Matteo Tesi

On the Proof Theory of Infinitary Modal Logic

Abstract. The article deals with infinitary modal logic. We first discuss the difficulties related to the development of a satisfactory proof theory and then we show how to overcome these problems by introducing a labelled sequent calculus which is sound and complete with respect to Kripke semantics. We establish the structural properties of the system, namely admissibility of the structural rules and of the cut rule. Finally, we show how to embed common knowledge in the infinitary calculus and we discuss first-order extensions of infinitary modal logic.

Keywords: Proof theory, Modal logic, Infinitary logic, Common knowledge .

1. Introduction

Infinitary modal logic is modal logic extended with countable conjunctions and disjunctions. Infinitary languages are particularly relevant in the modal setting, because various interesting fixpoint modal operators, e.g. common knowledge [9], can be defined by means of infinitary modal logic.

An axiomatization of infinitary modal logic which is sound and complete with respect to Kripke semantics augmented with natural—so to say—truth conditions for infinitary conjunctions and disjunctions has been provided in [19]. However, an axiomatic presentation is not suitable to reason on the structure of the derivations due to the lack of analyticity of the system. On the contrary, sequent calculi and their generalizations are often preferred in order to obtain analytic proof systems for non-classical logics which allow for backward reasoning.

It is well known that the standard sequent calculus for classical propositional logic **LK** with the addition of the rule:

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} \; \kappa$$

yields an axiomatization of the minimal normal modal propositional logic K [16]. Moreover, the calculus is cut free, in the sense that there is a syntactic procedure for cut elimination.

The extension of the basic system **LK** with infinitary countable conjunctions and disjunctions is obtained by adding the following rules for the infinitary conjunction:

$$\frac{A, \Gamma \Rightarrow \Delta}{\bigwedge \Theta, \Gamma \Rightarrow \Delta} \; L \bigwedge, \text{ for some } A \in \Theta \qquad\qquad \frac{\text{...} \qquad \Gamma \Rightarrow \Delta, A \qquad \text{...}}{\Gamma \Rightarrow \Delta, \bigwedge \Theta} \; R \bigwedge, \text{ for every } A \in \Theta$$

the rules for the infinitary disjunction connective are dual. A derivation is now defined as a well-founded rooted tree and rules are allowed to have countably many premises (for an overview on the history of infinitary logic the reader is referred to [22]). Such infinitary system that we shall denote by **LK**$_\omega$ enjoys a syntactic cut-elimination procedure arguing by transfinite induction [8].

The system **LK**$_\omega^\square$, which is obtained by **LK**$_\omega$ by adding the rule K for the modal operator, enjoys cut-elimination (with minor differences in the strategy), but it is not complete with respect to Kripke semantics. In particular, the sequent $\Rightarrow \bigwedge \square \Gamma \rightarrow \square \bigwedge \Gamma$—an infinitary version of the Barcan formula—is indeed valid, but is not derivable due to the context restriction imposed on the rule for the modal operator. In fact, as observed by Minari in [11], such system proves complete with respect to a suitable generalization of Kripkean semantics. Minari establishes some other negative results concerning the proof theory for infinitary modal logic.

In particular, he considers some alternative sequent calculi obtained by considering sequents built from infinite (multi)sets of formulas, rather than finite (multi)sets. The system thus obtained allows to regain soundness and completeness with respect to Kripke semantics, but it provably does not admit cut-elimination.

A cut-free sequent calculus was introduced by Tanaka in [25]. His system is called **TLM**$_\omega$ and consists of finite trees of (prefixed) sequents (possibly) countably infinite. However, Tanaka does not present a syntactic proof of cut elimination, but a semantic proof of completeness, which entails closure under cut. Furthermore, a purely syntactic proof of cut-elimination does not seem to be easily obtainable due to the presence of sequents with infinite multisets of formulas which hinder the application of standard cut-elimination strategies.

In the present paper we solve the problem of finding a syntactic cut-elimination procedure for infinitary modal logic [11]. In order to obtain the desired result we generalize the structure of the calculus. In particular, we introduce a labelled sequent [13] calculus for infinitary modal logic based on Kripke semantics. The use of a $G3$-style labelled sequent calculus allows

to regain the structural properties which are lost with more standard formalisms. Furthermore, the sequents are still built from finite multisets of formulas and this enables a structural analysis of the system. We establish the usual proof theoretic results of admissibility of weakening, contraction and cut as well as invertibility of every rule with preservation of height by means of (transfinite) induction.

We believe that a syntactic proof of cut-elimination for infinitary modal logic can be indeed regarded as a *desideratum* for at least three different reasons. First, a syntactic proof of cut-elimination is more informative as it enables to extract some bounds out of the procedure of removal of the cuts (which are not obtained via semantic methods). Second, it is a stronger result with respect to closure under cut, because there are various systems for which we have a semantic proof of completeness which entails closure under cut, but not a syntactic proof thereof. Third, a syntactic proof of cut-elimination is more informative both from a conceptual and a computational point of view. From a conceptual point of view, cut-elimination expresses a syntactic property of the system and therefore it is natural to ask whether it can be established by using syntactic methods. From a technical perspective, a syntactic proof of cut-elimination gives an effective procedure to transform a derivation containing cuts into a cut-free one and thus allows for the possibility of coding the derivations and the proof transformation (as in [21]).

Another interesting aspect to be remarked is that the formulation of a sequent calculus for infinitary modal logic admitting a syntactic proof of cut-elimination seems to *essentially* require the adoption of a generalization of the standard sequent calculus and a careful formulation of the rules of the system. In particular, the key to obtain a syntactic proof of cut-elimination is to work with finite sequents and to opt for an invertible formulation of the rule for the modal operator which requires a generalization of the structure of the sequents.

Furthermore, labelled sequent calculi allow for a uniform and modular presentation of every modal logic based on Kripke frames whose conditions are expressible in first-order geometric logic. As a consequence, we obtain as a payoff cut-free labelled sequent calculi for every infinitary modal logic characterized by Kripke frames with a first-order geometric condition. In this paper we focus on infinitary modal logics which have an axiomatic Hilbert style presentation, but lack an analytic sequent calculus with a cut-elimination procedure. Thus soundness and completeness are established for every labelled sequent system corresponding to an infinitary modal logic **L**

which is sound and complete with respect to a class of frames characterized by geometric conditions.

We exploit the calculi to show that the logic of common knowledge can be embedded in infinitary modal logic by exploiting the infinitary interpretation of the fixpoint operator for common knowledge [9]. The result is not new (the proof theory of common knowledge has been investigated in [10,27]), but we present a direct proof which explicitly shows all the steps required for the embedding. Our proof is obtained by purely syntactic means, by showing the cut-free derivations of the axioms of the logic of common knowledge in the labelled sequent calculus $\mathbf{G3K}_\omega$.

Finally, we discuss the extension of the present approach to first-order infinitary modal logics. As proved by Tanaka in [24], every first-order infinitary modal logic characterized by a class of Kripke frames with a universal frame condition and constant domains allows for a Hilbert-style axiomatization. We introduce labelled sequent calculi for every logic $\mathbf{L}$ with a Hilbert style axiomatization and characterized by a class of frames with a geometric frame condition. The calculi enjoy a syntactic cut-elimination proof which is carried out along the lines of the one for infinitary propositional modal logic.

The structure of the paper is as follows. In Section 2 we introduce the labelled sequent calculus $\mathbf{G3K}_\omega$ and we analyze the differences with respect to the unlabelled sequent calculus. Section 3 deals with the structural analysis of $\mathbf{G3K}_\omega$ and we present a syntactic proof of cut-elimination. Section 4. is devoted to show the embedding of common knowledge into infinitary modal logic. In Section 5 we offer a brief overview of the first-order extensions of the system to first-order infinitary modal logics. Finally, in Section 6 we conclude the paper by sketching some themes which may be object of future research.

2.   Labelled Sequent Calculus for Infinitary Modal Logic

We introduce the language of the infinitary propositional modal logic in which we admit countable conjunctions and disjunctions. The language contains a denumerable set of propositional atoms $p_0, p_1, \ldots$ which we denote by AT, the connectives $\neg, \square$ and $\rightarrow$, the infinitary connectives $\bigwedge$ and $\bigvee$ and the parentheses. Complex formulas are built up from propositional atoms using the finitary and the infinitary connectives. In particular, given a (possibly countably infinite) multiset of formulas Φ, the formulas $\bigwedge \Phi$ and $\bigvee \Phi$ denote the (infinitary) conjunction and disjunction of the elements in Φ. We

denote by $\mathsf{FOR}_\omega^\square$ the set of formulas of infinitary modal logic. We now recall the definition of Kripkean semantics [1] extended with truth conditions for countable conjunctions and disjunctions [19].

DEFINITION 2.1. A Kripke frame $\mathcal{F}$ is an ordered pair $\langle W, R \rangle$, where W is a non-empty set and R is a binary relation on W. A Kripke model is an ordered pair $\mathcal{M} = \langle \mathcal{F}, v \rangle$, where $\mathcal{F}$ is a Kripke frame and v is a function from AT to $\mathcal{P}(W)$. Given a world $x \in W$ and a formula $A \in \mathsf{FOR}_\omega^\square$, the satisfiability relation $x \vDash_\mathcal{M} A$, A is true at the world x in the model $\mathcal{M}$, is inductively defined:

- $x \vDash_\mathcal{M} p$ iff $x \in v(p)$
- $x \vDash_\mathcal{M} \neg B$ iff $x \nvDash_\mathcal{M} B$
- $x \vDash_\mathcal{M} B \to C$ iff $x \nvDash_\mathcal{M} B$ or $x \vDash_\mathcal{M} C$
- $x \vDash_\mathcal{M} \bigwedge \Phi$ iff $x \vDash_\mathcal{M} B$ for every $B \in \Phi$
- $x \vDash_\mathcal{M} \bigvee \Phi$ iff $x \vDash_\mathcal{M} B$ for some $B \in \Phi$
- $x \vDash_\mathcal{M} \square B$ iff $y \vDash_\mathcal{M} B$ for every $y \in W$ s.t. xRy

A formula $A \in \mathsf{FOR}_\omega^\square$ is true in a model $\mathcal{M}$, in symbols $\vDash_\mathcal{M} A$, iff $x \vDash_\mathcal{M} A$ for every x in the model $\mathcal{M}$. A formula $A \in \mathsf{FOR}_\omega^\square$ is valid in a frame $\mathcal{F}$ iff $\vDash_\mathcal{M} A$ for every model $\mathcal{M}$ based on $\mathcal{F}$. A formula $A \in \mathsf{FOR}_\omega^\square$ is valid in a class of frames $\mathcal{C}$ if and only if it is valid in every frame in the class.

We will often drop the subscript $\mathcal{M}$ when no confusion arises. Now we introduce some notational conventions which will be useful in the following. The modular extensions are defined as usual by imposing further conditions on the accessibility relations, see [1]. We will focus on the minimal infinitary modal logic $\mathbf{K}_\omega$ which is characterized by the class of all Kripke frames.

