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Group Representation for Even and Odd Involutive Commutative Residuated Chains

Abstract. For odd and for even involutive, commutative residuated chains a representation theorem is presented in this paper by means of direct systems of abelian \mathcal{o} -groups equipped with further structure. This generalizes the corresponding result of J. M. Dunn about finite Sugihara monoids.

Keywords: Involutive residuated lattices, Construction, Representation, Abelian \mathcal{o} -groups.

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1. Introduction

J. M. Dunn has described the structure of finite Sugihara monoids by showing in [2] that subdirectly irreducible Sugihara monoids are linearly ordered and by describing the structure of finite Sugihara chains in [15]. We generalize Dunn's results by dropping the idempotence and the finiteness axioms.

In line with the tradition of representing classes of residuated lattices by simpler and better known structures such as groups or Boolean algebras, Mundici's celebrated categorical equivalence theorem represents MV-algebras – a variety which corresponds to Łukasiewicz logic \mathbb{L} [12] – by ℓ -groups with strong units using a truncation construction [43]. By dropping the divisibility axiom of MV-algebras one obtains the variety of IMTL-algebras which correspond to the logic \mathbb{IMTL} [5, 9, 10, 13, 14, 16, 17, 24, 37, 44, 45]. The non-integral analogue of the class of IMTL-algebras shall be represented by \mathcal{o} -groups in this paper. By replacing the integrality axiom of IMTL-algebras by one of its two natural non-integral analogues one obtains the class of odd and the class of even involutive semilinear FL_e -algebras, the former of which is also known as the class of IUL^{fp} -algebras corresponding

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to the Involutive Uninorm Logics with fixed point (\mathbb{IUL}^{fp}) introduced by G. Metcalfe in [38], see also [39]. We shall represent all subdirectly irreducible members of these two classes by means of abelian \mathcal{o} -groups. Subdirectly irreducible even or odd involutive FL_e -algebras are linearly ordered. Little is known about the structure of IMTL-algebras and an effective structural description for this class seems to be out of reach at present, arguably, such a structural description is impossible. However, as shown in this paper, the non-integral counterpart of the class of IMTL-algebras possesses a representation by groups, similar to MV-algebras or to cancellative commutative residuated lattices [41].

Whether Involutive Uninorm Logic (\mathbb{IUL}) is standard complete or not is a problem posed by G. Metcalfe and F. Montagna [39] and has become a long-standing open problem which has invoked considerable effort to solve it with no avail¹. The representation theorem in this paper may serve as a step toward settling the problem of standard completeness of \mathbb{IUL} in an algebraic manner.

Group theory is the science of symmetries whereas the theory of inverse semigroups is the science of partial symmetries [35]. Our main result is reminiscent to the representation theorem of a subclass of inverse semigroups, called Clifford semigroups. Every Clifford semigroup is isomorphic to a strong semilattice of groups [35, Theorem 12 in Section 5.2] (see also [51]), and the starting point of the strong semilattice construction is a direct system of groups indexed by a semilattice. In our case the starting point is a direct system of abelian \mathcal{o} -groups indexed by a chain and equipped with some extra structure. The definition of the *product* in the strong semilattice construction is of the same Płonka sum fashion (see [48]) as in our construction in (4.8). However, in our case we need to modify the abelian \mathcal{o} -groups prior to applying (4.8), and we also need to handle the ordering, the residual operation, the isotonicity of the product with respect to the ordering etc. In particular, we introduce in this paper an ordering on Płonka sums, called the directed lexicographic order.

Płonka sums have found applications not only in computer science, in particular in the theory of program semantics [49], but recently its corresponding regular varieties have been unexpectedly connected to logic. They

¹Recently it is claimed to be proved in [52] by using proof-theoretic methods. However, the scientific community has doubts about the correctness of the proof, see the remark of the author himself in [52, second section in page 43].

provide algebraic semantics to logics obtained by imposing a deductive filter to other logics; for instance, Paraconsistent Weak Kleene logic coincides with the regularization of the variety of Boolean algebras [7, 8].

Integral residuated lattices have been widely studied in the literature [1, 4, 6, 12, 22, 25, 34]. However, as noted by N. Galatos and J. G. Raftery in [20], non-integral residuated structures and consequently, substructural logics without the weakening rule are less understood than their integral counterparts. To overcome this, they established category equivalences to carry over the algebraic knowledge on integral structures to non-integral ones [20, 21]. These kind of categorical approaches use some algebraic background knowledge which is available if idempotency is postulated. Our paper makes a step in the direction of providing an algebraic insight into the non-idempotent, non-integral, non-divisible setting.

Prominent examples of odd involutive FL_e -algebras are abelian ℓ -groups [3] and odd Sugihara monoids. These two classes of algebras represent two extremities: there is a single idempotent element in any lattice-ordered abelian group, whereas all elements are idempotent in any odd Sugihara monoid. The former class constitutes an algebraic semantics of Abelian Logic [11, 40, 47] while the latter constitutes an algebraic semantics of $IUML^*$, which is a logic at the intersection of relevance logic and many-valued logic [20]. Our representation theorem puts these two rather distant logics (and more) under the same methodological umbrella.

Despite the extensive literature devoted to classes of residuated lattices, there are still few effective structural descriptions. All these results in [1, 12, 22, 23, 25, 33, 34, 36, 42, 46, 50] (and also others where the focus is not a structural description in the first place such as in [20, 21] for example) postulate semilinearity which renders the subdirectly irreducible members linearly ordered, and either integrality together with the naturally ordered condition² or idempotency. Our study contributes to the structural description of residuated lattices which are semilinear but neither integral nor naturally ordered nor idempotent; a first such effective structural description to the best of our knowledge besides [31, 32].

A representation theorem has been presented in [31, 32] for those odd involutive FL_e -chains where the number of idempotent elements of the algebra is finite by means of partial sublex products of abelian o -groups, which are well understood mathematical objects that are much more regular than what had been expected to need for describing these particular FL_e -chains. In the present paper we prove a representation theorem for both even and

²Or its dual notion, called divisibility.

odd involutive FL_e -chains without assuming any constraints on the set of their idempotent elements. The conic case is also settled. While the construction in the representation of [31, 32] is done by starting with an abelian \mathcal{o} -group and iteratively enlarging it by other abelian \mathcal{o} -groups until the obtained structure becomes isomorphic to the given algebra, here we present a structural description using direct systems, without referring to iteration. To this end, a core auxiliary result concerns a one-to-one correspondence between two lattice ordered classes: odd involutive FL_e -algebras and even involutive FL_e -algebras with an idempotent falsum constant. Here we make use of and further develop ideas of E. Casari [11], see Section 6.

Since ℓ -groups are very specific mathematical objects compared to residuated lattices, having a representation of a class of residuated lattices by subclasses of ℓ -groups is a rarity. It is reasonable to expect that the inherent larger complexity of the more general classes of residuated lattices renders such a representation, if exists, quite involved in its constructional part. This phenomenon can be observed in [31, 32], for example, where the notions of two different partial sublex product constructions together with the technique of iteration have been used to describe the class of residuated lattices which is in our focus in the present paper with the additional assumption that the number of idempotent elements of the algebra is finite. In the present paper the complexity of the residuated structure is coded in the system of homomorphisms and in the delicate way of constructing the algebra from the direct system. Applications of the main result of this paper to amalgamation and densification problems are foreshadowed in [30].

2. Preliminaries

An FL_e -algebra is a structure $(X, \wedge, \vee, \cdot, \rightarrow, t, f)$ such that (X, \wedge, \vee) is a lattice, (X, \leq, \cdot, t) is a commutative, residuated monoid (the unit element t is also referred to as the *truth* constant), and f is an arbitrary constant, called the *falsum* constant. Being residuated means that there exists a binary operation \rightarrow , called the residual operation of \cdot , such that $xy \leq z$ if and only if $x \rightarrow z \geq y$. This equivalence is called adjointness condition, (\cdot, \rightarrow) is called an adjoint pair. Equivalently, for any x, z , the set $\{v \mid xv \leq z\}$ has its greatest element, and $x \rightarrow z$, the residuum of x and z , is defined as this element: $x \rightarrow z := \max\{v \mid xv \leq z\}$; this is called the residuation condition. Being residuated implies that \cdot is increasing. One defines the *residual complement operation* by $x' = x \rightarrow f$ and calls an FL_e -algebra *involutive* if $(x')' = x$ holds. An involutive FL_e -algebra is called *odd* if the residual complement operation

leaves the unit element fixed, that is, $t' = t$, and *even* if the following (two) quasi-identities hold: $x < t \Leftrightarrow x \leq f$. The former condition is equivalent to $f = t$, while the latter quasi-identities are equivalent to assuming that f is the lower cover of t (and t is the upper cover of f) if chains (or more generally conic algebras) are considered. An FL_e -algebra is called *integral* if the unit element of the multiplication is the top element of its universe. Commutative residuated lattices are the f -free reducts of FL_e -algebras. The geometric meaning of $t' = f$, which is valid in all involutive FL_e -algebras, is that the two constants are positioned symmetrically inside the underlying set. Therefore, one extremal setting is the integral case, when t and f are the top and bottom elements of X , respectively, and the other extremal one is the even or odd case when the two constants are both “in the middle of X ”. A bottom-top setup is not possible: if a residuated lattice has a bottom element then that element is annihilating, hence cannot be the unit of the multiplication³. Both odd or even involutive FL_e -chains and IMTL-chains are involutive residuated chains with a single additional postulate: the integrality condition of IMTL-chains postulates the unit element to be in one of its possible extremal positions (top), whereas the odd or even condition postulates the unit element to be in its other extremal position (in the middle).

Section 3 contains results on the local unit element function of involutive FL_e -algebras, a key concept for our representation theorem. Even involutive FL_e -algebras with non-idempotent and idempotent falsum constants will be characterized with respect to odd involutive FL_e -algebras in Sections 5 and 6, respectively. In Section 4 even and odd involutive FL_e -chains will be partitioned into their so-called layer algebras, which are also even or odd involutive FL_e -chains but more specific ones than the original algebra in the sense that they are either cancellative or are close to being such: in the latter case there exists a canonical homomorphism which maps the layer algebra into a cancellative one from which the layer algebra can be uniquely recovered. This specificity allows for establishing a connection between them and abelian \mathcal{o} -groups in Section 7, by using the characterizations of Sections 5 and 6. These lead to the main result of the paper in Section 8: a one-to-one correspondence in a constructive manner between the class of all even or odd involutive commutative residuated chains and the class of bunches of layer groups. Layer groups are direct systems of abelian \mathcal{o} -groups equipped with further structure.

³Unless the algebra is trivial.

3. The Local Unit Element Function

Let (X, \leq) be a poset. For $x, y \in X$ we say that y is a cover of x if $y > x$ and there exists no $z \in X$ such that $y > z > x$. For $x \in X$ let x_\uparrow be the unique cover of x if such exists, and let $x_\uparrow = x$ otherwise. Define x_\downarrow dually. Call \downarrow and \uparrow the neighbour operations of (X, \leq) . A partially ordered algebra with a poset reduct is called *discretely ordered* if for any element x , $x_\downarrow < x < x_\uparrow$ holds. If $'$ is an order-reversing involution of X then it holds true that

$$x'_\uparrow = (x_\downarrow)' \quad \text{and} \quad x'_\downarrow = (x_\uparrow)' \tag{3.1}$$

Algebras will be denoted by bold capital letters, their underlying sets by the same regular letter unless otherwise stated. Sometimes the lattice operators of an FL_e -algebra will be replaced by their induced ordering \leq in the signature, in particular, if an FL_e -chain is considered, that is, if the ordering is total. Call the elements $x \geq t$ of an FL_e -algebra $\mathbf{X} = (X, \wedge, \vee, \cdot, \rightarrow, t, f)$ *positive*. Call \mathbf{X} *conic* if all elements of X are comparable with t . Assume \mathbf{X} is involutive. For $x \in X$ let

$$\tau(x) = x \rightarrow x,$$

or equivalently, define $\tau(x)$ to be the greatest element of $Stab_x = \{z \in X \mid zx = x\}$. A key step toward our representation theorem is to understand the role of the τ function. In investigating more specific odd involutive FL_e -algebras $\tau(x)$ was recognized to play the role of the ‘absolute value of x ’ and was denoted by $|x|$, see [50]. The definition of layers in (3.2) and Lemma 3.1 reveal the true nature of τ in the present, more general setting: τ can be view as a ‘local unit element’ function. For any positive idempotent element u define

$$X_u = \{x \in X : \tau(x) = u\} \tag{3.2}$$

and call it the u -layer of \mathbf{X} . The X_u ’s form a partition of X by claim (vii) below, and if $x \in X_u$ then $\tau(x)$ is the unit element for the subset X_u of X by claim (v)⁴. Corresponding to the rotation-annihilation construction the idea of *decomposing* a class of FL_e -algebras with the help of a closely related

⁴For a first glance it might occur to the reader that the definition of X_u is equivalent to the well-known way of localizing in semigroup theory, namely, for an idempotent u , to the universe $M_u = \{x \in X : xu = x\}$ of the greatest unitary subsemigroup of \mathbf{X} with unit u . It is not the case if $u > t$. Rather, X_u can be regarded as the “outer edge” of M_u in the following sense: it holds true that $M_u = \bigcup_{v \leq u} X_v$ and $X_u = M_u \setminus \bigcup_{v < u} X_v$ (u and v are positive idempotents).

notion, the *skeleton* function of the algebra⁵ has been presented already in [27]. The name local unit function and the corresponding idea of using it to *localize* elements of the algebra has been introduced in [31], and the way of localizing as done in (3.2) is being introduced in the present paper, see also [29]. Some of the statements of the following lemma can be found in [31] or in [19], too. We include their proofs to keep the paper self-contained. As usual, when multiplication is denoted by \cdot , we write xy instead of $x \cdot y$.

LEMMA 3.1. *Let $\mathbf{X} = (X, \wedge, \vee, \cdot, \rightarrow, t, f)$ be an involutive FL_e -algebra. The following statements hold true.*

- (i) $t' = f$,
- (ii) $x \rightarrow y = (xy)'$,
- (iii) if $t \geq f$ then $xy \leq (x'y)'$,
- (iv) if the algebra is conic and either even or $t \leq f$ then $y_1 > y$ implies $(x'y_1)' \leq xy_1$,
- (v) $\tau(x)x = x$ and $\tau(x) \geq t$,
- (vi) $u \geq t$ is idempotent if and only if $\tau(u) = u$,
- (vii) $\{\tau(x) : x \in X\}$ is equal to the set of positive idempotent elements of \mathbf{X} ,
- (viii) $x_1y > xy$ holds whenever $x_1 > x$ and y is invertible,
- (xi) $\tau(x) = \tau(x')$,
- (x) if \cdot is cancellative then for $x \in X$, $xx' = f$,
- (xi) if \mathbf{X} is odd then the $(X, \wedge, \vee, \cdot, t)$ -reduct of \mathbf{X} is a lattice-ordered abelian group if and only if \cdot is cancellative,⁶
- (xii) for $x \geq t$, $\tau(x) \leq x$ holds,
- (xiii) if \mathbf{X} is an odd chain then X_t contains all the invertible elements of X , and it is a universe of an abelian o-group, its inverse operation is the residual complement operation.
- (xvi) if \mathbf{X} is an odd or even, cancellative, discretely ordered chain then $zt_{\downarrow} = z_{\downarrow}$,

⁵The skeleton function was defined there as the residual complement of what we coined local-unit function.

⁶When we (loosely) speak about a subgroup of odd involutive FL_e -algebra \mathbf{X} in the sequel, we shall mean a cancellative subalgebra of \mathbf{X} .

- (xv) if \mathbf{X} is an odd or even chain then $xy < x_1y_1$ holds whenever $x < x_1$ and $y < y_1$,
- (xvi) if \mathbf{X} is an odd or even chain and A is an \mathbf{X} -term which contains only the operations \cdot, \rightarrow and $'$ then for any evaluation e of the variables of A into X , $\tau(e(A))$ equals the maximum of the τ -values of the variables and constants of A under e ,

PROOF.

