Jeroen P. Goudsmit

Finite Frames Fail: How Infinity Works Its Way into the Semantics of Admissibility

Abstract.   Many intermediate logics, even extremely well-behaved ones such as IPC, lack the finite model property for admissible rules. We give conditions under which this failure holds. We show that frames which validate all admissible rules necessarily satisfy a certain closure condition, and we prove that this condition, in the finite case, ensures that the frame is of width 2. Finally, we indicate how this result is related to some classical results on finite, free Heyting algebras.

Keywords: Intermediate logics, Admissible rules, Finite model property, Projective Heyting algebras.

1.   Introduction

How does one recognise the non-theorems of a logic? In many intermediate logics, it suffices to inspect all finite models of a certain size, bounded in some way by the formula under consideration. This desirable result is unattainable in many an intermediate logic when generalising from the *derivability* of *propositions* to the *admissibility* of *rules*.

The admissible rules of a logic are those rules under which the logic's theorems are closed. Friedman [8, Problem 4] asked whether the set of all admissible rules of IPC is decidable, which was confirmed by Rybakov [17]. Much later, an alternative proof making use of a characterisation of finitely presented projective Heyting algebras was given by Ghilardi [9]. If the admissible rules of IPC, like the theorems of IPC, would be both sound and complete with respect to a certain set of finite frames, then the above could surely have been attained by simpler means.

The aim of this paper is to provide a new proof showing that, in particular, there exists no class of finite frames with respect to which the admissible rules of IPC are both sound and complete. We give sufficient conditions

under which the same conclusion can be attained for many intermediate logics. Our conditions are more lenient than those given earlier by Fedorishin and Ivanov [7].

The first hint that finite frames are unlikely to suffice as semantics for the admissible rules of IPC came from Citkin [4]. He showed that any finite model of all admissible rules necessarily is of an extremely restricted shape, making use of a particular generalisation of a rule introduced by Mints [16].

$$(z_1 \rightarrow x) \rightarrow z_1 \vee z_2 \big/ \big((z_1 \rightarrow x) \rightarrow z_1\big) \vee \big((z_1 \rightarrow x) \rightarrow z_2\big) \tag{1}$$

More precisely, a finite Heyting algebra that validates all admissible rules of IPC must be a subdirect product of projective Heyting algebras. This connection was not made explicit by Citkin, but it does follow readily from [2]. Baker [1, Corollary 3.10] had already shown that the equational theory of the class of finite projective Heyting algebras strictly extends that of all Heyting algebras. These observations combined lead to a proof of the failure of the finite model property for admissibility, as we indicate in Section 4.

Rybakov, Kiyatkin and Oner [18] proved the failure of the finite model property for admissible rules in a great variety of modal logics, including K4, S4 and GL. Moreover, they describe conditions under which the finite model property for admissible rules does hold. Building on these techniques, Fedorishin and Ivanov [7] proved the failure of the finite model property for admissibility in many intermediate logics. Their argument requires a fair amount of machinery, which we are able to hide away. Distilling some of the ingredients from the (admittedly quite colloquial) historical reconstruction above, we present an elementary proof of the failure of finite frames. It is our conviction that the argument, as presented below, can be readily understood by undergraduate students with a cursory background in modal or intuitionistic logic.

In Section 2, we introduce the basic notation, and give a semantic description of a modest generalisation of Mints' rule. This description is quite similar to that of Iemhoff [14, Theorem 4.6], who introduces the "offspring property". The difference between this description and ours is twofold. First, our description is given on a fixed frame, as opposed to on the totality of all models. Second, we stratify the offspring property over the natural numbers, so as to be able to retrieve Mints' rule from the more general scheme under consideration.

In Section 3, we explain how the binary case of the offspring property, coupled with the presence of certain configurations of points, is sufficient to construct an infinity of points within a model. We, quite literally, show that the validity of a certain admissible rule on a frame induces infinity within it.

