\neg(A \vee (B \wedge_{\mathrm{d}} C)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(A \vee B) \vee \neg(A \vee \neg(C \rightarrow_{\mathrm{d}} \bot_p)))$

with the following rules:

$$\frac{\top_p \wedge_{\mathrm{d}} B}{B} \qquad (\wedge_{\mathrm{d}}{}_{\mathrm{r}}^{-})$$

$$\frac{A \rightarrow_{\mathrm{d}} B; \quad A}{B} \qquad (\mathrm{MP}^{\rightarrow_{\mathrm{d}}})$$

$$\frac{A \rightarrow_{\mathrm{d}} B \wedge_{\mathrm{d}} C}{(C \rightarrow_{\mathrm{d}} D) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \rightarrow_{\mathrm{d}} D)} \qquad (\mathrm{Tr}_1^{\mathrm{ax}})$$

$$\frac{B \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} C}{(A \rightarrow_{\mathrm{d}} B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \rightarrow_{\mathrm{d}} C)} \qquad (\mathrm{Tr}_2^{\mathrm{ax}})$$

$$\frac{B \rightarrow_{\mathrm{d}} C}{(A \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \rightarrow_{\mathrm{d}} C)} \qquad (\mathrm{Tr}_3^{\mathrm{ax}})$$

$$\frac{\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p}{A \wedge_{\mathrm{d}} C \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (B \wedge_{\mathrm{d}} C)} \qquad (\mathrm{Tr}_4^{\mathrm{ax}})$$

$$\frac{A}{B \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg A \vee \neg B)} \qquad (\mathrm{Add}^{\wedge_{\mathrm{c}}})$$

$$\frac{A \rightarrow_{\mathrm{d}} B}{\top_p \wedge_{\mathrm{d}} A \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B} \qquad (\mathrm{Mon})$$

The proofs presented below are partially adapted proofs given in [4, Part II], but some other had to be done independently, given the very weak tools available in the considered system.

First, let us observe that the following rules are provable:

$$\frac{A \wedge_{\mathrm{d}} B}{B} \qquad (\wedge_{\mathrm{d}}{}_{\mathrm{rg}}^{-})$$

$$\frac{A \rightarrow_{\mathrm{d}} B \wedge_{\mathrm{d}} C; \quad C \rightarrow_{\mathrm{d}} D}{A \rightarrow_{\mathrm{d}} D} \qquad (\mathrm{Tr}^{-})$$

$$\frac{A \rightarrow_{\mathrm{d}} B; \quad C \rightarrow_{\mathrm{d}} B}{A \vee C \rightarrow_{\mathrm{d}} B} \qquad (\mathrm{Syl}^{\vee})$$

One can see that ($\wedge_{\mathrm{d}}{}_{\mathrm{rg}}^{-}$) follows by (D$_0$3) and ($\wedge_{\mathrm{d}}{}_{\mathrm{r}}^{-}$).

To obtain the second inference, it is enough to apply ($\mathrm{Tr}_1^{\mathrm{ax}}$), ($\mathrm{MP}^{\to_{\mathrm{d}}}$) and ($\wedge_{\mathrm{d}}{}_{\mathrm{rg}}^{-}$).

For the case of ($\mathrm{Syl}^{\vee}$) we give the following inference:

1. $A \to_{\mathrm{d}} B$ Asm.
2. $C \to_{\mathrm{d}} B$ Asm.
3. $(C \to_{\mathrm{d}} B) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(A \to_{\mathrm{d}} B) \vee \neg(C \to_{\mathrm{d}} B))$ 1 and ($\mathrm{Add}^{\wedge_{\mathrm{c}}}$)
4. $\neg(\neg(A \to_{\mathrm{d}} B) \vee \neg(C \to_{\mathrm{d}} B)) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \vee C \to_{\mathrm{d}} B)$ ($\mathrm{D}_0 10$)
5. $(C \to_{\mathrm{d}} B) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \vee C \to_{\mathrm{d}} B)$ 3, 4 and Tr^{-}
6. $\top_p \wedge_{\mathrm{d}} (A \vee C \to_{\mathrm{d}} B)$ 2, 5 and ($\mathrm{MP}^{\to_{\mathrm{d}}}$)
7. $A \vee C \to_{\mathrm{d}} B$ 6 and ($\wedge_{\mathrm{d}}{}_{\mathrm{r}}^{-}$)

The below formula ($\mathrm{D}_0^{\vdash}1$) will be used in the very same Lemma 6, but also in lemmas 7, 8 and 10. The formula ($\mathrm{D}_0^{\vdash}2$) will be used to obtain ($\mathrm{D}_0^{\vdash}3$) but also applied in Lemma 9. The formulas ($\mathrm{D}_0^{\vdash}3$) and ($\mathrm{D}_0^{\vdash}4$) will be needed in the proof of Lemma 8.

LEMMA 6. *The following discussive formulas are theses of* $\mathsf{D}_0^{\Vdash}$.

($D_0^{\vdash}1$) $\bot_A \to_{\mathrm{d}} B$

($D_0^{\vdash}2$) $(\neg(\neg A \vee B) \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B)$

($D_0^{\vdash}3$) $\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} A$

($D_0^{\vdash}4$) $\neg(A \vee B) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg B$

PROOF. *Ad* ($\mathrm{D}_0^{\vdash}1$)—follows by ($\mathrm{D}_0 6$), ($\mathrm{D}_0 13$) and (Tr^{-}).

Ad ($\mathrm{D}_0^{\vdash}2$)

1. $(\neg(\neg A \vee B) \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B) \vee \bot_p)$ ($\mathrm{D}_0 19$)
2. $(\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B)$ ($\mathrm{D}_0 12$)
3. $\bot_p \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B)$ ($\mathrm{D}_0^{\vdash}1$)
4. $((\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B) \vee \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B)$ 2, 3 and ($\mathrm{Syl}^{\vee}$)
5. $(\neg(\neg A \vee B) \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B)$ 1, 4 and (Tr^{-})

Ad ($\mathrm{D}_0^{\vdash}3$)

1. $\neg(\neg A \vee A) \to_{\mathrm{d}} \bot_p$ ($\mathrm{D}_0^{\vdash}1$)
2. $(\neg(\neg A \vee A) \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} A)$ ($\mathrm{D}_0^{\vdash}2$)
3. $\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} A$ 1, 2, ($\mathrm{MP}^{\to_{\mathrm{d}}}$), and ($\wedge_{\mathrm{d}}{}_{\mathrm{r}}^{-}$)

Ad ($\mathrm{D}_0^{\vdash}4$)—follows by ($\mathrm{D}_0 8$), ($\mathrm{D}_0 12$) and (Tr^{-}). ■

The next lemma provides some inferable rules. In particular, ($\bot_{\mathrm{p}}^{-}$) will be used in the proof of Lemma 8, while ($\mathrm{Weak}^{\bot}$) and ($\mathrm{Weak}^{\to_{\mathrm{d}}}$) in the proof of Lemma 9.

LEMMA 7. *The following rules are inferable on the basis of* $\Vdash$*:*

$$\frac{A \to_{\mathrm{d}} \bot_p \vee B}{A \to_{\mathrm{d}} B} \qquad (\bot_{\mathrm{p}}^-)$$

$$\frac{A \to_{\mathrm{d}} \bot_B}{\top_p \wedge_{\mathrm{d}} A \to_{\mathrm{d}} \bot_B} \qquad (\mathrm{Weak}^{\bot})$$

$$\frac{A \to_{\mathrm{d}} B}{A \to_{\mathrm{d}} (C \to_{\mathrm{d}} B)} \qquad (\mathrm{Weak}^{\to_{\mathrm{d}}})$$

PROOF. Consider the following inferences.

1.	$A \to_{\mathrm{d}} \bot_p \vee B$	Asm.
2.	$(A \to_{\mathrm{d}} (\bot_p \vee B)) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\bot_p \vee (A \to_{\mathrm{d}} B))$	$(\mathrm{D_0}26)$
3.	$\top_p \wedge_{\mathrm{d}} (\bot_p \vee (A \to_{\mathrm{d}} B))$	1, 2 and $(\mathrm{MP}^{\to_{\mathrm{d}}})$
4.	$\bot_p \vee (A \to_{\mathrm{d}} B)$	3 and $(\wedge_{\mathrm{d}_{\mathrm{rg}}}^-)$
5.	$\bot_p \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \to_{\mathrm{d}} B)$	$(\mathrm{D}_0^{\vdash}1)$
6.	$(A \to_{\mathrm{d}} B) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \to_{\mathrm{d}} B)$	$(\mathrm{D_0}12)$
7.	$\bot_p \vee (A \to_{\mathrm{d}} B) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \to_{\mathrm{d}} B)$	1, 2 and $(\mathrm{Syl}^{\vee})$
8.	$(A \to_{\mathrm{d}} (\bot_p \vee B)) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \to_{\mathrm{d}} B)$	2, 7 and (Tr^-)
9.	$\top_p \wedge_{\mathrm{d}} (A \to_{\mathrm{d}} B)$	1, 8 and $(\mathrm{MP}^{\to_{\mathrm{d}}})$
10.	$A \to_{\mathrm{d}} B$	9 and $(\wedge_{\mathrm{d}_{\mathrm{rg}}}^-)$

1.	$A \to_{\mathrm{d}} \bot_B$	Asm.
2.	$\top_p \wedge_{\mathrm{d}} A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \bot_B$	1 and (Mon)
3.	$\bot_B \to_{\mathrm{d}} \bot_B$	$(\mathrm{D}_0^{\vdash}1)$
4.	$\top_p \wedge_{\mathrm{d}} A \to_{\mathrm{d}} \bot_B$	2, 3 and (Tr^-)

1.	$A \to_{\mathrm{d}} B$	Asm.
2.	$B \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (C \to_{\mathrm{d}} B)$	$(\mathrm{D_0}1)$
3.	$(A \to_{\mathrm{d}} B) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \to_{\mathrm{d}} (C \to_{\mathrm{d}} B))$	2 and $(\mathrm{Tr}_2^{\mathrm{ax}})$
4.	$\top_p \wedge_{\mathrm{d}} (A \to_{\mathrm{d}} (C \to_{\mathrm{d}} B))$	1, 3 and $(\mathrm{MP}^{\to_{\mathrm{d}}})$
5.	$A \to_{\mathrm{d}} (C \to_{\mathrm{d}} B)$	4 and $(\wedge_{\mathrm{d}_{\mathrm{r}}}^-)$

■

The provability of $(\mathrm{D}_0^{\vdash}14)$–$(\mathrm{D}_0^{\vdash}16)$, $(\mathrm{A1}^{\mathrm{tr}})$–$(\mathrm{A12}^{\mathrm{tr}})$ will be needed for the proof of Theorem 13. Formulas $(\mathrm{D}_0^{\vdash}8)$–$(\mathrm{D}_0^{\vdash}13)$, $(\mathrm{D}_0^{\vdash}19)$–$(\mathrm{D}_0^{\vdash}21)$, $(\mathrm{D}_0^{\vdash}24)$–$(\mathrm{D}_0^{\vdash}26)$ are used for the proofs of subsequent formulas in the below lemma, while $(\mathrm{D}_0^{\vdash}6)$, $(\mathrm{D}_0^{\vdash}7)$, $(\mathrm{D}_0^{\vdash}17)$ will be used for other formulas within this lemma but also in Lemma 10. Formulas $(\mathrm{D}_0^{\vdash}22)$, $(\mathrm{D}_0^{\vdash}23)$ and $(\mathrm{D}_0^{\vdash}27)$ will be used in Lemma 9. Finally, $(\mathrm{D}_0^{\vdash}18)$ and $(\mathrm{D}_0^{\vdash}28)$ will be needed in Lemma 10 while $(\mathrm{D}_0^{\vdash}5)$ will be used in Lemma 11.

LEMMA 8. *The following discussive formulas are theses of* $\mathsf{D}_0^{\Vdash}$*:*

$(D_0^{\vdash}5)$ $\neg(\neg A \vee A) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}6)$ $\neg(A \vee B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg A$

$(D_0^{\vdash}7)$ $\neg(A \vee (B \vee C)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((A \vee B) \vee C)$

$(D_0^{\vdash}8)$ $\neg(\neg\neg A \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}9)$ $\neg(\neg\neg B \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}10)$ $\neg(\neg C \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}11)$ $\neg(\neg(\neg B \vee C) \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}12)$ $\neg(\neg A \vee (C \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C)))) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}13)$ $\neg(\neg B \vee (C \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C)))) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}14)$ $\neg(\neg(\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}15)$ $\neg\neg(\neg(\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg\neg A \rightarrow_{\mathrm{d}} \bot_p)) \vee$
$\vee \neg(\neg\neg(\neg\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}16)$ $\neg(\neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg(\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg B \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}17)$ $\neg(\neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee (A \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}18)$ $\neg(\neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$

$(D_0^{\vdash}19)$ $(A \rightarrow_{\mathrm{d}} B) \vee C \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \rightarrow_{\mathrm{d}} B \vee C)$

$(D_0^{\vdash}20)$ $(\neg(A \vee \bot_p) \rightarrow_{\mathrm{d}} C) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee C)$

$(D_0^{\vdash}21)$ $\neg A \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_p)$

$(D_0^{\vdash}22)$ $\neg(\neg A \rightarrow_{\mathrm{d}} \bot_p) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg A)$

$(D_0^{\vdash}23)$ $\neg(A \rightarrow_{\mathrm{d}} \bot_p) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} A)$

$(D_0^{\vdash}24)$ $A \vee \bot_p \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} A$

$(D_0^{\vdash}25)$ $(\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B))$

$(D_0^{\vdash}26)$ $\neg B \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(\neg A \vee A) \vee B)$

$(D_0^{\vdash}27)$ $\neg B \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg A \vee \neg(\neg A \vee B))$

$(D_0^{\vdash}28)$ $\neg(A \vee \neg B) \wedge_{\mathrm{d}} C \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (B \wedge_{\mathrm{d}} C)$.

and

$(A1^{tr})$ $\neg(\neg A \vee (\neg B \vee A)) \rightarrow_{\mathrm{d}} \bot_p$

$(A2^{tr})$ $\neg(\neg(\neg A \vee (\neg B \vee C)) \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \rightarrow_{\mathrm{d}} \bot_p$

$(A3^{tr})$ $\neg(\neg\neg(\neg A \vee \neg B) \vee A) \rightarrow_{\mathrm{d}} \bot_p$

$(A4^{tr})$ $\neg(\neg\neg(\neg A \vee \neg B) \vee B) \rightarrow_{\mathrm{d}} \bot_p$

$(A5^{tr})$ $\neg(\neg A \vee (\neg B \vee \neg(\neg A \vee \neg B))) \rightarrow_{\mathrm{d}} \bot_p$

$(A6^{tr})$ $\neg(\neg A \vee (A \vee B)) \rightarrow_{\mathrm{d}} \bot_p$

$(A7^{tr})$ $\neg(\neg B \vee (A \vee B)) \rightarrow_{\mathrm{d}} \bot_p$

$(A8^{tr})$ $\neg(\neg(\neg A \vee C) \vee (\neg(\neg B \vee C) \vee (\neg(A \vee B) \vee C))) \rightarrow_{\mathrm{d}} \bot_p$

$(A9^{tr})$ $\neg(\neg\neg(\neg(\neg A \vee B) \vee \neg(\neg B \vee A)) \vee (\neg A \vee B)) \rightarrow_{\mathrm{d}} \bot_p$

$(A10^{tr})$ $\neg(\neg\neg(\neg(\neg A \vee B) \vee \neg(\neg B \vee A)) \vee (\neg B \vee A)) \rightarrow_{\mathrm{d}} \bot_p$

$(A11^{tr})$ $\neg(\neg(\neg A \vee B) \vee (\neg(\neg B \vee A) \vee \neg(\neg(\neg A \vee B) \vee \neg(\neg B \vee A)))) \rightarrow_{\mathrm{d}} \bot_p$

$(A12^{tr})$ $\neg(\neg(\neg\neg A \vee \neg B) \vee (\neg B \vee A)) \rightarrow_{\mathrm{d}} \bot_p$

PROOF. *Ad* ($\mathrm{D}_0^{\vdash}5$)—follows by ($\mathrm{D}_0 6$), ($\mathrm{D}_0 13$) and (Tr^-).