- (i) Obvious by residuation.
- (ii) Using that $'$ is an involution one obtains $(xy')' = (xy') \rightarrow f = x \rightarrow (y' \rightarrow f) = x \rightarrow y$ (folklore).
- (iii) Next, $(xy)(x'y') = [x(x \rightarrow f)][y(y \rightarrow f)] \leq ff \leq tf = f$, hence $xy \leq (x'y') \rightarrow f$ follows by adjointness.
- (iv) Since the algebra is conic and involutive, every element is comparable with f , too. Indeed, if for any $a \in X$, a were not comparable with f then, since $'$ is an order reversing involution, a' were not comparable with f' , and $f' = t$ since the algebra is involutive, a contradiction. Therefore, by residuation, $y_1 > (y')' = y' \rightarrow f$ implies $y_1y' \not\leq f$, that is, $y_1y' > f$. If $t \leq f$ then $y_1y' \geq t$ follows. Likewise, if the algebra is even then $y_1y' \not\leq f$ implies $y_1y' \not\leq t$, that is, $y_1y' \geq t$, since the algebra is conic. Therefore, $(xy_1)' = (xy_1)'t \leq (xy_1)'y_1y' \stackrel{(ii)}{=} (y_1(y_1 \rightarrow x'))y' \leq x'y'$ follows.
- (v) Since t is the unit element, $Stab_x$ is nonempty. Therefore, by residuation $\tau(x) = x \rightarrow x$ is its largest element, hence $\tau(x)x = x$ holds. Since $tx = x$, $\tau(x) \geq t$ follows by residuation.
- (vi) If $u \geq t$ is idempotent then from $uu = u$, $u \rightarrow u \geq u$ follows by adjointness. But for any $z > u$, $uz \geq tz = z > u$, hence $\tau(u) = u$ follows. On the other hand, by claim (v), $\tau(u) = u$ implies $u \geq t$, and also the idempotency of u follows since $uu = u\tau(u) = u$.
- (vii) If $u > t$ is idempotent then claim (vi) shows that u is in the range of τ . If u is in the range of τ , that is $\tau(x) = u$ for some $x \in X$ then if $\tau(\tau(x)) = \tau(x)$ then it implies $\tau(u) = u$, hence u is a positive idempotent element by claim (vi), and we are done. Hence it suffices to prove $\tau(\tau(x)) = \tau(x)$ for all x . By claim (ii), $x \rightarrow x = \tau(x)$ is equivalent to $xx' = \tau(x)'$. Hence, $\tau(x)\tau(x)' = \tau(x)(xx') = (\tau(x)x)x' \stackrel{(v)}{=} xx' = \tau(x)'$ follows, which is equivalent to $\tau(\tau(x)) = \tau(x) \rightarrow \tau(x) = \tau(x)$.

- (viii) $x_1y \geq xy$ holds by monotonicity, and if $x_1y = xy$ then $x_1 = x_1yy^{-1} = xyy^{-1} = x$, a contradiction.
- (ix) By claim (ii) and the involutivity of $'$, $\tau(x) = x \rightarrow x = (xx')' = (x'x'')' = x' \rightarrow x' = \tau(x')$.
- (x) Since \cdot is cancellative, the strictly increasing nature of \cdot clearly follows: if $u < v$ then $uw < vw$. Therefore, $xa > xt = x$ for any $a > t$, and hence $x \rightarrow x = t$. An application of claim (ii) ends the proof.
- (xi) Necessity is straightforward, sufficiency follows from claim (x) since $f = t$.
- (xii) For $x \geq t$, $x \stackrel{(v)}{=} x\tau(x) \geq t\tau(x) = \tau(x)$.
- (xiii) If $x \in X_t$ then $\tau(x) = x \rightarrow x = t$ and hence $xx' \stackrel{(ii)}{=} t' \stackrel{(i)}{=} f = t$ holds since the algebra is odd. This shows that all elements of X_t are invertible in X and the inverse operation is $'$. On the other hand, if $x \in X$ is invertible, that is, $xy = t$ for some $y \in X$ then $\tau(x) = t$: for every $z > t$ it holds true that $xz > x$ since the opposite, that is, $xz = x$ would imply $z = tz = (yx)z = y(xz) = yx = t$, a contradiction. Invertible elements are clearly closed under \cdot and $'$. Since the order is total, the meet and the join of two invertible elements are also invertible.
- (xiv) Since \cdot is cancellative, \cdot is strictly increasing; we shall use it without further mention. Now, $zt_{\downarrow} < zt = z$ holds since $t_{\downarrow} < t$ follows from the algebra being discretely ordered. Contrary to the statement, assume that there exists a such that $zt_{\downarrow} < a < z$. Multiplying with z' , $ft_{\downarrow} < az' < f$ follows by claim (x). In the odd case it yields $t_{\downarrow} < az' < t$, a contradiction to the definition of t_{\downarrow} . In the even case it yields $t_{\downarrow}t_{\downarrow} < az' < t_{\downarrow}$. Since $t_{\downarrow} = tt_{\downarrow} < t_{\uparrow}t_{\downarrow} < t_{\uparrow}t = t_{\uparrow}$, we obtain $t_{\uparrow}t_{\downarrow} = t$, and hence multiplication by t_{\uparrow} implies $t_{\downarrow} < t_{\uparrow}az' < t$, a contradiction to the definition of t_{\downarrow} .
- (xv) By claim (iii), $(x'y')' \geq xy$ holds, hence it suffices to prove $x_1y_1 > (x'y')'$. Assume the opposite, which is $x_1y_1 \leq (x'y')'$ since (X, \leq) is a chain. By adjointness we obtain $(x'x_1)(y'y_1) = (x_1y_1)(x'y') \leq f$, and from $x_1 > x = (x')'$, $x'x_1 > f$ follows by residuation since (X, \leq) is a chain.

In the odd case these reduce to $(x'x_1)(y'y_1) \leq t$ and $x'x_1 > t$. Analogously we obtain $y'y_1 > t$. Therefore $(x'x_1)(y'y_1) \geq (x'x_1)t = x'x_1 > t$

follows, a contradiction.

In the even case these reduce to $(x'x_1)(y'y_1) < t$ and $x'x_1 \geq t$. Analogously we obtain $y'y_1 \geq t$. Therefore $(x'x_1)(y'y_1) \geq tt = t$ follows, a contradiction.

(xvi) We have already seen $\tau(x) = \tau(x')$ in claim (ix).

Next, we claim $\tau(xy) = \max(\tau(x), \tau(y)) (= \tau(x)\tau(y)^7)$. Indeed, $\tau(xy) \geq \tau(x)$ holds by residuation since $xy \rightarrow xy \geq x \rightarrow x$ is equivalent to $yx(x \rightarrow x) \leq xy$. Assume $z := \tau(xy) > \max(\tau(x), \tau(y))$. Since τ assigns to x the greatest element of the stabilizer set of x , therefore $z > \max(\tau(x), \tau(y))$ implies that z does not stabilize x neither y , hence $x < zx$ and $y < zy$ holds by the monotonicity of \cdot . On the other hand, $(zx)(zy) = ((xy)z)z \stackrel{(v)}{=} xy$ follows, a contradiction to claim (xv). This settles the claim.

By claim (ii), any term which contains only the connectives \cdot, \rightarrow and $'$ can be represented by an equivalent term using the same variables and constants but containing only \cdot and $'$. An easy induction on the recursive structure of this equivalent term using the two claims above concludes the proof. ■

The next lemma states that for an odd involutive FL_e -algebra, it is exactly cancellativity which is needed to make it a lattice ordered group.

LEMMA 3.2.

(1) For a cancellative odd involutive FL_e -algebra

$$\mathbf{X} = (X, \wedge, \vee, \cdot, \rightarrow, t, t)$$

with residual complement $'$, $\lambda(\mathbf{X}) = (X, \wedge, \vee, \cdot, {}^{-1}, t)$ is a lattice-ordered abelian group, called the lattice-ordered abelian group induced by \mathbf{X} , where

$$x^{-1} = x' = x \rightarrow t, \tag{3.3}$$

(2) For a lattice-ordered abelian group

$$\mathbf{G} = (G, \leq, \cdot, {}^{-1}, t),$$

⁷For positive idempotents $u \leq v$ it holds true that $uv = \max(u, v)$ since $v = tv \leq uv \leq vv = v$.

$\iota(\mathbf{G}) = (G, \wedge, \vee, \cdot, \rightarrow, t, t)$ is a cancellative odd involutive FL_e -algebra, called the cancellative odd involutive FL_e -algebra induced by \mathbf{G} , where

$$x \rightarrow y = x^{-1}y, \tag{3.4}$$

$$x' = x^{-1}, \tag{3.5}$$

(3) With the above assumptions it holds true that $\iota(\lambda(\mathbf{X})) = \mathbf{X}$ and $\lambda(\iota(\mathbf{G})) = \mathbf{G}$.

PROOF. Claim (xi) in Lemma 3.1 confirms the first statement, the rest is folklore or obvious. ■

The following lemma will simplify the proof of Theorem 6.3 and Lemma 4.2. Let $\mathcal{M} = (M, \leq, \cdot)$ be a structure such that (M, \leq) is a poset and (M, \cdot) is a commutative semigroup. Call $c \in M$ a dualizing element⁸ of \mathcal{M} , if (i) for $x \in M$ there exists, $x \rightarrow c$ ⁹ and (ii) for $x \in M$, $(x \rightarrow c) \rightarrow c = x$.

LEMMA 3.3. If there exists a dualizing element c of \mathcal{M} then \cdot is residuated and its residual operation is given by $x \rightarrow y = (x(y \rightarrow c)) \rightarrow c$.

PROOF. $zx \leq y$ is equivalent to $zx \leq (y \rightarrow c) \rightarrow c$. By adjointness it is equivalent to $(zx)(y \rightarrow c) \leq c$. By associativity it is equivalent to $z(x(y \rightarrow c)) \leq c$, which is equivalent to $z \leq (x(y \rightarrow c)) \rightarrow c$ by adjointness. By residuation $x \rightarrow y = (x(y \rightarrow c)) \rightarrow c$ follows. ■

4. Odd and Even Involutive FL_e -chains vs. Bunches of Layer Algebras

We shall prove the main theorem of the paper for three different kinds of involutive FL_e -chains: for odd involutive FL_e -chains, for even involutive FL_e -chains with an idempotent falsum, and for even involutive FL_e -chains with a non-idempotent falsum.

DEFINITION 4.1. Let (κ, \leq_κ) be a totally ordered set with least element t , and let an ordered triple $\langle \bar{\kappa}_I, \bar{\kappa}_J, \{t\} \rangle$ be a partition of κ , where $\bar{\kappa}_I$ and $\bar{\kappa}_J$ can also be empty. Define κ_o , κ_J , and κ_I by one of the rows of Table 1.¹⁰

⁸Dualizing elements have been defined only in *residuated* structures in the literature, see e.g. [19, Section 3.4.17].

⁹That is, there exists the greatest element of the set $\{z \in M \mid xz \leq c\}$.

¹⁰Explanation: if the first, the second, or the third row is used then the algebra corresponding to the bunch will be odd, even with a non-idempotent falsum constant, or even with an idempotent falsum constant, respectively.

Table 1. Three cases—three classes

κ_o	κ_J	κ_I
$\{t\}$	$\bar{\kappa}_J$	$\bar{\kappa}_I$
\emptyset	$\bar{\kappa}_J \cup \{t\}$	$\bar{\kappa}_I$
\emptyset	$\bar{\kappa}_J$	$\bar{\kappa}_I \cup \{t\}$

Let $\mathbf{X}_u = (X_u, \leq_u, \cdot_u, \rightarrow_u, u, u^\mu)$ be a family of involutive FL_e -chains indexed by elements of κ (let $^\mu$ denote the residual complement operation, \downarrow_u and \uparrow_u the neighbour κ operations of \mathbf{X}_u), such that \mathbf{X}_u is

$$\left\{ \begin{array}{ll} \text{cancellative and odd} & \text{if } u \in \kappa_o \\ \text{discretely ordered, cancellative and even} & \text{if } u \in \kappa_J \\ \text{even with an idempotent falsum satisfying } x \cdot_u x^\mu = u^\mu & \text{if } u \in \kappa_I \end{array} \right. \tag{4.1}$$

¹¹ and such that for $u, v \in \kappa, u \leq_\kappa v$, there exist a

$$\text{homomorphism } \rho_{u \rightarrow v} \tag{4.2}$$

from the residuated lattice reduct of \mathbf{X}_u to the residuated lattice reduct of \mathbf{X}_v satisfying

- (A1) $\rho_{u \rightarrow u} = id_{X_u}$ and $\rho_{v \rightarrow w} \circ \rho_{u \rightarrow v} = \rho_{u \rightarrow w}$ (direct system property),
- (A2) for $u <_\kappa v, \rho_{u \rightarrow v}(u) = \rho_{u \rightarrow v}(u^\mu)$ ¹² (constants' collision condition).

Call $\mathcal{A} = \langle \mathbf{X}_u, \rho_{u \rightarrow v} \rangle_{\langle \kappa_o, \kappa_J, \kappa_I, \leq_\kappa \rangle}$ a *bunch of layer algebras*. Call the \mathbf{X}_u 's the layer algebras, call $\langle \kappa, \leq_\kappa \rangle$ the *skeleton*, call $\langle \kappa_o, \kappa_J, \kappa_I \rangle$ the *partition* of the skeleton, and call $\langle \mathbf{X}_u, \rho_{u \rightarrow v} \rangle_\kappa$ the direct system of layer algebras over κ . Note that κ can be recovered from its partition, (and ultimately, from \mathcal{A}) via $\kappa = \kappa_o \cup \kappa_J \cup \kappa_I$.

We prove that every odd or even involutive FL_e -chain can be represented by a unique bunch of layer algebras. Later, in Section 7 we prove that every bunch of layer algebras can be represented by a unique bunch of layer groups.

LEMMA 4.2. *The following statements hold true.*

- (1) *Given an odd or an even involutive FL_e -chain $\mathbf{X} = (X, \leq, \cdot, \rightarrow, t, f)$ with residual complement operation $'$,*

$$\mathcal{A}_\mathbf{X} = \langle \mathbf{X}_u, \rho_{u \rightarrow v} \rangle_\kappa$$

¹¹Hence with a non-idempotent falsum.

¹²If $u \in \kappa_o$ then $\rho_{u \rightarrow v}(u) = \rho_{u \rightarrow v}(u^\mu)$ trivially holds since $t^\mu = t$.

is a bunch of layer algebras, called the bunch of layer algebras of \mathbf{X} , where $\tau(x) = x \rightarrow x$, $\kappa = \{\tau(x) : x \in X\}$, $\leq_\kappa = \leq \cap (\kappa \times \kappa)$, $\bar{\kappa}_I = \{u \in \kappa \setminus \{t\} : u' \text{ is idempotent}\}$, $\bar{\kappa}_J = \{u \in \kappa \setminus \{t\} : u' \text{ is not idempotent}\}$, κ_o , κ_J , κ_I are defined by Table 2,

$\kappa = \langle \kappa_o, \kappa_J, \kappa_I, \leq_\kappa \rangle$, for $u \in \kappa$,

$$\mathbf{X}_u = (X_u, \leq_u, \cdot_u, \rightarrow_u, u, u'), \tag{4.3}$$

where

$$X_u = \{x \in X : \tau(x) = u\},$$

\leq_u , \cdot_u , and \rightarrow_u are restrictions of \leq , \cdot , and \rightarrow to X_u , for $x \in X_u$, $x^u = x \rightarrow u'$, and for $u, v \in \kappa$, $u \leq_\kappa v$, $\rho_{u \rightarrow v} : X_u \rightarrow X_v$ is given by

$$\rho_{u \rightarrow v}(x) = vx. \tag{4.4}$$

(2) Given a bunch of layer algebras $\mathcal{A} = \langle \mathbf{X}_u, \rho_{u \rightarrow v} \rangle_\kappa$ with $\kappa = \langle \kappa_o, \kappa_J, \kappa_I, \leq_\kappa \rangle$, $\mathbf{X}_u = (X_u, \leq_u, \cdot_u, \rightarrow_u, u, u^u)$, and $x^u = x \rightarrow_u u^u$,

$$\mathcal{X}_\mathcal{A} = (X, \leq, \cdot, \rightarrow, t, t')$$

is an involutive FL_e -chain, called the involutive FL_e -chain derived from \mathcal{A} , where

$$X = \bigcup_{u \in \kappa} X_u, \tag{4.5}$$

for $v \in \kappa$, $\rho_v : X \rightarrow X$ is defined by

$$\rho_v(x) = \begin{cases} \rho_{u \rightarrow v}(x) & \text{if } u <_\kappa v \text{ and } x \in X_u \\ x & \text{if } u \geq_\kappa v \text{ and } x \in X_u \end{cases}, \tag{4.6}$$

for short, for $x \in X_u$ and $y \in X_v$,¹³

$$x < y \text{ iff } \rho_{uv}(x) <_{uv} \rho_{uv}(y) \text{ or } \rho_{uv}(x) = \rho_{uv}(y) \text{ and } u <_\kappa v \tag{4.7}$$

$$xy = \rho_{uv}(x) \cdot_{uv} \rho_{uv}(y), \tag{4.8}$$

$$x' = x^u, \tag{4.9}$$

$$x \rightarrow y = (xy)'. \tag{4.10}$$

¹³Note that for $u, v \in \kappa$, $uv = \max_\kappa(u, v)$ since u, v are positive idempotents.