This observation is then employed in Section 4 to prove our main theorem, which shows that finite frames fail to be both sound and complete with respect to the admissible rules of any intermediate logic that admits Mints' rule and that is either of width greater than two or is strictly below Scott's logic.

2. Semantics for Admissible Rules

We consider the propositional language generated by a fixed, countable infinity of variables, denoted P, as defined by the BNF below. We denote propositional variables by x, y, z.

$$\mathcal{L} ::= \top \mid \bot \mid \mathsf{P} \mid \mathcal{L} \vee \mathcal{L} \mid \mathcal{L} \wedge \mathcal{L} \mid \mathcal{L} \to \mathcal{L}$$

Formulae are elements of $\mathcal{L}$, and will be denoted by ϕ, ψ. A *substitution* is a mapping from formulae to formulae that respects all connectives.

The logics under consideration are consistent axiomatic extensions of the intuitionistic propositional calculus IPC, as specified fully below. For convenience's sake, we think of a logic as determined by its theorems.

DEFINITION 2.1. (Intermediate Logic) An *intermediate logic* Λ is a proper subset of the set of all formulae that contains all theorems of IPC, satisfying the following two conditions.

   modus ponens if $\phi \to \psi \in \Lambda$ and $\phi \in \Lambda$, then $\psi \in \Lambda$.

   substitution if $\phi \in \Lambda$ and σ is a substitution, then $\sigma(\phi) \in \Lambda$.

The elements of an intermediate logic Λ are said to be the *theorems* of Λ. We say that two formulae ϕ, ψ are *equivalent* in an intermediate logic Λ whenever $\phi \to \psi$ and $\psi \to \psi$ are both theorems.

There are several ways one can think of admissible rules. In the introduction, we alluded to the definition as given below, albeit informally. Were the logic at hand to be defined by means of a consequence relation, one could have defined a rule to be admissible precisely when adjoining it to said consequence relation does not lead to an enlargement of the set of theorems. When suitably formalised, these two descriptions coincide, as explained in [15, Corollary 4.2].

DEFINITION 2.2. (Admissible Rule) A rule ϕ/ψ is said to be *admissible in* Λ whenever $\sigma(\phi) \in \Lambda$ entails $\sigma(\psi) \in \Lambda$ for all substitutions σ.

The following definition is wholly standard, we merely give it to fix notation. For more details and background we refer to [20]. We write X to mean a set of variables, and $\mathsf{P}(X)$ denotes its power set.

DEFINITION 2.3. (Frames and Models) A *Kripke frame* (or frame, for short) is a partial order $K = \langle K, \leq \rangle$. A *model* is a pair $\langle K, V \rangle$, where K is a frame and $V : K \to \mathsf{P}(X)$ is a monotonic map, called the *valuation*. We define the validity of a formula ϕ at a point k, denoted $V, k \Vdash \phi$, as usual. Whenever ϕ is such that $V, k \Vdash \phi$ holds for all $k \in K$, we denote this by $V \Vdash \phi$.

DEFINITION 2.4. (Semantics) A rule ϕ/ψ is said to be *valid* on a frame K whenever $V, K \Vdash \phi$ implies $V, K \Vdash \psi$ for all valuations $V : K \to \mathsf{P}(X)$, where X is the set of variables contained in either ϕ or ψ.

The notion below is taken from [10, p. 107]. It is helpful to think of the following two examples. First, if $k \in K$ is a node, and W are its immediate successors (the nodes immediately following it in the order of K), then one can readily infer that $W \kappa k$ ought to hold. Second, $\emptyset \kappa k$ holds whenever k is a maximal node in k. Finally, note that $\{k\} \kappa k$ holds regardless of the choice of $k \in K$. This in contrast to the notion of a total cover as employed in [12].