Notational conventions
The letters $p, q, r \ldots$, $A, B, C \ldots$ and $\Gamma, \Delta, \Sigma \ldots$ are metavariables for atomic formulas, formulas and multisets of formulas, respectively. We write $A \wedge B$ instead of $\bigwedge \{A, B\}$ and the same dually holds for $A \vee B$ and $\lozenge$ abridges $\neg \square \neg$. Finally, $\square \Gamma$ indicates the multiset $[\square A \mid A \in \Gamma]$.

A complete and sound axiomatization for the modal logic $\mathbf{K}_\omega$ was presented in [19], see Figure 1 ($\mathbf{C}$ is the axiomatic calculus for classical propositional logic).

The modular extensions are defined by adding the corresponding axiom schemata, for instance $\mathbf{T}_\omega = \mathbf{K}_\omega + \square A \to A$.

We now introduce the labelled sequent calculus $\mathbf{G3K}_\omega$ in Figure 2. The freshness condition on the rule R$\square$ indicates that y does not occur in the

Axioms

The axioms of **C**

$\Box(A \to B) \to (\Box A \to \Box B)$ distribution

$\bigwedge \Box \Theta \to \Box \bigwedge \Theta$ Barcan formula

$\Box \bigwedge \Theta \to \bigwedge \Box \Theta$ Converse Barcan formula

$\bigwedge \Theta \to A$ for every A in Θ infinitary conjunction

Inference Rules

$$\dfrac{\vdash A \quad \vdash A \to B}{\vdash B} \text{ MP} \qquad\qquad \dfrac{\vdash A}{\vdash \Box A} \text{ RN} \qquad\qquad \dfrac{\vdash A \text{ for every } A \text{ in } \Theta}{\vdash \bigwedge \Theta} \text{ CONJ}$$

Figure 1. The axiomatic calculus $\mathbf{K}_\omega$

Initial Sequents

$$\dfrac{}{x : p, \Gamma \Rightarrow \Delta, x : p} \; Ax$$

Logical Rules

$$\dfrac{\Gamma \Rightarrow \Delta, x : A}{x : \neg A, \Gamma \Rightarrow \Delta} \; L\neg \qquad\qquad\qquad \dfrac{x : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : \neg A} \; R\neg$$

$$\dfrac{x : A, x : \bigwedge \Theta, \Gamma \Rightarrow \Delta}{x : \bigwedge \Theta, \Gamma \Rightarrow \Delta} \; L\bigwedge, A \in \Theta \qquad\qquad \dfrac{...\Gamma \Rightarrow \Delta, x : A...\text{for every } A \in \Theta}{\Gamma \Rightarrow \Delta, x : \bigwedge \Theta} \; R\bigwedge$$

$$\dfrac{...x : A, \Gamma \Rightarrow \Delta...\text{for every } A \in \Theta}{x : \bigvee \Theta, \Gamma \Rightarrow \Delta} \; L\bigvee \qquad\qquad \dfrac{\Gamma \Rightarrow \Delta, x : \bigvee \Theta, x : A}{\Gamma \Rightarrow \Delta, x : \bigvee \Theta} \; R\bigvee, A \in \Theta$$

$$\dfrac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \to B, \Gamma \Rightarrow \Delta} \; L\to \qquad \dfrac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \to B} \; R\to$$

$$\dfrac{xRy, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{xRy, x : \Box A, \Gamma \Rightarrow \Delta} \; L\Box \qquad \dfrac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A} \; R\Box, y \text{ fresh}$$

$$\dfrac{\Gamma \Rightarrow \Delta, x : A \quad x : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; \text{Cut}$$

Figure 2. The labelled sequent calculus $\mathbf{G3K}_\omega$

conclusion of the rule. Labelled systems are a wide class of calculi which explicitly internalize semantic features by enriching the syntax of the calculus, for an overview see [7,13,29].

The language of $\mathbf{G3K}_\omega$ contains a denumerable set of labels $x_0, x_1, \ldots$, the sequent arrow symbol $\Rightarrow$, a relational symbol R, the symbol :. A labelled formula is an object $x : A$, where x is a label and A is a formula in $\mathsf{FOR}_\omega^\square$, a relational atom is an object of the form xRy. A labelled sequent is an object $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of relational atoms and labelled formulas and of labelled formulas, respectively.

DEFINITION 2.2. A derivation $\mathcal{D}$ in $\mathbf{G3K}_\omega$ is a (possibly infinitely branching) well-founded tree of sequents locally correct with respect to the rules, i.e.:

- The leaves in $\mathcal{D}$ are initial sequents;
- Every sequent that occupies a node that is not a leaf is the conclusion of an application of a rule in $\mathbf{G3K}_\omega$ that has as (possibly infinite) premise(s) the sequent(s) that occupies its parent(s) node(s).
- The root is the conclusion

Notice that our calculus $\mathbf{G3K}_\omega$ deals with finite sequents and it is an extension of the labelled sequent calculus for the minimal normal modal logic $\mathbf{G3K}$ [14]. The rules for the infinitary conjunctions and disjunctions are inspired by the rules for the quantifiers in classical first order logic and follow the design proposed in [15,23]. In particular, the repetition of the principal formula in the premise of the rules $L \bigwedge$ and $R \bigvee$ enables us to avoid the use of sequents with (possibly) countably infinite formulas. However, as we will see, contrarily to the unlabelled systems, this restriction does not impair the completeness of the calculus.

2.1. Extensions of G3K$_\omega$

The formulation of a suitable labelled sequent calculus for a base logic can be extended with *relational rules* in order to obtain a sound and complete system for wide families of logics.

A relational rule is a rule whose principal formulas are exclusively relational atoms. They directly stem from the frame conditions of the corresponding Kripke frame applying the method of conversion of axioms into rules. In particular, the methodology can be employed to convert *geometric axioms* into relational rules [12].

DEFINITION 2.3. A geometric formula is a formula in the language of first-order classical logic of the shape $A \to B$, where A and B do not contain $\forall$ and $\to$.

Every geometric formula can be shown to equivalent to a formula of the shape [18]:

$$\forall \overline{x}(P_1 \wedge \ldots \wedge P_m \to \exists \overline{y}_1 \mathbf{Q_1} \vee \ldots \vee \exists \overline{y}_n \mathbf{Q_n})$$

where P_i and $\mathbf{Q_n}$ are atomic formulas and finite conjunctions of atomic formulas, respectively. Every geometric formula can be converted into an n-ary rule of the form:

$$\frac{Q_{11}[x_1/y_1], ..., Q_{1k_1}[x_1/y_1], \overline{P}, \Gamma \Rightarrow \Delta \quad\quad ... \quad\quad Q_{n1}[x_n/y_n], ..., Q_{nk_m}[x_n/y_n], \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta} \text{ Geom}$$

where $\overline{P}$ corresponds to $P_1, \ldots, P_m$ and $[x_i/y_i]$ is the replacement of the label x_i with a fresh label y_i which does not occur in the conclusion of the rule.

Let us consider the case of the frame condition of *strong directedness*, i.e.

$$\forall x \forall y \forall z(xRy \wedge xRz \to yRz \vee zRy)$$

This formula is converted into the binary rule:

$$\frac{xRy, xRz, yRz, \Gamma \Rightarrow \Delta \quad\quad xRy, xRz, zRy, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta} \text{ Lin}$$

The addition of this rule to the base calculus $\mathbf{G3K}_\omega$ along with the rules for transitivity and reflexivity yields a calculus for the infinitary extension of $\mathbf{S4.3}$ with the infinitary Barcan formula.

We give an example of a derivation in $\mathbf{G3S4.3}$ which is the system obtained from $\mathbf{G3K}_\omega$ by adding the rules corresponding to transitivity, $\forall xyz(xRy \wedge yRz \to xRz)$, reflexivity $\forall x(xRx)$ and strong directedness, Lin:

$$\frac{\dfrac{\dfrac{xRy, xRz, yRz, z:p, y:\Box p, z:\Box q \Rightarrow y:q, z:p}{xRy, xRz, yRz, y:\Box p, z:\Box q \Rightarrow y:q, z:p} \text{ L}\Box \quad\quad \dfrac{xRy, xRz, zRy, y:q, y:\Box p, z:\Box q \Rightarrow y:q, z:p}{xRy, xRz, zRy, y:\Box p, z:\Box q \Rightarrow y:q, z:p} \text{ L}\Box}{\dfrac{\dfrac{\dfrac{\dfrac{xRy, xRz, y:\Box p, z:\Box q \Rightarrow y:q, z:p}{xRy, xRz, y:\Box p \Rightarrow y:q, z:\Box q \to p} \text{ R}\to}{xRy, xRz \Rightarrow y:\Box p \to q, z:\Box q \to p} \text{ R}\to}{\dfrac{xRy \Rightarrow y:\Box p \to q, x:\Box(\Box q \to p)}{\Rightarrow x:\Box(\Box p \to q), x:\Box(\Box q \to p)} \text{ R}\Box}}{\Rightarrow x:\Box(\Box p \to q) \vee \Box(\Box q \to p)} \text{ R}\vee \text{ R}\Box} \text{ Lin}$$

To obtain the modular extensions of the sequent calculus $\mathbf{G3K}_\omega$ we simply add the rules corresponding to the conditions on the accessibility relation.

We write $\mathbf{G3K}_\omega^*$ to refer to the sequent calculus $\mathbf{G3K}_\omega$ extended with a set of relational rules $*$ obtained from geometric frame conditions. We

Initial Sequents

$$\frac{}{p, \Gamma \Rightarrow p, \Delta} \; Ax$$

Logical Rules

$$\frac{A, \bigwedge \Theta, \Gamma \Rightarrow \Delta}{\bigwedge \Theta, \Gamma \Rightarrow \Delta} \; L\bigwedge, \, A \in \Theta \qquad \frac{...\Gamma \Rightarrow \Delta, A \,...\text{for every } A \in \Theta}{\Gamma \Rightarrow \Delta, \bigwedge \Theta} \; R\bigwedge$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \; L\neg \qquad\qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \; R\neg$$

$$\frac{\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta} \; \Box$$

Figure 3. The sequent calculus $\mathbf{TK}_\omega$

observe that universal frame conditions form a proper subset of geometric frame conditions.

2.2. Comparing Labelled and Unlabelled Calculi

In this section we draw a comparison between our labelled sequent calculus and the sequent calculus $\mathbf{TK}_\omega$ in Figure 3 in which sequents are built up from finite sets of formulas. The system $\mathbf{TK}_\omega$ is the two-sided version of the calculus discussed in the paper by Minari and is a variant of the system $\mathbf{LK}_\omega^\Box$ that we described in the introduction. The system $\mathbf{G3K}_\omega$ shares some features with the system $\mathbf{TK}_\omega$ as the rules for the infinitary connectives are exactly the same (*modulo* the labelling). We observe that $\mathbf{TK}_\omega$ is not complete with respect to the semantics given in Definition 2.1, because the infinitary Barcan formula is valid, but it is not provable.