Table 2. Three cases—three classes

κ_o	κ_J	κ_I	
$\{t\}$	$\bar{\kappa}_J$	$\bar{\kappa}_I$	if \mathbf{X} is odd
\emptyset	$\bar{\kappa}_J \cup \{t\}$	$\bar{\kappa}_I$	if \mathbf{X} is even and f is not idempotent
\emptyset	$\bar{\kappa}_J$	$\bar{\kappa}_I \cup \{t\}$	if \mathbf{X} is even and f is idempotent

¹⁴ and t is the least element of κ . \mathcal{X}_A is odd if $t \in \kappa_o$, even with a non-idempotent falsum if $t \in \kappa_J$, and even with an idempotent falsum if $t \in \kappa_I$.

(3) For a bunch of layer algebras \mathcal{A} , $\mathcal{A}_{(\mathcal{X}_A)} = \mathcal{A}$, and for an odd or even involutive FL_e -chain \mathbf{X} , $\mathcal{X}_{A_X} = \mathbf{X}$.

PROOF. (1): κ is the set of positive idempotent elements of \mathbf{X} by claim (vii) in Lemma 3.1. Therefore, the least element of κ is t , and κ , being a subset of X , is totally ordered. The ordered triple $\langle \bar{\kappa}_I, \bar{\kappa}_J, \{t\} \rangle$ is clearly a partition of κ , where $\bar{\kappa}_I$ and $\bar{\kappa}_J$ can also be empty.

Let $u \in \kappa$. X_u is nonempty since $u \in X_u$ holds by claim (vi) in Lemma 3.1, and, being a subset of X , X_u is totally ordered by \leq_u . X_u is closed under \cdot_u , \rightarrow_u , and $'$ by claim (xvi) in Lemma 3.1, and thus $u^\mu = u \rightarrow u' \in X_u$. Since $\tau(x) = u$ holds for $x \in X_u$, therefore $xu = x\tau(x) \stackrel{L3.1.(v)}{=} x$ shows that u is the unit element of \mathbf{X}_u . For $x \in X_u$,

$$x^\mu = x' \tag{4.11}$$

holds since $x^\mu = x \rightarrow u' = x \rightarrow (u \rightarrow f) \stackrel{\text{is residuated}}{=} (xu) \rightarrow f = x \rightarrow f = x'$. Therefore, $u^\mu = u'$, and hence $x^\mu = x \rightarrow u^\mu$. Summing up,

$$\mathbf{X}_u = (X_u, \leq_u, \cdot_u, \rightarrow_u, u, u') \text{ is an involutive } FL_e\text{-chain.} \tag{4.12}$$

Next we prove that the \mathbf{X}_u 's satisfy (4.1).

If $u \in \kappa_o$ then \mathbf{X} is odd by Table 2. By claim (xiii) in Lemma 3.1 and by Lemma 3.2, \mathbf{X}_u is a cancellative odd involutive FL_e -chain.

If $u \in \kappa_J$ then by Table 2, $u = t$ or $u \in \bar{\kappa}_J$. In both cases u' is not idempotent.

We prove that X_u is discretely ordered by showing

$$xu' = x_{\downarrow u} < x \tag{4.13}$$

¹⁴Alternatively, we may write $x \leq y$ iff $\rho_{uv}(x) \leq_{uv} \rho_{uv}(y)$ except if $u >_\kappa v$ and $\rho_{uv}(x) = \rho_{uv}(y)$.

for $x \in X_u$, where \downarrow_u denotes the neighbour operation on X_u . It holds true that

$$u' < t. \tag{4.14}$$

Indeed, if $u \in \bar{\kappa}_J$ then $u > t$ and the involutivity of $'$ on X implies $u' < t' \stackrel{L3.1(i)}{=} f^{\mathbf{X} \text{ is odd or even}} \leq t$, whereas if $u = t$ then $u' = t' < t$ since the second row of Table 2 shows that \mathbf{X} is even. Therefore, by denoting $y = u'u'$,

$$y < u' \tag{4.15}$$

holds since $y = u'u' \stackrel{(4.14)}{\leq} u't = u'$ and equality cannot hold since u' is not idempotent. Now $xu' \stackrel{(4.14)}{\leq} xt = x$ follows. Assume, by contradiction $xu' = x$. It would imply $xy = x(u'u') = (xu')u' = x$, hence by claim (iii) in Lemma 3.1, $(x'y')' \geq xy = x$, and in turn $x'y' \leq x'$ would follow on the one hand. On the other hand, from $y' \stackrel{(4.15)}{>} u \geq t$, by monotonicity $x'y' \geq x't = x'$ follows, thus we obtain $x'y' = x'$, and hence $y' \leq \tau(x') \stackrel{L3.1(ix)}{=} u$, a contradiction to (4.15). We have just seen that $xu' < x$. Next, assume that there exists $z \in X_u$ such that $xu' < z < x$ holds. Since $z < x$, $xu' \geq (z'u)' = z'' = z$ follows by claim (iv) in Lemma 3.1, a contradiction, so (4.13) is confirmed.

Next we show that \mathbf{X}_u is even: $u^\mu \stackrel{(4.11)}{=} u' = uu' \stackrel{(4.13)}{=} u_{\downarrow_u} < u$.

Finally, we show that \mathbf{X}_u is cancellative, by showing that every element of X_u has inverse, that is, for $x \in X_u$, $x \cdot_u x'_{\uparrow_u} = u$. On the one hand, referring to (4.12), $x \cdot_u x'_{\uparrow_u} >_u u^\mu$ holds by residuation since X is a chain. It is equivalent to $x \cdot_u x'_{\uparrow_u} \geq_u u$ since \mathbf{X}_u is even, yielding $x \cdot_u x'_{\uparrow_u} \geq u$. On the other hand, $x \cdot_u x'_{\uparrow_u} = xx'_{\uparrow_u} \stackrel{(4.11)}{=} xx'_{\uparrow_u} \stackrel{(3.1)}{=} x(x_{\downarrow_u})' \stackrel{(4.13)}{=} x(xu')' \stackrel{L3.1(ii)}{=} x(x \rightarrow u) \stackrel{\cdot \text{ is residuated}}{\leq} u$.

Summing up, \mathbf{X}_u is a discretely ordered cancellative even involutive FL_e -chain.

If $u \in \kappa_I$ then by Table 2, $u = t$ or $u \in \bar{\kappa}_I$. In both cases u' is idempotent. We show that \mathbf{X}_u is even: first, $u^\mu < u$ holds since if $u \in \bar{\kappa}_I$ then $u > t$ and hence $u^\mu \stackrel{(4.11)}{=} u' < t' \stackrel{L3.1(i)}{=} f^{\mathbf{X} \text{ is odd or even}} \leq t < u$, whereas if $u = t$ then \mathbf{X} is

even by the third row of Table 2 and $u^\mu \stackrel{(4.11)}{=} u' = t' \stackrel{L3.1(ii)}{=} f \stackrel{\mathbf{X} \text{ is even}}{<} t = u$, and second, by claims (xii) and (xvi) in Lemma 3.1, no element $x \in X$ such that $u' < x < u$ can be in X_u (if $u' < x < u$ then $u' < x' < u$, so we may safely assume $x \geq t$, and then $x \in X_u$ implies $u = \tau(x) \leq x < u$, a contradiction); thus $u^\mu = u_{\downarrow u}$. Summing up, \mathbf{X}_u is an even involutive FL_e -chain with an idempotent falsum. It remains to prove $x \cdot_u x^\mu = u^\mu$ for $x \in X_u$, which follows from $u^\mu \stackrel{(4.11)}{=} u' = \tau(x)' = (x \rightarrow x)'\stackrel{L3.1(ii)}{=} xx' \stackrel{(4.11)}{=} xx^\mu = x \cdot_u x^\mu$.

Next we prove that $\rho_{u \rightarrow v}$ is a homomorphism from the residuated lattice reduct of \mathbf{X}_u to the residuated lattice reduct of \mathbf{X}_v . Let $u, v \in X$ be positive idempotent elements of \mathbf{X} such that $u < v$. $\rho_{u \rightarrow v}$ maps X_u to X_v by claim (xvi) in Lemma 3.1. $\rho_{u \rightarrow v}$ preserves the ordering since \cdot is monotone. $\rho_{u \rightarrow v}$ preserves products since \cdot is associative and v is idempotent.

To show that $\rho_{u \rightarrow v}$ preserves the residual operation we proceed as follows. Let $x, y \in X_u$. It holds true that $v(xy)'(vy)'\stackrel{L3.1(ii)}{=} x(x \rightarrow y)(y \rightarrow v')v \leq f$ since \cdot is residuated, hence by adjointness, $v(xy)' \leq (x(vy))'$ follows. On the other hand, $v(xy)'\stackrel{L3.1(iii)}{\geq} v(x'y) = x'(vy)$. Now $x = tx \leq vx$, and since $v > u = \tau(x)$ and hence v does not stabilize x , $x < vx$ follows. Therefore, $x'(vy)\stackrel{L3.1(iv)}{\geq} ((vx)(vy))' = (x(v(vy)))'\stackrel{(vy)' \in X_v}{=} (x(vy))'$. Summing up, $v(xy)' = (x(vy))'$. Therefore, $\rho_{u \rightarrow v}(x \rightarrow_u y)\stackrel{L3.1(ii)}{=} v(x \cdot_u y^\mu)^\mu \stackrel{(4.11)}{=} v(xy)' = (x(vy))'\stackrel{(vy)' \in X_v}{=} (x(v(vy)))' = ((vx)(vy))'\stackrel{(4.11)}{=} ((vx) \cdot_v (vy))^\mu \stackrel{L3.1(ii)}{=} (vx) \rightarrow_v (vy) = \rho_{u \rightarrow v}(x) \rightarrow_v \rho_{u \rightarrow v}(y)$ and we are done.

Finally, $\rho_{u \rightarrow v}$ preserves the unit element:

$$\rho_{u \rightarrow v}(u) = vu = v \tag{4.16}$$

holds since $v = vv \geq vu \geq vt = v$. Summing up, $\rho_{u \rightarrow v}$ is a homomorphism from the residuated lattice reduct of \mathbf{X}_u to the residuated lattice reduct of \mathbf{X}_v .

To conclude the proof of claim (1) it only remains to prove

- (A1) For $u, v, w \in X$ positive idempotent elements such that $u \leq v \leq w$, and for $x \in X_u$, $\rho_{u \rightarrow u}(x) = ux \stackrel{L3.1(v)}{=} x$ and $(\rho_{v \rightarrow w} \circ \rho_{u \rightarrow v})(x) \stackrel{(4.4)}{=} w(vx) = (wv)x \stackrel{(4.16)}{=} wx = \rho_{u \rightarrow w}(x)$.

(A2) Let $u, v \in X$ positive idempotent elements such that $u < v$. By claim (iv) in Lemma 3.1, $v = vt \geq vt' \geq (v't)' = \tau(v) = v$, yielding $\rho_{u \rightarrow v}(u') \stackrel{(4.4)}{=} vt' = v \stackrel{(4.16)}{=} vt \stackrel{(4.4)}{=} \rho_{u \rightarrow v}(u)$.

(2): We conclude by a series of claims.

\leq is a total ordering on X .

- Irreflexivity of $<$ is immediate from the irreflexivity of $<_{\kappa}$.
- Connectedness of $<$ is obvious, too: either $\rho_{uv}(x) <_{uv} \rho_{uv}(y)$ or $\rho_{uv}(y) <_{uv} \rho_{uv}(x)$ or $\rho_{uv}(x) = \rho_{uv}(y)$ holds, since \leq_{uv} is total. The first two cases yield $x <_{(4.7)} y$ and $y <_{(4.7)} x$, respectively. If $\rho_{uv}(x) = \rho_{uv}(y)$ then either $u <_{\kappa} v$ or $v <_{\kappa} u$ or $u =_{\kappa} v$ holds since \leq_{κ} is total. Here $u =_{\kappa} v$ yields $x \stackrel{(A1)}{=} \rho_{uu}(x) = \rho_{vv}(y) \stackrel{(A1)}{=} y$, whereas $u <_{\kappa} v$ and $v <_{\kappa} u$ yields $x <_{(4.7)} y$ and $y <_{(4.7)} x$, respectively.
- $<$ is transitive: Let $x \in X_u, y \in X_v, z \in X_w$, and assume $x < y < z$. From $x < y$ it follows that $\rho_{uv}(x) \leq_{uv} \rho_{uv}(y)$, hence by preservation of the ordering, $\rho_{uvw}(x) \leq_{uvw} \rho_{uvw}(y)$ holds. Analogously we obtain $\rho_{uvw}(y) \leq_{uvw} \rho_{uvw}(z)$, hence $\rho_{uvw}(x) \leq_{uvw} \rho_{uvw}(z)$ follows by the transitivity of \leq_{uvw} . Therefore either $\rho_{uvw}(x) <_{uvw} \rho_{uvw}(z)$ and we conclude $x < z$, or $\rho_{uvw}(x) = \rho_{uvw}(z)$. The latter implies $\rho_{uvw}(x) = \rho_{uvw}(y) = \rho_{uvw}(z)$, and also $u <_{\kappa} v$ and $v <_{\kappa} w$. Therefore, by the transitivity of $<_{\kappa}$, $u <_{\kappa} w$ follows and thus $x < z$.

(X, \cdot, t) is a commutative monoid.

- Commutativity of \cdot is straightforward.
- Let $x \in X_u, y \in X_v, z \in X_w$. Then $(xy)z = (\rho_{uv}(x) \cdot_{uv} \rho_{uv}(y))z$, and the latest is equal to $\rho_{uvw}(\rho_{uv}(x) \cdot_{uv} \rho_{uv}(y)) \cdot_{uvw} \rho_{uvw}(z)$ since $\rho_{uv}(x) \cdot_{uv} \rho_{uv}(y) \in X_{uv}$. Since the ρ 's preserve products and \cdot is idempotent, the latest is equal to $(\rho_{uvw}(x) \cdot_{uvw} \rho_{uvw}(y)) \cdot_{uvw} \rho_{uvw}(z)$. Analogously follows that $x(yz)$ is equal to $\rho_{uvw}(x) \cdot_{uvw} (\rho_{uvw}(y) \cdot_{uvw} \rho_{uvw}(z))$, and hence the associativity of \cdot_{uvw} implies the associativity of \cdot .
- For $x \in X_u, tx \stackrel{(4.8)}{=} \rho_u(t) \cdot_u \rho_u(x) \stackrel{(4.6)}{=} \rho_u(t) \cdot_u x \stackrel{(4.2)}{=} u \cdot_u x = x$ holds using that u is the unit element of \mathbf{X}_u .

' is an order reversing bijection on X .

We start with two claims.