DEFINITION 2.5. (Cover) Let K be a partial order, let $W \subseteq K$ be an arbitrary subset, and let $k \in K$ be a point. We say that W is a *cover* of k, denoted $W \kappa k$, whenever the following equivalence holds for all $l \in K$:

$$k \leq l \text{ if and only if } k = l \text{ or } w \leq l \text{ for some } w \in W. \tag{2}$$

Mints [16] introduced a particular rule that is admissible in IPC, but the implication from its antecedent to its conclusion is most certainly not a theorem. This rule was shortly thereafter generalised by Citkin [4] into something similar to the rule below, when one instantiates n as 2.[1]

Variants of this rule occur throughout the literature on admissibility. Let us but mention three quite distinct representatives. Citkin [5] was the first to consider it in its full generality. A similar rule was considered by Skura [19], who used it to characterise IPC among all intermediate logics. Finally, Iemhoff [13] gave a variant of this rule scheme, and showed that all admissible rules of IPC can be generated on the basis of this scheme.

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[1] Note that, when $n = 2$, (3) can clearly be retrieved from (1), when adding $y\vee$ to both this latter rule's antecedent and conclusion. The converse is also possible, by observing that $z_1 \to x$ is equivalent to $(z_1 \vee z_2) \to (z_1 \to x)$, as has been remarked by T. Skura.

$$y \vee \left(\left(\bigvee_{i=1}^{n} z_i \to x \right) \to \left(\bigvee_{j=1}^{n} z_j \right) \right) \Bigg/ y \vee \left(\bigvee_{j=1}^{n} \left(\bigvee_{i=1}^{n} z_i \to x \right) \to z_j \right) \quad (3)$$

The rule (3) is admissible in IPC, as follows immediately from [14, Theorem 4.6]. The same holds for BB_n, the logic of branching degree at most n as specified by [3, Proposition 2.41], for each $n \geq 2$.

Any frame that validates (3) reflects this in its order structure. In Theorem 2.7 below, we pin down this reflection, and show it to be both a necessary and sufficient condition for validating (3). Before we proceed, we first introduce the following property inspired by Iemhoff [14, p. 68].

DEFINITION 2.6. (Offspring Property) Let K be a frame. We say that K enjoys the *n-ary offspring property* whenever for every $l \in K$ and all $l \leq w_1, \ldots, w_n$ there exist $k, k^+ \in K$ such that $\{w_1, \ldots, w_n\} \, \kappa \, k$ and $k^+ \leq k, l$. In particular, if $n = 2$ then we speak of the *binary offspring property*.

THEOREM 2.7. *Let K be a finite frame. The following statements are equivalent for any $n \in \mathbb{N}$.*

1. *The rule (3) is valid on the frame K;*
2. *The frame K enjoys the binary offspring property.*

We briefly go over two special cases. In the case that $n = 0$, both the antecedent and the consequent of the rule (3) are equivalent to x. The 0-ary offspring property can be seen to hold, as there surely is a $k \in K$ satisfying both $l \leq k$ and $\emptyset \, \kappa \, k$. The case that $n = 1$ is trivial as well, for the antecedent and the consequent of the rule (3) are equivalent to one another. In this case, the 1-ary offspring property can be satisfied by choosing $k, k^+ = l$. Hence, the nullary and unary offspring property are always satisfied.

Proof of Theorem 2.7 Suppose that statement 2 holds. We proceed by contraposition, so suppose there exists some valuation $V : K \to \mathsf{P}(X)$ such that:

$$V \Vdash y \vee \left(\left(\bigvee_{i=1}^{n} z_i \to x \right) \to \left(\bigvee_{j=1}^{n} z_j \right) \right), \quad (4)$$

$$V \nVdash y \vee \left(\bigvee_{j=1}^{n} \left(\bigvee_{i=1}^{n} z_i \to x \right) \to z_j \right). \quad (5)$$

Through (5), we are guaranteed the existence of points $k \leq w_1, \ldots, w_n$ such that $k \not\Vdash y$ and

$$V, w_j \Vdash \left(\bigvee_{i=1}^{n} z_i \to x \right) \text{ and } V, w_j \not\Vdash z_j \text{ for each } j = 1, \ldots, n. \qquad (6)$$

As K enjoys the n-ary offspring property, we know there exist $k, k^+ \in K$ such that $\{w_1, \ldots, w_n\} \,\kappa\, k$ and $k^+ \leq k, l$. One may verify that V falsifies the antecedent of (3) at k^+, contradicting (4). This proves that statement 1 holds.