We first show that the Barcan formula is derivable in $\mathbf{G3K}_\omega$.

LEMMA 2.1. *The sequent* $\Rightarrow x : \bigwedge \Box\Theta \to \Box \bigwedge \Theta$ *is provable in* $\mathbf{G3K}_\omega$.

PROOF. We construct the following derivation.

$$\frac{\dfrac{\dfrac{\dfrac{xRy, x : \bigwedge \Box\Theta, x : \Box A, y : A \Rightarrow y : A}{xRy, x : \bigwedge \Box\Theta, x : \Box A \Rightarrow y : A} \; L\Box}{xRy, x : \bigwedge \Box\Theta \Rightarrow y : A} \; L\bigwedge \qquad ...}{\dfrac{\dfrac{xRy, x : \bigwedge \Box\Theta \Rightarrow y : \bigwedge \Theta}{x : \bigwedge \Box\Theta \Rightarrow x : \Box \bigwedge \Theta} \; R\Box}{\Rightarrow x : \bigwedge \Box\Theta \to \Box \bigwedge \Theta} \; R\to} \; R\bigwedge}$$

∎

We now show the underivability of an instance of the Barcan formula in the system $\mathbf{TK}_\omega$. Let us suppose $\Phi = \mathsf{AT}$.

LEMMA 2.2. *The sequent* $\Rightarrow \neg \bigwedge \Box \Phi, \Box \bigwedge \Phi$ *is not derivable in* $\mathbf{TK}_\omega$.

PROOF. Suppose towards a contradiction that there is a derivation of $\Rightarrow \neg \bigwedge \Box \Phi, \Box \bigwedge \Phi$. Hence it will have the following form:

$$
\vdots
$$
$$
\frac{\bigwedge \Box \Phi \Rightarrow \Box \bigwedge \Phi}{\Rightarrow \neg \bigwedge \Box \Phi, \Box \bigwedge \Phi} \; R\neg
$$

Due to the formulation of the rules, in order to obtain a derivation we need to apply the rule $\Box$. Hence we have the following situation:

$$
\vdots
$$
$$
\frac{p_1, ..., p_n \Rightarrow \bigwedge \Phi}{\bigwedge \Box \Phi, \Box p_1, ..., \Box p_n \Rightarrow \Box \bigwedge \Phi} \; \Box
$$

$$
\vdots \pi
$$
$$
\frac{\bigwedge \Box \Phi \Rightarrow \Box \bigwedge \Phi}{\Rightarrow \neg \bigwedge \Box \Phi, \Box \bigwedge \Phi} \; R\neg
$$

where π contains only applications of the rule $L\bigwedge$. The topmost sequent displayed in the above derivation is clearly not derivable. ∎

The underivabiliry of the Barcan formula depends on the context restriction imposed on the rule $\Box$ which removes the formula $\bigwedge \Box \Phi$ from the antecedent. Such restriction cannot be avoided, because otherwise the rule would turn out to be unsound (for example the formula $p \rightarrow \Box p$ would become provable). It is worth observing that the finite version of the schema $\bigwedge \Box \Phi \rightarrow \Box \bigwedge \Phi$, i.e. $\Box A \wedge \Box B \rightarrow \Box (A \wedge B)$, is indeed derivable, because we can adopt the multiplicative version of the rule $L\wedge$, that is:

$$
\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}
$$

On the contrary, the multiplicative version of the rule $L\bigwedge$ is not avalaible unless we admit multisets (or sets) of countably infinite formulas.

We could believe that the difference between the systems $\mathbf{G3K}_\omega$ and $\mathbf{TK}_\omega$ lies in the treatment of the infinitary rules. However, on the basis of the above considerations, we have shown that the problem actually arises due to the different formulations of the modal rules. Labelled calculi enjoy good structural properties and the rules themselves satisfy various *desiderata*, both from a conceptual and practical perspective. In particular, the two

rules for the modal operator $\Box$ are symmetric and they are both invertible as we will show in the following sections.

In standard sequent calculi the rule $\Box$ removes from the antecedent and the succedent every unboxed formula, whereas in labelled calculi the rule $L\Box$ can be applied to analyze a labelled formula $x : \Box A$ whenever we have a relational atom xRy in the antecedent. In a sense it could be argued that the relational atom keeps track of the previous application of $R\Box$, thus interpreting the formula $y : A$ as a boxed formula.

3. Structural Analysis of G3K$_\omega^*$

The calculus **G3K**$_\omega^*$ has good structural properties, in what follows we show the admissibility of substitution of labels, weakening, contraction and of the cut rule. Since we are working with an infinitary proof system we have to modify accordingly the standard definition of height of a derivation as well as the definition of degree of a formula. We essentially follow the presentation given in [2,23].

DEFINITION 3.1. OT is a constructive system of ordinal terms with an associated well-ordering $<$ such that it is decidable whether $\alpha < \beta$, $\beta < \alpha$ or $\alpha = \beta$ for every $\alpha, \beta \in$ OT. There are functions φ and $\#$ (the natural sum) such that:

- $\varphi_0(\alpha) = \omega^\alpha$ and $\varphi_\gamma(\alpha)$ for $\gamma > 0$ be the $\alpha^{th} + 1$ fixpoint of φ_β for every $\beta < \gamma$.

- For every ordinal term α, β and γ, the natural sum, $\alpha\#\beta$, has the following properties:
  - $\alpha\#\beta = \beta\#\alpha$
  - $\alpha\#(\beta\#\gamma) = (\alpha\#\beta)\#\gamma$
  - If $\alpha < \beta$, then $\gamma\#\alpha < \gamma\#\beta$.

There are various constructive systems of ordinal terms with the properties described above. In particular, we refer the reader to [20,23] for further details. The proofs which follow are carried by standard or double induction on the well-ordering relation $<$ on the elements of OT. We start by assigning to formulas and to derivations ordinal terms in order to measure their complexity and height, respectively.

DEFINITION 3.2. For every formula A the relation $A \leqslant \alpha$, i.e. A is of degree $\leqslant \alpha$ is thus defined:

- $p \leqslant \alpha$ for every $p \in$ AT and α ordinal.

- $\bigwedge \Phi \leqslant \alpha$, $\bigvee \Phi \leqslant \alpha$ if for each $B \in \Phi$, there is an ordinal β such that $B \leqslant \beta$ and $\beta < \alpha$.

- $B \to C \leqslant \alpha$ if there are β, β' such that $B \leqslant \beta, C \leqslant \beta'$ and $\beta, \beta' < \alpha$.

- $\Box B \leqslant \alpha$ if there is β such that $B \leqslant \beta$ and $\beta < \alpha$.

A labelled formula $x : A$ is of degree $\leqslant \alpha$ if and only if $A \leqslant \alpha$.

DEFINITION 3.3. A derivation $\mathcal{D}$ in $\mathbf{G3K}_\omega^*$ is of cut rank $\leqslant \gamma$ if for every principal formula $x : A$ in every application of the cut rule in $\mathcal{D}$, $A \leqslant \delta$ for some $\delta < \gamma$.

DEFINITION 3.4. For every derivation $\mathcal{D}$, the relation $\mathcal{D} \leqslant \alpha$, i.e. $\mathcal{D}$ is of height $\leqslant \alpha$ (where α is an ordinal), is inductively defined. If $\mathcal{D}$ is an instance of Ax, then $\mathcal{D} \leqslant \alpha$ for every α, i.e. $\mathcal{D}$ is of height 0. If $\mathcal{D}$ is obtained from an application of a rule, then $\mathcal{D} \leqslant \alpha$ if every subderivation $\mathcal{D}'$ of every premise and every β s.t. $\mathcal{D}' \leqslant \beta$, $\beta < \alpha$.

Notational convention We write $\Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta$ to indicate that the labelled sequent $\Gamma \Rightarrow \Delta$ has a derivation of height $\leqslant \alpha$ and cut rank $\leqslant \gamma$.

We say that a rule is height (rank) preserving admissible if given a proof of each of the premises, there is a proof of its conclusion with at most the same height (rank).

LEMMA 3.1. (Tautology) *For every finite multiset of labelled formulas Γ, Δ, for every formula $x : A$, the sequent $\Gamma, x : A \Rightarrow \Delta, x : A$ is derivable in* $\mathbf{G3K}_\omega^*$.

PROOF. The proof is by induction on the complexity of the formula A. We limit ourselves to consider the case of $A \equiv \bigwedge \Theta$.

$$
\cfrac{
\cdots \quad
\cfrac{
\cfrac{\vdots_{IH}}{\Gamma, \bigwedge \Theta, x : A \Rightarrow \Delta, x : A}
}{\Gamma, x : \bigwedge \Theta \Rightarrow \Delta, x : A} \, {\scriptstyle L\wedge}
\quad \cdots
}{\Gamma, x : \bigwedge \Theta \Rightarrow \Delta, \bigwedge \Theta} \, {\scriptstyle R\wedge}
$$

∎

The substitution of labels is defined as usual and is height preserving admissible.

DEFINITION 3.5. Given labels x, y the operation $[x/y]$ is thus defined:

- $z : A[x/y] := z : A$ if $x \not\equiv z$
- $z : A[x/y] := y : A$ if $x \equiv z$

- $zRw[x/y]:=zRw$ if $z \not\equiv x, w \not\equiv x$
- $zRw[x/y]:=yRw$ if $z \equiv x, w \not\equiv x$
- $zRw[x/y]:=zRy$ if $z \not\equiv x, w \equiv x$
- $zRw[x/y]:=yRy$ if $z \equiv x, w \equiv x$

LEMMA 3.2. (Substitution) *Let x, y labels be given. If $\vdash_{\mathbf{G3K}_\omega^*} \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta$, then $\vdash_{\mathbf{G3K}_\omega^*} \Gamma[x/y] \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta[x/y]$.*

PROOF. The proof is by induction on the height of the derivations. In the case of a rule with countably many premises we apply the induction hypothesis to each premise and then we apply the rule again. ∎

The rule of weakening is height-preserving and rank-preserving admissible as well.

LEMMA 3.3. (Weakening) *For every $\Gamma, \Gamma', \Delta, \Delta'$ finite multisets of labelled formulas, the rule:*

$$\frac{\Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta}{\Gamma', \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, \Delta'} \; Weak$$

is admissible in $\mathbf{G3K}_\omega^*$.

PROOF. We proceed by induction on the height of derivations possibly applying the substitution lemma in order to avoid clashes with the eigenvariables in case the last rule applied is $R\square$ or a relational rule with a freshness condition. ∎

The calculus $\mathbf{G3K}_\omega^*$ enjoys height-preserving invertibility of every rule, differently from the unlabelled modal calculi. In particular, the relational rules are height-preserving admissible by height-preserving admissibility of weakening. As regards the other rules we discuss them in the following lemma.