- (C1) If $u <_{\kappa} v$ then for $x \in X_u$, $\rho_v(x^{\mu})$ is the inverse of $\rho_v(x)$ in \mathbf{X}_v . Indeed, $\rho_v(x) \cdot_v \rho_v(x^{\mu}) \stackrel{(4.6)}{=} \rho_{u \rightarrow v}(x) \cdot_v \rho_{u \rightarrow v}(x^{\mu}) \stackrel{(4.2)}{=} \rho_{u \rightarrow v}(x \cdot_u x^{\mu})$. Note that $x \cdot_u x^{\mu} = u^{\mu}$ holds not only if $u \in \kappa_I$ (see (4.1)), but also if $u \in \kappa_o \cup \kappa_J$, since due to the cancellativity of \mathbf{X}_u , $x \cdot_u x^{\mu} = x \cdot_u (x \rightarrow_u u^{\mu}) \stackrel{(3.4)}{=} x \cdot_u (x^{-1_u} \cdot_u u^{\mu}) = (x \cdot_u x^{-1_u}) \cdot_u u^{\mu} = u \cdot_u u^{\mu} = u^{\mu}$. Therefore, $\rho_{u \rightarrow v}(x \cdot_u x^{\mu}) = \rho_{u \rightarrow v}(u^{\mu})$, which is equal to $\rho_{u \rightarrow v}(u) \stackrel{(4.2)}{=} v$ if $u \in \kappa_o$ (since then $u^{\mu} = u$), and is equal to $\stackrel{(A2)}{=} \rho_{u \rightarrow v}(u) \stackrel{(4.2)}{=} v$ if $u \notin \kappa_o$.
- (C2) If $u <_{\kappa} v$ then for $x \in X_u$, $\rho_v(x)^{\flat} = \rho_v(x^{\mu})_{\downarrow_v} < \rho_v(x^{\mu})$. First we prove $\rho_v(x)^{\flat} \cdot_v \rho_v(x) = v_{\downarrow_v}$: if $v \in \kappa_J$ then due to the cancellativity of \mathbf{X}_v (see (4.1)), $\rho_v(x)^{\flat} \cdot_v \rho_v(x) = \rho_v(x) \cdot_v (\rho_v(x) \rightarrow_v v^{\flat}) \stackrel{(3.4)}{=} \rho_v(x) \cdot_v (\rho_v(x)^{-1} \cdot_v v^{\flat}) = (\rho_v(x) \cdot_v \rho_v(x)^{-1}) \cdot_v v^{\flat} = v \cdot_v v^{\flat} = v^{\flat}$ \mathbf{X}_v is even, see (4.1) v_{\downarrow_v} , whereas if $v \in \kappa_I$ then $\rho_v(x)^{\flat} \cdot_v \rho_v(x) = v^{\flat} = v_{\downarrow_v}$ holds by (4.1), and we are done. Now, multiplying both sides by $\rho_v(x^{\mu})$ yields $\rho_v(x)^{\flat} = v_{\downarrow_v} \cdot_v \rho_v(x^{\mu})$ using (C1). If $v \in \kappa_J$ then by (4.1) we can apply claim (xiv) in Lemma 3.1 resulting in $\rho_v(x^{\mu})_{\downarrow_v} = v_{\downarrow_v} \cdot_v \rho_v(x^{\mu}) < v \cdot_v \rho_v(x^{\mu}) = \rho_v(x^{\mu})$, so we are done. If $v \in \kappa_I$ then by (4.1), \mathbf{X}_v is an even involutive FL_e -chain with an idempotent falsum. Therefore, $\mathbf{X}_v = Sp(\mathbf{A}, \mathbf{H})$ by Theorem 6.3, and since $\rho_v(x^{\mu})$ is invertible by (C1), $\rho_v(x^{\mu})$ is an element of \mathbf{H} by claim (c) in Theorem 6.3. On the other hand, v_{\downarrow_v} is clearly in H^{\bullet} . Therefore, $v_{\downarrow_v} \cdot_v \rho_v(x^{\mu}) \stackrel{(6.8)}{=} (v \cdot_v \rho_v(x^{\mu}))_{\downarrow_v} = \rho_v(x^{\mu})_{\downarrow_v} \stackrel{\rho_v(x^{\mu}) \in H}{<} \rho_v(x^{\mu})$ holds.
- Since for $u \in \kappa$, $^{\mu}$ is of order 2, so is $'$ over X by (4.9), hence $'$ is a bijection. It remains to prove that $'$ is order reversing. Let $X_u \ni x \leq y \in X_v$. If $u = v$ then $x \leq_u y$ holds by (4.7), hence $y^{\mu} \leq_u x^{\mu}$ follows since \mathbf{X}_u is involutive, thus $y' \leq x'$ holds by (4.7) and (A1). If $u <_{\kappa} v$ then $x \leq y$ implies $\rho_v(x) \leq_{uv} y$ by (4.7), hence $y^{\flat} \leq_v \rho_v(x)^{\flat}$ follows since \mathbf{X}_v is involutive. Therefore, $y^{\flat} \leq_v \rho_v(x)^{\flat} \stackrel{(C2)}{<} \rho_v(x^{\mu})$, and thus $y^{\flat} <_v \rho_v(x^{\mu})$ implies $y^{\flat} \leq x^{\mu}$ by (4.7), yielding $y' \leq x'$ by (4.9). If $u >_{\kappa} v$ then $x \leq y$ implies $x <_u \rho_u(y)$ by (4.7), which is equivalent to $\rho_u(y)^{\mu} <_u x^{\mu}$ since \mathbf{X}_u is involutive. By (C2), $\rho_u(y)^{\flat} \leq_u x^{\mu}$ follows, which yields $y^{\flat} \leq x^{\mu}$ by (4.7), and hence $y' \leq x'$ follows by (4.9).

t' is a dualizing element of (X, \leq) .

- It suffices to prove that for $x \in X$, there exists $x \rightarrow t'$ and

$$x \rightarrow t' = x', \tag{4.17}$$

since the involutivity of $'$ then ensures $(x \rightarrow t') \rightarrow t' = x$. Equivalently, that for $x, y \in X$, $xy \leq t'$ if and only if $x \leq y'$. Let $x \in X_u, y \in X_v$. Since \cdot is commutative and since $'$ is an order reversing bijection, we may safely assume $u \leq_\kappa v$. Since $t' \stackrel{(4.9)}{=} t^\dagger \in X_t$ and $t \leq_\kappa v$, by (4.8) $xy \leq t^\dagger$ is equivalent to $\rho_v(x) \cdot_v \rho_v(y) \leq t^\dagger$. Since $\rho_v(x) \cdot_v \rho_v(y) \in X_v$, $\rho_v(x) \cdot_v \rho_v(y) \leq t^\dagger$ is equivalent to $\rho_v(x) \cdot_v \rho_v(y) \leq_v v^\dagger$: indeed, if $t = v$ then $\rho_v(x) \cdot_v \rho_v(y) \leq t^\dagger$ is equivalent to $\rho_v(x) \cdot_v \rho_v(y) \stackrel{(4.7)}{\leq_v} t^\dagger = v^\dagger$, whereas if $t <_\kappa v$ then $v \notin \kappa_o$ and $\rho_v(x) \cdot_v \rho_v(y) \leq t^\dagger$ is equivalent to $\rho_v(x) \cdot_v \rho_v(y) \stackrel{(4.7)}{<_v} \rho_v(t^\dagger) \stackrel{(4.6)}{=} \rho_{t \rightarrow v}(t^\dagger) \stackrel{(A2)}{=} \rho_{t \rightarrow v}(t) \stackrel{(4.2)}{=} v$, that is, equivalent to $\rho_v(x) \cdot_v \rho_v(y) \stackrel{v \notin \kappa_o, (4.1)}{\leq_v} v \downarrow_v \stackrel{(4.1)}{=} v^\dagger$. Since \mathbf{X}_v is residuated, $\rho_v(x) \cdot_v \rho_v(y) \leq_v v^\dagger$ is equivalent to $\rho_v(x) \leq_v y \rightarrow_v v^\dagger = y^\dagger$, and by (4.7) it is equivalent to $x \leq y^\dagger \stackrel{(4.9)}{=} y'$.

Summing up, we have shown that (X, \leq) is a chain and (X, \cdot, t) is a commutative monoid. Since t' is a dualizing element of (X, \leq, \cdot) , Lemma 3.3 shows that (X, \leq, \cdot) is residuated and $x \rightarrow y = (xy)'$. Since $'$ (given in (4.9)) coincides with the residual complement of \mathbf{X} (given by $x \rightarrow t'$, see (4.17)), and since $'$ is an order reserving involution on X , it follows that \mathbf{X} is involutive. Finally, by (4.1), \mathbf{X}_t and hence also \mathbf{X} is odd if $t \in \kappa_o$, \mathbf{X} is even with a non-idempotent falsum if $t \in \kappa_J$, and \mathbf{X} is even with an idempotent falsum if $t \in \kappa_I$.

(3): Let $\mathcal{A} = \langle \mathbf{X}_u, \rho_{u \rightarrow v} \rangle_\kappa$ be a bunch of layer algebras, and adapt the notations in Definition 4.1 and the definitions in claim (2). To see that the universe of the u^{th} layer algebra of $\mathcal{A}_{(\mathcal{X}_\mathcal{A})}$ is equal to X_u which is the universe of the u^{th} layer algebra \mathbf{X}_u of \mathcal{A} , we need to prove that for $x \in X$ (where X is given in (4.5)), $x \rightarrow x = u$ if and only if x is in X_u , the universe of the u^{th} -layer algebra: $x \in X$ implies that $x \in X_v$ for some $v \in \kappa$. Now $u = x \rightarrow x \stackrel{(4.10)}{=} (xx)'$ $\stackrel{(4.9)(4.8)}{=} (x \cdot_v x^\dagger)'$ $\stackrel{(4.9)}{=} (x \cdot_v x^\dagger)^\dagger$, where the last equality holds since X_v is closed under † and \cdot_v and thus $x \cdot_v x^\dagger$ is in X_v . Therefore, $u \in X_v$ follows. Hence $u = v$ must hold since $u \in X_u$ and X is the disjoint union of the X_u 's by (4.5). The definition of the ordering relation in claim (1) and (4.7) show that the ordering of the u^{th} layer algebra of $\mathcal{A}_{(\mathcal{X}_\mathcal{A})}$ is the same as the ordering \leq_u of the u^{th} layer algebra \mathbf{X}_u of \mathcal{A} . Likewise show (4.8) and the definition of the monoidal

operation in claim (1) that the monoidal operation of the u^{th} layer algebra of $\mathcal{A}_{(\mathcal{X}_{\mathcal{A}})}$ is the same as the monoidal operation \cdot_u of the u^{th} layer algebra \mathbf{X}_u of \mathcal{A} . Since both the u^{th} layer algebra of $\mathcal{A}_{(\mathcal{X}_{\mathcal{A}})}$ and the u^{th} layer algebra \mathbf{X}_u of \mathcal{A} are involutive FL_e -chains over the same universe, equipped with the same ordering relation and the same product operation, their residual operations – which are uniquely determined by these – must coincide, too. The unit element of \mathbf{X}_u is u , therefore u acts as the unit element of the u^{th} layer algebra of $\mathcal{A}_{(\mathcal{X}_{\mathcal{A}})}$ (which is over the same set X_u and is equipped with the same monoidal operation, as we have seen above). Finally, the falsum constant of \mathbf{X}_u is u'' . On the other hand, the falsum constant of the u^{th} layer algebra of $\mathcal{A}_{(\mathcal{X}_{\mathcal{A}})}$ is $u \rightarrow u' \stackrel{(4.9)}{=} u \rightarrow u'' = u \rightarrow_u u'' = u''$, where the last equality holds by residuation since u is the unit element over X_u . Summing up, $\mathcal{A}_{(\mathcal{X}_{\mathcal{A}})} = \mathcal{A}$.

Let $\mathbf{X} = (X, \leq, \cdot, \rightarrow, t, f)$ be an odd or even involutive FL_e -chain, and adapt the definitions in claim (1). We have seen in the proof of claim (1) that the X_u 's are nonempty. It is straightforward that they are disjoint, too, and their union, which is the universe of $\mathcal{X}_{\mathcal{A}_{\mathbf{X}}}$, see (4.5), is equal to X . To prove that the ordering of \mathbf{X} and of $\mathcal{X}_{\mathcal{A}_{\mathbf{X}}}$ coincide, first we prove the following statement. If $v < u$ and $x \in X_v$ then

$$\rho_u(x) = \min\{z \in X_u : z \geq x\} > x. \tag{4.18}$$

Indeed, $ux \in X_u$ by claim (xvi) in Lemma 3.1, and $ux \geq x$ holds since $ux \geq tx = x$. By contradiction, assume that there exists $z \in X_u$ such that $x < z < ux$. Since \leq is total, by adjointness $z'' < ux$ is equivalent to $f < z'(ux) = (z'u)x \stackrel{z' \in X_u}{=} z'x$. Finally, since \leq is total, by adjointness $f < z'x$ is equivalent to $x > z'' = z$, a contradiction. Referring to (4.18) a moment's reflection shows that the ordering \leq of X coincides with the ordering of $\mathcal{X}_{\mathcal{A}_{\mathbf{X}}}$ given by (4.7). Since for $x \in X_u$ and $y \in X_v$, $xy \stackrel{L3.1(v)}{=} (uux)(vvy) = ((uv)x)((uv)y) = \rho_{uv}(x)\rho_{uv}(y) = \rho_{uv}(x) \cdot_{uv} \rho_{uv}(y)$, the monoidal operation \cdot (of X) coincides with \cdot given in (4.8). Since both \mathbf{X} and $\mathcal{X}_{\mathcal{A}_{\mathbf{X}}}$ are involutive FL_e -chains over the same universe, equipped with the same ordering relation and the same product operation, their residual operations – which are uniquely determined by these – must coincide, too. By (4.3), the unit element of \mathbf{X}_t is t , and hence the unit element of $\mathcal{X}_{\mathcal{A}_{\mathbf{X}}}$ is also t . Finally, the falsum constant of \mathbf{X} is f , hence the falsum constant of \mathbf{X}_t is $t^f = t \rightarrow t' \stackrel{L3.1(i)}{=} t \rightarrow f \stackrel{L3.1(i)}{=} f$. Therefore, the falsum constant of $\mathcal{X}_{\mathcal{A}_{\mathbf{X}}}$ is also $t^f = f$. Summing up, $\mathcal{X}_{\mathcal{A}_{\mathbf{X}}} = \mathbf{X}$. ■

REMARK 4.3. Two elements of the construction of \mathcal{X}_A in Lemma 4.2/(2) are reminiscent to the Płonka sum construction [48]. In the Płonka sum construction the universe of the algebra is the disjoint union of the universes of the algebras in the direct system. So is in our construction, c.f. (4.5). In the Płonka sum construction every binary operator $\circ \in \{\cdot, \rightarrow, \wedge, \vee\}$ would be given in terms of the direct system as follows: for $x \in X_u, y \in X_v$,

$$x \circ y = \rho_{u \rightarrow v}(x) \circ_{uv} \rho_{u \rightarrow v}(y) \tag{4.19}$$

In our construction it is the case only for $\circ \in \{\cdot\}$, c.f. (4.8). As for $\circ \in \{\rightarrow\}$, if \rightarrow were given by (4.19) then for any positive idempotent elements $u < v$, it would hold true that $v \rightarrow u = u \rightarrow v$ since $v \rightarrow u \stackrel{(4.19)}{=} v \rightarrow_v v \stackrel{(4.19)}{=} u \rightarrow v$. But in an involutive FL_e -chain it would be equivalent to $vu' = uv'$ by Lemma 3.1/(ii), a contradiction, since $vu' = v$ and $uv' = v'$ (as shown by $v \stackrel{L3.1(vi)}{=} \tau(v) \stackrel{L3.1(ii)}{=} (vv')' \stackrel{L3.1(iv)}{\leq} vu' \leq vt = v$ and $v' = tv' \leq uv' \leq vv' \stackrel{L3.1(ii)}{=} \tau(v)' \stackrel{L3.1(vi)}{=} v'$, respectively). Finally, notice that Płonka's definition doesn't work for $\circ \in \{\wedge, \vee\}$ since (4.19) would render different elements of the universe equal. Indeed, if $u < v$ are positive idempotent elements, $x \in X_u$ and $\rho_{u \rightarrow v}(x) = y \in X_v$ then according to (4.19), $x \wedge y = \rho_{u \rightarrow v}(x) \wedge_v y = y = \rho_{u \rightarrow v}(x) \vee_v y = x \vee y$ yielding $x = y$, a contradiction since x and y are in different layers. Even though our definition is applied in our specific (residuated lattice) setting only, we have introduced in (4.7) a general way of ordering Płonka sums, call it the *directed lexicographic order*. Making use of the (slight) similarity between our construction and that of Płonka's would have made the related proof only a few lines shorter. Therefore, we have included those few lines, too, to make the treatment self-contained.

Bunches of layer algebras will, in turn, be represented by bunches of layer groups in Section 7.1. To that end, in the following two sections the necessary auxiliary results (along with some which are more general than what we actually need in this paper) will be developed.

5. Even Involutive FL_e -Chains with Non-idempotent Falsum Constants – Changing the Falsum Constant

An abelian o -group is called *discrete*, if there exists the smallest positive element greater than the unit element. It is equivalent to saying that the abelian o -group is discretely ordered, or that its induced cancellative odd involutive FL_e -algebra is discretely ordered. In the following definition the

residuated chain reduct of the FL_e -chain is left unchanged, and only the falsum constant, and thus also the residual complement are changed slightly.

DEFINITION 5.1. For a discretely ordered, cancellative, odd, involutive FL_e -chain $\mathbf{X} = (X, \leq, \cdot, \rightarrow, t, t)$, let

$$\mathbf{X}_\downarrow = (X, \leq, \rightarrow, t, t_\downarrow).$$

For a discretely ordered, cancellative, even, involutive FL_e -chain $\mathbf{Y} = (Y, \leq, \cdot, \rightarrow, t, t_\downarrow)$, let

$$\mathbf{Y}_\uparrow = (Y, \leq, \cdot, \rightarrow, t, t).$$

LEMMA 5.2. Let \mathbf{X} and \mathbf{Y} be discretely ordered, cancellative, involutive FL_e -chains, \mathbf{X} is odd, \mathbf{Y} is even. Then

- (1) \mathbf{X}_\downarrow is a discretely ordered, cancellative, even, involutive FL_e -chain,
- (2) \mathbf{Y}_\uparrow is a discretely ordered, cancellative, odd, involutive FL_e -chain,
- (3) $\mathbf{X}_{\downarrow\uparrow} = \mathbf{X}$ and $\mathbf{Y}_{\uparrow\downarrow} = \mathbf{Y}$.