The implication from statement 2 to statement 1 is straightforward. Suppose statement 1 holds, and let $l \in K$ and $l \leq w_1, \ldots, w_n$ be arbitrary. We define a valuation $V : K \to \mathsf{P}(K)$ such that:

$$V, h \Vdash x \text{ iff } w_i \leq h \text{ for some } i = 1, \ldots, n,$$
$$V, h \Vdash y \text{ iff } h \not\leq l,$$
$$V, h \Vdash z_i \text{ iff } h \not\leq w_i.$$

It is an easy matter to verify that $V, w_j \not\Vdash (\bigvee_{i=1}^{n} z_i \to x) \to z_j$ for each $j = 1, \ldots, n$. It is, moreover, clear that $V, l \not\Vdash y$. As a consequence, we know that V does not validate the consequent of (3). By assumption, the antecedent of this rule must also be false.

We thus obtain a point $k^+ \in K$ such that

$$V, k^+ \not\Vdash y \vee \left(\bigvee_{i=1}^{n} z_i \to x \right) \to \bigvee_{j=1}^{n} z_j.$$

From the falsehood of the left-hand disjunct, we can deduce that $k^+ \leq l$. Similarly, the falsehood of the right-hand disjunct yields a point $k' \geq k^+$ satisfying:

$$V, k' \Vdash \bigvee_{i=1}^{n} z_i \to x \text{ and } V, k' \not\Vdash \bigvee_{j=1}^{n} z_j.$$

The latter conjunct entails that $k' \leq w_j$ for all $j = 1, \ldots, n$. Consider a maximal such point, above k', and call it k. As we assumed K to be finite, such a point must exist. We claim that $\{w_1, \ldots, w_n\} \,\kappa\, k$.

In order to prove this claim, assume that $h \geq k$ and $w_i \not\leq h$ for all $i = 1, \ldots, n$. If $h \leq w_i$ holds for all $i = 1, \ldots, n$, then the maximality condition on k ensures that $h = k$. We can thus assume this not to hold, which, in turn, shows that $V, h \Vdash \bigvee_{i=1}^{n} z_i$. By assumption, we now obtain $V, h \Vdash x$, a contradiction. We have thus constructed the desired points k and k^+, proving statement 2. ∎

3. Sufficient Conditions for Infinity

Many properties of frames can be codified by means of the validity of axioms. In this section, we investigate two distinct such conditions, whose corresponding axioms axiomatize the well-known *intermediate logic of bounded width three* and *Scott's logic*. Both of these conditions are shown to *not hold* on finite frames satisfying the binary offspring property. In the subsequent section, we use this information to show that certain classes of intermediate logics can not have the finite model property for their admissible rules.

The first property we investigate is the *width* of a frame, as defined below. The axiomatic counterpart of this semantic property is given in Definition 3.2. We omit the proof as it is wholly standard.

DEFINITION 3.1. (Width) A frame K is said to be of *width n* whenever for each $k \in K$ and all $l_0, l_1, \ldots, l_n \geq k$ there exist some $i \neq j$ such that $l_i \leq l_j$.