LEMMA 3.4. (Inversion) *The following statements hold:*

1. If $\vdash_{\mathbf{G3K}_\omega^*} x : \bigwedge \Theta, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta$, then $\vdash_{\mathbf{G3K}_\omega^*} x : \bigwedge \Theta, x : A, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta$.

2. If $\vdash_{\mathbf{G3K}_\omega^*} \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : \bigwedge \Theta$, then $\vdash_{\mathbf{G3K}_\omega^*} \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : A$ for every A in Θ.

3. If $\vdash_{\mathbf{G3K}_\omega^*} x : \bigvee \Theta, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta$, then $\vdash_{\mathbf{G3K}_\omega^*} x : A, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta$ for every A in Θ.

4. *If* $\vdash_{\mathbf{G3K^*_\omega}} \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : \bigvee \Theta$, *then* $\vdash_{\mathbf{G3K^*_\omega}} \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : \bigvee \Theta, x : A$

5. *If* $\vdash_{\mathbf{G3K^*_\omega}} x : A \to B, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta$, *then* $\vdash_{\mathbf{G3K^*_\omega}} \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : A$ *and* $\vdash_{\mathbf{G3K^*_\omega}} x : B, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta$

6. *If* $\vdash_{\mathbf{G3K^*_\omega}} \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : A \to B$, *then* $\vdash_{\mathbf{G3K^*_\omega}} x : A, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : B$

7. *If* $\vdash_{\mathbf{G3K^*_\omega}} xRy, x : \Box A, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta$, *then* $\vdash_{\mathbf{G3K^*_\omega}} xRy, x : \Box A, y : A, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta$

8. *If* $\vdash_{\mathbf{G3K^*_\omega}} \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : \Box A$, *then* $\vdash_{\mathbf{G3K^*_\omega}} xRy, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, y : A$, *for any* y *which does not occur in* $\Gamma \Rightarrow \Delta, x : \Box A$.

PROOF. We limit ourselves to discuss the items 2. and 8., the other cases are similar. As regards item 2. we reason by transfinite induction on the height of derivations. If $\alpha = 0$, then $\Gamma \Rightarrow \Delta, x : \bigwedge \Phi$ is an initial sequent and for every $A \in \Phi$ $\Gamma \Rightarrow \Delta, x : A$ is an initial sequent as well. If $\alpha > 0$ and $x : \bigwedge \Phi$ is principal we take the premises. If $\alpha > 0$ and the last inference rule is $L \to, R \to, L\Box, R\Box, L \bigwedge$ or $R \bigvee$ we apply the induction hypothesis to each premise and we apply again the rule. If $\alpha > 0$ and the last rule applied is $L \bigvee, R \bigvee$ we proceed as follows (we deal with $L \bigvee$):

$$\frac{\dots \qquad y : B, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, x : \bigwedge \Theta \qquad \dots}{y : \bigvee \Psi, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : \bigwedge \Theta} \, L \bigvee$$

For every A in Θ by induction hypothesis we obtain a derivation of $y : B, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, x : A$ for every $B \in \Psi$ and then we conclude by $L \bigvee$.

With respect to item 8. we argue again by transfinite induction on the height of the derivation. If $\alpha = 0$, then $\Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : \Box A$ is an initial sequent and so is $xRy, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, y : A$. If $m > 0$ and $x : \Box A$ is principal, we take the premise. Otherwise we distinguish cases according to the last rule applied. If the last rule applied is $R\Box$ we can assume (by height-preserving substitution) that the eigenvariable of the rule is different from y, so we apply the induction hypothesis to the premise and we apply the rule again. In every other case we apply the induction hypothesis to the premises and then we apply again the rule. For example if the last rule applied is $L\bigvee$, we have:

$$\frac{\dots \qquad z : B, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, x : \Box A \qquad \dots}{z : \bigvee \Psi, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : \Box A} \, L \bigvee$$

We transform the derivation into:

$$
\cfrac{
\cfrac{
\cfrac{z : B, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, x : \Box A}
{\ \cdots\ \quad xRy, z : B, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, y : A \quad \cdots\ } \ {\scriptstyle IH}
}
{xRy, z : \bigvee \Psi, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, y : A} \ {\scriptstyle L\bigvee}
}{}
$$

∎

We now show that the rules of contraction are height-preserving admissible.

LEMMA 3.5. (Contraction) *The rules:*

$$
\cfrac{x : A, x : A, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta}{x : A, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta} \ {\scriptstyle LC}
\qquad
\cfrac{\Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : A, x : A}{\Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : A} \ {\scriptstyle RC}
\qquad
\cfrac{xRy, xRy, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta}{xRy, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta} \ {\scriptstyle LC_{Rel}}
$$

are height-preserving admissible in $\mathbf{G3K}^*_\omega$.

PROOF. The proof is by simultaneous transfinite induction on the height of derivations, we deal with the LC rule. If $\alpha = 0$ then it is an initial sequent and so is $x : A, \Gamma \Rightarrow \Delta$. If $\alpha > 0$ and $x : A$ is not principal we apply the induction hypothesis to the (possibly countably infinite) premise(s) and we apply again the rule.

If $\alpha > 0$ and $x : A$ is principal we distinguish cases according to the last rule applied. Cases $L \to, L\Box$ are standard. The case of $L \bigwedge$ is trivial due to the repetition of the principal formula in the premise. We deal with the case of $L \bigvee$.

$$
\cfrac{\cdots \quad x : A, x : \bigvee \Phi, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta \quad \cdots}{x : \bigvee \Phi, x : \bigvee \Phi, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta} \ {\scriptstyle L\bigvee}
$$

We apply height-preserving invertibility of $L\bigvee$ to every premise and we obtain the derivation of the following sequents: $x : A, x : A, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta$. To each of them we apply the induction hypothesis and we conclude the proof by an application of $L\bigvee$. ∎

We now prove the central syntactic result of the work, namely the reduction lemma.

DEFINITION 3.6. A derivation in $\mathbf{G3K}_\omega^*$ is *distinct* whenever the eigenvariables occur only in the subderivation above the rule which introduce them.

It is straightforward to prove that every derivation in $\mathbf{G3K}_\omega^*$ can be transformed in a distinct derivation preserving the height and the rank. Given two premises of a cut on $x : A$, we show how to construct a derivation in which the cut is replaced by cuts on smaller formulas at the cost of an increase of the length of the derivation.

LEMMA 3.6. (Reduction lemma) *Given derivations $\mathcal{D}'$, $\mathcal{D}''$ of*

$$\Gamma \underset{\gamma}{\overset{\alpha}{\Rightarrow}} \Delta, x : A \text{ and } x : A, \Gamma \underset{\gamma}{\overset{\beta}{\Rightarrow}} \Delta$$

with $A \leqslant \gamma$, respectively, there is a derivation $\mathcal{D}$ of $\Gamma, \Gamma' \underset{\gamma}{\overset{\alpha\#\beta}{\Rightarrow}} \Delta, \Delta'$ in $\mathbf{G3K}_\omega^*$.

PROOF. We proceed by transfinite induction on the natural sum of the height of derivations $\alpha\#\beta$. We distinguish five cases:

1. $(Ax, ?)$, the left premise is an initial sequent or a zeroary relational rule.

2. $(?, Ax)$, the right premise is an initial sequent or a zeroary relational rule.

3. (Pr, Pr), the cut formula is principal in both premises.

4. $(\neg Pr, ?)$, the cut formula is not principal in the left premise.

5. $(?, \neg Pr)$, the cut formula is not principal in the right premise.

For each case we discuss we first show the derivation obtained via a cut, which would increase the cut rank of the derivation and then we give the reduction strategy. Cases 1. and 2. are straightforward.

We deal with case 3. We deal with the case in which the modal operator $\square$ is principal in both premises:

$$\cfrac{\cfrac{xRz, xRy, \Gamma \underset{\gamma}{\overset{\alpha'}{\Rightarrow}} \Delta, y : A}{xRz, \Gamma \underset{\gamma}{\overset{\alpha}{\Rightarrow}} \Delta, x : \square A} \, R\square \qquad \cfrac{xRz, z : A, x : \square A, \Gamma \underset{\gamma}{\overset{\beta'}{\Rightarrow}} \Delta}{xRz, x : \square A, \Gamma \underset{\gamma}{\overset{\beta}{\Rightarrow}} \Delta} \, L\square}{xRz, \Gamma \Rightarrow \Delta} \, Cut$$

We construct the following derivation:

We consider the case in which the principal formula is $x : \bigwedge \Phi$. If we applied cut we would have:

$$\frac{\dfrac{\quad \dots \quad \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, x : B \quad \dots \quad}{\Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : \bigwedge \Phi} R\wedge \qquad \dfrac{x : \bigwedge \Phi, x : A, \Gamma \overset{\beta'}{\underset{\gamma}{\Rightarrow}} \Delta}{x : \bigwedge \Phi, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta} L\wedge}{\Gamma \Rightarrow \Delta} Cut$$

We observe that since A is in Φ, then the sequent $\Gamma \overset{\beta'}{\underset{\gamma}{\Rightarrow}} \Delta, x : A$ has to be among the premises of the left premise. Hence we construct the following derivation:

$$\frac{\Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, x : A \qquad \dfrac{\dfrac{\dfrac{\Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : \bigwedge \Phi}{x : A, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : \bigwedge \Phi} Weak \qquad x : \bigwedge \Phi, x : A, \Gamma \overset{\beta'}{\underset{\gamma}{\Rightarrow}} \Delta}{x : A, \Gamma \overset{\alpha\#\beta'}{\underset{\gamma}{\Rightarrow}} \Delta} IH}{}}{\Gamma \overset{\alpha\#\beta}{\underset{\gamma}{\Rightarrow}} \Delta} Cut$$

Since we have $\alpha', \alpha\#\beta' < \alpha\#\beta$, the desired conclusion follows.

Cases 4 and 5 are symmetric. We deal exclusively with 4. We have to distinguish subcases according to the last rule applied.