PROOF. The rest being obvious we prove the involutivity of \mathbf{X}_\downarrow and \mathbf{Y}_\uparrow .

(1): To see that \mathbf{X}_\downarrow is involutive, denote the residual complement operation of \mathbf{X} by $'$, and the inverse operation of $\lambda(\mathbf{X})$ by $^{-1}$. Then $x \rightarrow t_\downarrow \stackrel{(3.4)}{=} x^{-1}t_\downarrow \stackrel{(3.3)}{=} x't_\downarrow \stackrel{L3.1(xiv)}{=} (x')_\downarrow$, and hence (3.1) confirms involutivity. (2): To see that \mathbf{Y}_\uparrow is involutive, denote the residual complement operation of \mathbf{Y} by \bullet . Then $y \rightarrow t \stackrel{L3.1(ii)}{=} (yt') \bullet \stackrel{L3.1(i)}{=} (yf) \bullet \stackrel{\mathbf{Y} \text{ is even}}{=} (yt_\downarrow) \bullet \stackrel{L3.1(xiv)}{=} y_\downarrow \bullet$ holds, and hence (3.1) confirms involutivity. ■

6. Even Involutive FL_e -Algebras with Idempotent Falsum Constants – Subgroup Splits of Odd Involutive FL_e -Algebras

As an investigation into the structure of residuated lattices which are not necessarily totally ordered, a one-to-one correspondence between pairs of an odd involutive FL_e -algebra and a cancellative subalgebra of it, and even involutive FL_e -algebras where the residual complement of the unit element is idempotent will be proved in this section. In groups the unit element has two different roles to play. It serves as the unit element of the multiplication, and also the product of any element by its inverse is equal to it. We shall replace the unit element of any odd involutive FL_e -algebra by two elements,

each will inherit a single role. This way both the unit element itself and its two roles will be “split” into two. Some anticipatory examples: the unit-split algebra of the one-element group will be the two-element Boolean algebra, and the unit-split algebra of the Sugihara lattice $\mathbf{S}_{\mathbb{Z}}$ will be *the* Sugihara lattice $\mathbf{S}_{\mathbb{Z}^*}$, (both named by Meyer, see [2][p. 414]). Moreover, not only the unit element, but in fact any subgroup of an odd involutive FL_e -algebra can be “split”, that is, each element of the subgroup of an odd involutive FL_e -algebra will be replaced by two elements. We prove that by splitting (and thus “doubling”) a subgroup of any odd involutive FL_e -algebra we obtain an even involutive FL_e -algebra with an idempotent falsum constant, and each even involutive FL_e -algebra with an idempotent falsum constant arises this way in a unique manner (Theorem 6.3).

Such a splitting method has roots in the literature. In [11] an analogous idea has been applied to abelian pregroups. Our method here is both more general and more specific: the algebras that we split are more general than that of [11], however, in our approach the (second) algebra we use in the splitting procedure is the two element Boolean algebra (each element is split into two elements only) whereas in [11] it is taken from the more general class of zero-pregroups. The notion of the melting structure (of a pregroup) in [11] is analogous to the image (of the even involutive FL_e -algebra with idempotent falsum constant) under what we call its canonical homomorphism. The splitting procedure in [11] is precursor also for the partial lex-product construction of [31], which when further generalized to partial sublex-products has been capable to describe the structure of all odd, involutive FL_e -chains which have only finitely many idempotent elements [31,32].

DEFINITION 6.1. Let $\mathbf{Y} = (Y, \wedge, \vee, \star, \rightarrow_{\star}, t, f)$ be an even FL_e -algebra with an idempotent falsum constant. Denote its residual complement by $\overset{\uparrow}{\cdot}$. Let

$$\pi_1(\mathbf{Y}) = \mathbf{X} = (X, \dot{\wedge}, \dot{\vee}, \cdot, \rightarrow, t, t) \text{ and } \pi_2(\mathbf{Y}) = \mathbf{H} = (H, \dot{\wedge}, \dot{\vee}, \cdot, \rightarrow, t, t)$$

be given by

$$H = \{x \in Y : x \star f < x\}, H^{\bullet} = \{x \star f : x \in H\}, X = Y \setminus H^{\bullet}, \quad (6.1)$$

for $y \in Y$,

$$h_{\mathbf{Y}}(x) = \begin{cases} x & \text{if } x \in X \\ x_{\uparrow} & \text{if } x \in H^{\bullet} \end{cases}, \quad (6.2)$$

and for $x, y \in X$,

$$\begin{aligned} x \dot{\wedge} y &= h_{\mathbf{Y}}(x \wedge y), \\ x \dot{\vee} y &= h_{\mathbf{Y}}(x \vee y), \\ xy &= h_{\mathbf{Y}}(x \star y), \end{aligned} \tag{6.3}$$

$$x \rightarrow y = h_{\mathbf{Y}}(x \rightarrow_{\star} y). \tag{6.4}$$

DEFINITION 6.2. Let $\mathbf{X} = (X, \wedge, \vee, \cdot, \rightarrow, t, t)$ be an odd involutive FL_e -algebra with residual complement $'$. Let $\mathbf{H} \leq \mathbf{X}$ (over $H \subseteq X$), \mathbf{H} cancellative¹⁵. Let $Sp(\mathbf{X}, \mathbf{H})$, the \mathbf{H} -split of \mathbf{X} be

$$\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \rightarrow_{\star}, t, t^{\bullet}),$$

where $H^{\bullet} = \{h^{\bullet} : h \in H\}$ is a copy of H disjoint from X ,

$$Y = X \cup H^{\bullet}, \tag{6.5}$$

the lattice ordering \leq of X is extended to Y by letting

$$a^{\bullet} <_Y b^{\bullet} \text{ and } x <_Y a^{\bullet} <_Y y \text{ for } a, b \in H, a < b, x, y \in X, x < a \leq y, \tag{6.6}$$

$h : Y \rightarrow X$,

$$h(x) = \begin{cases} x & \text{if } x \in X \\ x_{\uparrow} & \text{if } x \in H^{\bullet} \end{cases}, \tag{6.7}$$

where \uparrow and \downarrow denote the neighbour operations of \mathbf{Y} ,

$$x \star y = \begin{cases} h(x)h(y) & \text{if } h(x)h(y) \notin H \text{ or } x, y \in H \\ (h(x)h(y))_{\downarrow} & \text{if } \neg(x, y \in H) \text{ and } h(x)h(y) \in H \end{cases}, \tag{6.8}$$

$\overset{\star}{t} : Y \rightarrow Y$,

$$\begin{aligned} x^{\overset{\star}{t}} &= \begin{cases} x' & \text{if } x \in X \setminus H \\ (x')_{\downarrow} & \text{if } x \in H \\ (x_{\uparrow})' & \text{if } x \in H^{\bullet} \end{cases}, \\ x \rightarrow_{\star} y &= (x \star y^{\overset{\star}{t}})^{\overset{\star}{t}}. \end{aligned} \tag{6.9}$$

THEOREM 6.3. *The following statements hold true.*

(1) *Let \mathbf{X} be an odd involutive FL_e -algebra, \mathbf{H} be a cancellative subalgebra of it, and \mathbf{Y} be $Sp(\mathbf{X}, \mathbf{H})$, the \mathbf{H} -split of \mathbf{X} . Then*

(a) *\mathbf{Y} is an even FL_e -algebra with an idempotent falsum constant.*

(b) *If \mathbf{X} is cancellative then for $x \in Y$, $x \star x^{\overset{\star}{t}} = t^{\bullet}$ holds.*

¹⁵Equivalently, $\lambda(\mathbf{H})$ is a lattice ordered abelian group by Lemma 3.2.

- (c) $\{x \in Sp(\mathbf{X}, \mathbf{H}) : x \text{ is invertible in } Sp(\mathbf{X}, \mathbf{H})\} = \mathbf{H}$.
- (2) Let \mathbf{Y} be an even involutive FL_e -algebra with an idempotent falsum constant. Then
 - (a) there exists a unique pair $\langle \mathbf{X}, \mathbf{H} \rangle$ of an odd involutive FL_e -algebra $\mathbf{X} = (X, \wedge, \vee, \cdot, \rightarrow, t, t)$ and a cancellative subalgebra \mathbf{H} of \mathbf{X} such that \mathbf{Y} is $Sp(\mathbf{X}, \mathbf{H})$, the \mathbf{H} -split of \mathbf{X} . \mathbf{X} and \mathbf{H} are given by $\pi_1(\mathbf{Y})$ and $\pi_2(\mathbf{Y})$, respectively.
 - (b) If for $x \in Y$, $x \star x^{\check{}}$ = f holds then \mathbf{X} is cancellative.
 - (c) $h_{\mathbf{Y}}$ is a surjective homomorphism (called the canonical homomorphism of \mathbf{Y}) from \mathbf{Y} onto \mathbf{X} , that is, it holds true that

$$\mathbf{X} = h_{\mathbf{Y}}(Sp(\mathbf{X}, \mathbf{H})).$$

PROOF. (a): It is obvious from (6.6) that

$$\text{for } a \in H, a \text{ is the unique cover of } a^{\bullet}. \tag{6.10}$$

Therefore, by letting $k : X \rightarrow Y$,

$$k(x) = \begin{cases} x & \text{if } x \in X \setminus H \\ x^{\bullet} & \text{if } x \in H \end{cases},$$

a moment's reflection shows that \leq_Y is a lattice ordering on Y and the corresponding lattice operations are given by

$$x \wedge_Y y = y \wedge_Y x = \begin{cases} y & \text{if } h(y) \leq h(x), y \in H^{\bullet} \\ h(x) \wedge h(y) & \text{otherwise} \end{cases},$$

$$x \vee_Y y = y \vee_Y x = \begin{cases} h(x) \vee h(y) & \text{if } h(x) \vee h(y) \notin H \\ k(h(x) \vee h(y)) & \text{if } \{h(x), h(y)\} \not\preceq h(x) \vee h(y) \in H \\ h(x) \vee h(y) & \text{if } h(x) \leq h(y), y \in H \\ k(h(x) \vee h(y)) & \text{if } h(x) \leq h(y), y \in H^{\bullet} \end{cases}.$$

It is straightforward from (6.8) that \star is commutative. t is the unit element of \star : using that t is the unit element over X ,

$$x \star t \stackrel{(6.8), (6.7)}{=} \begin{cases} h(x)t = h(x) = x & \text{if } h(x) \notin H \text{ or } x \in H \\ (h(x)t)_{\downarrow} = h(x)_{\downarrow} = x & \text{if } x \notin H \text{ and } h(x) \in H \end{cases}.$$

We obtain $a^{\bullet} = a_{\downarrow} < a$ for $a \in H$ from (6.10), therefore, $\check{}$ in (6.9) is clearly an order reversing involution by (3.1).

As for the associativity of \star , first notice that

$$x \star y \in H \text{ if and only if } x, y \in H. \tag{6.11}$$

Indeed, if $x, y \in H$ then $x \star y \stackrel{(6.8)}{=} h(x)h(y) \stackrel{(6.7)}{=} xy \in H$ since H is closed under \cdot . If $\neg(x, y \in H)$ then either $h(x)h(y) \in H$ in which case $x \star y \stackrel{(6.8)}{\in} H^\bullet$ implies $x \star y \notin H$, or $h(x)h(y) \notin H$ in which case $x \star y \stackrel{(6.8)}{=} h(x)h(y) \notin H$, hence we are done. Next notice that

$$\text{for } x, y \in Y, h(x \star y) = h(x)h(y). \tag{6.12}$$

Indeed, if the first row of (6.8) defines the value of $x \star y$ then $x \star y = (h(x)h(y))_\downarrow$ and $h(x)h(y) \in H$, and hence $h(x \star y) = h((h(x)h(y))_\downarrow) \stackrel{(6.7)}{=} (h(x)h(y))_{\downarrow\uparrow} = h(x)h(y)$, whereas if $x \star y \stackrel{(6.8)}{=} h(x)h(y)$ then $h(x \star y) = h(h(x)h(y)) = h(x)h(y)$ since h maps to X , X is closed under \cdot , and h is the identity on X .

Now, $h((x \star y) \star z) = h(x \star (y \star z))$ readily follows from (6.12) and the associativity of \cdot . Therefore, by (6.7), $(x \star y) \star z \neq x \star (y \star z)$ can only be possible if one side is in H and the other side is in H^\bullet . However, (6.11) shows that if one side is in H then $x, y, z \in H$, and hence, since H is closed under \star by (6.11), also the other side must be in H .

It is easily seen that

$$\star \text{ is isotone with respect to } \leq_Y. \tag{6.13}$$

Indeed, let $x, y, z \in Y$ with $y <_Y z$. Since h and \cdot are increasing with respect to \leq , it follows that $h(x)h(y) \leq h(x)h(z)$. Now by (6.8), $x \star y \leq_Y x \star z$ clearly holds if $h(x)h(y) < h(x)h(z)$, hence we may assume $h(x)h(y) = h(x)h(z)$. But then the only way for $x \star y \not\leq_Y x \star z$ to hold is if $x \star y = h(x)h(y)$ and $x \star z = (h(x)h(z))_\downarrow$, which by (6.8) leads to assuming $\neg(x, z \in H)$, $h(x)h(z) \in H$ and either $h(x)h(y) \notin H$ or $x, y \in H$. It follows that $x, y \in H$. But then $h(x)h(y) = xy < xh(z) \leq h(x)h(z)$ is a contradiction, where the strict inequality follows from $y < h(z)$ and since x has an inverse.

To prove that \star is residuated, and that $x \rightarrow_\star y = (x \star y^\dagger)^\dagger$, by Lemma 3.3 it suffices to verify that t^\dagger is a dualizing element of (Y, \leq, \star) . Since \dagger is clearly an order reversing involution, it suffices to verify that that $x \rightarrow_\star t_\downarrow$ exists and is equal to x^\dagger . It amounts to verifying only three cases. (i) If $x \in H$ then $x^\dagger = (x')_\downarrow \in H^\bullet$ and $x \star x^\dagger \stackrel{(6.9)}{=} x \star (x')_\downarrow \stackrel{(6.8)}{=} (xx')_\downarrow = t_\downarrow$, where in the last equality we used that \mathbf{H} is a subgroup, hence x' is the inverse of x in \mathbf{X} . On the other hand, for $z > x^\dagger$ it follows that $z \geq x'$ and hence $x \star z \stackrel{(6.13)}{\geq} x \star x' \stackrel{x' \in H, (6.8)}{=} xx' = t > t_\downarrow$. Therefore, by residuation, $x \rightarrow_\star t_\downarrow$ exists and is equal to x^\dagger . (ii) If $x \in H^\bullet$ then $x = y_\downarrow$ for some $y \in H$ and

$x^{\uparrow} = y' = (x_{\uparrow})' \stackrel{(3.1)}{=} x'_{\downarrow} > x'$. We obtain $x \star x^{\uparrow} = y_{\downarrow} \star y' \stackrel{(6.8)}{=} (yy')_{\downarrow} = t_{\downarrow}$.

On the other hand, for $z > x^{\uparrow}$ it follows that $z \geq x'$ and hence $x \star z \stackrel{(6.13)}{\geq} y_{\downarrow} \star x' \stackrel{(6.8)}{=} (yx')_{\downarrow} = (yy'_{\uparrow})_{\downarrow}$. Here $yy'_{\uparrow} > t$ holds by residuation since $y'_{\uparrow} > y'$, hence $x \star z > t_{\downarrow}$ follows. (iii) Finally, if $x \in X \setminus H$ then $x^{\uparrow} = x' \in X \setminus H$. The case $x \star x^{\uparrow} = x \star x' \in H$ leads to $x \star x^{\uparrow} = x \star x' \stackrel{(6.8)}{=} (xx')_{\downarrow} \leq t_{\downarrow}$, whereas if $x \star x^{\uparrow} = x \star x' \in X \setminus H$ then $x \star x^{\uparrow} = x \star x' \stackrel{(6.8)}{=} xx' \leq t$, but due to $x \star x^{\uparrow} \in X \setminus H$ and $t \in H$ equality cannot hold, hence here too, $x \star x^{\uparrow} \leq t_{\downarrow}$ follows. On the other hand if $z > x^{\uparrow} = x'$ then $x \star z \geq (xz)_{\downarrow} > t_{\downarrow}$ holds by (6.8) and by residuation, respectively.