THEOREM 3.2. ([3, Proposition 2.3.9]) *A frame is of width n exactly if every valuation on it is a model of the logic* $\mathsf{BW}_n := \mathsf{IPC} + \mathsf{bw}_n$.

$$\mathsf{bw}_n := \bigvee_{i=0}^{n} \left(x_i \to \bigvee_{j \neq i} x_j \right) \tag{7}$$

Consider a frame that is not of width 2. By the very definition of width, we now know that the three-fork, as depicted in Fig. 1, must occur within this frame. The following lemma shows that the presence of the three-fork yields an infinity of points whenever the frame enjoys the binary offspring property. Pictorially, the proof of the lemma is straightforward; Fig. 2 says it all.

LEMMA 3.3. *Let K be a frame that has the binary offspring property and that is not of width 2. Now, K must be infinite.*

PROOF. We will prove that there exists a sequence of subsets $W_0, W_1, \ldots \subseteq K$ satisfying, for each n, the following three properties:

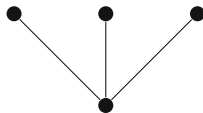1. The elements in W_i are pairwise incomparable
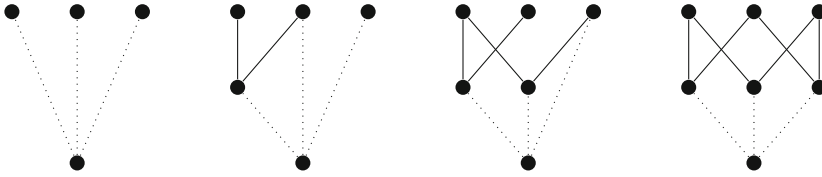

Figure 1. The three-fork

Figure 2. Extending the three-fork with one layer

2. There exists a $k \in K$ such that $W_i \subseteq \uparrow k$ for each $i = 0, 1, \ldots, n$

3. $W_j \subset \uparrow W_i$ for all $j < i \leq n$

Assuming such a sequence, it is immediate that $\bigcup_i W_i$ is infinite, proving the desired result.

We need but show that one can construct said sequence, which we readily do by induction along its length. In the base case there is little to do. As K is not of width two, there must be points $k, w_0, w_1, w_2 \in K$ such that $k \leq w_0, w_1, w_2$, satisfying the additional condition that $W_0 := \{w_0, w_1, w_2\}$ be an anti-chain. Properties 1 and 2 follow immediately from our assumption, and property 3 holds vacuously.

Now, suppose we have such a chain $W_0, \ldots, W_n$, satisfying the three conditions. By property 2, there exists a point $k_0^+ \leq w$ for all $w \in W_n$. Note that there are precisely three subsets of size two within W_n, let us call these S_1, S_2 and S_3. Through the offspring property, we know of six points k_j, k_j^+ with $j = 1, 2, 3$, satisfying $k_{i+1}^+ \leq k_i, k_i^+$ and $W_i \,\kappa\, k_i$ for $i = 1, 2, 3$. We claim that $W_{n+1} = \{k_1, k_2, k_3\}$ does the trick.

Property 1 holds through immediate verification. Indeed, suppose that $k_i < k_j$ holds for some $j \neq i$. Recall that $S_i \,\kappa\, k_i$, hence there must be some $s \in S_i$ such that $s \leq k_j$. Furthermore, we observe that $S_j \subseteq \uparrow k_j \subseteq \uparrow s$ and $S_j - \{s\} \neq \emptyset$. These two observations combine to prove that two elements in W_n are comparable, a contradiction by induction.