4.1. The cut formula is not principal and it has been obtained by a rule with finitely many premises (including relational rules). The strategy essentially consists of permuting the cut upwards, applying the induction hypothesis and then the rule again. We consider as an example the case of the rule L→:

$$\frac{\dfrac{y : C, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, x : A \qquad \Gamma \overset{\alpha''}{\underset{\gamma}{\Rightarrow}} \Delta, y : B, x : A}{y : B \to C, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : A} L\to \qquad y : B \to C, x : A, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta}{y : B \to C, \Gamma \Rightarrow \Delta} Cut$$

We construct the following derivation:

$$\frac{\dfrac{\Gamma \overset{\alpha''}{\underset{\gamma}{\Rightarrow}} \Delta, y : B, x : A \qquad \dfrac{x : A, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta}{x : A, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta, y : B} Weak}{\Gamma \overset{\alpha''\#\beta}{\underset{\gamma}{\Rightarrow}} \Delta, y : B} IH \qquad \dfrac{y : C, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, x : A \qquad \dfrac{x : A, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta}{x : A, y : C, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta} Weak}{y : C, \Gamma \overset{\alpha'\#\beta}{\underset{\gamma}{\Rightarrow}} \Delta} IH}{y : B \to C, \Gamma \overset{\alpha\#\beta}{\underset{\gamma}{\Rightarrow}} \Delta} L\to$$

4.2 The cut formula is not principal and the last rule applied is either $R \bigwedge$ or $L \bigvee$, we deal with the latter case.

$$
\cfrac{
\cfrac{\ldots \qquad y : B, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, x : A \qquad \ldots}{y : \bigvee \Phi, \Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : A} \; L \bigvee \qquad x : A, y : \bigvee \Phi, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta
}{y : \bigvee \Phi, \Gamma \Rightarrow \Delta} \; Cut
$$

For every B in Φ we construct the following derivation:

$$
\cfrac{
\ldots \qquad \cfrac{
\cfrac{y : B, \Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, x : A \qquad \cfrac{x : A, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta}{=========== \; Weak}{x : A, y : B, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta}}{=========================== \; IH}{y : B, \Gamma \overset{\alpha' \# \beta}{\underset{\gamma}{\Rightarrow}} \Delta} \qquad \ldots
}{y : \bigvee \Phi, \Gamma \overset{\alpha \# \beta}{\underset{\gamma}{\Rightarrow}} \Delta} \; L \bigvee
$$

This concludes the proof.   ∎

We write φ^n to denote the application of the function φ n-times.

THEOREM 3.7. (Cut elimination) *Given a derivation in* $\mathbf{G3K}_\omega^*$ *of* $\Gamma \overset{\alpha}{\underset{\gamma + \omega^\delta}{\Rightarrow}} \Delta$, *there is a derivation of* $\Gamma \overset{\varphi_\delta(\alpha)}{\underset{\gamma}{\Rightarrow}} \Delta$.

PROOF. We argue by double transfinite induction with main induction hypothesis on δ and secondary induction hypothesis on α, making use of the reduction lemma, see [23]. In particular, if $\alpha = 0$, then the derivation is already cut-free. If $\alpha > 0$ we distinguish cases according to the last rule applied. If the last rule is any rule different from the cut rule, then we simply apply the secondary induction hypothesis and then we apply the rule again. If the last rule applied is the cut rule, we have:

$$
\cfrac{\Gamma \overset{\alpha'}{\underset{\gamma + \omega^\delta}{\Rightarrow}} \Delta, x : A \qquad x : A, \Gamma \overset{\alpha''}{\underset{\gamma + \omega^\delta}{\Rightarrow}} \Delta}{\Gamma \overset{\alpha}{\underset{\gamma + \omega^\delta}{\Rightarrow}} \Delta} \; Cut
$$

First, we apply the induction hypothesis to the two premises and we get: $\Gamma \overset{\varphi_\delta(\alpha')}{\underset{\gamma}{\Rightarrow}} \Delta, x : A$ and $x : A, \Gamma \overset{\varphi_\delta(\alpha'')}{\underset{\gamma}{\Rightarrow}} \Delta$.

In this case we distinguish two further subcases. If $\delta = 0$, then $\gamma + \omega^\delta = \gamma + 1$ and the degree of A is $\leqslant \gamma$. Thus we apply the reduction lemma to obtain the desired conclusion.

If $\delta > 0$, let us fix $\beta = max(\alpha', \alpha'')$. Since the degree of A is $< \gamma + \omega^\delta$, there are $\eta < \delta$ and y such that $A < \gamma + \omega^\eta \cdot y$. Therefore we apply a cut and we obtain a proof of

$$\Gamma \underset{\gamma + \omega^\eta \cdot y}{\overset{\varphi_\delta(\beta) + 1}{\Rightarrow}} \Delta$$

We are now in the position to apply the primary induction hypothesis y-times. This yields a derivation $\mathcal{D}$ of

$$\Gamma, \underset{\gamma}{\overset{\varphi_\eta^y(\varphi_\delta(\beta) + 1)}{\Rightarrow}} \Delta$$

However, $\varphi_\eta^y(\varphi_\delta(\beta) + 1) < \varphi_\delta(\alpha)$ and thus we have obtained the desired conclusion. ∎

As a corollary we obtain the cut-elimination theorem.

COROLLARY. *Given a derivation in* $\mathbf{G3K}_\omega^*$ *of* $\Gamma \underset{\omega \cdot n}{\overset{\alpha}{\Rightarrow}} \Delta$*, there is a derivation of* $\Gamma \underset{0}{\overset{\varphi_1^n(\alpha)}{\Rightarrow}} \Delta$*.*

3.1. Applications of Cut Elimination

We sketch some immediate consequences of the theorem of cut-elimination. First, we can show that the rule of modus ponens can be simulated in the system which yields a completeness result. In the present section we assume that our system $\mathbf{G3K}^*$ is cut-free.

LEMMA 3.8. *The rule:*

$$\frac{\Rightarrow x : A \to B \qquad \Rightarrow x : A}{\Rightarrow x : B} \ MP$$

is admissible in $\mathbf{G3K}_\omega^*$*.*

PROOF. Straightforward by the invertibility of the rule $R \to$, weakening and an application of cut. ∎

In order to prove the completeness with respect to the standard Kripke semantics we proceed indirectly by showing the embedding of the axiomatization $\mathbf{K}_\omega$ presented above.

LEMMA 3.9. *The infinitary Barcan formula and converse Barcan formula are derivable in* $\mathbf{G3K}_\omega^*$*.*

PROOF. The direction from left to right was already proved in Lemma 2.1. We prove the converse Barcan formula, namely $\Rightarrow x : \Box \bigwedge \Theta \to \bigwedge \Box \Theta$:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{xRy, x : \Box \bigwedge \Phi, y : \bigwedge \Phi, y : B \Rightarrow y : B}{xRy, x : \Box \bigwedge \Phi, y : \bigwedge \Phi \Rightarrow y : B} \; {\scriptstyle L \wedge}
}{xRy, x : \Box \bigwedge \Phi \Rightarrow y : B} \; {\scriptstyle L \Box}
}{x : \Box \bigwedge \Phi \Rightarrow x : \Box B} \; {\scriptstyle R \Box} \qquad \dots
}{
\qquad \dots \qquad x : \Box \bigwedge \Phi \Rightarrow x : \bigwedge \Box \Phi
} \; {\scriptstyle R \wedge}
}{\Rightarrow x : \Box \bigwedge \Phi \to \bigwedge \Box \Phi} \; {\scriptstyle R \to}
$$

∎

Notice that this sequent (modulo the labelling) was already derivable in $\mathbf{TK}_\omega$ because in this case it is possible to apply the rule K. This yields the following result:

LEMMA 3.10. *For every formula A: if $\vdash_{\mathbf{K}_\omega} A$, then $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : A$.*

PROOF. Immediate by induction on the height of derivations in $\mathbf{K}_\omega$: the axioms are derivable and the rules are admissible. ∎

We are now going to prove the soundness and completeness of the systems $\mathbf{G3K}^*$. Before proving soundness we will give a definition of truth conditions for labelled sequents.

DEFINITION 3.7. Let $\mathcal{M} = \langle W, R, N, v \rangle$ be a model. An interpretation is a mapping $\| \ \|$ from the set of labels to W. A labelled sequent is valid in $\mathcal{M}$ with respect to an interpretation $\| \ \|$ if for every labelled formula $x : A$ and for every relational atom yRz in Γ, whenever $\| x \| \Vdash A$ and $\| y \| R \| z \|$, then for some $u : B$ in Δ, $\| u \| \vDash B$.

A labelled sequent $\Gamma \Rightarrow \Delta$ is valid in a model $\mathcal{M}$ if it is valid under every interpretation. A sequent is valid in a class of frames with properties $*$ if for every frame $\mathfrak{F}$ with properties $*$ and for every model $\mathcal{M}$ based on $\mathfrak{F}$, $\Gamma \Rightarrow \Delta$ is valid in $\mathcal{M}$.

Let $\mathbf{L}_\omega$ be a Hilbert-style presentation of an infinitary modal logic which is sound and complete with respect to a class of geometric frames, we denote by $\mathbf{G3KL}_\omega$ the system obtained by adding to $\mathbf{G3K}_\omega$ the relational rules corresponding to the frame conditions.

THEOREM 3.11. (Soundness and completeness) *For every infinitary modal logic $\mathbf{L}$ characterized by a class of geometric frames and every formula A:*

$$\vdash_{\mathbf{G3KL}_\omega} \Rightarrow x : A \text{ iff } \vDash A.$$

PROOF. From left to right we proceed by induction on the height of derivations. From right to left we assume that $\vDash A$, hence $\vdash_{\mathbf{K}_\omega} A$ and by the embedding of $\mathbf{L}_\omega$ into $\mathbf{G3KL}^*_\omega$ we conclude that $\vDash_{\mathbf{G3KL}^*_\omega} \Rightarrow x : A$. ∎

The above theorem holds in particular for $\mathbf{T}_\omega$, $\mathbf{S4}_\omega$, $\mathbf{S5}_\omega$, $\mathbf{D}_\omega$ and also other systems. A more direct completeness proof which is independent of a specific axiomatization can be obtained by defining a suitable modification of the standard reduction tree for labelled systems as in [14]. The method of reduction trees establishes the completeness of the system with respect to the corresponding class of frames regardless of the existence of the existence of an axiomatic calculus.

REMARK. A standard application of the cut-elimination theorem for intuitionistic logic is a proof of the disjunction property. In particular, given a proof of $\vdash A \vee B$ in the intuitionistic calculus, then $\vdash A$ or $\vdash B$. In the context of modal logic a similar property—called *modal primality*—holds for some modal systems. The existence of a cut-free sequent calculus enables us to obtain a simple proof of this result (for a model-theoretic proof the reader is referred to [4]). Notice that this cannot be achieved working with the standard Gentzen formalism, as there are no cut-free sequent calculi for infinitary modal logic with the axiom $\bigwedge \Box \Phi \Rightarrow \Box \bigwedge \Phi$.

THEOREM 3.12. *The logics* $\mathbf{K}_\omega$, $\mathbf{KD}_\omega$, $\mathbf{K4}_\omega$, $\mathbf{KT}_\omega$, $\mathbf{K4D}_\omega$, $\mathbf{S4}_\omega$ *enjoy the modal disjunction property.*

PROOF. We work in the corresponding labelled sequent calculus $\mathbf{G3K}^*_\omega$. We have a proof of $\vdash \Rightarrow x : \Box A \vee \Box B$. By applying invertibility of the rules R$\vee$ and R$\Box$ we get a derivation of the sequent:

$$xRy, xRz \Rightarrow y : A, z : B$$

By inspection of the rules of the calculi we can see that the only rules which can be applied to xRy and xRz are Ref and Trs (in the systems in which Trs is present). These applications are redundant, so we can remove xRy and xRz, thus we have a derivation of $\Rightarrow y : A, z : B$.