Summing up, the falsum-free reduct of \mathbf{Y} is an involutive commutative residuated lattice with residual complement operation $\overset{\uparrow}{\cdot}$. By the second row of (6.9), $t^{\uparrow} = (t')_{\downarrow} = t_{\downarrow}$, and t_{\downarrow} is idempotent by the second row of (6.8). A particular instance of (6.10) shows that t is the unique cover of t^{\bullet} , hence $t^{\bullet} = t_{\downarrow}$ and thus \mathbf{Y} is even.

(b): Since

$$h(x)h(x') = xx' \stackrel{L3.1(x)}{=} f = t \in H, \tag{6.14}$$

for $x \in Y$, $x \star x^{\uparrow} \stackrel{(6.9)}{=} \begin{cases} x \star x' \stackrel{(6.8), (6.14)}{=} (h(x)h(x'))_{\downarrow} = t^{\bullet} & \text{if } x \in X \setminus H \\ x \star (x')_{\downarrow} \stackrel{(6.8)}{=} (h(x)h(x'_{\downarrow}))_{\downarrow} \stackrel{x' \in H, (6.7)}{=} (xx')_{\downarrow} \stackrel{(3.5)}{=} (xx^{-1})_{\downarrow} = t^{\bullet} & \text{if } x \in H \\ x \star (x_{\uparrow})' \stackrel{(6.8)}{=} (h(x)h(x_{\uparrow}')')_{\downarrow} \stackrel{(6.7)}{=} (x_{\uparrow}x_{\uparrow}')_{\downarrow} \stackrel{(3.5)}{=} (x_{\uparrow}x_{\uparrow}^{-1})_{\downarrow} = t^{\bullet} & \text{if } x \in H^{\bullet} \end{cases}$

(c): Since $t \in H$ and H^{\bullet} is disjoint from H , referring to (6.7), it follows from (6.8) that $x \star y$ can be equal to t only if $x, y \in H$. On the other hand, every element of H is invertible in \mathbf{X} by claim (xi) in Lemma 3.1, since \mathbf{H} is cancellative by assumption. Hence $x \star x^{-1} \stackrel{(6.8)}{=} h(x)h(x^{-1}) \stackrel{(6.7)}{=} xx^{-1} = t$.

(a): Unicity of \mathbf{H} follows from claim (c), and it readily implies the unicity of \mathbf{X} , too, by (6.5). Let $\mathbf{X} = \pi_1(\mathbf{Y})$ and $\mathbf{H} = \pi_2(\mathbf{Y})$. Denote $x^{\bullet} = x \star f$ for $x \in H$.

(i) Any element x of H is invertible, that is, $x \star x^{\uparrow} = t$ holds, where $\overset{\uparrow}{\cdot}$ is given by

$$x^{\uparrow} = x \rightarrow_{\star} t.$$

Indeed, $t \geq x \star (x \rightarrow_{\star} t) \stackrel{L3.1(ii)}{=} x \star (x \star t^{\uparrow})^{\uparrow} \stackrel{L3.1(i)}{=} x \star (x \star f)^{\uparrow} \not\leq f$, where the latest step holds by residuation since $x \in H$, that is $x \star f <$

Table 3. Product Table

\star	$y \in X \setminus H$	$l \in H$	$l^\bullet \in H^\bullet$
$x \in X \setminus H$	$\in (X \setminus H) \cup H^\bullet$	$\in X \setminus H$	$x \star l \in X \setminus H$
$m \in H$	$\in (X \setminus H)$	$\in H$	$(m \star l)^\bullet \in H^\bullet$
$m^\bullet \in H^\bullet$	$m \star y \in X \setminus H$	$(m \star l)^\bullet \in H^\bullet$	$(m \star l)^\bullet \in H^\bullet$

x , and it implies $(x \star f)^\dagger > x^\dagger$. Since \mathbf{Y} is even, $t \geq x \star x^\dagger \not\leq f$ implies $x \star x^\dagger = t$.

- (ii) $H \cap H^\bullet = \emptyset$. Indeed, if $x \in H^\bullet$, that is, if $x = y \star f$ for some $y \in H$ then $x \star f = (y \star f) \star f = y \star (f \star f) = y \star f = x$ ensures $x \notin H$.
- (iii) $t \in H \subseteq X$. Indeed, $t \in H$ because of $t \star f = f < t$, and $H \subseteq X$ readily follows from (6.1) and claim (ii). Hence it holds true that

$$Y = (X \setminus H) \dot{\cup} H \dot{\cup} H^\bullet. \tag{6.15}$$

- (iv) Next we prove $m^\bullet = m_\downarrow < m$ for $m \in H$. Indeed, the assumption $m \star f < z < m$ would yield $f = t \star f \stackrel{(i)}{=} (m^\dagger \star m) \star f = m^\dagger \star (m \star f) \stackrel{L3.1(viii)}{<} m^\dagger \star z \stackrel{L3.1(viii)}{<} m^\dagger \star m = t$, a contradiction to t covering f .

- (v) For $m \in H$ and $y \in (X \setminus H) \cup H^\bullet$ it holds true that

$$m^\bullet \star y = m \star y.$$

Indeed, $(X \setminus H) \cup H^\bullet = Y \setminus H$ holds by (6.15), therefore $y \star f = y$. We obtain $m \star y = m \star (y \star f) = (m \star f) \star y = m^\bullet \star y$, as stated.

- (vi) For $m \in H, m^\dagger \in H: m^\dagger \notin H$ would imply $t \stackrel{(i)}{=} m \star m^\dagger \stackrel{(v)}{=} m^\bullet \star m^\dagger = (m \star f) \star m^\dagger = (m \star m^\dagger) \star f \stackrel{(i)}{=} t \star f = f$, a contradiction.

The following product table holds true¹⁶, see Table 3.

$\star_{(2,2)}$: Since m is invertible by claim (i), $(m \star l) \star f = m \star (l \star f) < m \star l$ follows by claim (viii) in Lemma 3.1.

$\star_{(2,3)}$: $m \star l^\bullet = m \star (l \star f) = (m \star l) \star f \stackrel{\star_{(2,2)}}{=} (m \star l)^\bullet \in H^\bullet$.

$\star_{(3,3)}$: $m^\bullet \star l^\bullet = (m \star f) \star (l \star f) = (m \star l) \star (f \star f) = (m \star l) \star f \stackrel{\star_{(2,2)}}{=} (m \star l)^\bullet \in H^\bullet$.

¹⁶We shall refer to the (i, j) cell of this product table by $\star_{(i,j)}$.

$\star_{(1,2)}$: By (6.15) the opposite of the statement is $x \star l \in H \cup H^\bullet$. Then, by claims (i) and (vi) , $x = x \star t = x \star (l \star l^\dagger) = (x \star l) \star l^\dagger \in (H \cup H^\bullet) \star H \subseteq H \cup H^\bullet$ follows using $\star_{(2,2)}$ and $\star_{(2,3)}$, a contradiction to (6.15).

$\star_{(1,3)}$: It follows from claim (v) and $\star_{(1,2)}$.

$\star_{(1,1)}$: Since $y \notin H$, $y = y \star f$ holds. Therefore, $x \star y = x \star (y \star f) = (x \star y) \star f$ follows, hence $x \star y$ cannot be in H .

We are ready to prove that $\mathbf{X} = (X, \hat{\wedge}, \hat{\vee}, \cdot, \rightarrow, t, t)$ is an odd involutive FL_e -algebra. $(X, \hat{\wedge}, \hat{\vee})$ is a lattice. Indeed, all elements of H^\bullet are meet-irreducible because of claim (iv), hence X is closed under $\hat{\wedge}$ (clearly, $\hat{\wedge}$ is the restriction of \wedge to X). Commutativity of $\hat{\vee}$ is straightforward, and using claim (iv) a moment's reflection shows that $\hat{\vee}$ is associative, too, and the absorption law holds for $\hat{\vee}$ and $\hat{\wedge}$. Commutativity of \cdot is straightforward. X is closed under \cdot since by claim (iv), $(x \star y)_\uparrow \in H$ if $x \star y \in H^\bullet$. $t \in X$ holds by claim (iii). Since $t \in H$, $\star_{(2,1)}$ and $\star_{(2,2)}$ show that for $y \in X$, $t \star y \notin H^\bullet$ holds, hence $ty \stackrel{(6.3)}{=} h_Y(t \star y) \stackrel{(6.2)}{=} t \star y$ and thus t is the unit element for \cdot over X since it is the unit element for \star over Y .

As for the associativity of \cdot , notice that

$$\text{for } x, y \in Y, h_Y(x \star y) = h_Y(x)h_Y(y) \text{ holds.} \tag{6.16}$$

Indeed, by (6.3), $h_Y(x)h_Y(y) = h_Y(h_Y(x) \star h_Y(y))$, and Table 3 readily confirms that $h_Y(h_Y(x) \star h_Y(y)) = h_Y(x \star y)$. Hence, (X, \cdot) being the homomorphic image of a semigroup, is a semigroup.

Next we prove that \cdot is residuated. For $x, y \in X$, $x \rightarrow y = \max\{z \in X : xz \leq y\}$. Here

$$xz \stackrel{(6.3)}{=} \begin{cases} x \star z & \text{if } x \star z \notin H^\bullet \\ (x \star z)_\uparrow & \text{if } x \star z \in H^\bullet \end{cases} .$$

If $x \star z \in H^\bullet$ then since $y \in X$, $(x \star z)_\uparrow \leq y$ holds if and only if $x \star z \leq y$ holds by claim (iv). Therefore, $\max\{z \in X : xz \leq y\} = \max\{z \in X : x \star z \leq y\}$ holds yielding $x \rightarrow y = x \rightarrow_\star y$. Since $x \rightarrow y \in X$, it also follows that $x \rightarrow y \stackrel{(6.2)}{=} h_Y(x \rightarrow y) = h_Y(x \rightarrow_\star y)$, as stated.

Involutivity of $\hat{\cdot}$ is seen as follows. We will verify that

$$x^\hat{\cdot} = x_\uparrow^\star \text{ if } x \in H, \text{ and } x^\hat{\cdot} = x^\star \text{ if } x \in X \setminus H.$$

This, combined with (3.1) and that $x^\hat{\cdot} \in H$ if and only if $x \in H$ (shown by claims (vi) and (i)) concludes the proof of the statement. Clearly, $x^\hat{\cdot} =$

$x \rightarrow_\star t \geq x \rightarrow_\star f = x^{\uparrow}$. Let $x \in H$. Equality cannot hold since $x \star x^{\uparrow} = t$ by claim (i), whereas $x \star x^{\uparrow} \leq f < t$ by residuation and since \mathbf{Y} is even. Assume that there exists $a \in X$ such that $x^{\uparrow} > a > x^{\downarrow}$. Then $x^{\uparrow} = x \rightarrow_\star t \stackrel{L3.1(ii)}{=} (x \star t^{\downarrow})^{\uparrow} \stackrel{L3.1(i)}{=} (x \star f)^{\uparrow} > a > x^{\downarrow}$, and hence $x \star f < a^{\downarrow} < x$, a contradiction to claim (iv). Let $x \in X \setminus H$. If $x^{\uparrow} > x^{\downarrow}$ then, as above, $x \star f < x$ follows, a contradiction to $x \notin H$.

Finally, \mathbf{X} is clearly odd, since the constant which defines the involution \uparrow is the unit element.

Next we prove that \mathbf{H} is a cancellative subalgebra of \mathbf{X} . Indeed, H is closed under \star , shown by $\star_{(2,2)}$, $t \in H$ holds by claim (iii), and \mathbf{H} has an inverse operation \uparrow , see claims (i) and (vi), hence \mathbf{H} is cancellative by claim (xi) in Lemma 3.1. It is a subalgebra of \mathbf{X} by claim (iii).

Finally, we verify that \mathbf{Y} is the \mathbf{H} -split of \mathbf{X} . Indeed, the universe is as expected, see (6.15). The elements of H^\bullet are just below the respective elements of H , as they should be, see claim (iv). Finally we verify that \star coincides with the product operation of the \mathbf{H} -split of \mathbf{X} .

If $x, y \in H$ then $x \star y \in H$ by $\star_{(2,2)}$. Hence $x \star y \notin H^\bullet$ by claim (ii), yielding $x \star y \stackrel{(6.3)}{=} xy \stackrel{(6.2)}{=} h_{\mathbf{Y}}(x)h_{\mathbf{Y}}(y)$, as required in (6.8).

If $h_{\mathbf{Y}}(x)h_{\mathbf{Y}}(y) \notin H$ then since $h_{\mathbf{Y}}(x), h_{\mathbf{Y}}(y) \in X$ and X is closed under \cdot , $h_{\mathbf{Y}}(x)h_{\mathbf{Y}}(y) \in X \setminus H$ follows. Therefore, by $\star_{(1-1,2-2)}$ ¹⁷, either $h_{\mathbf{Y}}(x)$ or $h_{\mathbf{Y}}(y)$ (say $h_{\mathbf{Y}}(x)$) must be in $X \setminus H$, yielding $h_{\mathbf{Y}}(x) \stackrel{(6.2)}{=} x \in X \setminus H$. By the first row of Table 3, for any $y \in Y$, $x \star y = x \star h_{\mathbf{Y}}(y)$. Since $H \not\ni h_{\mathbf{Y}}(x)h_{\mathbf{Y}}(y) \stackrel{(6.16)}{=} h_{\mathbf{Y}}(x \star y)$, it follows that $x \star y \notin H^\bullet$ and hence $x \star h_{\mathbf{Y}}(y) \notin H^\bullet$. Therefore, $x \star y = x \star h_{\mathbf{Y}}(y) \stackrel{(6.2)}{=} h_{\mathbf{Y}}(x \star h_{\mathbf{Y}}(y)) \stackrel{(6.3)}{=} xh_{\mathbf{Y}}(y) \stackrel{(6.2)}{=} h_{\mathbf{Y}}(x)h_{\mathbf{Y}}(y)$, as required in (6.8).

Assume $\neg(x, y \in H)$ and $h_{\mathbf{Y}}(x)h_{\mathbf{Y}}(y) \in H$. By (6.16), $h_{\mathbf{Y}}(x \star y) \in H$, hence $x \star y \in H \cup H^\bullet$. Since $\neg(x, y \in H)$, by Table 3 it follows that $x \star y \in H^\bullet$. Table 3 and $x \star y \in H^\bullet$ also implies that either $x, y \in X \setminus H$, or at least one of x and y is in H^\bullet and the other is in $H \cup H^\bullet$. In all these cases, by Table 3 and (6.2) it follows that $x \star y = h_{\mathbf{Y}}(x) \star h_{\mathbf{Y}}(y)$.

Hence $h_{\mathbf{Y}}(x) \star h_{\mathbf{Y}}(y) \in H^\bullet$ and it yields $(h_{\mathbf{Y}}(x) \star h_{\mathbf{Y}}(y))_{\uparrow} \stackrel{(6.2)}{=} h_{\mathbf{Y}}(h_{\mathbf{Y}}(x) \star h_{\mathbf{Y}}(y)) \stackrel{(6.3)}{=} h_{\mathbf{Y}}(x)h_{\mathbf{Y}}(y)$, that is, $x \star y = h_{\mathbf{Y}}(x) \star h_{\mathbf{Y}}(y) = (h_{\mathbf{Y}}(x)h_{\mathbf{Y}}(y))_{\downarrow}$, as required in (6.8).

¹⁷This notation refers to the submatrix containing $\star_{(1,1)}$, $\star_{(1,2)}$, $\star_{(2,1)}$, $\star_{(2,2)}$.

(b): Since $h_{\mathbf{Y}}$ maps onto \mathbf{X} , to prove that \mathbf{X} is cancellative it suffices to prove that each element of $h_{\mathbf{Y}}(\mathbf{Y})$ has inverse, which holds since for $x \in Y$, $h_{\mathbf{Y}}(x)h_{\mathbf{Y}}(x^{\dagger}) \stackrel{(6.16)}{=} h_{\mathbf{Y}}(x \star x^{\dagger}) = h_{\mathbf{Y}}(f) \stackrel{\mathbf{Y} \text{ is even}}{=} h_{\mathbf{Y}}(t_{\downarrow}) \stackrel{t \in H, (6.7)}{=} t$.

(c): It is obvious that $h_{\mathbf{Y}}$ preserves the unit element and the falsum constant. The definitions in (6.2)-(6.4) readily yield that $h_{\mathbf{Y}}$ preserves the meet, the join, the product and the residual operation. ■

7. Bunches of Layer Algebras vs. Bunches of Layer Groups

We introduce the notion of bunches of layer groups, and show that every bunch of layer algebras can be represented by a unique bunch of layer groups.