To show property 2, simply observe that $k = k_3^+$ does the trick. Finally, property 3 holds immediately by construction. ∎

The property "being of width 2" is formulated in a negative manner: a frame satisfies this property in case a certain configuration of nodes does not exist within it. We now turn to the second property we consider in this section, which is also formulated negatively. To state the definition, we introduce the frame F_9 as in Fig. 3 and we define what it means to be a morphism of frames. We subsequently state the well-known Theorem 3.6, see for instance [3, Exercise 2.12].
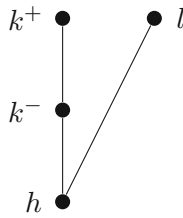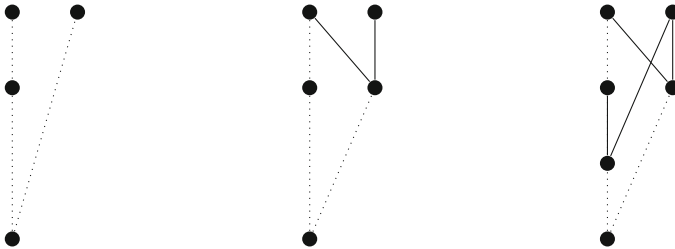
Figure 3. The frame F_9

Figure 4. Extending F_9 with one layer

DEFINITION 3.4. (Morphism of Frames) A morphism of frames $f : K \to L$ is a monotonic map function where $f(U)$ is an upset for each upset $U \subseteq K$.

DEFINITION 3.5. (Scott Frame) A frame K is said to be a *Scott frame* if for all upsets $U \subseteq K$ there exists no morphism of frames $f : U \to F_9$.

THEOREM 3.6. *A frame is a Scott frame precisely if every valuation on it is a model of the intermediate logic* $\mathsf{ST} := \mathsf{IPC} + \mathsf{st}$, *where:*

$$\mathsf{st} := \big((\neg\neg x \to x) \to x \vee \neg x \big) \to \neg x \vee \neg\neg x.$$

Analogous to Lemma 3.3, we can prove that no finite frame with the binary offspring property can be a Scott frame. We do not directly prove this lemma, but instead proceed via Lemma 3.8. Subsequently, we show how this more general lemma can be applied to prove Lemma 3.7. The reader will be able to readily see how the same lemma can be used to prove Lemma 3.3 as well. Moreover, the reader can reconstruct a direct proof of Lemma 3.7 following the steps sketched in Fig. 4.

LEMMA 3.7. *Let K be a frame that has the binary offspring property and that is not a Scott frame. Now, K must be infinite.*

LEMMA 3.8. *Let K be a frame with the binary offspring property. Suppose there exist points k^+, k^-, l and h such that:*

1. k^+ *and l are incomparable;*

2. k^- *and l are incomparable;*

3. $k^+ \leq k^-$ *does not hold;*

4. $k^+, k^-, l \geq h$;

Now, K is infinite.

PROOF. We construct four sequences of points:

$$(k_i^+)_{i \in \mathbb{N}}, (k_i^-)_{i \in \mathbb{N}}, (l)_{i \in \mathbb{N}}, (h)_{i \in \mathbb{N}},$$

satisfying the obvious adaptation of the criteria mentioned above and the additional constraint that $l_0 > l_1 > \ldots$. This latter condition immediately entails the desired. Suppose the sequences have been constructed for $i \in \mathbb{N}$ satisfying $i \leq n$. Because $h_n \leq k_n^-, l_n$, we know there to exist points k_{n+1}^- and h_{n+1}' such that:

$$\left\{ k_n^-, l_n \right\} \kappa \, k_{n+1}^- \text{ and } h_{n+1}' \leq k_{n+1}^-, h_n. \tag{8}$$

Moreover, because $h_{n+1}' \leq k_n^+, l$ is known to hold, there must exist points l_{n+1} and h_{n+1} satisfying:

$$\left\{ k_n^+, l_n \right\} \kappa \, l_{n+1} \text{ and } h_{n+1} \leq l_{n+1}, h_{n+1}'. \tag{9}$$

We define $k_{n+1}^+ := k_n^-$, which finishes the definition of the $(n+1)^{\text{th}}$ elements of the four sequences. Let us now verify that the four conditions hold.