Ad ($\mathrm{D}_0^{\vdash}6$)—follows by ($\mathrm{D}_0 6$), ($\mathrm{D}_0^{\vdash}4$) and (Tr^-).

Ad ($\mathrm{D}_0^{\vdash}7$)—follows standardly by ($\mathrm{D}_0 6$), ($\mathrm{D}_0 20$), ($\mathrm{D}_0 6$), ($\mathrm{D}_0 20$), ($\mathrm{D}_0 6$), and (Tr^-).

Ad ($\mathrm{A1}^{\mathrm{tr}}$).

1. $\neg(\neg A \vee (\neg B \vee A)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee \neg B) \vee A)$ — Ax. ($\mathrm{D}_0^{\vdash}7$)
2. $\neg((\neg A \vee \neg B) \vee A) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee (\neg A \vee \neg B))$ — Ax. ($\mathrm{D}_0 6$)
3. $\neg(A \vee (\neg A \vee \neg B)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((A \vee \neg A) \vee \neg B))$ — Ax. ($\mathrm{D}_0^{\vdash}7$)
4. $\neg(\neg A \vee (\neg B \vee A)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((A \vee \neg A) \vee \neg B))$ — 1, 2, 3 and (Tr^-)
5 $\neg((A \vee \neg A) \vee \neg B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee \neg A)$ — ($\mathrm{D}_0^{\vdash}6$)
6. $\neg(A \vee \neg A) \rightarrow_{\mathrm{d}} \bot_p$ — ($\mathrm{D}_0 13$)
7. $\neg(\neg A \vee (\neg B \vee A)) \rightarrow_{\mathrm{d}} \bot_p$ — 4–6 and (Tr^-)

Ad ($\mathrm{D}_0^{\vdash}8$).

1. $\neg(\neg\neg A \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg\neg A \vee \neg(\neg A \vee B)) \vee (\neg A \vee C))$ — Ax. ($\mathrm{D}_0^{\vdash}7$)
2. $\neg((\neg\neg A \vee \neg(\neg A \vee B)) \vee (\neg A \vee C)) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(((\neg\neg A \vee \neg(\neg A \vee B)) \vee \neg A) \vee C)$ — Ax. ($\mathrm{D}_0^{\vdash}7$)
3. $\neg(((\neg\neg A \vee \neg(\neg A \vee B)) \vee \neg A) \vee C) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg\neg A \vee \neg(\neg A \vee B)) \vee \neg A)$ — ($\mathrm{D}_0^{\vdash}6$)
4. $\neg((\neg\neg A \vee \neg(\neg A \vee B)) \vee \neg A) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg A \vee (\neg\neg A \vee \neg(\neg A \vee B)))$ — ($\mathrm{D}_0 6$)
5. $\neg(\neg A \vee (\neg\neg A \vee \neg(\neg A \vee B))) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee \neg\neg A) \vee \neg(\neg A \vee B))$ — ($\mathrm{D}_0^{\vdash}7$)
6. $\neg((\neg A \vee \neg\neg A) \vee \neg(\neg A \vee B)) \rightarrow_{\mathrm{d}} \wedge_{\mathrm{d}} \neg(\neg A \vee \neg\neg A)$ — ($\mathrm{D}_0^{\vdash}6$)
7. $\neg(\neg A \vee \neg\neg A) \rightarrow_{\mathrm{d}} \bot_p$ — ($\mathrm{D}_0 13$)
8. $\neg(\neg\neg A \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \rightarrow_{\mathrm{d}} \bot_p$ — 1–7 and (Tr^-)

Ad ($\mathrm{D}_0^{\vdash}9$).

1. $\neg(\neg\neg B \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \to_{\mathrm{d}}$
$\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(B \vee (\neg(\neg A \vee B) \vee (\neg A \vee C)))$ ($\mathrm{D}_0 14$)
2. $\neg(B \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \to_{\mathrm{d}}$
$\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((B \vee \neg(\neg A \vee B)) \vee (\neg A \vee C))$ ($\mathrm{D}_0^{\vdash}7$)
3. $\neg((B \vee \neg(\neg A \vee B)) \vee (\neg A \vee C)) \to_{\mathrm{d}}$
$\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(((B \vee \neg(\neg A \vee B)) \vee \neg A) \vee C)$ ($\mathrm{D}_0^{\vdash}7$)
4. $\neg(((B \vee \neg(\neg A \vee B)) \vee \neg A) \vee C) \to_{\mathrm{d}}$
$\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((B \vee \neg(\neg A \vee B)) \vee \neg A)$ ($\mathrm{D}_0^{\vdash}6$)
5. $\neg((B \vee \neg(\neg A \vee B)) \vee \neg A) \to_{\mathrm{d}}$
$\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg A \vee (B \vee \neg(\neg A \vee B)))$ ($\mathrm{D}_0 6$)
6. $\neg(\neg A \vee (B \vee \neg(\neg A \vee B))) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee B) \vee \neg(\neg A \vee B))$ ($\mathrm{D}_0^{\vdash}7$)
7. $\neg((\neg A \vee B) \vee \neg(\neg A \vee B)) \to_{\mathrm{d}} \bot_p$ ($\mathrm{D}_0 13$)
8. $\neg(\neg\neg B \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \to_{\mathrm{d}} \bot_p$ 1–7 and (Tr^-)

Ad ($\mathrm{D}_0^{\vdash}10$).

1. $\neg(\neg C \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \to_{\mathrm{d}}$
$\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg C \vee \neg(\neg A \vee B)) \vee (\neg A \vee C))$ ($\mathrm{D}_0^{\vdash}7$)
2. $\neg((\neg C \vee \neg(\neg A \vee B)) \vee (\neg A \vee C)) \to_{\mathrm{d}}$
$\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(((\neg C \vee \neg(\neg A \vee B)) \vee \neg A) \vee C))$ ($\mathrm{D}_0^{\vdash}7$)
3. $\neg(((\neg C \vee \neg(\neg A \vee B)) \vee \neg A) \vee C) \to_{\mathrm{d}}$
$\top_p \wedge_{\mathrm{d}} \neg(C \vee ((\neg C \vee \neg(\neg A \vee B)) \vee \neg A))$ ($\mathrm{D}_0 6$)
4. $\neg(C \vee ((\neg C \vee \neg(\neg A \vee B)) \vee \neg A)) \to_{\mathrm{d}}$
$\top_p \wedge_{\mathrm{d}} \neg((C \vee (\neg C \vee \neg(\neg A \vee B))) \vee \neg A)$ ($\mathrm{D}_0^{\vdash}7$)
5. $\neg((C \vee (\neg C \vee \neg(\neg A \vee B))) \vee \neg A) \to_{\mathrm{d}}$
$\top_p \wedge_{\mathrm{d}} \neg(C \vee (\neg C \vee \neg(\neg A \vee B)))$ ($\mathrm{D}_0^{\vdash}6$)
6. $\neg(C \vee (\neg C \vee \neg(\neg A \vee B))) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((C \vee \neg C) \vee \neg(\neg A \vee B))$ ($\mathrm{D}_0^{\vdash}7$)
7. $\neg((C \vee \neg C) \vee \neg(\neg A \vee B)) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(C \vee \neg C)$ ($\mathrm{D}_0^{\vdash}6$)
8. $\neg(C \vee \neg C) \to_{\mathrm{d}} \bot_p$ ($\mathrm{D}_0 13$)
9. $\neg(\neg C \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \to_{\mathrm{d}} \bot_p$ 1–8 and (Tr^-)

Ad ($\mathrm{D}_0^{\vdash}11$).

1. $\neg(\neg(\neg B \vee C) \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \to_{\mathrm{d}}$
$\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\neg\neg B \vee (\neg(\neg A \vee B) \vee (\neg A \vee C)))\vee$
$\vee\neg(\neg C \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))))$ Ax. ($\mathrm{D}_0 22$)
2. $(\neg(\neg\neg B \vee (\neg(\neg A \vee B) \vee (\neg A \vee C)))\vee$
$\vee\neg(\neg C \vee (\neg(\neg A \vee B) \vee (\neg A \vee C)))) \to_{\mathrm{d}} \bot_p$ ($\mathrm{D}_0^{\vdash}9$), ($\mathrm{D}_0^{\vdash}10$) and ($\mathrm{Syl}^{\vee}$)
3. $\neg(\neg(\neg B \vee C) \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \to_{\mathrm{d}} \bot_p$ 1–2 and (Tr^-)

Ad (A2$^{\mathrm{tr}}$).

1. $\neg(\neg(\neg A \vee (\neg B \vee C)) \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\neg\neg A \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \vee$
$\vee \neg(\neg(\neg B \vee C) \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))))$ (D$_0$22)
2. $(\neg(\neg\neg A \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \vee$
$\vee \neg(\neg(\neg B \vee C) \vee (\neg(\neg A \vee B) \vee (\neg A \vee C)))) \rightarrow_{\mathrm{d}} \bot_p$
(D$_0^{\vdash}$8), (D$_0^{\vdash}$11) and (Syl$^{\vee}$)
3. $\neg(\neg(\neg A \vee (\neg B \vee C)) \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))) \rightarrow_{\mathrm{d}} \bot_p$ 1–2 and (Tr$^-$)

Ad (A3$^{\mathrm{tr}}$).

1. $\neg(\neg\neg(\neg A \vee \neg B) \vee A) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee \neg B) \vee A)$ (D$_0$14)
2. $\neg((\neg A \vee \neg B) \vee A)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee (\neg A \vee \neg B)))$ (D$_0$6)
3. $\neg(A \vee (\neg A \vee \neg B)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((A \vee \neg A) \vee \neg B)$ (D$_0^{\vdash}$7)
4. $\neg((A \vee \neg A) \vee \neg B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee \neg A)$ (D$_0^{\vdash}$6)
5. $\neg(A \vee \neg A) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0$13)
6. $\neg(\neg\neg(\neg A \vee \neg B) \vee A) \rightarrow_{\mathrm{d}} \bot_p$ 1–5 and (Tr$^-$)

Ad (A4$^{\mathrm{tr}}$).

1. $\neg(\neg\neg(\neg A \vee \neg B) \vee B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee \neg B) \vee B)$ (D$_0$14)
2. $\neg((\neg A \vee \neg B) \vee B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee (\neg B \vee B))$ (D$_0$20)
3. $\neg(A \vee (\neg B \vee B)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg B \vee B)$ (D$_0^{\vdash}$4)
4. $\neg(\neg B \vee B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(B \vee \neg B)$ (D$_0$6)
5. $\neg(B \vee \neg B) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0$13)
6. $\neg(\neg\neg(\neg A \vee \neg B) \vee B) \rightarrow_{\mathrm{d}} \bot_p$ 1–5 and (Tr$^-$)

Ad (A5$^{\mathrm{tr}}$).

1. $\neg(\neg A \vee (\neg B \vee \neg(\neg A \vee \neg B))) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee \neg B) \vee \neg(\neg A \vee \neg B))$
(D$_0^{\vdash}$7)
2. $\neg((\neg A \vee \neg B) \vee \neg(\neg A \vee \neg B))) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0$13)
3. $\neg(\neg A \vee (\neg B \vee \neg(\neg A \vee \neg B))) \rightarrow_{\mathrm{d}} \bot_p$ 1–2 and (Tr$^-$)

Ad (A6$^{\mathrm{tr}}$).

1. $\neg(\neg A \vee (A \vee B)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee A) \vee B))$ (D$_0^{\vdash}$7)
2. $\neg((\neg A \vee A) \vee B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg A \vee A)$ (D$_0^{\vdash}$6)
3. $\neg(\neg A \vee A) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee \neg A)$ (D$_0$6)
4. $\neg(A \vee \neg A) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0$13)
5. $\neg(\neg A \vee (A \vee B)) \rightarrow_{\mathrm{d}} \bot_p$ 1–4 and (Tr$^-$)

Ad (A7$^{\mathrm{tr}}$).

1. $\neg(\neg B \vee (A \vee B)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((A \vee B) \vee \neg B)$ (D$_0$6)
2. $\neg((A \vee B) \vee \neg B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee (B \vee \neg B))$ (D$_0$20)
3. $\neg(A \vee (B \vee \neg B)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(B \vee \neg B)$ (D$_0^{\vdash}$4)
4. $\neg(B \vee \neg B) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0$13)
5. $\neg(\neg B \vee (A \vee B)) \rightarrow_{\mathrm{d}} \bot_p$ 1–4 and (Tr$^-$)

Ad (D$_0^{\vdash}$12).

1. $\neg(\neg A \vee (C \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C))))$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee C) \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C)))$ (D$_0^{\vdash}$7)
2. $\neg((\neg A \vee C) \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C))) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(((\neg A \vee C) \vee \neg(\neg A \vee C)) \vee \neg(\neg B \vee C))$ (D$_0^{\vdash}$7)
3. $\neg(((\neg A \vee C) \vee \neg(\neg A \vee C)) \vee \neg(\neg B \vee C)) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee C) \vee \neg(\neg A \vee C))$ (D$_0^{\vdash}$6)
4. $\neg((\neg A \vee C) \vee \neg(\neg A \vee C)) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0$13)
5. $\neg(\neg A \vee (C \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C)))) \rightarrow_{\mathrm{d}} \bot_p$ 1–4 and (Tr$^-$)

Ad (D$_0^{\vdash}$13)—similarly as above: 2 × (D$_0^{\vdash}$7), (D$_0$6), (D$_0^{\vdash}$7), (D$_0^{\vdash}$6), (D$_0$6), (D$_0$13) and (Tr$^-$).
Ad (A8$^{\mathrm{tr}}$).

1. $\neg(\neg(\neg A \vee C) \vee (\neg(\neg B \vee C) \vee (\neg(A \vee B) \vee C))) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg(\neg A \vee C) \vee \neg(\neg B \vee C)) \vee (\neg(A \vee B) \vee C))$ (D$_0^{\vdash}$7)
2. $\neg((\neg(\neg A \vee C) \vee \neg(\neg B \vee C)) \vee (\neg(A \vee B) \vee C)) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg(A \vee B) \vee C) \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C)))$ (D$_0$6)
3. $\neg((\neg(A \vee B) \vee C) \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C))) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(A \vee B) \vee (C \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C))))$ (D$_0$20)
4. $\neg(\neg(A \vee B) \vee (C \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C)))) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg A \vee (C \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C))))\vee$
   $\vee\neg(\neg B \vee (C \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C))))$ (D$_0$22)
5. $\neg(\neg A \vee (C \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C))))\vee$
   $\vee\neg(\neg B \vee (C \vee (\neg(\neg A \vee C) \vee \neg(\neg B \vee C)))) \rightarrow_{\mathrm{d}} \bot_p$
   (D$_0^{\vdash}$12), (D$_0^{\vdash}$13) and (Syl$^{\vee}$)
6. $\neg(\neg(\neg A \vee C) \vee (\neg(\neg B \vee C) \vee (\neg(A \vee B) \vee C))) \rightarrow_{\mathrm{d}} \bot_p$ 1–5 and (Tr$^-$)

(A9$^{\mathrm{tr}}$), (A10$^{\mathrm{tr}}$) and (A11$^{\mathrm{tr}}$) are special cases respectively of (A3$^{\mathrm{tr}}$), (A4$^{\mathrm{tr}}$) and (A5$^{\mathrm{tr}}$).
Ad (A12$^{\mathrm{tr}}$).