Hence, since the two labels y and z are not connected by any relational atom and by the subterm property the only labels occurring in the derivation are either y, z or eigenvariables, we get that $\Rightarrow y : A$ or $\Rightarrow z : B$ is derivable which yields the desired conclusion. ∎

COROLLARY. *The denecessitation rule:*

$$\frac{\Rightarrow x : \Box A}{\Rightarrow x : A} \; Den$$

is admissible in the logics $\mathbf{K}_\omega$, $\mathbf{KD}_\omega$, $\mathbf{K4}_\omega$, $\mathbf{KT}_\omega$, $\mathbf{K4D}_\omega$, $\mathbf{S4}_\omega$.

PROOF. The proof is trivial for every logic which is an extension of $\mathbf{KT}_\omega$. In fact:

$$\cfrac{\cfrac{= = \cfrac{\Rightarrow x:\Box A}{\Rightarrow x:\Box A, x:A}\ \text{Weak}= = = = = = = = =}{\Rightarrow x:A}\qquad x:\Box A \Rightarrow x:A}{\Rightarrow x:A}\ \text{Cut}$$

For $\mathbf{K}_\omega$, $\mathbf{KD}_\omega$, $\mathbf{K4D}_\omega$ and $\mathbf{K4}_\omega$ we proceed as follows. Suppose $\Rightarrow x:\Box A$ is derivable. By height-preserving admissibility of weakening we get $\Rightarrow x:\Box A, x:\Box A$, thus by the modal disjunction property $\Rightarrow x:A$. ∎

REMARK. We would like to briefly explain the reason why we chose to work with finite sequents instead of infinite ones. Had we opted for the following *multiplicative* version of the left rule for the infinitary conjunction:

$$\frac{x:\Phi, \Gamma \Rightarrow \Delta}{x:\bigwedge\Phi, \Gamma \Rightarrow \Delta}\ \text{L}\bigwedge$$

we would not have been able to prove syntactic cut-elimination with the methodology here presented. To witness this, consider the crucial case in which the cut-formula is principal in both the premises of the cut:

$$\frac{\cfrac{\dots \quad \Gamma \Rightarrow \Delta, x:\varphi \quad \dots}{\Gamma \Rightarrow x:\bigwedge\Phi, \Delta}\ \text{R}\bigwedge \qquad \cfrac{x:\Phi, \Gamma \Rightarrow \Delta}{x:\bigwedge\Phi, \Gamma \Rightarrow \Delta}\ \text{L}\bigwedge}{\Gamma \Rightarrow \Delta}\ \text{Cut}$$

Clearly, every formula $x:\varphi$ has a lower degree with respect to $x:\bigwedge\Phi$. However, in order to eliminate the cut we would need to replace it with infinitely many other cuts. However, this reduction is problematic, insofar as we are dealing with infinitary branching derivation in which every branch is of finite length.

4.   Embedding of Common Knowledge into G3K$_\omega$

We extend our previous considerations to a multi-agent setting. In particular, the multimodal propositional infinitary language contains—in addition to the base language of infinitary modal logic—a non-empty set of agents AG denoted by the letters $a, b, c, ...$, a modal operator $\Box_a$ for every agent a. The set of well formed formulas is defined accordingly as usual.

Since we will be concerned with the embedding of the logic of common knowledge into infinitary modal logic, we recall the definition of the corresponding language. The language of common knowledge extends the

Axioms

The axioms of **K**

$\mathsf{C}(A \to B) \to (\mathsf{C}A \to \mathsf{C}B)$ distribution

$\mathsf{C}A \to (\mathsf{E}A \wedge \mathsf{E}\mathsf{C}A)$ closure

$\mathsf{E}A \wedge \mathsf{C}(A \to \mathsf{E}A) \to \mathsf{C}A$ induction

Inference Rules

$$\frac{\vdash A \qquad \vdash A \to B}{\vdash B} \ \text{MP} \qquad\qquad \frac{\vdash A}{\vdash \Box_a A} \ RN_a \qquad \frac{\vdash A}{\vdash \mathsf{C}A} \ RC$$

Figure 4. The axiomatic calculus $\mathbf{KC}_\omega$

language of finitary (i.e. not containing infinitary conjunctions and disjunctions) multimodal propositional logic by adding a unary modal operator C. In order to present the axiomatization of common knowledge [5,9,10], we first recall some preliminary definitions.

DEFINITION 4.1. Given a formula A, $\mathsf{E}^n A$ for $n \geqslant 1$ (everybody knows A) is inductively defined: $\mathsf{E}A := \Box_{a_1} A \wedge ... \wedge \Box_{a_n} A$ where $AG = \{a_1, ..., a_n\}$ and $\mathsf{E}^{n+1}A := \mathsf{E}\mathsf{E}^n A$.

In infinitary modal logic $\mathsf{C}A$ (common knowledge that A) is defined as $\bigwedge_{n=1}^{\omega} \mathsf{E}^n A$. The axiomatization of the logic of common knowledge is presented in Figure 4. We now show that $\mathbf{G3K}_\omega$ proves the infinitary translation of every axiom of the calculus. This, together with the theorem of cut elimination, yields an embedding of the logic of common knowledge into $\mathbf{G3K}_\omega$.

As remarked by a referee, the result is immediately entailed by the Kripke completeness of the systems. However, we believe that a purely syntactic proof is interesting as it shows the precise steps required in order to construct the proofs of the translation of the axioms. Furthermore, we believe that the proofs are not always straightforward, as they require the use of inductive arguments. Moreover, by a purely syntactic approach one can also extract some bounds on the height of the derivations of the translated proofs (although it is not among our present purposes).

If we chose to work with the system $\mathbf{G3S5}_\omega$, i.e. the labelled sequent calculi for the infinitary modal logic **S5** obtained by adding to $\mathbf{G3K}_\omega$ the

relational rules corresponding to transitivity, reflexity and symmetry of the accessibility relation, (which is usually preferred as a basis for common knowledge, see [5]), the following results would still hold *modulo* some slight changes in the axiomatization of the system of common knowledge.

LEMMA 4.1. *The following statements hold:*

1. $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : \mathsf{C}A \to \mathsf{E}A \wedge \mathsf{EC}A$

2. $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : \mathsf{E}A \wedge \mathsf{C}(A \to \mathsf{E}A) \to \mathsf{C}A$

3. $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : \mathsf{C}(A \to B) \to (\mathsf{C}A \to \mathsf{C}B)$

4. *If* $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : A$, *then* $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : \mathsf{C}A$

PROOF. In order to prove 1. we construct the following derivation:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{
            \cfrac{xR_ay, x:\mathsf{C}A, x:\mathsf{E}^{m+1}, x:\square_a\mathsf{E}^m, y:\mathsf{E}^mA \Rightarrow y:\mathsf{E}^mA}{xR_ay, x:\mathsf{C}A, x:\mathsf{E}^{m+1}, x:\square_a\mathsf{E}^m \Rightarrow y:\mathsf{E}^mA}\ L\square
          }{xR_ay, x:\mathsf{C}A, x:\mathsf{E}^{m+1} \Rightarrow y:\mathsf{E}^mA}\ L\wedge
        }{xR_ay, x:\mathsf{C}A \Rightarrow y:\mathsf{E}^mA}
      }{xR_ay, x:\mathsf{C}A \Rightarrow y:\mathsf{C}A}\ R\square
    }{x:\mathsf{C}A \Rightarrow x:\square_a\mathsf{C}A} \quad \cdots
  }{x:\mathsf{C}A \Rightarrow x:\mathsf{EC}A}\ R\wedge
}{}
$$

The topmost sequents are provable.

In order to prove 2. we reason by transfinite induction. We claim that for every $n \geqslant 1$ the sequent $x : \mathsf{E}A \wedge \mathsf{E}^n(A \to \mathsf{E}A) \Rightarrow x : \mathsf{E}^nA$ is derivable in $\mathbf{G3K}_\omega$. If $n = 1$, the proof is immediate. If $n = k + 1$, then by induction hypothesis we have that the sequent $x : \mathsf{E}A \wedge \mathsf{E}^k(A \to \mathsf{E}A) \Rightarrow x : \mathsf{E}^kA$ is derivable. Hence we construct the following derivation:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{\vdots\ \text{IH}}{xR_ay, x:\mathsf{E}A, x:\mathsf{E}^{k+1}(A \to \mathsf{E}A), x:\square_a\mathsf{E}^k(A \to \mathsf{E}A), y:\mathsf{E}^k(A \to \mathsf{E}A) \Rightarrow y:\mathsf{E}^kA}
        }{xR_ay, x:\mathsf{E}A, x:\mathsf{E}^{k+1}(A \to \mathsf{E}A), x:\square_a\mathsf{E}^k(A \to \mathsf{E}A) \Rightarrow y:\mathsf{E}^kA}\ L\square
      }{xR_ay, x:\mathsf{E}A, x:\mathsf{E}^{k+1}(A \to \mathsf{E}A) \Rightarrow y:\mathsf{E}^kA}\ L\wedge
    }{xR_ay, x:\mathsf{E}A \wedge \mathsf{E}^{k+1}(A \to \mathsf{E}A) \Rightarrow y:\mathsf{E}^kA}
  }{x:\mathsf{E}A \wedge \mathsf{E}^{k+1}(A \to \mathsf{E}A) \Rightarrow x:\square_a\mathsf{E}^kA}\ R\square \quad \cdots
}{x:\mathsf{E}A \wedge \mathsf{E}^{k+1}(A \to \mathsf{E}A) \Rightarrow x:\mathsf{E}^{k+1}A}\ R\wedge
$$

Hence we construct the following derivation:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{x:\mathsf{E}A, x:\mathsf{C}(A \to \mathsf{E}A), x:\mathsf{E}^n(A \to \mathsf{E}A) \Rightarrow x:\mathsf{E}^nA}{x:\mathsf{E}A, x:\mathsf{C}(A \to \mathsf{E}A) \Rightarrow x:\mathsf{E}^nA}\ L\wedge \quad \cdots
    }{x:\mathsf{E}A, x:\mathsf{C}(A \to \mathsf{E}A) \Rightarrow x:\mathsf{C}A}\ R\wedge
  }{x:\mathsf{E}A \wedge \mathsf{C}(A \to \mathsf{E}A) \Rightarrow x:\mathsf{C}A}\ L\wedge
}{\Rightarrow x:\mathsf{E}A \wedge \mathsf{C}(A \to \mathsf{E}A) \to \mathsf{C}A}\ R\to
$$

where the topmost sequent is derivable due to the previous considerations.