DEFINITION 7.1. Let (κ, \leq_{κ}) be a totally ordered set with least element t , and let an ordered triple $\langle \bar{\kappa}_I, \bar{\kappa}_J, \{t\} \rangle$ be a partition of κ , where $\bar{\kappa}_I$ and $\bar{\kappa}_J$ can also be empty. Define κ_o, κ_J , and κ_I by one of the rows of Table 1, and let $\boldsymbol{\kappa} = \langle \kappa_o, \kappa_J, \kappa_I, \leq_{\kappa} \rangle$. Let $\mathbf{G}_u = (G_u, \preceq_u, \cdot_u, {}^{-1}_u, u)$ be a family of abelian o -groups indexed by elements of κ , an let $\mathbf{H}_u = (H_u, \preceq_u, \cdot_u, {}^{-1}_u, u)$ be a family of abelian o -groups indexed by elements of κ_I , such that

$$\begin{aligned} &\text{for } u \in \kappa_J, \mathbf{G}_u \text{ is discrete,} \\ &\text{for } u \in \kappa_I, \mathbf{H}_u \leq \mathbf{G}_u, \end{aligned} \tag{7.1}$$

and such that for $u, v \in \kappa, u \leq_{\kappa} v$, there exist

$$\text{homomorphisms } \varsigma_{u \rightarrow v} : G_u \rightarrow G_v$$

satisfying

- (G1) $\varsigma_{u \rightarrow u} = id_{G_u}$ and $\varsigma_{v \rightarrow w} \circ \varsigma_{u \rightarrow v} = \varsigma_{u \rightarrow w}$ (direct system property),
- (G2) for $v >_{\kappa} u \in \kappa_J, \varsigma_{u \rightarrow v}(u) = \varsigma_{u \rightarrow v}(u_{\downarrow u})$,
- (G3) for $u <_{\kappa} v \in \kappa_I, \varsigma_{u \rightarrow v}$ maps into H_v .

Call $\mathcal{G} = \langle \mathbf{G}_u, \mathbf{H}_u, \varsigma_{u \rightarrow v} \rangle_{\boldsymbol{\kappa}}$ a bunch of layer groups.

LEMMA 7.2. The following statements hold true.

- (1) Given a bunch of layer algebras $\mathcal{A} = \langle \mathbf{X}_u, \rho_{u \rightarrow v} \rangle_{\boldsymbol{\kappa}}$ with $\boldsymbol{\kappa} = \langle \kappa_o, \kappa_J, \kappa_I, \leq_{\kappa} \rangle$,

$$\mathcal{G}_{\mathcal{A}} = \langle \mathbf{G}_u, \mathbf{H}_u, \varsigma_{u \rightarrow v} \rangle_{\boldsymbol{\kappa}}$$

is bunch of layer groups, where

$$\mathbf{G}_u = (G_u, \preceq_u, \cdot_u, {}^{-1}_u, u) = \begin{cases} \lambda(\mathbf{X}_u) & \text{if } u \in \kappa_o \\ \lambda(\mathbf{X}_{u\uparrow}) & \text{if } u \in \kappa_J, \\ \lambda(\pi_1(\mathbf{X}_u)) & \text{if } u \in \kappa_I \end{cases}, \quad (7.2)$$

for $u \in \kappa_I$,

$$\mathbf{H}_u = (H_u, \preceq_u, \cdot_u, {}^{-1}_u, u) = \lambda(\pi_2(\mathbf{X}_u)), \quad (7.3)$$

$\kappa = \kappa_o \cup \kappa_J \cup \kappa_I$, and for $u, v \in \kappa$ such that $u \leq_\kappa v$, $\varsigma_{u \rightarrow v} : G_u \rightarrow G_v$ is defined by

$$\varsigma_{u \rightarrow v} = \rho_{u \rightarrow v}|_{G_u}. \quad (7.4)$$

Call \mathcal{G}_A the bunch of layer groups derived from \mathcal{A} .

- (2) Given a bunch of layer groups $\mathcal{G} = \langle \mathbf{G}_u, \mathbf{H}_u, \varsigma_{u \rightarrow v} \rangle_\kappa$ with $\kappa = \langle \kappa_o, \kappa_J, \kappa_I, \leq_\kappa \rangle$,

$$\mathcal{A}\mathcal{G} = \langle \mathbf{X}_u, \rho_{u \rightarrow v} \rangle_\kappa$$

is bunch of layer algebras, called the bunch of layer algebras derived from \mathcal{G} , where

$$\mathbf{X}_u = (X_u, \leq_u, \cdot_u, \rightarrow_u, u, u^\mu) = \begin{cases} \iota(\mathbf{G}_u) & \text{if } u \in \kappa_o \\ \iota(\mathbf{G}_u)\downarrow & \text{if } u \in \kappa_J, \\ Sp(\iota(\mathbf{G}_u), \iota(\mathbf{H}_u)), & \text{if } u \in \kappa_I \end{cases}, \quad (7.5)$$

$\kappa = \kappa_o \cup \kappa_J \cup \kappa_I$, and for $u, v \in \kappa$ such that $u \leq_\kappa v$, $\rho_{u \rightarrow v} : X_u \rightarrow X_v$ is defined by

$$\rho_{u \rightarrow v} = \begin{cases} \varsigma_{u \rightarrow v} & \text{if } u \notin \kappa_I \\ \varsigma_{u \rightarrow v} \circ h_u & \text{if } v > u \in \kappa_I, \\ id_{X_u} & \text{if } v = u \in \kappa_I \end{cases}, \quad (7.6)$$

where h_u is the canonical homomorphism of \mathbf{X}_u .

- (3) Given a bunch of layer groups \mathcal{G} , it holds true that $\mathcal{G}_{(\mathcal{A}\mathcal{G})} = \mathcal{G}$, and given a bunch of layer algebras \mathcal{A} , it holds true that $\mathcal{A}_{(\mathcal{G}\mathcal{A})} = \mathcal{A}$.

PROOF. (1): For $u \in \kappa$, \mathbf{G}_u defined by (7.2) is an abelian o -group. Indeed, being totally ordered is granted since so is the original algebra. If $u \in \kappa_o$ then see (4.1) and Lemma 3.2, if $u \in \kappa_J$ then see (4.1) and Lemmas 5.2 and 3.2, it also confirms (7.1), whereas if $u \in \kappa_I$ then see (4.1), Theorem 6.3 and Lemma 3.2. The ς 's defined in (7.4) satisfy

- (G1) for $x \in G_u$, $\varsigma_{u \rightarrow u}(x) \stackrel{(7.4)}{=} \rho_{u \rightarrow u}(x) \stackrel{(A1)}{=} x$ and $(\varsigma_{v \rightarrow w} \circ \varsigma_{u \rightarrow v})(x) \stackrel{(7.4)}{=} (\rho_{v \rightarrow w} \circ \rho_{u \rightarrow v})(x) \stackrel{(A1)}{=} \rho_{u \rightarrow w}(x) \stackrel{(7.4)}{=} \varsigma_{u \rightarrow w}(x)$.
- (G2) Let $u \in \kappa_J$. Then $\varsigma_{u \rightarrow v}(u_{\downarrow u}) \stackrel{(7.4)}{=} \rho_{u \rightarrow v}(u_{\downarrow u}) \stackrel{(4.1)}{=} \rho_{u \rightarrow v}(u^u) \stackrel{(A2)}{=} \rho_{u \rightarrow v}(u) \stackrel{(7.4)}{=} \varsigma_{u \rightarrow v}(u)$.
- (G3) Let $v \in \kappa_I$. Then $\mathbf{X}_v = Sp(\pi_1(\mathbf{X}_v), \pi_2(\mathbf{X}_v))$ holds by Theorem 6.3. For $x \in G_u$, $\varsigma_{u \rightarrow v}(x) \stackrel{(7.4)}{=} \rho_{u \rightarrow v}(x) \in X_v$, and by claim (C1) in the proof of claim (2) of Lemma 4.2, $\rho_{u \rightarrow v}(x)$ is invertible in \mathbf{X}_v . Therefore, $\rho_{u \rightarrow v}(x)$ is an element of $\pi_2(\mathbf{X}_v) \stackrel{(7.3)}{=} \mathbf{H}_u$ by claim (c) in Theorem 6.3.

(2): \mathbf{X}_u defined in (7.5) is an involutive FL_e -chain satisfying (4.1): if $u \in \kappa_o$ then see Lemma 3.2, if $u \in \kappa_J$ then see Lemmas 3.2 and 5.2, if $u \in \kappa_I$ then see Lemma 3.2 and Theorem 6.3.

The ρ 's defined in (7.6) are well defined since $X_u = G_u$ holds by (7.5) if $u \notin \kappa_I$, and if $u \in \kappa_I$ then h_u maps to the universe of $\iota(G_u)$ (see (6.2)), which is G_u .

The ρ 's are residuated lattice homomorphisms, since the (totally ordered group) homomorphisms $\varsigma_{u \rightarrow v}$ from G_u to G_v naturally extend to homomorphisms from the residuated lattice reduct of $\iota(G_u)$ to the residuated lattice reduct of $\iota(G_v)$ via claim (2) of Lemma 3.3, and hence $\rho_{u \rightarrow v}$ can be regarded as the composition of residuated lattice homomorphisms (ς 's and h 's).

The ρ 's satisfy

(A1): Notice that for $u < v$,

$$\rho_{u \rightarrow v} \text{ maps } X_u \text{ to } G_v, \tag{7.7}$$

since so does $\varsigma_{u \rightarrow v}$. Over X_u ,

$$\rho_{u \rightarrow u} \stackrel{(7.6)}{=} \begin{cases} \varsigma_{u \rightarrow u} \stackrel{(G1)}{=} id_{G_u} \stackrel{(7.5)}{=} id_{X_u} & \text{if } u \notin \kappa_I \\ id_{X_u} & \text{if } v = u \in \kappa_I \end{cases}.$$

Therefore, it suffices to prove the other condition in (A1) for $u < v < w$ only:

$$\begin{aligned} \rho_{v \rightarrow w} \circ \rho_{u \rightarrow v} &\stackrel{(7.6)}{=} \left\{ \begin{array}{ll} \varsigma_{v \rightarrow w} \circ \rho_{u \rightarrow v} & \text{if } v \notin \kappa_I \\ \varsigma_{v \rightarrow w} \circ h_v \circ \rho_{u \rightarrow v} & \stackrel{(7.7)(6.7)}{=} \varsigma_{v \rightarrow w} \circ \rho_{u \rightarrow v} \text{ if } v \in \kappa_I \end{array} \right\} = \\ \varsigma_{v \rightarrow w} \circ \rho_{u \rightarrow v} &\stackrel{(7.6)}{=} \left\{ \begin{array}{ll} \varsigma_{v \rightarrow w} \circ \varsigma_{u \rightarrow v} \stackrel{(G1)}{=} \varsigma_{u \rightarrow w} & \text{if } u \notin \kappa_I \\ \varsigma_{v \rightarrow w} \circ \varsigma_{u \rightarrow v} \circ h_u \stackrel{(G1)}{=} \varsigma_{u \rightarrow w} \circ h_u & \text{if } u \in \kappa_I \end{array} \right\} \stackrel{(7.6)}{=} \rho_{u \rightarrow w}. \end{aligned}$$

(A2): For $u \notin \kappa_o$ we have already seen that

$$\mathbf{X}_u \text{ is even,} \tag{7.8}$$

therefore, $\rho_{u \rightarrow v}(u^\mu) \stackrel{(7.6)}{=} \dots$

$$\left\{ \begin{array}{ll} \begin{array}{l} \rho_{u \rightarrow v}(u^\mu) \stackrel{(7.8)}{=} \rho_{u \rightarrow v}(u_{\downarrow u}) \stackrel{(G2)}{=} \rho_{u \rightarrow v}(u) \stackrel{(7.6)}{=} \rho_{u \rightarrow v}(u) \text{ if } u \in \kappa_J, \\ (\rho_{u \rightarrow v} \circ h_u)(u^\mu) \stackrel{(6.7)}{=} \rho_{u \rightarrow v}(u_{\uparrow}^\mu) \stackrel{(7.8)}{=} \rho_{u \rightarrow v}(u) \\ \stackrel{(6.7)}{=} (\rho_{u \rightarrow v} \circ h_u)(u) \stackrel{(7.6)}{=} \rho_{u \rightarrow v}(u) \end{array} & \text{if } u \in \kappa_I. \end{array} \right.$$

(3): If $u = \kappa_o$ then $\lambda(\iota(\mathbf{G}_u)) = \mathbf{G}_u$ and $\iota(\lambda(\mathbf{X}_u)) = \mathbf{X}_u$ by Lemma 3.2. If $u = \kappa_J$ then $\lambda(\iota(\mathbf{G}_u)_{\downarrow \uparrow}) = \lambda(\iota(\mathbf{G}_u)) = \mathbf{G}_u$ and $\iota(\lambda(\mathbf{X}_u)_{\uparrow \downarrow}) = \mathbf{X}_u$ follow from Lemma 5.2 and Lemma 3.2. In these two cases $G_u = X_u$, thus it is obvious from (7.6) and (7.4) that $\mathcal{G}_{(\mathcal{A}_G)}$ and \mathcal{G} have the same homomorphisms (the same ς 's) from the u^{th} -layer, and that $\mathcal{A}_{(\mathcal{G}_A)}$ and \mathcal{A} have the same homomorphisms (the same ρ 's) from the u^{th} -layer. If $u \in \kappa_I$ then $\lambda(\pi_1(Sp(\iota(\mathbf{G}_u), \iota(\mathbf{H}_u)))) = \lambda(\iota(\mathbf{G}_u)) = \mathbf{G}_u$, $\lambda(\pi_2(Sp(\iota(\mathbf{G}_u), \iota(\mathbf{H}_u)))) = \lambda(\iota(\mathbf{H}_u)) = \mathbf{H}_u$ and $Sp(\iota(\lambda(\pi_1(\mathbf{X}_u))), \iota(\lambda(\pi_2(\mathbf{X}_u)))) = Sp(\pi_1(\mathbf{X}_u), \pi_2(\mathbf{X}_u)) = \mathbf{X}_u$ follow from Theorem 6.3 and Lemma 3.2. As for the homomorphisms, h_u maps X_u to G_u by (7.2), hence the composition $\varsigma_{u \rightarrow v} \circ h_u$ is well-defined. By the construction in Definition 6.1, G_u can also be regarded as a subset of X_u , and h_u is the identity mapping on G_u by (6.2). Therefore, $(\varsigma_{u \rightarrow v} \circ h_u)|_{G_u} = \rho_{u \rightarrow v}$ holds on the one hand. On the other hand, to prove $\rho_{u \rightarrow v}|_{G_u} \circ h_u = \rho_{u \rightarrow v}$, first notice that for $x \in X_u \setminus G_u$,

$$\rho_{u \rightarrow v}(x_{\uparrow u}) = \rho_{u \rightarrow v}(x). \tag{7.9}$$

Indeed, $\rho_{u \rightarrow v}(x_{\uparrow u}) = \rho_{u \rightarrow v}(x_{\uparrow u} \cdot_u u) \stackrel{(4.2)}{=} \rho_{u \rightarrow v}(x_{\uparrow u}) \cdot_v \rho_{u \rightarrow v}(u) \stackrel{(A2)}{=} \rho_{u \rightarrow v}(x_{\uparrow u}) \cdot_v \rho_{u \rightarrow v}(u^\mu) \stackrel{X_u \text{ is even, see (4.1)}}{=} \rho_{u \rightarrow v}(x_{\uparrow u}) \cdot_v \rho_{u \rightarrow v}(u_{\downarrow u}) \stackrel{(4.2)}{=} \rho_{u \rightarrow v}(x_{\uparrow u} \cdot_u u_{\downarrow u}) \stackrel{(6.8)}{=} \rho_{u \rightarrow v}(x_{\uparrow u \downarrow u}) = \rho_{u \rightarrow v}(x)$. Therefore, $(\rho_{u \rightarrow v}|_{G_u} \circ h_u)(x) \stackrel{(6.7)}{=} \dots$

$$\stackrel{(6.7)}{=} \left\{ \begin{array}{ll} (\rho_{u \rightarrow v}|_{G_u})(x) = \rho_{u \rightarrow v}(x) & \text{if } x \in G_u \\ (\rho_{u \rightarrow v}|_{G_u})(x_{\uparrow u}) \stackrel{x_{\uparrow u} \in H_u \subseteq G_u}{=} \rho_{u \rightarrow v}(x_{\uparrow u}) \stackrel{(7.9)}{=} \rho_{u \rightarrow v}(x) & \text{if } x \in X_u \setminus G_u \end{array} \right. .$$

8. The Representation Theorem

The main theorem of the paper is a representation theorem of odd or even involutive FL_e -chains by bunches of layer groups. Lemmas 4.2 and 7.2 prove

Theorem 8.1. The second part of Theorem 8.1 presents the direct constructional correspondence between odd or even involutive FL_e -chains and bunches of layer-groups, that is, one without referring to the intermediate explanatory step of layer algebras.