1. $\neg(\neg(\neg\neg A \vee \neg B) \vee (\neg B \vee A)) \rightarrow_{\mathrm{d}}$
   $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\neg\neg\neg A \vee (\neg B \vee A)) \vee \neg(\neg\neg B \vee (\neg B \vee A)))$ (D$_0$22)

2. $\neg(\neg\neg\neg A \vee (\neg B \vee A)) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg A \vee (\neg B \vee A))$ (D$_0$14)
3. $\neg(\neg A \vee (\neg B \vee A)) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee \neg B) \vee A))$ (D$_0^{\vdash}$7)
4. $\neg((\neg A \vee \neg B) \vee A) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee (\neg A \vee \neg B))$ (D$_0$6)
5. $\neg(A \vee (\neg A \vee \neg B)) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((A \vee \neg A) \vee \neg B)$ (D$_0^{\vdash}$7)
6. $\neg((A \vee \neg A) \vee \neg B) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee \neg A)$ (D$_0^{\vdash}$6)
7. $\neg(A \vee \neg A) \to_{\mathrm{d}} \bot_p$ (D$_0$13)
8. $\neg(\neg\neg\neg A \vee (\neg B \vee A)) \to_{\mathrm{d}} \bot_p$ 2–7 and (Tr$^-$)
9. $\neg(\neg\neg B \vee (\neg B \vee A))) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(B \vee (\neg B \vee A))$ (D$_0$14)
10. $\neg(B \vee (\neg B \vee A))) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((B \vee \neg B) \vee A))$ (D$_0^{\vdash}$7)
11. $\neg((B \vee \neg B) \vee \neg A) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(B \vee \neg B)$ (D$_0^{\vdash}$6)
12. $\neg(B \vee \neg B) \to_{\mathrm{d}} \bot_p$ (D$_0$13)
13. $\neg(\neg\neg B \vee (\neg B \vee A))) \to_{\mathrm{d}} \bot_p$ 9–12 and (Tr$^-$)
14. $\neg(\neg\neg\neg A \vee (\neg B \vee A)) \vee \neg(\neg\neg B \vee (\neg B \vee A)) \to_{\mathrm{d}} \bot_p$ 8, 13 and (Syl$^{\vee}$)
15. $\neg(\neg(\neg\neg A \vee \neg B) \vee (\neg B \vee A)) \to_{\mathrm{d}} \bot_p$ 1, 14 and (Tr$^-$)

Ad (D$_0^{\vdash}$14).

1. $\neg(\neg(\neg A \to_{\mathrm{d}} \bot_p) \vee \neg(A \to_{\mathrm{d}} \bot_p)) \to_{\mathrm{d}}$
   $\to_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg(\neg(A \to_{\mathrm{d}} \bot_p) \vee \neg(\neg A \to_{\mathrm{d}} \bot_p)))$ (D$_0$6)
2. $\neg(\neg(A \to_{\mathrm{d}} \bot_p) \vee \neg(\neg A \to_{\mathrm{d}} \bot_p)) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (A \vee \neg A \to_{\mathrm{d}} \bot_p)$ (D$_0$10)
3. $(A \vee \neg A \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \bot_p$ (D$_0$25)
4. $\neg(\neg(\neg A \to_{\mathrm{d}} \bot_p) \vee \neg(A \to_{\mathrm{d}} \bot_p)) \to_{\mathrm{d}} \bot_p$ 1–3 and (Tr$^-$)

Ad (D$_0^{\vdash}$15).

1. $\neg\neg(\neg(\neg\neg(A \to_{\mathrm{d}} \bot_p) \vee \neg(\neg\neg A \to_{\mathrm{d}} \bot_p)) \vee$
   $\vee \neg(\neg\neg(\neg\neg A \to_{\mathrm{d}} \bot_p) \vee \neg(A \to_{\mathrm{d}} \bot_p))) \to_{\mathrm{d}}$
   $\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\neg\neg(A \to_{\mathrm{d}} \bot_p) \vee \neg(\neg\neg A \to_{\mathrm{d}} \bot_p)) \vee$
   $\vee \neg(\neg\neg(\neg\neg A \to_{\mathrm{d}} \bot_p) \vee \neg(A \to_{\mathrm{d}} \bot_p)))$ (D$_0^{\vdash}$3)
2. $\neg(\neg\neg(A \to_{\mathrm{d}} \bot_p) \vee \neg(\neg\neg A \to_{\mathrm{d}} \bot_p)) \to_{\mathrm{d}}$
   $\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((A \to_{\mathrm{d}} \bot_p) \vee \neg(\neg\neg A \to_{\mathrm{d}} \bot_p))$ (D$_0$14)
3. $\neg((A \to_{\mathrm{d}} \bot_p) \vee \neg(\neg\neg A \to_{\mathrm{d}} \bot_p)) \to_{\mathrm{d}}$
   $\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(\neg\neg A \to_{\mathrm{d}} \bot_p) \vee (A \to_{\mathrm{d}} \bot_p))$ (D$_0$6)
4. $\neg(\neg(\neg\neg A \to_{\mathrm{d}} \bot_p) \vee (A \to_{\mathrm{d}} \bot_p)) \to_{\mathrm{d}}$
   $\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(A \to_{\mathrm{d}} \bot_p) \vee (A \to_{\mathrm{d}} \bot_p))$ (D$_0$15)
5. $\neg(\neg\neg(A \to_{\mathrm{d}} \bot_p) \vee \neg(\neg\neg A \to_{\mathrm{d}} \bot_p)) \to_{\mathrm{d}} \bot_p$
   (D$_0$6), (D$_0$13), 2, 3, 4 and (Tr$^-$)
6. $\neg(\neg\neg(\neg\neg A \to_{\mathrm{d}} \bot_p) \vee \neg(A \to_{\mathrm{d}} \bot_p)) \to_{\mathrm{d}}$
   $\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg\neg A \to_{\mathrm{d}} \bot_p) \vee \neg(A \to_{\mathrm{d}} \bot_p))$ (D$_0$14)
7. $\neg((\neg\neg A \to_{\mathrm{d}} \bot_p) \vee \neg(A \to_{\mathrm{d}} \bot_p)) \to_{\mathrm{d}}$
   $\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(A \to_{\mathrm{d}} \bot_p) \vee (\neg\neg A \to_{\mathrm{d}} \bot_p))$ (D$_0$6)
8. $\neg(\neg(A \to_{\mathrm{d}} \bot_p) \vee (\neg\neg A \to_{\mathrm{d}} \bot_p)) \to_{\mathrm{d}}$

$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(\neg\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg\neg A \rightarrow_{\mathrm{d}} \bot_p))$ (D$_0$16)

9. $\neg(\neg\neg(\neg\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0$6), (D$_0$13), 6, 7, 8 and (Tr$^-$)

10. $(\neg(\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg\neg A \rightarrow_{\mathrm{d}} \bot_p)) \vee$
$\vee \neg(\neg\neg(\neg\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$ 5, 9 and (Syl$^\vee$)

11. $\neg\neg(\neg(\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg\neg A \rightarrow_{\mathrm{d}} \bot_p)) \vee$
$\vee \neg(\neg\neg(\neg\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$ 1, 10 and (Tr$^-$)

Ad (D$_0^{\vdash}$16).

1. $\neg(\neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg(\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg B \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg(\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg B \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p))$ (D$_0$6)

2. $\neg((\neg(\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg B \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee ((\neg B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p)))$ (D$_0$20)

3. $\neg(\neg(\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee ((\neg B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg\bot_p \wedge_{\mathrm{d}} \neg A) \vee ((\neg B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p)))$ (D$_0$23)

4. $\neg((\neg\bot_p \wedge_{\mathrm{d}} \neg A) \vee ((\neg B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(((\neg\bot_p \wedge_{\mathrm{d}} \neg A) \vee (\neg B \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p))$ (D$_0^{\vdash}$7)

5. $\neg(((\neg\bot_p \wedge_{\mathrm{d}} \neg A) \vee (\neg B \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p))$ (D$_0$18)

6. $\neg((\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0$13)

7. $\neg(\neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg(\neg A \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg B \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$ 1–6 and (Tr$^-$)

Ad (D$_0^{\vdash}$17).

1. $\neg(\neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee (A \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee (A \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p))$ (D$_0$6)

2. $\neg((\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee (A \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee ((A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p)))$ (D$_0$20)

3. $\neg(\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee ((A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg\bot_p \wedge_{\mathrm{d}} B) \vee ((A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p)))$ (D$_0$23)

4. $\neg((\neg\bot_p \wedge_{\mathrm{d}} B) \vee ((A \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(((\neg\bot_p \wedge_{\mathrm{d}} B) \vee (A \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p))$ (D$_0^{\vdash}$7)

5. $\neg(((\neg\bot_p \wedge_{\rm d} B) \vee (A \to_{\rm d} \bot_p)) \vee \neg(\neg(\neg A \vee B) \to_{\rm d} \bot_p)) \to_{\rm d}$
$\to_{\rm d} \top_p \wedge_{\rm d} \neg(((\neg\bot_p \wedge_{\rm d} B) \vee (\neg\neg A \to_{\rm d} \bot_p)) \vee \neg(\neg(\neg A \vee B) \to_{\rm d} \bot_p))$
($\mathrm{D_0}$20), ($\mathrm{D_0}$6), ($\mathrm{D_0}$20), ($\mathrm{D_0}$17), ($\mathrm{D_0^{\vdash}}$7) , ($\mathrm{D_0}$6), ($\mathrm{D_0^{\vdash}}$7) and (Tr^-)
6. $\neg(((\neg\bot_p \wedge_{\rm d} B) \vee (\neg\neg A \to_{\rm d} \bot_p)) \vee \neg(\neg(\neg A \vee B) \to_{\rm d} \bot_p)) \to_{\rm d}$
$\to_{\rm d} \top_p \wedge_{\rm d} \neg((\neg(B \vee \neg A) \to_{\rm d} \bot_p) \vee \neg(\neg(\neg A \vee B) \to_{\rm d} \bot_p))$ ($\mathrm{D_0}$18)
7 $\neg((\neg(B \vee \neg A) \to_{\rm d} \bot_p) \vee \neg(\neg(\neg A \vee B) \to_{\rm d} \bot_p))$
$\to_{\rm d} \top_p \wedge_{\rm d} \neg((\neg(\neg A \vee B) \to_{\rm d} \bot_p) \vee \neg(\neg(\neg A \vee B) \to_{\rm d} \bot_p))$ ($\mathrm{D_0}$7)
8. $\neg((\neg(\neg A \vee B) \to_{\rm d} \bot_p) \vee \neg(\neg(\neg A \vee B) \to_{\rm d} \bot_p)) \to_{\rm d} \bot_p$ ($\mathrm{D_0}$13)
9. $\neg(\neg(\neg(\neg A \vee B) \to_{\rm d} \bot_p) \vee (\neg(B \to_{\rm d} \bot_p) \vee (A \to_{\rm d} \bot_p))) \to_{\rm d} \bot_p$
1–8 and (Tr^-)

Ad ($\mathrm{D_0^{\vdash}}$18)—easily follows from ($\mathrm{D_0^{\vdash}}$17) by ($\mathrm{D_0}$14), ($\mathrm{D_0}$6), ($\mathrm{D_0}$20) and (Tr^-).
Ad ($\mathrm{D_0^{\vdash}}$19).

1. $C \to_{\rm d} \top_p \wedge_{\rm d} (B \vee C)$ ($\mathrm{D_0}$5)
2. $(B \vee C) \to_{\rm d} \top_p \wedge_{\rm d} (A \to_{\rm d} (B \vee C))$ ($\mathrm{D_0}$1)
3. $C \to_{\rm d} \top_p \wedge_{\rm d} (A \to_{\rm d} (B \vee C))$ 1. 2 and (Tr^-)
4. $B \to_{\rm d} \top_p \wedge_{\rm d} (B \vee C)$ ($\mathrm{D_0}$4)
5. $(A \to_{\rm d} B) \to_{\rm d} \top_p \wedge_{\rm d} (A \to_{\rm d} (B \vee C))$ 4 and ($\mathrm{Tr}_2^{\rm ax}$)
6. $((A \to_{\rm d} B) \vee C) \to_{\rm d} \top_p \wedge_{\rm d} (A \to_{\rm d} (B \vee C))$ 3, 5 and ($\mathrm{Syl}^{\vee}$)

Ad ($\mathrm{D_0^{\vdash}}$20).

1. $(\neg(A \vee \bot_p) \to_{\rm d} C) \to_{\rm d} \top_p \wedge_{\rm d} ((\neg A \to_{\rm d} (\top_p \wedge_{\rm d} \bot_p)) \vee C)$ ($\mathrm{D_0}$19)
2. $\bot_p \to_{\rm d} \bot_p$ ($\mathrm{D_0^{\vdash}}$1)
3. $(\neg A \to_{\rm d} (\top_p \wedge_{\rm d} \bot_p)) \to_{\rm d} \top_p \wedge_{\rm d} (\neg A \to_{\rm d} \bot_p)$ 2 and ($\mathrm{Tr}_3^{\rm ax}$)
4. $(\neg A \to_{\rm d} \bot_p) \to_{\rm d} \top_p \wedge_{\rm d} ((\neg A \to_{\rm d} \bot_p) \vee C)$ ($\mathrm{D_0}$4)
5. $(\neg A \to_{\rm d} (\top_p \wedge_{\rm d} \bot_p)) \to_{\rm d} \top_p \wedge_{\rm d} ((\neg A \to_{\rm d} \bot_p) \vee C)$ 3, 4 and (Tr^-)
6. $C \to_{\rm d} \top_p \wedge_{\rm d} ((\neg A \to_{\rm d} \bot_p) \vee C)$ ($\mathrm{D_0}$5)
7. $(\neg A \to_{\rm d} (\top_p \wedge_{\rm d} \bot_p)) \vee C \to_{\rm d} \top_p \wedge_{\rm d} ((\neg A \to_{\rm d} \bot_p) \vee C)$ 5, 6 and ($\mathrm{Syl}^{\vee}$)
8. $(\neg(A \vee \bot_p) \to_{\rm d} C) \to_{\rm d} \top_p \wedge_{\rm d} ((\neg A \to_{\rm d} \bot_p) \vee C)$ 1, 7 and (Tr^-)

Ad ($\mathrm{D_0^{\vdash}}$21).

1. $\neg(A \vee \bot_p) \to_{\rm d} \top_p \wedge_{\rm d} \neg(A \vee \bot_p)$ ($\mathrm{D_0}$12)
2. $(\neg(A \vee \bot_p) \to_{\rm d} (\top_p \wedge_{\rm d} \neg(A \vee \bot_p))) \to_{\rm d}$
$\to_{\rm d} \top_p \wedge_{\rm d} ((\neg A \to_{\rm d} \bot_p) \vee (\top_p \wedge_{\rm d} \neg(A \vee \bot_p)))$ ($\mathrm{D_0^{\vdash}}$20)
3. $((\neg A \to_{\rm d} \bot_p) \vee (\top_p \wedge_{\rm d} \neg(A \vee \bot_p))) \to_{\rm d}$
$\to_{\rm d} \top_p \wedge_{\rm d} (\neg A \to_{\rm d} (\bot_p \vee (\top_p \wedge_{\rm d} \neg(A \vee \bot_p))))$ ($\mathrm{D_0^{\vdash}}$19)
4. $\neg A \to_{\rm d} (\bot_p \vee (\top_p \wedge_{\rm d} \neg(A \vee \bot_p)))$ 2–3, (Tr^-), 1, ($\mathrm{MP}^{\to_{\rm d}}$) and ($\wedge_{\mathrm{d}_{\rm rg}}^-$)
5. $\neg A \to_{\rm d} \top_p \wedge_{\rm d} \neg(A \vee \bot_p))$ 4 and ($\bot_{\rm p}^-$)

Ad ($\mathrm{D}_0^{\vdash}22$). Similarly we prove ($\mathrm{D}_0^{\vdash}23$).