To prove 3. we prove by induction that for every $n \geqslant 1$ the sequent $x : \mathsf{E}^n(A \to B), x : \mathsf{E}^n A \Rightarrow x : \mathsf{E}^n B$ is derivable in $\mathbf{G3K}_\omega$. If $n = 1$, then we proceed as follows:

$$
\cfrac{
  \cdots
  \quad
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{xR_a y, x : \mathsf{E}(A \to B), x : \mathsf{E}A, x : \square_a(A \to B), x : \square_a A \Rightarrow y : B \dots}{xR_a y, x : \mathsf{E}(A \to B), x : \mathsf{E}A, x : \square_a A \Rightarrow y : B} \; {\scriptstyle L \bigwedge}
      }{xR_a y, x : \mathsf{E}(A \to B), x : \mathsf{E}A \Rightarrow y : B} \; {\scriptstyle L \bigwedge}
    }{x : \mathsf{E}(A \to B), x : \mathsf{E}A \Rightarrow x : \square_a B} \; {\scriptstyle R \square}
  }{}
  \quad
  \cdots
}{x : \mathsf{E}(A \to B), x : \mathsf{E}A \Rightarrow x : \mathsf{E}B} \; {\scriptstyle R \bigwedge}
$$

Notice that the topmost sequent is clearly derivable. If $n = k+1$, then by induction hypothesis we have that the sequent $x : \mathsf{E}^k(A \to B), x : \mathsf{E}^k A \Rightarrow x : \mathsf{E}^k B$ is derivable in $\mathbf{G3K}_\omega$.

$$
\cfrac{
  \cdots
  \quad
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{\vdots \, {\scriptstyle IH}}{xR_a y, x : \mathsf{E}^{k+1}(A \to B), x : \square_a \mathsf{E}^k(A \to B), x : \mathsf{E}^{k+1} A, x : \square_a \mathsf{E}^k A, y : \mathsf{E}^k A \Rightarrow y : \mathsf{E}^k B}
        }{xR_a y, x : \mathsf{E}^{k+1}(A \to B), x : \square_a \mathsf{E}^k(A \to B), x : \mathsf{E}^{k+1} A, x : \square_a \mathsf{E}^k A \Rightarrow y : \mathsf{E}^k B} \; {\scriptstyle L \square}
      }{xR_a y, x : \mathsf{E}^{k+1}(A \to B), x : \square_a \mathsf{E}^k(A \to B), x : \mathsf{E}^{k+1} A \Rightarrow y : \mathsf{E}^k B} \; {\scriptstyle L \bigwedge}
    }{xR_a y, x : \mathsf{E}^{k+1}(A \to B), x : \mathsf{E}^{k+1} A \Rightarrow y : \mathsf{E}^k B} \; {\scriptstyle L \bigwedge}
  }{x : \mathsf{E}^{k+1}(A \to B), x : \mathsf{E}^{k+1} A \Rightarrow x : \square_a \mathsf{E}^k B} \; {\scriptstyle R \square}
  \quad
  \cdots
}{x : \mathsf{E}^{k+1}(A \to B), x : \mathsf{E}^{k+1} A \Rightarrow x : \mathsf{E}^{k+1} B} \; {\scriptstyle R \bigwedge}
$$

Then we construct the following derivation:

$$
\cfrac{
  \cfrac{
    \cdots
    \quad
    \cfrac{
      \cfrac{x : \mathsf{C}(A \to B), x : E^n(A \to B), x : E^n A \Rightarrow x : E^n B}{x : \mathsf{C}(A \to B), x : \mathsf{C}A, x : \mathsf{E}^n A \Rightarrow x : \mathsf{E}^n B} \; {\scriptstyle L \bigwedge}
    }{x : \mathsf{C}(A \to B), x : \mathsf{C}A \Rightarrow x : \mathsf{E}^n B} \; {\scriptstyle L \bigwedge}
    \quad
    \cdots
  }{
    \cfrac{
      \cfrac{x : \mathsf{C}(A \to B), x : \mathsf{C}A \Rightarrow x : \mathsf{C}B}{x : \mathsf{C}(A \to B) \Rightarrow x : \mathsf{C}A \to \mathsf{C}B} \; {\scriptstyle R \to}
    }{\Rightarrow x : \mathsf{C}(A \to B) \to (\mathsf{C}A \to \mathsf{C}B)} \; {\scriptstyle R \to}
  } \; {\scriptstyle R \bigwedge}
}{}
$$

the topmost sequent is derivable due to the above argument.

To prove 4. we show that if $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : A$, then $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : \mathsf{E}^n A$ for every $n \geqslant 1$. We argue by induction on n. If $n = 1$, then for every $a \in AG$ we apply admissibility of the rule of necessitation and thus we obtain $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : \square_a A$, hence we apply $R \bigwedge$ and we conclude $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : \mathsf{E}A$. If $n = k+1$, then by induction hypothesis we have $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : \mathsf{E}^k A$. Therefore we apply the admissibility of the necessitation rule and we apply $R \bigwedge$ as before and thus we obtain $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : \mathsf{E}^{k+1} A$. Hence given $\vdash_{\mathbf{G3K}_\omega} \Rightarrow x : A$ we construct a proof of $\Rightarrow x : \mathsf{E}^n A$ for every $n \geqslant 1$ and we apply $R \bigwedge$, thus we obtain $\Rightarrow x : \mathsf{C}A$. $\blacksquare$

THEOREM 4.2. (Embedding of common knowledge) *If* $\vdash_{\mathsf{KC}} A$, *then* $\vdash_{\mathbf{G3K}_\omega} \Rightarrow$ $x : A^{fix}$, *where* A^{fix} *is the translation of* A *in the language of infinitary modal logic.*

PROOF. By induction on the height of derivation in the axiomatic calculus KC. The axioms are derivable by Lemma 4.1. The rule of modus ponens is admissible by the cut-elimination theorem. ∎

The embedding theorem suggests how to extract an infinitary labelled sequent calculus for the logic of common knowledge. In particular, it is enough to remove the rules for the infinitary conjunctions and disjunctions and adding the infinitary rule for the common knowledge operator which naturally stems from its infinitary interpretation. The resulting system—let us denote it by **G3KC**—enjoys good structural properties and cut-elimination and the proofs follow the pattern detailed for the system $\mathbf{G3K}_\omega^*$, thus we omit the details. We observe that there are various sequent systems which are sound and complete with respect to commmon knowledge, see [2,26].

5. First-order Infinitary Modal Logic

In this section we briefly discuss the extension of the present framework to the context of first-order modal logic. Thus we consider a first-order language with variables and constants and which contains the modal operator, the usual connectives, infinitary conjunctions and disjunctions and also existential and universal quantifiers. We will work with constant domain first-order infinitary modal logics. We recall the natural generalization of Kripke-style semantics for first-order infinitary modal logic (see also [24] and [6]).

DEFINITION 5.1. A Kripke frame for infinitary modal first-order logic is a quadruple $\langle W, R, D, v \rangle$, where:

- $\langle W, R \rangle$ is a Kripke frame.
- D is a non-empty set.
- v is a mapping from W such that:
  - $v_w(c) \in D$ for every $w \in W$ and for every constant c of the language and $v_{w'}(c) = v_w(c)$ for every world w'.
  - $v_w(P) \subseteq D$ for every world w and every predicate P.

  An *assignment* σ is a function from the set of variables to D. The function $f_{v_w,\sigma} : TER \to D$ is thus defined for every valuation v and assignment σ:

$$\frac{w : A[x/t], w : \forall xA, \Gamma \Rightarrow \Delta}{w : \forall xA, \Gamma \Rightarrow \Delta} \text{ L}\forall \qquad \frac{\Gamma \Rightarrow \Delta, w : A[x/a]}{\Gamma \Rightarrow \Delta, w : \forall xA} \text{ R}\forall, \, a \text{ fresh}$$

$$\frac{w : A[x/a], \Gamma \Rightarrow \Delta}{w : \exists xA, \Gamma \Rightarrow \Delta} \text{ L}\exists, \, a \text{ fresh} \qquad \frac{\Gamma \Rightarrow \Delta, w : \exists xA, w : A[x/t]}{\Gamma \Rightarrow \Delta, w : \exists xA} \text{ R}\exists$$

Figure 5. The quantifier rules for constant domains

$$f_{v_w,\sigma}(t) = \sigma(t) \text{ if } t \text{ is a variable, } f_{v_w,\sigma}(t) = v_{w'}(t), \text{ if } t \text{ is a constant.}$$

The truth conditions are inductively defined (we omit the ones for propositional and infinitary connectives which are identical):

- $w \Vdash_\sigma P(t_1, ..., t_n)$ if and only if $P(f_{v_w,\sigma}(t_1), ..., f_{v_w,\sigma}(t_n)) \in v_w(P)$.
- $w \Vdash_\sigma \forall xA$ if and only if $w \Vdash_{\sigma'} A$ for every σ' such that $\sigma'(y) = \sigma(y)$ for every variable $y \not\equiv x$.
- $w \Vdash_\sigma \exists xA$ if and only if $w \Vdash_{\sigma'} A$ for some σ' such that $\sigma'(y) = \sigma(y)$ for every variable $y \not\equiv x$.

This presentation of the semantics assumes that the domain set D is constant. A labelled sequent calculus for infinitary first-order modal logic immediately stems from the truth conditions of the quantifiers. The only difference with respect to the formulation of **G3K*** is that sequents are objects of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of relational atoms and first-order infinitary modal labelled formula.

Hence we add to the calculus **G3K*** the rules in Figure 5 and we refer to the calculus thus obtained as **G3K**$_\omega{}^{fo*}$.

The notions of degree of a formulas and of height of a derivation are complemented with obvious additions to deal with the quantifiers. Substitution $[x/t]$ of a term t in place of a variable x is defined as usual. The structural lemmata established in Section 3 hold also for the systems **G3K**$_\omega{}^{fo*}$: we limit ourselves to stating the Lemma.

LEMMA 5.1. *The rules of substitution of labels and of terms, weakening and contraction are height and rank preserving admissible in* **G3K**$_\omega{}^{fo*}$. *Every rule is height and rank preserving invertible in* **G3K**$_\omega{}^{fo*}$.

The cut-elimination then follows according to the usual procedure.

THEOREM 5.2. *Given derivations $\mathcal{D}'$, $\mathcal{D}''$ of*

$$\Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, x : A \ \textit{and} \ x : A, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta$$

with $A \leqslant \gamma$, respectively, there is a derivation $\mathcal{D}$ of $\Gamma \overset{\alpha \# \beta}{\underset{\gamma}{\Rightarrow}} \Delta$ in $\mathbf{G3K}^{\mathbf{fo}*}_\omega$.