THEOREM 8.1. *For every odd or even involutive FL_e -chain \mathbf{X} there exists a unique bunch of layer groups \mathcal{G} such that \mathbf{X} is the involutive FL_e -chain derived from the bunch of layer algebras derived from \mathcal{G} , in notation, $\mathbf{X} = \mathcal{X}_{\mathcal{A}_{\mathcal{G}}}$. Conversely, for every bunch of layer groups \mathcal{G} there exists a unique odd or even involutive FL_e -chain \mathbf{X} such that \mathcal{G} is the bunch of layer groups derived from the bunch of layer algebras of \mathbf{X} , in notation, $\mathcal{G} = \mathcal{G}_{\mathcal{A}_{\mathbf{X}}}$. In more details:*

(A) *Given an odd or an even involutive FL_e -chain $\mathbf{X} = (X, \leq, \cdot, \rightarrow, t, f)$ with residual complement operation $'$,*

$$\mathcal{G}_{\mathbf{X}} = \langle \mathbf{G}_u, \mathbf{H}_u, \varsigma_{u \rightarrow v} \rangle_{\langle \kappa_o, \kappa_J, \kappa_I, \leq \kappa \rangle}$$

is bunch of layer groups, called the bunch of layer groups of \mathbf{X} , where

$\kappa = \{x \rightarrow x : x \in X\} = \{u \geq t : u \text{ is idempotent}\}$ *is ordered by \leq ,*

$\bar{\kappa}_I = \{u \in \kappa \setminus \{t\} : u' \text{ is idempotent}\}$,

$\bar{\kappa}_J = \{u \in \kappa \setminus \{t\} : u' \text{ is not idempotent}\}$,

$\kappa_o, \kappa_J, \kappa_I$ *are defined by Table 4,*

$$\begin{aligned} \mathbf{G}_u &= (G_u, \leq, \cdot, {}^{-1}, u) && \text{if } u \notin \kappa_I, \\ \mathbf{G}_u &= (G_u, \leq, \cdot_u, {}^{-1}, u) && \text{if } u \in \kappa_I, \\ \mathbf{H}_u &= (H_u, \leq, \cdot, {}^{-1}, u) && \text{if } u \in \kappa_I, \end{aligned}$$

where $X_u = \{x \in X : x \rightarrow x = u\}$, $H_u = \{x \in X_u : xu' < x\} = \{x \in X_u : x \text{ is } u\text{-invertible in } X_u\}$ ¹⁸, $\dot{H}_u = \{xu' : x \in H_u\}$,

$$G_u = \begin{cases} X_u & \text{if } u \notin \kappa_I \\ X_u \setminus H_u^\bullet & \text{if } u \in \kappa_I \end{cases}$$

$$x \cdot_u y = (xy \rightarrow u) \rightarrow u$$

$$x^{-1} = x \rightarrow u,$$

and for $u, v \in \kappa$ such that $u \leq v$, $\varsigma_{u \rightarrow v} : G_u \rightarrow G_v$ is defined by

$$\varsigma_{u \rightarrow v}(x) = vx.$$

(B) *Given a bunch of layer groups $\mathcal{G} = \langle G_u, H_u, \varsigma_{u \rightarrow v} \rangle_{\langle \kappa_o, \kappa_J, \kappa_I, \leq \kappa \rangle}$ with $\mathbf{G}_u = (G_u, \leq_u, \cdot_u, {}^{-1}_u, u)$*

¹⁸There exists $y \in X_u$ such that $xy = u$.

$$\mathbf{X}_G = (X, \leq, \cdot, \rightarrow, t, t')$$

is an involutive FL_e -chain with residual complement $'$, called the involutive FL_e -chain of \mathcal{X} , where $\kappa = \kappa_o \cup \kappa_J \cup \kappa_I$, for $u \in \kappa$,

$$X_u = \begin{cases} G_u & \text{if } u \notin \kappa_I, \\ G_u \cup H_u^\bullet & \text{if } u \in \kappa_I, \end{cases}$$

(where $H_u^\bullet = \{h^\bullet : h \in H_u\}$ is a copy of H_u which is disjoint from G_u),

$$X = \bigcup_{u \in \kappa} X_u,$$

if $u \notin \kappa_I$ then $\leq_u = \preceq_u$, if $u \in \kappa_I$ then \leq_u extends \preceq_u to X_u by letting

$a^\bullet <_u b^\bullet$ and $x <_u a^\bullet <_u y$ if $a, b \in H_u, x, y \in G_u, a \prec_u b, x \prec_u a \preceq_u y$,

for $v \in \kappa, \rho_v : X \rightarrow X$ is defined by

$$\rho_v(x) = \begin{cases} s_{u \rightarrow v}(x) & \text{if } x \in G_u \text{ and } u <_\kappa v, \\ x & \text{if } x \in G_u \text{ and } u \geq_\kappa v, \end{cases}$$

$$\rho_v(x^\bullet) = \begin{cases} s_{u \rightarrow v}(x) & \text{if } x^\bullet \in H_u^\bullet \text{ and } \kappa_I \ni u <_\kappa v, \\ x^\bullet & \text{if } x^\bullet \in H_u^\bullet \text{ and } \kappa_I \ni u \geq_\kappa v, \end{cases}$$

by denoting for $u, v \in \kappa, uv = \max_\kappa(u, v)$, for $x \in X_u$ and $y \in X_v$,

$x < y$ iff $\rho_{uv}(x) <_{uv} \rho_{uv}(y)$ or $\rho_{uv}(x) = \rho_{uv}(y)$ and $u <_\kappa v$,

for $u \in \kappa_I, h_u : X_u \rightarrow G_u$,

$$h_u(x) = x \quad \text{if } x \in G_u,$$

$$h_u(x^\bullet) = x \quad \text{if } x^\bullet \in H_u^\bullet,$$

for $x, y \in X_u$,

$$x \bullet_u y = \begin{cases} (h_u(x) \cdot_u h_u(y))^\bullet & \text{if } u \in \kappa_I, h_u(x) \cdot_u h_u(y) \in H_u \text{ and } \neg(x, y \in H_u) \\ h_u(x) \cdot_u h_u(y) & \text{if } u \in \kappa_I, h_u(x) \cdot_u h_u(y) \notin H_u \text{ or } x, y \in H_u \\ x \cdot_u y & \text{if } u \notin \kappa_I \end{cases},$$

for $x \in X_u$ and $y \in X_v$,

$$xy = \rho_{uv}(x) \bullet_{uv} \rho_{uv}(y),$$

for $x \in X$,

$$(x^\bullet)' = \begin{cases} x^{-1} & \text{if } u \in \kappa_I \text{ and } x^\bullet \in H_u^\bullet \\ (x^{-1})^\bullet & \text{if } u \in \kappa_I \text{ and } x \in H_u \\ x^{-1} & \text{if } u \in \kappa_I \text{ and } x \in G_u \setminus H_u \\ x^{-1} & \text{if } u \in \kappa_o \text{ and } x \in G_u \\ x^{-1}_\downarrow & \text{if } u \in \kappa_J \text{ and } x \in G_u \end{cases},$$

Table 4. Three cases—three classes

κ_o	κ_J	κ_I	
$\{t\}$	$\bar{\kappa}_J$	$\bar{\kappa}_I$	if \mathbf{X} is odd
\emptyset	$\bar{\kappa}_J \cup \{t\}$	$\bar{\kappa}_I$	if \mathbf{X} is even and f is not idempotent
\emptyset	$\bar{\kappa}_J$	$\bar{\kappa}_I \cup \{t\}$	if \mathbf{X} is even and f is idempotent

for $x, y \in X$,

$$x \rightarrow y = (xy')'$$

\rightarrow is the residual operation of \cdot ,

t is the least element of κ ,

f is the residual complement of t ,

and is given by

$$t' = \begin{cases} (t^{-1})^\bullet & \text{if } u \in \kappa_I \\ t^{-1} & \text{if } u \in \kappa_o \\ t^{-1}_\downarrow & \text{if } u \in \kappa_J \end{cases} .$$

In addition,

$$\rho_v(x) = vx \text{ for } v \in \kappa \text{ and } x \in X,$$

$\mathbf{X}_{\mathcal{X}}$ is odd if $t \in \kappa_o$, even with a non-idempotent falsum if $t \in \kappa_J$, and even with an idempotent falsum if $t \in \kappa_I$.

(C) Items (A) and (B) describe a one-to-one correspondence between the class containing all odd and all even involutive FL_e -chains and the class of bunches of layer groups: given a bunch of layer groups \mathcal{X} it holds true that $\mathcal{X}_{(\mathbf{X}_{\mathcal{X}})} = \mathcal{X}$, and given an odd or even involutive FL_e -chain \mathbf{X} it holds true that $\mathbf{X}_{(\mathcal{X}_{\mathbf{X}})} \simeq \mathbf{X}$.¹⁹ ■

EXAMPLE 8.2. We present the bunch representation of a few known structures, among which are the two extremal classes (abelian o -groups and odd

¹⁹If Definition 7.1 is slightly modified in such a way that the \dot{H}_u 's are "stored" in the definition of a bunch (like $\mathbf{X} = \langle G_u, H_u, \dot{H}_u, \varsigma_{u \rightarrow v} \rangle_{\langle \kappa_o, \kappa_J, \kappa_I, \leq \kappa \rangle}$) and instead of taking a copy \dot{H}_u of H_u , that stored copy is used in the construction of Theorem 8.1, then also $\mathbf{X}_{(\mathcal{X}_{\mathbf{X}})} = \mathbf{X}$ holds. Then, Theorem 8.1 describes a bijection, in a constructive manner, between the classes of odd or even involutive FL e-chains and the class of bunches of layer groups.

Sugihara chains) mentioned in the introduction. Denote by $\mathbb{1}$ the trivial (one-element) group.

- If \mathbf{G} is an abelian o -group then $\mathbf{G} = \mathbf{X}_{\mathcal{G}}$ where

$$\mathcal{G} = \langle \mathbf{G}, \emptyset, \emptyset \rangle_{\langle \{t\}, \emptyset, \emptyset, \leq_{\kappa} \rangle}.$$

- Even Sugihara chains are exactly the algebras $\mathbf{X}_{\mathcal{G}}$, where

$$\mathcal{G} = \langle \mathbb{1}_u, \mathbb{1}_u, \varsigma_{u \rightarrow v} \rangle_{\langle \emptyset, \emptyset, \kappa, \leq_{\kappa} \rangle}.$$

- Odd Sugihara chains are exactly the algebras $\mathbf{X}_{\mathcal{G}}$, where

$$\mathcal{G} = \langle \mathbb{1}_u, \mathbb{1}_u, \varsigma_{u \rightarrow v} \rangle_{\langle \{t\}, \emptyset, \kappa \setminus \{t\}, \leq_{\kappa} \rangle}.$$

- Finite partial sublex products of abelian o -groups have been shown in [31, 32] to be exactly those odd involutive FL_e -chains which have finitely many positive idempotent elements. These are exactly the algebras $\mathbf{X}_{\mathcal{G}}$, where κ is finite in

$$\mathcal{G} = \langle \mathbf{G}_u, \mathbf{H}_u, \varsigma_{u \rightarrow v} \rangle_{\langle \{t\}, \bar{\kappa}_J, \bar{\kappa}_I, \leq_{\kappa} \rangle}.$$

- Algebras which can be constructed by the involutive ordinal sum construction of [28] are exactly the algebras $\mathbf{X}_{\mathcal{G}}$, where

$$\mathcal{G} = \langle \mathbf{G}_u, \mathbb{1}_u, \varsigma_{u \rightarrow v} \rangle_{\langle \{t\}, \emptyset, \kappa \setminus \{t\}, \leq_{\kappa} \rangle}.$$

By (G3), also the homomorphisms are trivial.

- Algebras which can be constructed by the consecutive application of the ordinal sum construction as defined in [18] are exactly the algebras $\mathbf{X}_{\mathcal{G}}$, where κ is finite in

$$\mathcal{G} = \langle \mathbf{G}_u, \mathbb{1}_u, \varsigma_0^{u \rightarrow v} \rangle_{\langle \{t\}, \emptyset, \kappa \setminus \{t\}, \leq_{\kappa} \rangle}.$$

By (G3), also the homomorphisms are trivial.

REMARK 8.3. If \mathbf{X} is densely ordered in Theorem 8.1 then the \mathbf{H}_u 's in the representation $\mathcal{G} = \langle \mathbf{G}_u, \mathbf{H}_u, \varsigma_{u \rightarrow v} \rangle_{\kappa}$ of \mathbf{X} are uniquely determined by the rest of \mathcal{G} . Therefore, if \mathbf{X} is densely ordered then the representation of \mathbf{X} by layer groups can be written in a simpler form of $\langle \mathbf{G}_u, \varsigma_{u \rightarrow v} \rangle_{\langle \kappa_o, \kappa_J, \kappa_I, \leq_{\kappa} \rangle}$. To prove it we state that for $u \in \kappa_I$,

$$H_u = \bigcup_{\kappa \ni s <_{\kappa} u} \varsigma_{s \rightarrow u}(G_s).$$

Indeed, $H_u \supseteq \bigcup_{\kappa \ni s <_{\kappa} u} \varsigma_{s \rightarrow u}(G_s)$ follows from (G3). Since $u \in \kappa_I$, and since κ is the same in $\mathcal{G}_{\mathcal{A}_{\mathbf{X}}}$ and in $\mathcal{A}_{\mathbf{X}}$, the u^{th} -layer algebra \mathbf{X}_u of \mathbf{X}

has idempotent falsum constant u' by (4.1). Therefore, by Theorem 6.3, $\mathbf{X}_u = Sp(\pi_1(\mathbf{X}_u), \pi_2(\mathbf{X}_u))$, where $\pi_2(\mathbf{X}_u) = \mathbf{H}_u$ comprises the following elements $H_u = \{x \in X_u : xu' < x\} = \{x \in X : \tau(x) = u, xu' < x\}$. Hence proving $H_u \subseteq \bigcup_{\kappa \ni s < \kappa u} \varsigma_{s \rightarrow u}(G_s)$ amounts to showing that in every densely ordered, odd or even involutive FL_e -chain $\mathbf{X} = (X, \leq, \cdot, \rightarrow, t, f)$ with residual complement operation $'$, if $u \geq t$ and u' are idempotent, and $x \in X$ such that $\tau(x) = u$ and $xu' < x$ then there exist a positive idempotent element $X \ni s < u$ and $y \in X$ such that $\tau(y) = s$ and $yu = x$. Let $xu' < y < x$ (such y exists since X is densely ordered). Now, $y' < (xu')' \leq y'u$ follows by claim (iv) in Lemma 3.1. Therefore, $u \notin Stab_{y'}$ and $s := \tau(y) = \tau(y') < u$ follows. Since $yu \stackrel{(4.4)}{=} \rho_{s \rightarrow u}(y) \stackrel{(4.6)}{=} \rho_u(y) \stackrel{(4.18)}{=} \min\{z \in X_u : z \geq y\}$ and since $X_u \ni xu' < y < x \in X_u$, to see that $yu = x$, it suffices to prove that $xu' = x_{\downarrow u}$ ²⁰. But it holds true since $x \in H_u$ and $u' \in H_u^\bullet$, and hence $xu' = xu_{\downarrow u} = xu \cdot \overset{Table\ 3/\star(2,3)}{=} x^\bullet = x_{\downarrow u}$ follows from claim (iv) in the proof of Theorem 6.3/(a), and we are done.

REMARK 8.4. The easiest way of generalizing Theorem 8.1 to conic algebras is to observe that there exist no conic odd or even involutive FL_e -algebras which are not linearly ordered. Indeed, let $a, b \in X$. Proving that a and b are comparable, that is, $a \leq b$ or $b \leq a$ amounts to proving $a \rightarrow b \leq b$ or $b \rightarrow a \leq a$, or equivalently, $a \rightarrow b \geq t$ or $b \rightarrow a \geq t$ by adjointness. If $a \rightarrow b \not\geq t$ then $a \rightarrow b < t$ since X is conic, hence $a \rightarrow b \leq f$ since X is odd or even. By claims (i) and (ii) in Lemma 3.1, $ab' \geq t$ follows. Therefore, $(a'b)' \geq t$ holds by claim (iii) in Lemma 3.1, hence $b \rightarrow a \geq t$ holds by claim (ii) in Lemma 3.1.

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²⁰Recall that $x_{\downarrow u}$ is computed in X_u .

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