1. $\neg(\neg A \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg((\neg A \to_{\mathrm{d}} \bot_p) \vee \bot_p))$ ($\mathrm{D}_0^{\vdash}21$)
2. $\neg((\neg A \to_{\mathrm{d}} \bot_p) \vee \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\bot_p \vee \bot_p) \wedge_{\mathrm{d}} \neg A)$ ($\mathrm{D}_0 21$)
3. $(\neg(\bot_p \vee \bot_p) \wedge_{\mathrm{d}} \neg A) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg A)$ ($\mathrm{D}_0 3$)
4. $\neg(\neg A \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg A)$ 1–3 and (Tr^-)

Ad ($\mathrm{D}_0^{\vdash}24$).

1. $A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} A$ ($\mathrm{D}_0 12$)
2. $\bot_p \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} A$ ($\mathrm{D}_0^{\vdash}1$)
3. $A \vee \bot_p \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} A$ 1, 2 and ($\mathrm{Syl}^{\vee}$)

Ad ($\mathrm{D}_0^{\vdash}25$).

1. $\neg(A \vee \bot_B) \to_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B))$ ($\mathrm{D}_0 12$)
2. $(\neg(A \vee \bot_B) \to_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B))) \to_{\mathrm{d}}$
 $\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((\neg A \to_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \bot_B)) \vee (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B)))$ ($\mathrm{D}_0 19$)
3. $\bot_B \to_{\mathrm{d}} \bot_p$ ($\mathrm{D}_0^{\vdash}1$)
4. $(\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \bot_B) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg A \to_{\mathrm{d}} \bot_p)$ 3 and ($\mathrm{Tr}_3^{\mathrm{ax}}$)
5. $(\neg A \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((\neg A \to_{\mathrm{d}} \bot_p) \vee (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B)))$ ($\mathrm{D}_0 4$)
6. $(\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \bot_B) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((\neg A \to_{\mathrm{d}} \bot_p) \vee (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B)))$
 4, 5 and (Tr^-)
7. $(\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B)) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((\neg A \to_{\mathrm{d}} \bot_p) \vee (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B)))$
 ($\mathrm{D}_0 5$)
8. $(\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \bot_B) \vee (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B)) \to_{\mathrm{d}}$
 $\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((\neg A \to_{\mathrm{d}} \bot_p) \vee (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B)))$ 6, 7 and ($\mathrm{Syl}^{\vee}$)
9. $(\neg(A \vee \bot_B) \to_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B))) \to_{\mathrm{d}}$
 $\to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((\neg A \to_{\mathrm{d}} \bot_p) \vee (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B)))$ 2, 8 and (Tr^-)
10. $\top_p \wedge_{\mathrm{d}} ((\neg A \to_{\mathrm{d}} \bot_p) \vee (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B)))$ 1, 9 and ($\mathrm{MP}^{\to_{\mathrm{d}}}$)
11. $(\neg A \to_{\mathrm{d}} \bot_p) \vee (\top_p \wedge_{\mathrm{d}} \neg(A \vee \bot_B))$ 10 and ($\wedge_{\mathrm{d_r}}^-$)

Ad ($\mathrm{D}_0^{\vdash}26$).

1. $(\neg B \to_{\mathrm{d}} \bot_p) \vee (\top_p \wedge_{\mathrm{d}} \neg(B \vee \bot_A))$ ($\mathrm{D}_0^{\vdash}25$)
2. $\bot_p \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} (\bot_p \vee \neg(B \vee \bot_A)))$ ($\mathrm{D}_0^{\vdash}1$)
3. $(\neg B \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg B \to_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} (\bot_p \vee \neg(B \vee \bot_A))))$
 2 and($\mathrm{Tr}_2^{\mathrm{ax}}$)
4. $\neg(B \vee \bot_A) \to_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} (\bot_p \vee \neg(B \vee \bot_A)))$ ($\mathrm{D}_0 5$)
5. $\neg(B \vee \bot_A) \to_{\mathrm{d}} (\neg B \to_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} (\bot_p \vee \neg(B \vee \bot_A))))$ 4 and ($\mathrm{Weak}^{\to_{\mathrm{d}}}$)
6. $\top_p \wedge_{\mathrm{d}} \neg(B \vee \bot_A) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg B \to_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} (\bot_p \vee \neg(B \vee \bot_A))))$
 5 and (Mon)

7. $(\neg B \to_{\rm d} \bot_p) \vee \top_p \wedge_{\rm d} \neg(B \vee \bot_A) \to_{\rm d}$
   $\to_{\rm d} \top_p \wedge_{\rm d} (\neg B \to_{\rm d} (\top_p \wedge_{\rm d} (\bot_p \vee \neg(B \vee \bot_A))))$ 3, 6 and (Syl$^{\vee}$)
8. $\top_p \wedge_{\rm d} (\neg B \to_{\rm d} (\top_p \wedge_{\rm d} (\bot_p \vee \neg(B \vee \bot_A))))$ 1, 7 and (MP$^{\to_{\rm d}}$)
9. $\neg B \to_{\rm d} (\top_p \wedge_{\rm d} (\bot_p \vee \neg(B \vee \bot_A)))$ 8 and ($\wedge_{\rm d_r}^{-}$)
10. $\neg(B \vee \bot_A) \to_{\rm d} \top_p \wedge_{\rm d} \neg(\neg(\neg A \vee A) \vee B)$ (D$_0$6)
11. $\bot_p \to_{\rm d} \top_p \wedge_{\rm d} \neg(\neg(\neg A \vee A) \vee B)$ (D$_0^{\vdash}$1)
12. $\bot_p \vee \neg(B \vee \bot_A) \to_{\rm d} \top_p \wedge_{\rm d} \neg(\neg(\neg A \vee A) \vee B)$ 11, 10 and (Syl$^{\vee}$)
13. $\neg B \to_{\rm d} \top_p \wedge_{\rm d} \neg(\neg(\neg A \vee A) \vee B)$ 9, 12 and (Tr^{-})

Ad (D$_0^{\vdash}$27).

1. $\neg B \to_{\rm d} \top_p \wedge_{\rm d} \neg(\neg(\neg A \vee A) \vee B)$ (D$_0^{\vdash}$26)
2. $\neg(\neg(\neg A \vee A) \vee B) \to_{\rm d} \top_p \wedge_{\rm d} (\neg(\neg\neg A \vee B) \vee \neg(\neg A \vee B))$ (D$_0$22)
3. $\neg(\neg A \vee B) \to_{\rm d} \top_p \wedge_{\rm d} (\neg A \vee \neg(\neg A \vee B))$ (D$_0$5)
4. $\neg(\neg\neg A \vee B) \to_{\rm d} \top_p \wedge_{\rm d} \neg\neg\neg A$ (D$_0^{\vdash}$6)
5. $\neg\neg\neg A \to_{\rm d} \top_p \wedge_{\rm d} \neg A$ (D$_0^{\vdash}$3)
6. $\neg A \to_{\rm d} \top_p \wedge_{\rm d} (\neg A \vee \neg(\neg A \vee B))$ (D$_0$4)
7. $\neg(\neg\neg A \vee B) \to_{\rm d} \top_p \wedge_{\rm d} (\neg A \vee \neg(\neg A \vee B))$ 4–6 and (Tr^{-})
8. $\neg(\neg\neg A \vee B) \vee \neg(\neg A \vee B) \to_{\rm d} \top_p \wedge_{\rm d} (\neg A \vee \neg(\neg A \vee B))$ 7, 3 and (Syl$^{\vee}$)
9. $\neg B \to_{\rm d} \top_p \wedge_{\rm d} (\neg A \vee \neg(\neg A \vee B))$ 1, 2, 8 and (Tr^{-})

Ad (D$_0^{\vdash}$28).

1. $\neg(\neg\neg(A \vee \neg B) \vee B) \to_{\rm d} \top_p \wedge_{\rm d} \neg((A \vee \neg B) \vee B)$ (D$_0$14)
2. $\neg((A \vee \neg B) \vee B) \to_{\rm d} \top_p \wedge_{\rm d} \neg(A \vee (\neg B \vee B))$ (D$_0$20)
3. $\neg(A \vee (\neg B \vee B)) \to_{\rm d} \top_p \wedge_{\rm d} \neg(\neg B \vee B)$ (D$_0^{\vdash}$4)
4. $\neg(\neg B \vee B) \to_{\rm d} \bot_p$ (D$_0^{\vdash}$1)
5. $\neg(\neg\neg(A \vee \neg B) \vee B) \to_{\rm d} \bot_p$ 1–4 and (Tr^{-})
6. $\neg(A \vee \neg B) \wedge_{\rm d} C \to_{\rm d} \top_p \wedge_{\rm d} (B \wedge_{\rm d} C)$ 5 and (Tr$_4^{\rm ax}$)

■

The below lemma will indicate provability of rules that correspond to rules of $\Diamond$-**D** in the axiomatization provided for Lemma 3. In particular, inferability of rules ($\Box$nec$^{\rm tr}$), ($\Box$mp$^{\rm tr}$), ($\Box$mp$^{\rm tr}_{-}$) and (pos$^{\rm tr}_{\Leftarrow}$) will be used in the proof of the final Theorem 13.

Lemma 9. *The following rules are inferable on the basis of* $\Vdash$*:*

$$\frac{\neg A \to_{\rm d} \bot_p}{\neg(\neg A \to_{\rm d} \bot_p) \to_{\rm d} \bot_p} \qquad (\Box\text{nec}^{\rm tr})$$

$$\frac{\neg A \to_{\rm d} \bot_p; \quad \neg(\neg A \vee B) \to_{\rm d} \bot_p}{\neg B \to_{\rm d} \bot_p} \qquad (\Box\text{mp}^{\rm tr})$$

$$\frac{A; \quad \neg(\neg A \vee B) \to_{\mathrm{d}} \bot_p}{B} \qquad (\Box\mathrm{mp}^{\mathrm{tr}}_{-})$$

$$\frac{\neg(A \to_{\mathrm{d}} \bot_p)}{A} \qquad (\mathrm{pos}^{\mathrm{tr}}_{\Leftarrow})$$

Proof. *Ad* ($\Box$nec$^{\mathrm{tr}}$).

1. $\neg A \to_{\mathrm{d}} \bot_p$ — Asm.
2. $\neg(\neg A \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg A)$ — ($D_0^{\vdash}22$)
3. $(\top_p \wedge_{\mathrm{d}} \neg A) \to_{\mathrm{d}} \bot_p$ — 1 and (Weak$^{\bot}$)
4. $\neg(\neg A \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \bot_p$ — 2, 3 and (Tr^{-})

Ad ($\Box$mp$^{\mathrm{tr}}$).

1. $\neg A \to_{\mathrm{d}} \bot_p$ — Asm.
2. $\neg(\neg A \vee B) \to_{\mathrm{d}} \bot_p$ — Asm.
3. $\neg A \vee \neg(\neg A \vee B) \to_{\mathrm{d}} \bot_p$ — 1, 2 and (Syl$^{\vee}$)
4. $\neg B \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg A \vee \neg(\neg A \vee B))$ — ($D_0^{\vdash}27$)
5. $\neg B \to_{\mathrm{d}} \bot_p$ — 4, 3 and (Tr^{-})

Ad ($\Box$mp$^{\mathrm{tr}}_{-}$).

1. A — Asm.
2. $\neg(\neg A \vee B) \to_{\mathrm{d}} \bot_p$ — Asm.
3. $(\neg(\neg A \vee B) \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B)$ — ($D_0^{\vdash}2$)
4. $\top_p \wedge_{\mathrm{d}} (\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B)$ — 2, 3 and (MP$^{\to_{\mathrm{d}}}$)
5. $(\neg\neg A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B)$ — 4 and ($\wedge_{\mathrm{d_r}}^{-}$)
6. $A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg\neg A$ — ($D_0 11$)
7. $A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B$ — 6, 5 and (Tr^{-})
8. $\top_p \wedge_{\mathrm{d}} B$ — 1, 7 and (MP$^{\to_{\mathrm{d}}}$)
9. B — 8 and ($\wedge_{\mathrm{d_r}}^{-}$)

Ad (pos$^{\mathrm{tr}}_{\Leftarrow}$).

1. $\neg(A \to_{\mathrm{d}} \bot_p)$ — Asm.
2. $\neg(A \to_{\mathrm{d}} \bot_p) \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} A)$ — ($D_0^{\vdash}23$)
3. $\top_p \wedge_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} A)$ — 1, 2 and (MP$^{\to_{\mathrm{d}}}$)
4. $\top_p \wedge_{\mathrm{d}} A$ — 3 and ($\wedge_{\mathrm{d_{rg}}}^{-}$)
5. A — 4 and ($\wedge_{\mathrm{d_{rg}}}^{-}$)

■

The translation $\mathtt{i}_2$: $\mathrm{For}_{\mathrm{m}} \longrightarrow \mathrm{For}_{\mathrm{d}}$ given below is used in [4], another translation also denoted by $\mathtt{i}_2$ was considered in [11]. We refer to the axiomatization that arose from [4], thus, we also follow the respective translation.

1 $\mathtt{i}_2(a) = a$, for any propositional letter a,

2 for any $\varphi, \psi \in \mathrm{For_m}$:

(a) $\mathtt{i}_2(\neg\varphi) = \neg\mathtt{i}_2(\varphi)$,
(b) $\mathtt{i}_2(\Box\varphi) = \neg\mathtt{i}_2(\varphi) \rightarrow_{\mathrm{d}} \bot_p$,
(c) $\mathtt{i}_2(\Diamond\varphi) = \neg(\mathtt{i}_2(\varphi) \rightarrow_{\mathrm{d}} \bot_p)$,
(d) $\mathtt{i}_2(\varphi \vee \psi) = \mathtt{i}_2(\varphi) \vee \mathtt{i}_2(\psi)$,
(e) $\mathtt{i}_2(\varphi \wedge \psi) = \neg(\neg\mathtt{i}_2(\varphi) \vee \neg\mathtt{i}_2(\psi))$,
(f) $\mathtt{i}_2(\varphi \rightarrow \psi) = \neg\mathtt{i}_2(\varphi) \vee \mathtt{i}_2(\psi)$,
(g) $\mathtt{i}_2(\varphi \leftrightarrow \psi) = \neg(\neg(\neg\mathtt{i}_2(\varphi) \vee \mathtt{i}_2(\psi)) \vee \neg(\neg\mathtt{i}_2(\psi) \vee \mathtt{i}_2(\varphi)))$.

We will need the following lemma that will be used in the proof of Lemma 11:

Lemma 10. *For every $A, A', B, B' \in \mathrm{For_d}$, if*

$$\neg(\neg A \vee A') \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash} \tag{8}$$

$$\neg(\neg B \vee B') \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash} \tag{9}$$

$$\neg(\neg A' \vee A) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash} \tag{10}$$

$$\neg(\neg B' \vee B) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash} \tag{11}$$

then

$$\neg(\neg\neg(\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee (A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash} \tag{12}$$

$$\neg(\neg(A' \wedge_{\mathrm{d}} B') \vee \neg(\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash} \tag{13}$$

$$\neg(\neg(\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B) \vee (A' \rightarrow_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash} \tag{14}$$

$$\neg(\neg(A' \rightarrow_{\mathrm{d}} B') \vee (\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B)) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash} \tag{15}$$

$$\neg(\neg(A \vee B) \vee (A' \vee B')) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash} \tag{16}$$

Proof. Consider the following proof, where (8) and (9) are used as assumptions.