PROOF. The proof is by induction on $\alpha \# \beta$. The only new cases to detail are those involving the quantifiers. The cases involving quantifiers follow the standard pattern as in first-order classical (infinitary) logic *modulo* the labelling.

$$
\frac{
\Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, w : \exists x A, w : A[x/t]
}{
\Gamma \overset{\alpha}{\underset{\gamma}{\Rightarrow}} \Delta, w : \exists x A
} \text{R}\exists
\quad
\frac{
w : A[x/a], \Gamma \overset{\beta'}{\underset{\gamma}{\Rightarrow}} \Delta
}{
w : \exists x A, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta
} \text{L}\exists, \ a \text{ fresh}
$$
$$
\frac{}{\Gamma \Rightarrow \Delta} \text{Cut}
$$

We proceed as follows:

$$
\frac{
\frac{
\Gamma \overset{\alpha'}{\underset{\gamma}{\Rightarrow}} \Delta, w : \exists x A, w : A[x/t]
\qquad
\dfrac{\dfrac{w : \exists x A, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta}{w : \exists x A, \Gamma \overset{\beta}{\underset{\gamma}{\Rightarrow}} \Delta, w : A[x/t]} \text{Weak}}{}
}{
\Gamma \overset{\alpha' \# \beta}{\underset{\gamma}{\Rightarrow}} \Delta, w : A[x/t]
} \text{IH}
\qquad
\frac{
\dfrac{w : A[x/a], \Gamma \overset{\beta'}{\underset{\gamma}{\Rightarrow}} \Delta}{w : A[x/t], \Gamma \overset{\beta'}{\underset{\gamma}{\Rightarrow}} \Delta} Sub[a/t]
}{}
}{
\Gamma \Rightarrow \Delta
} \text{Cut}
$$

∎

We are now in the position to state and prove the soundness and completeness theorem for our calculi. It is known that every first-order infinitary modal logic characterized by a class of frames with universal frame conditions is Kripke complete if we add the infinitary Barcan formula $\bigwedge \Box \Phi \to \Box \bigwedge \Phi$ and the quantified Barcan formula $\forall x \Box A \to \Box \forall x A$ [24]. Therefore let $\mathbf{L}$ be a Hilbert-style presentation of an infinitary first-order modal logic which is sound and complete with respect to a class of universal frames, we denote by $\mathbf{G3KL}^{fo}_\omega$ the system obtained by adding to $\mathbf{G3K}^*_\omega$ the relational rules corresponding to the frame conditions.

THEOREM 5.3. *For every infinitary universal first-order modal logic $\mathbf{L}$: if $\mathbf{L} \vdash A$, then $\mathbf{G3KL}_\omega{}^{fo} \vdash \Rightarrow x : A$.*

PROOF. By induction on the height of the derivation in $\mathbf{L}$. We first show that all the axioms are provable. For example the Barcan formula $\forall x \Box A \to \Box \forall x A$ is provable as:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{wRo, o : A[x/a], w : \Box A[x/a], w : \forall x \Box A \Rightarrow o : A[x/a]}{wRo, w : \Box A[x/a], w : \forall x \Box A \Rightarrow o : A[x/a]} \text{ L}\Box}{wRo, w : \forall x \Box A \Rightarrow o : A[x/a]} \text{ L}\forall}{wRo, w : \forall x \Box A \Rightarrow o : \forall x A} \text{ R}\forall}{w : \forall x \Box A \Rightarrow \Box \forall x A} \text{ R}\Box}{\Rightarrow w : \forall x \Box A \rightarrow \Box \forall x A} \text{ R}{\rightarrow}$$

The rules of necessitation, modus ponens and for infinitary conjunction are trivially seen to be admissible.   ∎

THEOREM 5.4. *For every first-order infinitary modal logic* **L** *characterized by a class of universal frames, the corresponding calculus* $\mathbf{G3L}_{\omega}^{\mathbf{fo}*}$ *are sound and complete.*

PROOF. Soundness is proved by induction on the height of the derivation. Completeness follows from the embedding of the corresponding axiomatic calculus.   ∎

The class of logics **L** which $\mathbf{G3L}_{\omega}^{\mathbf{fo}*}$ is sound and complete includes the first-order extensions of the logics $\mathbf{T}_{\omega}$, $\mathbf{K4}_{\omega}$, $\mathbf{S4}_{\omega}$ and $\mathbf{S5}_{\omega}$ and $\mathbf{S4.3}_{\omega}$.

REMARK. This proof of completeness relies on the completeness of the relative axiomatic calculus. However, the methodology of labelled sequent calculi allows for a more direct approach to completeness. In particular, for every class of frames characterized by a geometric frame condition, the calculus $\mathbf{G3K}_{\omega}^{\mathbf{fo}*}$ can be shown to be sound and complete by a Tait-Schütte-Takeuti style technique, which consists in constructing a reduction tree applying the rules of the calculus to a sequent. The construction yields either a derivation or an infinite branch out of which a countermodel is extracted. Therefore the calculi are complete with respect to the corresponding class of geometric frames regardless of the existence of a Hilbert-style axiomatization.

6.   Concluding Remarks and Related Works

We have introduced a labelled sequent calculus for infinitary modal logic. The calculus enjoys good structural properties, such as height preserving admissibility of the rules of weakening, contraction and admissibility of cut. Our system is complete with respect to the standard kripkean semantics for infinitary modal logic, contrarily to the standard unlabelled calculus. Furthermore, the system can be modularly extended so as to cover all infinitary modal logics characterized by first-order frame conditions.

As regards other possible approaches to the proof theory of infinitary modal logic, we believe that nested sequent calculi [2] could be employed in

order to obtain a cut-free system. For example, in [3] an infinitary nested sequent has been exploited in order to obtain a cut-elimination theorem for the logic of common knowledge. A similar approach was pursued by Tanaka in [26], in which some sequent systems (with sequents built from sets) for predicate common knowledge have been introduced and studied from a semantic viewpoint.

There are also works which focussed on the semantics of the system $\mathbf{TK}_\omega$. An adequate semantics which generalizes Kripke semantics by allowing a set of accessibility relations has been proposed by Minari in [11]. Also neighborhood semantics [17]—which has recently been employed in order to obtain a soundness and completeness result with respect to infinitary intuitionistic logic [28]—has proved to be an adequate semantic [27].

Another aspect worth investigating are first-order infinitary modal logics with varying domains. First, it would be interesting to see whether it is possible to obtain an axiomatization of these logics. Second, the proof theory of these systems could be investigated.

Finally, we think that a proof theory based on sequent calculi with infinite multisets of formulas is worth investigating. In the case of unlabelled sequent calculi the adoption of infinite multisets allowed to regain completeness on pain of losing cut-elimination. We leave the problem of finding a syntactic proof of cut-elimination (or a counterexample to cut-elimination) for labelled sequent calculi with sequents with infinite multisets of formulas as a theme of future research.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] BLACKBURN, P., M. DE RIJKE, and Y. VENEMA, *Modal Logic*, Cambridge University Press, 2001.

[2] BRÜNNLER, K., Deep sequent systems for modal logic, *Archive for Mathematical Logic* 48:551–577, 2009.

[3] BRÜNNLER, K., and T. STUDER, Syntactic cut-elimination for common knowledge, *Annals of Pure and Applied Logic* 160(1):82–95, 2009.

[4] CHAGROV, A., and M. ZAKHARYASCHEV, *Modal Logic*, Oxford University Press, Clarendon Press, 1997

[5] VAN DITMARSCH, H., B. P. KOOI, and W. HOEK, *Dynamic Epistemic Logic*, Cambridge University Press, 2007.

[6] FITTING, M., and R. MENDELSOHN, *First-order modal logic*, Springer, Dordrecht, 1998.

[7] GABBAY, D., *Labelled Deductive Systems - Volume 1*, Oxford University Press, 1996.

[8] GIRARD, J. Y., *Proof Theory and Logical Complexity*, Bibliopolis, 1987

[9] HALPERN, J. Y., and Y. O. MOSES, Knowledge and common knowledge in a distibuted environment, *Journal of the Association for Computing Machinery* 37(3):549–58, 1990.

[10] MARTI, M., and T. STUDER, *The proof theory of common knowledge*, in: H. van Ditmarsch, and G. Sandu, (eds.), *Jaakko Hintikka on Knowledge and Game-Theoretical Semantics*, vol. 12 of *Outstanding Contributions to Logic*, Springer, Cham, 2018.

[11] MINARI, P., Some remarks on the proof-theory and the semantics of infinitary modal logic, in R. Kahle, T. Strahm, and T. Studer, (eds.), *Advances in Proof Theory*, vol. 28 of *Progress in Computer Science and Applied Logic*, Springer, 2016, pp. 291–318.

[12] NEGRI S., Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem, *Archive for Mathematical Logic* 42:389–401, 2003.

[13] NEGRI S., Proof analysis in modal logic, *Journal of Philosophical Logic* 34:507, 2005.

[14] NEGRI S., and J. VON PLATO, *Proof Analysis*, Cambridge University Press, 2001.

[15] NEGRI S., Geometric rules in infinitary logic, in O. Arieli, and A. Zamansky, (eds.), *Arnon Avron on Semantics and Proof Theory of Non-Classical Logics*, vol. 21 of *Outstanding Contributions to Logic*, Springer, 2021, pp. 265–293.

[16] OHNISHI M., and K. MATSUMOTO, Gentzen method in modal calculi, *Osaka Journal of Mathematics* 9:113–130, 1957

[17] PACUIT, E., *Neighborhood semantics for modal logic*, Springer, 2017.

[18] PALMGREN, E., An Intuitionistic Axiomatisation of Real Closed Fields, *Mathematical Logic Quarterly* 48(2):297–299, 2002.

[19] RADEV, S., Infinitary propositional normal modal logic, *Studia Logica* 46:291–309, 1987.

[20] SCHÜTTE, K., *Proof Theory*, Springer-Verlag, 1977.

[21] SCHWICHTENBERG, H., Proof theory: some applications of cut-elimination, in J. Barwise, (ed.), *Handbook of Mathematical Logic*, Elsevier, 1977, pp. 867–895.

[22] SUNDHOLM, B. G., *Proof Theory: a Survey of the Omega-Rule*, Ph.D. thesis, University of Oxford, 1983

[23] TAIT, W., Normal derivability in classical logic, in J. Barwise, (ed.), *The Syntax and Semantics of Infinitary Languages*, vol. 72 of *Lecture Notes in Mathematics*, Springer, 1968, pp. 204–236.

[24] TANAKA, Y., Kripke Completeness of Infinitary Predicate Multimodal Logics, Notre Dame *Journal of Formal Logic* 40(3):326–340, 1999.

[25] TANAKA, Y., Cut-elimination Theorems for Some Infinitary Modal Logics, *Mathematical Logic Quarterly* 47(3):327–339, 2001

[26] TANAKA, Y., Some Proof Systems for Predicate Common Knowledge Logic, *Reports on Mathematical Logic* 37:79–100, 2003.

[27] TANAKA, Y., Duality for κ-additive complete atomic modal algebras, *Algebra Universalis* 82:31, 2021.

[28] TESI, M., and S. NEGRI, Neighbourhood semantics and labelled calculus for intuitionistic infinitary logic, *Journal of Logic and Computation* 31(7):1608–1639, 2021

[29] VIGANÒ, L., *Labelled Non-Classical Logics*, Springer US, 2000

M. TESI
Scuola Normale Superiore di Pisa
Pisa
Italy
matteo.tesi@sns.it