1. $\neg(\neg\neg(\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee (A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee (A' \wedge_{\mathrm{d}} B'))$ (D$_0$14)
2. $\neg((\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee (A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg((\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee A') \vee$
 $\vee \neg((\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p)))$ (D$_0$27)
3. $\neg((\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee A') \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A' \vee (\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)))$ (D$_0$6)
4. $\neg(A' \vee (\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((A' \vee \neg A) \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p))$ (D$_0^{\vdash}$7)
5. $\neg((A' \vee \neg A) \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A' \vee \neg A)$ (D$_0^{\vdash}$6)
6. $\neg(A' \vee \neg A) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0$6), (8) and (Tr$^-$)
7. $\neg((\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee A') \rightarrow_{\mathrm{d}} \bot_p$ 3–6 and (Tr$^-$)

8. $\neg((\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg A \vee (\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p)))$ (D$_0$20)
9. $\neg(\neg A \vee (\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p))$ (D$_0^{\vdash}$4)
10. $\neg(\neg(\neg(\neg B \vee B') \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0^{\vdash}$18)
11. $(\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p))$ (9) and ($\Box$mp$^{\mathrm{tr}}_{-}$)
12. $(\neg(\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}}$
$\top_p \wedge_{\mathrm{d}} \neg(\neg(\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p)) \vee$
$\neg\neg(\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p)))$ 11 and (Add$^{\wedge_c}$)
13. $\neg(\neg(\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p)) \vee$
$\neg\neg(\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0$13)
14. $\neg((\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}} \bot_p$ 8–9, 12–13 and (Tr^{-})
15. $(\neg((\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee A') \vee$
$\vee \neg((\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(B' \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$ 7, 14 and (Syl$^{\vee}$)
16. $\neg(\neg\neg(\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee (A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \bot_p$ 1, 2, 15 and (Tr^{-})

For the case of (13) consider the following sequence.

1. $\neg(\neg(A' \wedge_{\mathrm{d}} B') \vee \neg(\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}}$
$\neg(\neg(\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(A' \wedge_{\mathrm{d}} B'))$ (D$_0$6)
2. $\neg(\neg(\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p)) \vee \neg(A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\neg\neg A \vee \neg(A' \wedge_{\mathrm{d}} B')) \vee \neg(\neg\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A' \wedge_{\mathrm{d}} B')))$
(D$_0$22)
3. $\neg(\neg\neg A \vee \neg(A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(A \vee \neg(A' \wedge_{\mathrm{d}} B'))$ (D$_0$14)
4. $\neg(A \vee \neg(A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(A' \wedge_{\mathrm{d}} B') \vee A)$ (D$_0$6)
5. $\neg(\neg(A' \wedge_{\mathrm{d}} B') \vee A) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\neg A' \vee A) \wedge_{\mathrm{d}} B')$ (D$_0$24)
6. $(\neg(\neg A' \vee A) \wedge_{\mathrm{d}} B') \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg A' \vee A)$ (D$_0$2)
7. $\neg(\neg A' \vee A) \rightarrow_{\mathrm{d}} \bot_p$ (10)
8. $\neg(\neg\neg A \vee \neg(A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \bot_p$ 3–7 and (Tr^{-})
9. $\neg(\neg\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}}$
$\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A' \wedge_{\mathrm{d}} B'))$ (D$_0$14)
10. $\neg(\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(A' \wedge_{\mathrm{d}} B') \vee \neg(B \rightarrow_{\mathrm{d}} \bot_p))$
(D$_0$6)
11. $\neg(\neg(A' \wedge_{\mathrm{d}} B') \vee \neg(B \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\neg A' \vee \neg(B \rightarrow_{\mathrm{d}} \bot_p)) \wedge_{\mathrm{d}} B')$
(D$_0$24)
12. $\neg(\neg A' \vee \neg(B \rightarrow_{\mathrm{d}} \bot_p)) \wedge_{\mathrm{d}} B' \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((B \rightarrow_{\mathrm{d}} \bot_p) \wedge_{\mathrm{d}} B')$ (D$_0^{\vdash}$28)
13. $\neg(\neg(\neg(\neg B' \vee B) \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee (B' \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$ (D$_0^{\vdash}$17)
14. $\neg(\neg B' \vee B) \rightarrow_{\mathrm{d}} \bot_p$ (11)
15. $\neg(\neg(\neg B' \vee B) \rightarrow_{\mathrm{d}} \bot_p) \rightarrow_{\mathrm{d}} \bot_p$ 14 and ($\Box$nec$^{\mathrm{tr}}$)
16. $\neg(\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee (B' \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}} \bot_p$ 15, 13 and ($\Box$mp$^{\mathrm{tr}}$)
17. $(B \rightarrow_{\mathrm{d}} \bot_p) \wedge_{\mathrm{d}} B' \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((B' \rightarrow_{\mathrm{d}} \bot_p) \wedge_{\mathrm{d}} B')$ 16 and (Tr$^{\mathrm{ax}}_4$)
18. $(B' \rightarrow_{\mathrm{d}} \bot_p) \wedge_{\mathrm{d}} B' \rightarrow_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \bot_p)$ (D$_0$9)
19. $\bot_p \rightarrow_{\mathrm{d}} \bot_p$ (D$_0^{\vdash}$1)
20. $\neg(\neg\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A' \wedge_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \bot_p$ 9–12, 17–19 and (Tr^{-})
21. $(\neg(\neg\neg A \vee \neg(A' \wedge_{\mathrm{d}} B')) \vee \neg(\neg\neg\neg(B \rightarrow_{\mathrm{d}} \bot_p) \vee \neg(A' \wedge_{\mathrm{d}} B'))) \rightarrow_{\mathrm{d}} \bot_p$
8, 20 (Syl$^{\vee}$)
22. $\neg(\neg(A' \wedge_{\mathrm{d}} B') \vee \neg(\neg A \vee \neg\neg(B \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$ 1, 2, 21 and (Tr^{-})

For the case of (14) consider the following sequence.

1. $\neg(\neg(\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B) \vee (A' \rightarrow_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(\neg\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee (A' \rightarrow_{\mathrm{d}} B')) \vee \neg(\neg B \vee (A' \rightarrow_{\mathrm{d}} B')))$ (D$_0$22)
2. $\neg(\neg\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee (A' \rightarrow_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee (A' \rightarrow_{\mathrm{d}} B'))$ (D$_0$14)
3. $\neg(\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee (A' \rightarrow_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((A' \rightarrow_{\mathrm{d}} B') \vee \neg(A \rightarrow_{\mathrm{d}} \bot_p))$ (D$_0$6)
4. $\neg((A' \rightarrow_{\mathrm{d}} B') \vee \neg(A \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(B' \vee \neg(A \rightarrow_{\mathrm{d}} \bot_p)) \wedge_{\mathrm{d}} A')$ (D$_0$21)
5. $\neg(B' \vee \neg(A \rightarrow_{\mathrm{d}} \bot_p)) \wedge_{\mathrm{d}} A' \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((A \rightarrow_{\mathrm{d}} \bot_p) \wedge_{\mathrm{d}} A')$ ($\mathrm{D}_0^{\vdash}28$)
6. $\neg(\neg(\neg(\neg A' \vee A) \rightarrow_{\mathrm{d}} \bot_p) \vee (\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee (A' \rightarrow_{\mathrm{d}} \bot_p))) \rightarrow_{\mathrm{d}} \bot_p$ ($\mathrm{D}_0^{\vdash}17$)
7. $\neg(\neg A' \vee A) \rightarrow_{\mathrm{d}} \bot_p$ (10)
8. $\neg(\neg(\neg A' \vee A) \rightarrow_{\mathrm{d}} \bot_p) \rightarrow_{\mathrm{d}} \bot_p$ 7 and ($\Box\mathrm{nec}^{\mathrm{tr}}$)
9. $\neg(\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee (A' \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}} \bot_p$ 8, 6 and ($\Box\mathrm{mp}^{\mathrm{tr}}$)
10. $((A \rightarrow_{\mathrm{d}} \bot_p) \wedge_{\mathrm{d}} A') \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} ((A' \rightarrow_{\mathrm{d}} \bot_p) \wedge_{\mathrm{d}} A')$ 9 and ($\mathrm{Tr}_4^{\mathrm{ax}}$)
11. $(A' \rightarrow_{\mathrm{d}} \bot_p) \wedge_{\mathrm{d}} A' \rightarrow_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \bot_p)$ (D$_0$9)
12. $\bot_p \rightarrow_{\mathrm{d}} \bot_p$ ($\mathrm{D}_0^{\vdash}1$)
13. $\neg(\neg\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee (A' \rightarrow_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \bot_p$ 2–5, 10–12 and (Tr^-)
14. $\neg(\neg B \vee (A' \rightarrow_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((A' \rightarrow_{\mathrm{d}} B') \vee \neg B)$ (D$_0$6)
15. $\neg((A' \rightarrow_{\mathrm{d}} B') \vee \neg B) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg(B' \vee \neg B) \wedge_{\mathrm{d}} A')$ (D$_0$21)
16. $(\neg(B' \vee \neg B) \wedge_{\mathrm{d}} A') \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(B' \vee \neg B)$ (D$_0$2)
17. $\neg(B' \vee \neg B) \rightarrow_{\mathrm{d}} \bot_p$ (11)
18. $\neg(\neg B \vee (A' \rightarrow_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \bot_p$ 14–17 and (Tr^-)
19. $(\neg(\neg\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee (A' \rightarrow_{\mathrm{d}} B')) \vee \neg(\neg B \vee (A' \rightarrow_{\mathrm{d}} B'))) \rightarrow_{\mathrm{d}} \bot_p$
 13, 18 and ($\mathrm{Syl}^{\vee}$)
20. $\neg(\neg(\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B) \vee (A' \rightarrow_{\mathrm{d}} B')) \rightarrow_{\mathrm{d}} \bot_p$ 1, 19 and (Tr^-)

For the case of (15) consider the following sequence.

1. $\neg(\neg(A' \rightarrow_{\mathrm{d}} B') \vee (\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B)) \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} \neg((\neg B' \wedge_{\mathrm{d}} A') \vee (\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B)))$ (D$_0$23)
2. $\neg((\neg B' \wedge_{\mathrm{d}} A') \vee (\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B)) \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg((\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B) \vee (\neg B' \wedge_{\mathrm{d}} A'))$ (D$_0$6)
3. $\neg((\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B) \vee (\neg B' \wedge_{\mathrm{d}} A')) \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\neg((\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B) \vee \neg B') \vee$
 $\vee \neg((\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B) \vee \neg(A' \rightarrow_{\mathrm{d}} \bot_p)))$ (D$_0$27)
4. $\neg((\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B) \vee \neg B') \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee (B \vee \neg B'))$ (D$_0$20)
5. $\neg(\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee (B \vee \neg B')) \rightarrow_{\mathrm{d}}$
 $\rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(B \vee \neg B')$ ($\mathrm{D}_0^{\vdash}4$)
6. $\neg(B \vee \neg B') \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\neg B' \vee B)$ (D$_0$6)
7. $\neg(\neg B' \vee B) \rightarrow_{\mathrm{d}} \bot_p$ (11)
8. $\neg((\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B) \vee \neg B') \rightarrow_{\mathrm{d}} \bot_p$ 4–7 and (Tr^-)
9. $\neg((\neg\neg(A \rightarrow_{\mathrm{d}} \bot_p) \vee B) \vee \neg(A' \rightarrow_{\mathrm{d}} \bot_p)) \rightarrow_{\mathrm{d}}$

$\to_{\rm d} \top_p \wedge_{\rm d} \neg(\neg\neg(A \to_{\rm d} \bot_p) \vee (B \vee \neg(A' \to_{\rm d} \bot_p)))$ (D$_0$20)
10. $\neg(\neg\neg(A \to_{\rm d} \bot_p) \vee (B \vee \neg(A' \to_{\rm d} \bot_p))) \to_{\rm d}$
$\to_{\rm d} \top_p \wedge_{\rm d} \neg((A \to_{\rm d} \bot_p) \vee (B \vee \neg(A' \to_{\rm d} \bot_p)))$ (D$_0$14)
11. $\neg((A \to_{\rm d} \bot_p) \vee (B \vee \neg(A' \to_{\rm d} \bot_p))) \to_{\rm d}$
$\to_{\rm d} \top_p \wedge_{\rm d} \neg((B \vee \neg(A' \to_{\rm d} \bot_p)) \vee (A \to_{\rm d} \bot_p))$ (D$_0$6)
12. $\neg((B \vee \neg(A' \to_{\rm d} \bot_p)) \vee (A \to_{\rm d} \bot_p)) \to_{\rm d}$
$\to_{\rm d} \top_p \wedge_{\rm d} \neg(B \vee (\neg(A' \to_{\rm d} \bot_p) \vee (A \to_{\rm d} \bot_p)))$ (D$_0$20)
13. $\neg(B \vee (\neg(A' \to_{\rm d} \bot_p) \vee (A \to_{\rm d} \bot_p))) \to_{\rm d}$
$\to_{\rm d} \top_p \wedge_{\rm d} \neg(\neg(A' \to_{\rm d} \bot_p) \vee (A \to_{\rm d} \bot_p))$ (D$_0^{\vdash}$4)
14. $\neg(\neg(\neg(\neg A \vee A') \to_{\rm d} \bot_p) \vee (\neg(A' \to_{\rm d} \bot_p) \vee (A \to_{\rm d} \bot_p))) \to_{\rm d} \bot_p$ (D$_0^{\vdash}$17)
15. $\neg(\neg A \vee A') \to_{\rm d} \bot_p$ (8)
16. $\neg(\neg(\neg A \vee A') \to_{\rm d} \bot_p) \to_{\rm d} \bot_p$ 15 and ($\Box$nec$^{\rm tr}$)
17. $\neg(\neg(A' \to_{\rm d} \bot_p) \vee (A \to_{\rm d} \bot_p)) \to_{\rm d} \bot_p$ 16, 14 and ($\Box$mp$^{\rm tr}$)
18. $\neg((\neg\neg(A \to_{\rm d} \bot_p) \vee B) \vee \neg(A' \to_{\rm d} \bot_p)) \to_{\rm d} \bot_p$ 9–17 and (Tr$^-$)
19. $(\neg((\neg\neg(A \to_{\rm d} \bot_p) \vee B) \vee \neg B') \vee$
$\vee \neg((\neg\neg(A \to_{\rm d} \bot_p) \vee B) \vee \neg(A' \to_{\rm d} \bot_p))) \to_{\rm d} \bot_p$ 8, 18 and (Syl$^{\vee}$)
20. $\neg(\neg(A' \to_{\rm d} B') \vee (\neg\neg(A \to_{\rm d} \bot_p) \vee B)) \to_{\rm d} \bot_p$ 1–3, 19 and (Tr$^-$)

And finally for the case of (16) we have:

1. $\neg(\neg(A \vee B) \vee (A' \vee B')) \to_{\rm d}$
$\to_{\rm d} \top_p \wedge_{\rm d} (\neg(\neg A \vee (A' \vee B')) \vee \neg(\neg B \vee (A' \vee B')))$ (D$_0$22)
2. $\neg(\neg A \vee (A' \vee B')) \to_{\rm d} \top_p \wedge_{\rm d} \neg((\neg A \vee A') \vee B')$ (D$_0^{\vdash}$7)
3. $\neg((\neg A \vee A') \vee B') \to_{\rm d} \top_p \wedge_{\rm d} \neg(\neg A \vee A')$ (D$_0^{\vdash}$6)
4. $\neg(\neg A \vee A') \to_{\rm d} \bot_p$ (8)
5. $\neg(\neg A \vee (A' \vee B')) \to_{\rm d} \bot_p$ 2–4 and (Tr$^-$)
6. $\neg(\neg B \vee (A' \vee B')) \to_{\rm d} \top_p \wedge_{\rm d} \neg((A' \vee B') \vee \neg B)$ (D$_0$6)
7. $\neg((A' \vee B') \vee \neg B) \to_{\rm d} \top_p \wedge_{\rm d} \neg(A' \vee (B' \vee \neg B))$ (D$_0$20)
8. $\neg(A' \vee (B' \vee \neg B)) \to_{\rm d} \top_p \wedge_{\rm d} \neg(B' \vee \neg B)$ (D$_0^{\vdash}$4)
9. $\neg(B' \vee \neg B) \to_{\rm d} \top_p \wedge_{\rm d} \neg(\neg B \vee B')$ (D$_0$6)
10. $\neg(\neg B \vee B') \to_{\rm d} \bot_p$ (9)
11. $\neg(\neg B \vee (A' \vee B')) \to_{\rm d} \bot_p$ 6–10 and (Tr$^-$)
12. $(\neg(\neg A \vee (A' \vee B')) \vee \neg(\neg B \vee (A' \vee B'))) \to_{\rm d} \bot_p$ 5, 11 and (Syl$^{\vee}$)
13. $\neg(\neg(A \vee B) \vee (A' \vee B')) \to_{\rm d} \bot_p$ 1, 12 and (Tr$^-$)

■

The next Lemma will be used in the proof of Theorem 13.

Lemma 11. *For every $A \in {\rm For_d}$ the following formulas are theses of $\mathsf{D}_0^{\Vdash}$:*

$$\neg(\neg \mathtt{i}_2(\mathtt{i}_1(A)) \vee A) \to_{\rm d} \bot_p \tag{17}$$

$$\neg(\neg A \vee \mathtt{i}_2(\mathtt{i}_1(A))) \to_{\rm d} \bot_p \tag{18}$$

Proof. The proof goes by induction on the complexity of a formula.

The case of variables: by the definitions of $\mathtt{i}_1$ and $\mathtt{i}_2$, $\mathtt{i}_2(\mathtt{i}_1(a)) = a$, hence due to Lemma 8 stating that (D$_0^{\vdash}$5): $\neg(\neg a \vee a) \to_{\rm d} \bot_p$ is a thesis of $\mathsf{D}_0^{\Vdash}$.

The case of negation: by the definitions of $\mathtt{i}_1$ and $\mathtt{i}_2$,

$$\mathtt{i}_2(\mathtt{i}_1(\neg A)) = \neg\mathtt{i}_2(\mathtt{i}_1(A)) \tag{19}$$

By the inductive hypothesis, we have both $\neg(\neg\mathtt{i}_2(\mathtt{i}_1(A)) \vee A) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash}$ and $\neg(\neg A \vee \mathtt{i}_2(\mathtt{i}_1(A))) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash}$. First, by (19) we have $\neg(\neg\mathtt{i}_2(\mathtt{i}_1(\neg A)) \vee \neg A) = \neg(\neg\neg\mathtt{i}_2(\mathtt{i}_1(A)) \vee \neg A)$. Hence, by ($\mathrm{D_0}$14) $\neg(\neg\mathtt{i}_2(\mathtt{i}_1(\neg A)) \vee \neg A) \rightarrow_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} \neg(\mathtt{i}_2(\mathtt{i}_1(A)) \vee \neg A)$ belongs to $\mathsf{D}_0^{\Vdash}$. So, $\neg(\neg\mathtt{i}_2(\mathtt{i}_1(\neg A)) \vee \neg A) \rightarrow_{\mathrm{d}} \bot_p$ follows by ($\mathrm{D_0}$6), the inductive hypothesis and (Tr^-). Similarly using the other inductive hypothesis we infer $\neg(\neg\neg A \vee \mathtt{i}_2(\mathtt{i}_1(\neg A))) \rightarrow_{\mathrm{d}} \bot_p$.

The case of conjunction. By the definitions of $\mathtt{i}_1$ and $\mathtt{i}_2$,

$$\mathtt{i}_2(\mathtt{i}_1(A \wedge_{\mathrm{d}} B)) = \neg(\neg\mathtt{i}_2(\mathtt{i}_1(A)) \vee \neg\neg(\mathtt{i}_2(\mathtt{i}_1(B)) \rightarrow_{\mathrm{d}} \bot_p)) \tag{20}$$

By the inductive hypothesis we also have $\neg(\neg\mathtt{i}_2(\mathtt{i}_1(B)) \vee B) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash}$ and $\neg(\neg B \vee \mathtt{i}_2(\mathtt{i}_1(B))) \rightarrow_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash}$. Hence the required conditions hold by (12) and (13) given in Lemma 10.

The case of implication. By the definitions of $\mathtt{i}_2$ and $\mathtt{i}_1$ we obtain:

$$\mathtt{i}_2(\mathtt{i}_1(A \rightarrow_{\mathrm{d}} B)) = \neg\neg(\mathtt{i}_2(\mathtt{i}_1(A)) \rightarrow_{\mathrm{d}} \bot_p) \vee \mathtt{i}_2(\mathtt{i}_1(B)) \tag{21}$$

Hence the required conditions hold by (14) and (15).

The case of disjunction. By the definitions of $\mathtt{i}_2$ and $\mathtt{i}_1$ we have:

$$\mathtt{i}_2(\mathtt{i}_1(A \vee B)) = \mathtt{i}_2(\mathtt{i}_1(A)) \vee \mathtt{i}_2(\mathtt{i}_1(B)) \tag{22}$$

Thus, the required fact follows by Lemma 10(16). ■

Theorem 12. (Soundness) *For every thesis A of $\mathsf{D}_0^{\Vdash}$, it belongs to D_0, i.e. $\mathsf{D}_0^{\Vdash} \subseteq \mathsf{D}_0$.*

Proof. First we will prove that each of the axioms belongs to D_0. So, for a given axiom A we show that $A \in \mathsf{D}_0$. By the condition (4) on page 6 it is enough to show that $\mathtt{i}_1(A) \in \mathbf{D}$.

To make notations shorter, i.e. to avoid the usage of values of the function $\mathtt{i}_1$, we will consider specific formulas, but not formula schemas.

$\mathtt{i}_1(\mathrm{D_0}1) = \Diamond q \rightarrow (\neg p \vee p) \wedge \Diamond(\Diamond r \rightarrow q)$, this formula belongs to $\mathbf{K}$.

$\mathtt{i}_1(\mathrm{D_0}2) = \Diamond(q \wedge \Diamond r) \rightarrow (\neg p \vee p) \wedge \Diamond q \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D_0}3) = \Diamond(q \wedge \Diamond r) \rightarrow (\neg p \vee p) \wedge \Diamond((\neg p \vee p) \wedge \Diamond r) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D_0}4) = \Diamond q \rightarrow (\neg p \vee p) \wedge \Diamond(q \vee r) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D_0}5) = \Diamond r \rightarrow (\neg p \vee p) \wedge \Diamond(q \vee r) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D_0}6) = \Diamond\neg(q \vee r) \rightarrow (\neg p \vee p) \wedge \Diamond\neg(r \vee q) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D_0}7) = \Diamond\neg((\Diamond\neg(q \vee r) \rightarrow s) \vee t) \rightarrow (\neg p \vee p) \wedge \Diamond\neg((\Diamond\neg(r \vee q) \rightarrow s) \vee t)$. On the basis of $\mathbf{K}$ it is equivalent to $\Box((\Diamond\neg(r \vee q) \rightarrow s) \vee t) \rightarrow \Box((\Diamond\neg(q \vee r) \rightarrow s) \vee t)$ which belongs to $\mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 8) = \Diamond\neg(q \vee r) \to (((\neg p \vee p) \wedge \Diamond\neg q) \wedge \Diamond\neg r) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 9) = \Diamond((\Diamond A \to B) \wedge \Diamond A) \to ((\neg p \vee p) \wedge \Diamond B) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 10) = \Diamond\neg(\neg(\Diamond q \to s) \vee \neg(\Diamond r \to s)) \to (\neg p \vee p) \wedge \Diamond(\Diamond(q \vee r) \to s)$. It is equivalent on the basis of $\mathbf{K}$ to $\Diamond((\Diamond q \to s) \wedge (\Diamond r \to s)) \to \Diamond(\Diamond(q \vee r) \to s)$, so belongs to $\mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 11) = \Diamond q \to (\neg p \vee p) \wedge \Diamond\neg\neg q \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 12) = \Diamond q \to (\neg p \vee p) \wedge \Diamond q \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 13) = \Diamond\neg(q \vee \neg q) \to r \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 14) = \Diamond\neg(\neg\neg q \vee r) \to ((\neg p \vee p) \wedge \Diamond\neg(q \vee r)) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 15) = \Diamond\neg(\neg(\Diamond\neg\neg q \to r) \vee s) \to (\neg p \vee p) \wedge \Diamond\neg(\neg(\Diamond q \to r) \vee s) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 16) = \Diamond\neg(\neg(\Diamond q \to r) \vee s) \to (\neg p \vee p) \wedge \Diamond\neg(\neg(\Diamond\neg\neg q \to r) \vee s) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 17) = \Diamond\neg((\Diamond q \to r) \vee s) \to (\neg p \vee p) \wedge \Diamond\neg((\Diamond\neg\neg q \to r) \vee s) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 18) = \Diamond\neg(((\neg\neg(\neg p \vee p) \wedge \Diamond q) \vee (\Diamond\neg r \to s)) \vee t) \to (\neg p \vee p) \wedge \Diamond\neg((\Diamond\neg(q \vee r) \to s) \vee t)$. On the basis of $\mathbf{K}$ it is equivalent to $\Box((\neg s \to \Box(\neg q \to r)) \vee t) \to \Box((\neg s \to (\Box\neg q \to \Box r)) \vee t)$, so belongs to $\mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 19) = \Diamond(\Diamond\neg(q \vee r) \to s) \to (\neg p \vee p) \wedge \Diamond((\Diamond\neg q \to ((\neg p \vee p) \wedge \Diamond r)) \vee s)$. It is equivalent on the basis of $\mathbf{K}$ to $\Diamond(\Diamond\neg(q \vee r) \to s) \to \Diamond((\Diamond\neg q \wedge \Box\neg r) \to s)$, so also belongs to $\mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 20) = \Diamond\neg((q \vee r) \vee s) \to (\neg p \vee p) \wedge \Diamond\neg(q \vee (r \vee s)) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 21) = \Diamond\neg((\Diamond q \to r) \vee s) \to (\neg p \vee p) \wedge \Diamond(\neg(r \vee s) \wedge \Diamond q)$. On the basis of $\mathbf{K}$ it is equivalent to $\Diamond((\Diamond q \wedge \neg r) \wedge \neg s) \to \Diamond(\neg(r \vee s) \wedge \Diamond q)$, so belongs to $\mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 22) = \Diamond\neg(\neg(q \vee r) \vee s) \to (\neg p \vee p) \wedge \Diamond(\neg(\neg q \vee s) \vee \neg(\neg r \vee s)) \in \mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 23) = \Diamond\neg(\neg(\Diamond q \to r) \vee s) \to ((\neg p \vee p) \wedge \Diamond\neg((\neg r \wedge \Diamond q) \vee s))$. It is equivalent on the basis of $\mathbf{K}$ to $\Diamond((\Diamond q \to r) \wedge \neg s) \to \Diamond(\neg(\Diamond q \wedge \neg r) \wedge \neg s)$, which belongs to $\mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 24) = \Diamond\neg(\neg(q \wedge \Diamond r) \vee s) \to (\neg p \vee p) \wedge \Diamond(\neg(\neg q \vee s) \wedge \Diamond r)$. By $\mathbf{K}$ it is equivalent to $\Diamond\neg(\neg(q \wedge \Diamond r) \vee s) \to \Diamond(\neg(\neg q \vee s) \wedge \Diamond r)$, so belongs to $\mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 25) = \Diamond(\Diamond(q \vee \neg q) \to \neg(\neg p \vee p)) \to \neg(\neg p \vee p)$ which is equivalent on the basis of $\mathbf{K}$ to $\Box\Diamond(q \vee \neg q)$, hence it belongs to $\mathbf{D}$.

$\mathtt{i}_1(\mathrm{D}_0 26) = \Diamond(\Diamond q \to (r \vee s)) \to (\neg p \vee p) \wedge \Diamond(r \vee (\Diamond q \to s))$ and it belongs to $\mathbf{K}$.

$\mathtt{i}_1(\mathrm{D}_0 27) = \Diamond\neg(q \vee (r \wedge \Diamond s)) \to (\neg p \vee p) \wedge \Diamond(\neg(q \vee r) \vee \neg(q \vee \neg(\Diamond s \to \neg(\neg p \vee p))))$. On the basis of $\mathbf{K}$ it is equivalent to $\Diamond\neg(q \vee (r \wedge \Diamond s)) \to \Diamond\neg((q \vee r) \wedge (q \vee \Diamond s))$, hence it belongs to $\mathbf{K}$.

Second, now we observe that each of the primitive rules leads from theses to theses of D_0.

For ($\wedge_{d_r}^-$) assume that $\top_p \wedge_d B \in \mathsf{D}_0$, by the condition ($\mathrm{def}_{\mathsf{D}_0}$) and Fact 2 it means that $\mathtt{i}_1(\top_p \wedge_d B) \in \mathsf{D}_0$, hence $\Diamond\mathtt{i}_1(B) \in \mathsf{D}_0$, so $B \in \mathsf{D}_0$.

The case of ($\mathrm{MP}^{\to_d}$) was considered in Fact 1. In what follows we skip the references to Fact 2.

The case of ($\mathrm{Tr}_1^{\mathrm{ax}}$). Assume that $A \to_d B \wedge_d C \in \mathsf{D}_0$, i.e. $\mathtt{i}_1(A \to_d B \wedge_d C) \in \mathbf{D}$. Hence $\Diamond\mathtt{i}_1(A) \to \mathtt{i}_1(B) \wedge \Diamond\mathtt{i}_1(C) \in \mathbf{D}$ and in particular, by monotonicity $\Box\Diamond\mathtt{i}_1(A) \to \Box\Diamond\mathtt{i}_1(C) \in \mathbf{D}$, but by classical logic, this means that also $(\Box\Diamond\mathtt{i}_1(C) \to \Diamond\mathtt{i}_1(D)) \to (\neg p \vee p) \wedge (\Box\Diamond\mathtt{i}_1(A) \to \Diamond\mathtt{i}_1(D)) \in \mathbf{D}$, so $(C \to_d D) \to_d \top_p \wedge_d (A \to_d D) \in \mathsf{D}_0$.

The case of ($\mathrm{Tr}_2^{\mathrm{ax}}$). Assume that $B \to_d \top_p \wedge_d C \in \mathsf{D}_0$, i.e. $\mathtt{i}_1(B \to_d \top_p \wedge_d C) \in \mathbf{D}$ and $\Diamond\mathtt{i}_1(B) \to \mathtt{i}_1(\top_p) \wedge \Diamond\mathtt{i}_1(C) \in \mathbf{D}$. Hence $(\Box\Diamond\mathtt{i}_1(A) \to \Diamond\mathtt{i}_1(B)) \to (\neg p \vee p) \wedge (\Box\Diamond\mathtt{i}_1(A) \to \Diamond\mathtt{i}_1(C)) \in \mathbf{D}$. Therefore, $(A \to_d B) \to_d \top_p \wedge_d (A \to_d C) \in \mathsf{D}_0$.

The case of ($\mathrm{Tr}_3^{\mathrm{ax}}$). Assume that $B \to_d C \in \mathsf{D}_0$, i.e. $\mathtt{i}_1(B \to_d C) \in \mathbf{D}$ and $\Diamond\mathtt{i}_1(B) \to \mathtt{i}_1(C) \in \mathbf{D}$. So, $(\Box\Diamond\mathtt{i}_1(A) \to \Diamond\Diamond\mathtt{i}_1(B)) \to (\Box\Diamond\mathtt{i}_1(A) \to \Diamond\mathtt{i}_1(C)) \in \mathbf{D}$. But from this follows $\Diamond(\Diamond\mathtt{i}_1(A) \to (\neg p \vee p) \wedge \Diamond\mathtt{i}_1(B)) \to (\neg p \vee p) \wedge \Diamond(\Diamond\mathtt{i}_1(A) \to \mathtt{i}_1(C)) \in \mathbf{D}$, i.e. $(A \to_d \top_p \wedge_d B) \to_d \top_p \wedge_d (A \to_d C) \in \mathsf{D}_0$.

For the case of ($\mathrm{Tr}_4^{\mathrm{ax}}$) assume $\neg(\neg A \vee B) \to_d \bot_p \in \mathsf{D}_0$, i.e. $\mathtt{i}_1(\neg(\neg A \vee B) \to_d \bot_p) \in \mathbf{D}$. By the definition of $\mathtt{i}_1$ $(\Diamond\neg(\neg\mathtt{i}_1(A) \vee \mathtt{i}_1(B)) \to \neg(\neg p \vee p)) \in \mathbf{D}$. So $\Box(\mathtt{i}_1(A) \to \mathtt{i}_1(B)) \in \mathbf{D}$. By using positive logic we have $(\mathtt{i}_1(A) \to \mathtt{i}_1(B)) \to (\mathtt{i}_1(A) \wedge \Diamond\mathtt{i}_1(C) \to \mathtt{i}_1(B) \wedge \Diamond\mathtt{i}_1(C))$, hence by necessitation, axioms ($\mathtt{K}$) and ($\mathtt{K}^\Diamond$) we obtain that $\Diamond(\mathtt{i}_1(A) \wedge \Diamond\mathtt{i}_1(C)) \to \Diamond(\mathtt{i}_1(B) \wedge \Diamond\mathtt{i}_1(C)) \in \mathbf{D}$, so $\Diamond(\mathtt{i}_1(A) \wedge \Diamond\mathtt{i}_1(C)) \to (\neg p \vee p) \wedge \Diamond(\mathtt{i}_1(B) \wedge \Diamond\mathtt{i}_1(C)) \in \mathbf{D}$. That is by the definition of $\mathtt{i}_1$, we have $\mathtt{i}_1(A \wedge_d C \to_d \top_p \wedge_d (B \wedge_d C)) \in \mathbf{D}$, i.e. $A \wedge_d C \to_d \top_p \wedge_d (B \wedge_d C) \in \mathsf{D}_0$.

The case of ($\mathrm{Syl}^\vee$). We assume that $A \to_d B \in \mathsf{D}_0$ and $C \to_d B \in \mathsf{D}_0$, i.e. $\mathtt{i}_1(A \to_d B) \in \mathbf{D}$ and $\mathtt{i}_1(C \to_d B) \in \mathbf{D}$. By the definition of $\mathtt{i}_1$, we have that $\Diamond\mathtt{i}_1(A) \to \mathtt{i}_1(B) \in \mathbf{D}$ and $\Diamond\mathtt{i}_1(C) \to \mathtt{i}_1(B) \in \mathbf{D}$. Therefore, by positive logic $\Diamond(\mathtt{i}_1(A) \vee \mathtt{i}_1(C)) \to \mathtt{i}_1(B) \in \mathbf{D}$, in other words $\mathtt{i}_1(A \vee C \to_d B) \in \mathbf{D}$, i.e. $A \vee C \to_d B \in \mathsf{D}_0$.

The case of ($\mathrm{Add}^{\wedge_c}$). Assume $A \in \mathsf{D}_0$, i.e. $\mathtt{i}_1(A) \in \mathbf{D}$ and by necessitation $\Box\mathtt{i}_1(A) \in \mathbf{D}$. Hence by positive logic $\Diamond\mathtt{i}_1(B) \to \Box\mathtt{i}_1(A) \wedge \Diamond\mathtt{i}_1(B) \in \mathbf{D}$. Thus, $\Diamond\mathtt{i}_1(B) \to \Diamond(\mathtt{i}_1(A) \wedge \mathtt{i}_1(B)) \in \mathbf{D}$. Therefore, $\Diamond\mathtt{i}_1(B) \to ((\neg p \vee p) \wedge \Diamond\neg(\neg\mathtt{i}_1(A) \vee \neg\mathtt{i}_1(B))) \in \mathbf{D}$. Hence, $\mathtt{i}_1(B \to_d (\top_p \wedge_d \neg(\neg A \vee \neg B))) \in \mathbf{D}$, so $B \to_d (\top_p \wedge_d \neg(\neg A \vee \neg B) \in \mathsf{D}_0$.

The case of (Mon). Assume $A \to_d B \in \mathsf{D}_0$, i.e. $\mathtt{i}_1(A \to_d B) \in \mathbf{D}$. Hence $\Diamond\mathtt{i}_1(A) \to \mathtt{i}_1(B) \in \mathbf{D}$ and also $(\neg p \vee p) \wedge \Diamond\mathtt{i}_1(A) \to \mathtt{i}_1(B) \in \mathbf{D}$, while by monotonicity $\Diamond((\neg p \vee p) \wedge \Diamond\mathtt{i}_1(A)) \to \Diamond\mathtt{i}_1(B) \in \mathbf{D}$, so also $\Diamond((\neg p \vee p) \wedge$

$\Diamond \mathtt{i}_1(A)) \to (\neg p \vee p) \wedge \Diamond \mathtt{i}_1(B) \in \mathbf{D}$ and $\mathtt{i}_1(\top_p \wedge_{\mathrm{d}} A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B) \in \mathbf{D}$. Thus $\top_p \wedge_{\mathrm{d}} A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} B \in \mathsf{D}_0$. ■

Taking into account that D_0 can be defined semantically, we could transfer the following theorem into a completeness or adequacy theorem for D_0. To be more strict, applying the condition defining D_0 on page 5, using standard Kripke-style semantics for the normal modal logic **D** and straightforward semantical reading of conditions defining the translation $\mathtt{i}_1$, one could treat definition ($\mathrm{def}_{\mathsf{D}_0}$) in semantic manners.

THEOREM 13. *For every thesis* $A \in \mathsf{D}_0$ *there is a proof on the basis of* $\Vdash$, *i.e.* $\mathsf{D}_0 \subseteq \mathsf{D}_0^{\Vdash}$.

PROOF. Let us consider a formula $A \in \mathsf{D}_0$. By definition ($\mathrm{def}_{\mathsf{D}_0}$), $\Diamond \mathtt{i}_1(A) \in \mathbf{D}$ and by Lemma 2, it is equivalent to the fact that $\mathtt{i}_1(A) \in \mathbf{D}$. By Lemma 3 there is a proof of the formula $\mathtt{i}_1(A) \in \mathbf{D}$ on the basis of the system $\mathbf{D}^{\vdash}$. Consider a respective proof $\varphi_1, \ldots, \varphi_k = \mathtt{i}_1(A) \in \mathbf{D}$. Now, let us consider the sequence of values of the function $\mathtt{i}_2$ of elements of the initial sequence: $(\mathtt{i}_2(\varphi_i))_{1 \leqslant i \leqslant k}$.

Observe that $\mathtt{i}_2(\Box \mathrm{A}i) = (Ai^{\mathrm{tr}})$ for $1 \leqslant i \leqslant 12$, but by Lemma 8, for every $1 \leqslant i \leqslant 12$, (Ai^{tr}) is a thesis of $\mathsf{D}_0^{\Vdash}$. Next one can see that $\mathtt{i}_2(\Box \mathtt{df}\,\Diamond) = (\mathrm{D}_0^{\vdash}15)$, $\mathtt{i}_2(\Box \mathtt{D}) = (\mathrm{D}_0^{\vdash}14)$, $\mathtt{i}_2(\Box \mathtt{K}) = (\mathrm{D}_0^{\vdash}16)$. Moreover, the translation of every rule among ($\Box$nec), ($\Box$mp), ($\Box\mathrm{mp}_{\_}$) and ($\mathrm{pos}_{\Leftarrow}$) gives respectively ($\Box\mathrm{nec}^{\mathrm{tr}}$), ($\Box\mathrm{mp}^{\mathrm{tr}}$), ($\Box\mathrm{mp}^{\mathrm{tr}}_{\_}$) and ($\mathrm{pos}^{\mathrm{tr}}_{\Leftarrow}$), but by Lemma 9, these rules are inferable for the considered system $\mathsf{D}_0^{\Vdash}$. So, by induction on the length of the proof we see that each element in the sequence $(\mathtt{i}_2(\varphi_i))_{1 \leqslant i \leqslant k}$ is a thesis of $\mathsf{D}_0^{\Vdash}$. In particular, for $i = k$, we have $(\mathtt{i}_2(\mathtt{i}_1(A))) \in \mathsf{D}_0^{\Vdash}$, but by Lemma 11, we have $\neg(\neg \mathtt{i}_2(\mathtt{i}_1(A)) \vee A) \to_{\mathrm{d}} \bot_p \in \mathsf{D}_0^{\Vdash}$, so by ($\Box\mathrm{mp}^{\mathrm{tr}}_{\_}$), we conclude that $A \in \mathsf{D}_0^{\Vdash}$. ■

5. Towards the Embedding Procedure

The reason to base the system on the rules directly relying on discussive connectives is to be close to the formulation of D_2 with modus ponens for discussive as the only rule of inference. Below we will indicate other syntactic analogies, in particular between axioms from the axiomatizations of D_0 and D_2.

Although only one axiom schema (D_{13}) from D_2-axiomatization, given in [21] is D_0-valid ($\mathrm{D}_0 13$): $\neg(A \vee \neg A) \to_d B$, other analogies are evident there. To explicate these analogies, let us denote a discussive formula of the form

$\top_p \wedge_{\rm d} A$ as $(A)^{\Diamond_{\rm d}}$. Using this shortcut we can rewrite the following axioms of the considered axiomatization of D_0:

(D$_0$1) $A \to_{\rm d} (B \to_{\rm d} A)^{\Diamond_{\rm d}}$ (D$_1$)

(D$_0$2) $A \wedge_{\rm d} B \to_{\rm d} A^{\Diamond_{\rm d}}$ (D$_4$)

(D$_0$3) $A \wedge_{\rm d} B \to_{\rm d} (B^{\Diamond_{\rm d}})^{\Diamond_{\rm d}}$ (D$_5$)

(D$_0$4) $A \to_{\rm d} (A \vee B)^{\Diamond_{\rm d}}$ (D$_7$)

(D$_0$5) $B \to_{\rm d} (A \vee B)^{\Diamond_{\rm d}}$ (D$_8$)

(D$_0$6) $\neg(A \vee B) \to_{\rm d} (\neg(B \vee A))^{\Diamond_{\rm d}}$ (D$_{13}$)

(D$_0$8) $\neg(A \vee B) \to_{\rm d} ((\neg A)^{\Diamond_{\rm d}} \wedge_{\rm d} \neg B)$ (D$_{14}$)

(D$_0$11) $A \to_{\rm d} (\neg\neg A)^{\Diamond_{\rm d}}$ (D$_{10}$)

(D$_0$14) $\neg(\neg\neg A \vee B) \to_{\rm d} (\neg(A \vee B))^{\Diamond_{\rm d}}$ (D$_{15}$)

(D$_0$19) $(\neg(A \vee B) \to_{\rm d} C) \to_{\rm d} ((\neg A \to_{\rm d} B^{\Diamond_{\rm d}}) \vee C)^{\Diamond_{\rm d}}$ (D$_{16}$)

(D$_0$20) $\neg((A \vee B) \vee C) \to_{\rm d} (\neg(A \vee (B \vee C)))^{\Diamond_{\rm d}}$ (D$_{17}$)

(D$_0$22) $\neg(\neg(A \vee B) \vee C) \to_{\rm d} (\neg(\neg A \vee C) \vee \neg(\neg B \vee C))^{\Diamond_{\rm d}}$ (D$_{20}$)

(D$_0$24) $\neg(\neg(A \wedge_{\rm d} B) \vee C) \to_{\rm d} (\neg(\neg A \vee C) \wedge_{\rm d} B)^{\Diamond_{\rm d}}$ (D$_{22}$)

The axiom:

(D$_0$7) $\neg((\neg(A \vee B) \to_{\rm d} C) \vee D) \to_{\rm d} (\neg((\neg(B \vee A) \to_{\rm d} C) \vee D))^{\Diamond_{\rm d}}$

serves as an additional variant of the axiom (D$_0$6) and also corresponds to (D$_{13}$) assuring it in the needed contexts.

In the context of (D$_{10}$), (D$_{11}$) and (D$_{15}$), a similar role is played by:

(D$_0$15) $\neg(\neg(\neg\neg A \to_{\rm d} B) \vee C) \to_{\rm d} (\neg(\neg(A \to_{\rm d} B) \vee C))^{\Diamond_{\rm d}}$

(D$_0$16) $\neg(\neg(A \to_{\rm d} B) \vee C) \to_{\rm d} (\neg(\neg(\neg\neg A \to_{\rm d} B) \vee C))^{\Diamond_{\rm d}}$

(D$_0$17) $\neg((A \to_{\rm d} B) \vee C) \to_{\rm d} (\neg((\neg\neg A \to_{\rm d} B) \vee C))^{\Diamond_{\rm d}}$

which together with (D$_0$11) and (D$_0$14) allow to handling double negations.

The axiom:

(D$_0$21) $\neg((A \to_{\rm d} B) \vee C) \to_{\rm d} (\neg(B \vee C) \wedge_{\rm d} A)^{\Diamond_{\rm d}}$

naturally uses the idea of (D$_{18}$), however, for D_2 the order of conjuncts can be changed.

Besides,

(D$_0$18) $\neg(((\neg\bot_p \wedge_{\mathrm{d}} A) \vee (\neg B \rightarrow_{\mathrm{d}} C)) \vee D) \rightarrow_{\mathrm{d}} (\neg((\neg(A \vee B) \rightarrow_{\mathrm{d}} C) \vee D))^{\Diamond_{\mathrm{d}}}$

corresponds in some way to (D$_{16}$) in the contexts of $\neg$ and $\vee$ with some additional needed transformations. In particular, due to (D$_{13}$) and positive logic, one can use as the consequent of (D$_{16}$) also $((\neg B \rightarrow_{\mathrm{d}} A) \vee C)$ (= E). On the other hand, the negated formula in the antecedent of (D$_0$18) corresponds to the formula $A \vee (\neg B \rightarrow_{\mathrm{d}} C))$ (= F), while one can see that on the basis of the axiomatic system of D_2, formulas E and F are equivalent in the sense that $(\neg E \vee F) \rightarrow_{\mathrm{d}} \bot_p$ is inferable on the basis of D_2.[6]

The case of

(D$_0$23) $\neg(\neg(A \rightarrow_{\mathrm{d}} B) \vee C) \rightarrow_{\mathrm{d}} (\neg((\neg B \wedge_{\mathrm{d}} A) \vee C))^{\Diamond_{\mathrm{d}}}$

is more complicated. In D_2 it corresponds to the formula $(A \rightarrow_{\mathrm{d}} B) \rightarrow_{\mathrm{d}} \neg(\neg B \wedge_{\mathrm{d}} A)$ in the context of $\vee$ and $\neg$. The proof of the sole formula is quite long, it requires the thesis $(\neg(\neg A \vee B) \rightarrow_{\mathrm{d}} \bot_p) \rightarrow_{\mathrm{d}} (A \rightarrow_{\mathrm{d}} B)$ (on the basis of the axiomatization of D_2 one can prove it using (D$_{16}$), (D$_{12}$), (D$_{13}$), (D$_{10}$) and positive logic), while the postulated schema can be inferred using the formula $\neg(\neg(A \rightarrow_{\mathrm{d}} B) \vee \neg(\neg B \wedge_{\mathrm{d}} A)) \rightarrow_{\mathrm{d}} \bot_p$ (its proof can be obtained with the help of (D$_{13}$), (D$_{22}$), (D$_{15}$), (D$_{21}$) and positive logic).

Similarly, on the basis of D_2, the counterpart of our axiom

(D$_0$27) $\neg(A \vee (B \wedge_{\mathrm{d}} C)) \rightarrow_{\mathrm{d}} (\neg(A \vee B) \vee \neg(A \vee \neg(C \rightarrow_{\mathrm{d}} \bot_p)))^{\Diamond_{\mathrm{d}}}$

could be proved by using, among others, (D$_{20}$), (D$_{18}$), (D$_{19}$), however due to limitations of D_0, this proof cannot be conducted.[7]

There are axioms whose content in D_2 is covered by positive logic, in particular the form of (D$_0$10) and is connected with 'the cost' of modalities involved in discussive functors, so in its antecedent the classical formulation of conjunction has been used.

The axiom:

(D$_0$10) $\neg(\neg(A \rightarrow_{\mathrm{d}} C) \vee \neg(B \rightarrow_{\mathrm{d}} C)) \rightarrow_{\mathrm{d}} (A \vee B \rightarrow_{\mathrm{d}} C)^{\Diamond_{\mathrm{d}}}$

[6]Of course, what we are presenting here is not a formal proof of the formula (D$_0$18) on the basis of D_2 but only some intuitions that show a kind of correspondence between the considered axioms. The full version of the proof of $\neg(((\neg\bot_p \wedge_{\mathrm{d}} A) \vee (\neg B \rightarrow_{\mathrm{d}} C)) \vee D) \rightarrow_{\mathrm{d}} (\neg((\neg(A \vee B) \rightarrow_{\mathrm{d}} C) \vee D))$ on the basis of D_2 requires quite few applications of axiom (D$_{16}$).

[7]Notice that due to Theorem 12 and definition of D_0, as well as the fact that none of values of the function $\mathtt{i}_1$ at (D$_{20}$), (D$_{18}$), (D$_{19}$) is a thesis of **D**, none of these formulas is a thesis of D_0.

corresponds to the axiom (D_9). Strictly speaking, it corresponds to $\neg(\neg(A \rightarrow_d C) \vee \neg(B \rightarrow_d C)) \rightarrow_d (A \vee B \rightarrow_d C))$. One can easily see that by positive logic valid in D_2 it is enough on the side of D_2 to refer next to (D_9), additionally to the scheme $\neg(\neg A \vee \neg B) \rightarrow_d (A \wedge_d B)$, which can be easily obtained on the basis of the axiomatization of D_2 by (D_{14}), (D_4)–(D_{16}), (D_{11}) and positive logic.

And there are also other schemas, which correspond to positive logic:

($\mathsf{D}_0$9) $(A \rightarrow_d B) \wedge_d A \rightarrow_d B^{\Diamond_d}$

($\mathsf{D}_0$12) $A \rightarrow_d A^{\Diamond_d}$

($\mathsf{D}_0$26) $(A \rightarrow_d (B \vee C)) \rightarrow_d (B \vee (A \rightarrow_d C))^{\Diamond_d}$

As regards the rules, their role is either to directly obtain positive inferences or—as in the case of ($\mathrm{Tr}_4^{\mathrm{ax}}$)—to simulate the use of positive logic. Interestingly, the rule ($\mathrm{Add}^{\wedge_c}$) is not valid for D_2 in general, but in the used context, needed cases are also legitimate for D_2.

Hence, despite the weakness of discussive implication in D_0 observed on page 6,[8] there are $\rightarrow_d$-theorems of D_2 that are provable on the basis of the given axiomatization of D_0, or saying more, since D_0 is contained in D_2, and due to the above-mentioned analogies, at least some proofs conducted on the basis of this axiomatization can be transferred into an inference on the basis of the axiomatization of D_0—as an example, one can mention the proof of ($\mathsf{D}_0^{\vdash}$3).

On the basis of D_2 the formula:

($\mathsf{D}_0$25) $(A \vee \neg A \rightarrow_d \bot_p) \rightarrow_d \bot_p$

follows from the above-mentioned formula $(\neg(\neg A \vee B) \rightarrow_d \bot_p) \rightarrow_d (A \rightarrow_d B)$, ($D_{14}$), ($D_{11}$), positive logic and the thesis $A \vee \neg A$. Again the proof cannot be repeated due to the weak part of positive logic that is valid for D_0.

As we mentioned, one of the reasons and the aim was also to identify the smallest part which is in the same language as D_2 formalized in the language with right discussive conjunction, since such a language is nowadays treated as the intended one by Jaśkowski after an amendment presented by him in 1949. The aim of the current paper seemed to us to give an axiomatization that would correspond (in some way) to the axiomatization of D_2 that is given by the correction in [21] of the proposal in [3]. Although, this 'correspondence' has not been defined, the presented in this section

[8]Let us recall the non-validity of (Syl) on the basis of D_0 observed there.

syntactic similarities between the given axiomatisation of D_0 and axiomatisation of D_2 is proposed by us as a small justification for postulating a kind of a correlation between both systems, including the role of rules of $\mathsf{D}_0^{\Vdash\vdash}$ needed to express in a way the behaviour of the positive part of classical logic. Notice, for example, that although $p \to_{\mathrm{d}} p$ is not a thesis of D_0, $p \to_d (p \vee \neg p) \wedge_{\mathrm{d}} p = p \to_d (p)^{\Diamond_{\mathrm{d}}}$ is.

Since D_0 is a proper subsystem of D_2, a natural question concerning the relationships between these two systems arises. Specifically, can we manage to embed D_2 into D_0 or is this impossible in principle? Having at hand axiomatizations of the above-mentioned systems we can try to address the problem.

First of all, in spite of the mentioned similarities between the two axiomatizations appearing in logical forms of axiom schemas, we cannot directly reuse any of D_2-schemas in axiomatizations of D_0. To see why this happens we shall extensively use modal counterparts of the systems. For instance, $A \to_{\mathrm{d}} (B \to_{\mathrm{d}} A)$ is not a D_0-thesis, because its translation into the modal language, $\Diamond \mathtt{i}_1(A) \to (\Diamond \mathtt{i}_1(B) \to \mathtt{i}_1(A))$, is not a thesis of the modal system $\Diamond$-**D**, since the scheme $\Diamond A \to (\Diamond B \to A)$ is not a valid schema on the basis of **D**.[9] Fortunately, $\Diamond$-$\mathbf{D} = \mathbf{D}$, so it is convenient to use the existing proof-theoretical tools for **D** to check D_0 related facts.

It appears that if we add the constant $\top_p$ to the consequent of the above schema, which results in $A \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (B \to_{\mathrm{d}} A)$, we obtain an expression which is still not too far away from the original form but fits better for the purposes of the axiomatization of D_0, since its translation, $\Diamond A \to \top_p \wedge \Diamond(\Diamond B \to A)$, *is* a thesis of **D**. We can rewrite it in an equivalent form $\Diamond A \to \Diamond(\Diamond B \to A)$ to see that the point here is the "compensation" of the presence of a diamond in the antecedent in front of A.

But the situation can be slightly more complicated. Consider the schema $A \wedge_{\mathrm{d}} B \to_{\mathrm{d}} B$ from the list of axioms of D_2. Its D_0-analogue is $A \wedge_{\mathrm{d}} B \to_{\mathrm{d}} \top_p \wedge_{\mathrm{d}} (\top_p \wedge_{\mathrm{d}} B)$. Why do we have a duplication of $\top_p$ now? Because $\mathtt{i}_1(A \wedge_{\mathrm{d}} B \to_{\mathrm{d}} B) = \Diamond(A \wedge \Diamond B) \to B$. As one can see, now the subformula B has a deeper "nested diamonds depth" in the antecedent, so we need $\Diamond(A \wedge \Diamond B) \to \Diamond\Diamond B$ to convert the translation into a thesis of **D**. However, we can observe that using the discussive translation of the modal scheme $A \wedge_{\mathrm{d}} B \to_{\mathrm{d}} B$ into the modal language, together with the application of simplifications valid for **S5**, we obtain $(\Diamond A \wedge \Diamond B) \to \Diamond B$. And this

[9] For this reason we do not need to bother about the concrete result of $\mathtt{i}_1(A)$, $\mathtt{i}_1(B)$ and so on. So, when there no potential confusion appears, we shall skip recursive calling of $\mathtt{i}_1$ when reasoning about translations of schemes.

formula is also a thesis of **D**. We can see that the usage of the mentioned simplifications of modal formulas can be repeated in the general way which leads to a transformation of a given discussive formula into a modal version without iterated modalities. Moreover, having modal formulas without iterated modalities we could use the results on relations between sets of theses without iterated modalities of subsystems of **S5** [22]. That's a guiding idea.

Following e.g., [22] we say that a modal formula involves iterated modalities iff some instance of '$\Box$' or '$\Diamond$' occurs within the scope of some other instance of '$\Box$' or '$\Diamond$'. We say $A \in \mathrm{For_m}$ is *at most of the first-degree*[10] iff it either does not contain any modal operator or contains a modal operator, but does not involve iterated modalities. Let ${}^1\mathrm{For_m}$ be the set of all at most the first-degree formulas.

A formula in $\mathrm{For_m}$ is said to be in Modal Conjunctive Normal Form iff it is a conjunction (possibly degenerated), each conjunct of which is a disjunction (possibly degenerated) of classical formulas or formulas of the form $\Box\alpha_i$, for some natural number i or a formula $\Diamond\alpha$, where α_i and α are classical formulas (see, e.g., [6]). Let MCNF be the set of all such formulas.

It is a well known fact that for any $\varphi \in \mathrm{For_m}$ there is $\varphi' \in {}^1\mathrm{For_m}$ such that $\varphi \leftrightarrow \varphi' \in \mathbf{S5}$ (see [6, p. 98]). One can easily see the same result holds for **KD45**. Although the above-mentioned φ' is not determined uniquely, taking into account that all these formulas are equivalent on the basis of **KD45**, we can assume that under some order on the set $\mathrm{For_m}$, we can take the earliest respective formula under the given order. So, for any φ, let the above-described formula in ${}^1\mathrm{For_m}$ be denoted as $\mathtt{m}(\varphi)$.

As it is known, to define D_2, one can use any modal logic which has the same theses beginning with '$\Diamond$' as **S5**. Let $\mathbf{S5}_\Diamond$ be the set of all modal logics such that $\boldsymbol{L} \in \mathbf{S5}_\Diamond$ iff $\forall_{A\in}(\ulcorner\Diamond A\urcorner \in \boldsymbol{L} \iff \ulcorner\Diamond A\urcorner \in \mathbf{S5})$. It is known [2,15,16,24] that the logic $\mathbf{S5^M}$—the smallest normal logic defining D_2 and simultaneously the smallest normal logic in $\mathbf{S5}_\Diamond$, is the smallest normal logic containing

$$\Box\Diamond\Diamond p \to \Diamond p$$
$$\Box\Diamond p \to \Diamond p$$

Moreover, since for any modal logic $\boldsymbol{L}$: if $\mathbf{S5^M} \subseteq \boldsymbol{L} \subseteq \mathbf{S5}$, then $\boldsymbol{L} \in \mathbf{S5}_\Diamond$, so $\Diamond\mathbf{S5} = \Diamond\mathbf{KD45}$ (see for example [16]). On the other hand, ${}^\mathbf{1}\mathbf{KD45} = {}^\mathbf{1}\mathbf{D}$ (see [22,23])[11]

[10] In [22] the term "first-degree" is used instead.

[11] Although in [22] the language with $\Box$ as the only modal operator is concerned, one can easily see that the respective result holds for the language with $\Box$ and $\Diamond$.

To be able to apply a result from [13], we recall a translation $\mathtt{i}_3\colon \mathrm{For}_{\mathrm{m}} \longrightarrow \mathrm{For}_{\mathrm{d}}$[12]:

1. $\mathtt{i}_3(a) = a$, for any $a \in \mathrm{At}$,
2. for any $\varphi, \psi \in \mathrm{For}_{\mathrm{m}}$:
   (a) $\mathtt{i}_3(\neg\varphi) = \neg\mathtt{i}_3(\varphi)$,
   (b) $\mathtt{i}_3(\Box\varphi) = \neg((\neg p \vee p) \wedge_{\mathrm{d}} \neg\mathtt{i}_3(\varphi))$,
   (c) $\mathtt{i}_3(\Diamond\varphi) = (\neg p \vee p) \wedge_{\mathrm{d}} \mathtt{i}_3(\varphi)$,
   (d) $\mathtt{i}_3(\varphi \vee \psi) = \mathtt{i}_3(\varphi) \vee \mathtt{i}_3(\psi)$,
   (e) $\mathtt{i}_3(\varphi \wedge \psi) = \neg(\neg\mathtt{i}_3(\varphi) \vee \neg\mathtt{i}_3(\psi))$,
   (f) $\mathtt{i}_3(\varphi \rightarrow \psi) = \neg\mathtt{i}_3(\varphi) \vee \mathtt{i}_3(\psi)$,
   (g) $\mathtt{i}_3(\varphi \leftrightarrow \psi) = \neg(\neg(\neg\mathtt{i}_3(\varphi) \vee \mathtt{i}_3(\psi)) \vee \neg(\neg\mathtt{i}_3(\psi) \vee \mathtt{i}_3(\varphi)))$.

We have:

LEMMA 14. ([13]) *For any* $\varphi \in \mathrm{For}_{\mathrm{m}}$, $\mathtt{i}_1(\mathtt{i}_3(\varphi)) \leftrightarrow \varphi \in \mathbf{D}$.

Hence we see that the following sequence holds: $A \in \mathsf{D}_2$ iff $\Diamond\mathtt{i}_1(A) \in \mathbf{S5}$ iff $\Diamond\mathtt{i}_1(A) \in \mathbf{KD45}$ iff $\mathtt{m}(\Diamond\mathtt{i}_1(A)) \in \mathbf{KD45}$ iff $\mathtt{m}(\Diamond\mathtt{i}_1(A)) \in \mathbf{D}$ iff $\mathtt{i}_1(\mathtt{i}_3(\mathtt{m}(\Diamond\mathtt{i}_1(A)))) \in \mathbf{D}$ iff $\mathtt{i}_3(\mathtt{m}(\Diamond\mathtt{i}_1(A))) \in \mathsf{D}_0$.

So, we have proven that:

THEOREM 15. *There is a function that translates all theses of* D_2 *into theses of* D_0 *and only them.*

6. Conclusion

These considerations can be treated as an initial step in the investigations on other variants of discussive logics obtained by other cases of relations that connect participants of a discussion. Following the given considerations, as a work for the future, the problem of axiomatizing a non-trivial minimal paracomplete discussive logic contained in the system $\mathbf{D}_2^{\mathsf{p}}$ considered in [14] can be formulated.

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[12] In [13] $\mathtt{i}_3$ is denoted as $\mathtt{i}_2$.

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