We start with property 1 and proceed by contradiction, so we assume that k_{n+1}^+ and l_{n+1} are comparable. We distinguish two cases: either $k_{n+1}^+ \leq l_{n+1}$, or $k_{n+1}^+ > l_{n+1}$. In the former case, note that $l_{n+1} \leq l_n$ holds, thus this ensures $k_{n+1}^+ = k_n^- \leq l_n$ which contradicts property 2. In the latter case, we observe that $k_n^- = k_{n+1}^+ \geq k_n^+$ or $k_n^- = k_{n+1}^+ \geq l_n$ which contradict properties 3 and 2 respectively.

Let us now turn to property 2. We proceed by contradiction and thus assume that k_{n+1}^- and l_{n+1} are comparable. We distinguish three cases: either $k_{n+1}^- = l_{n+1}$, or $k_{n+1}^- < l_{n+1}$, or $k_{n+1}^- > l_{n+1}$. In the first case, (8) proves that:

$$l_{n+1} = k_{n+1}^- \leq k_n^- = k_{n+1}^+,$$

contradicting property 1 as proven in the above paragraph. In the second case, one uses (8) to readily derive that at least one of $l_n \leq l_{n+1}$ or $k_n^- \leq l_{n+1}$ must hold. Yet the former ensures $l_n \leq k_n^+$ and the latter proves $k_n^- \leq l_n$, both through (9), contradicting properties 1 and 2 respectively. In the

third case, we observe that (9) leads to the validity of $k_n^+ \leq k_{n+1}^-$ or $l_n \leq k_{n+1}^-$. Because (8) shows that $k_{n+1}^- \leq k_n^-$, we know the former to contradict property 3. Similarly, the latter contradicts property 1 as $k_{n+1}^- \leq k_n^-$. These observations combined prove property 2.

To prove property 3, it suffices to note that $k_{n+1}^+ \leq k_{n+1}^-$ would yield

$$k_n^- = k_{n+1}^+ \leq k_{n+1}^- \leq l_n$$

by (8), violating property 2.

The validity of property 4 is straightforward. To conclude the argument, we prove that $l_{n+1} > l_n$. Indeed, suppose that $l_n = l_{n+1}$. It readily follows from (9) that $l_n \leq k_n^+$, a contradiction to property 1. We thus have shown K to be infinite. ∎

Proof of Lemma 3.7 Let $U \subseteq K$ be an upset and suppose that $f : U \to F_9$ is a morphism of frames. We know of a $h_o \in U$ such that $f(h_o) = h$. Subsequently, we choose $k_o^+, k_o^-, l_o \geq h_0$ such that:

$$f(k_o^+) = k^+, f(k_o^-) = k^-, f(l_o) = l.$$

We need but show that the conditions of Lemma 3.8 apply to the four points k_o^+, k_o^-, l_o, h_o. Yet this is immediate by the very definition of a morphism of frames, proving the desired. ∎

4. Finite Frames Fail

We now have sufficient machinery to prove the main theorem. Its proof is a simple composition of the elementary results gathered above.

THEOREM 4.1. *Let Λ be an intermediate logic in which the rule (3) is admissible. Suppose there exists a set of finite frames $\mathcal{K}$ with respect to which the admissible rules of Λ are both sound and complete.[2] Now, Λ extends bw_2 and ST.*

PROOF. We proceed by contradiction, so suppose that Λ does not extend both bw_2 and ST. Consider the rule $\top/\chi$ for $\chi = \mathsf{bw}_2, \mathsf{st}$. Note that this rule is admissible precisely if $\chi \in \Lambda$, which we know to not be the case for at least one choice of $\chi = \mathsf{bw}_2, \mathsf{st}$. Fix this choice in the following.

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[2] Note that one could strengthen the theorem by merely assuming that $\mathcal{K}$ is complete with respect to the theorems of Λ and sound with respect to the admissible rules of Λ.

We assumed that $\mathcal{K}$ is a class of finite frames such that the following holds for all formulae ϕ and ψ.

$$K \Vdash \phi/\psi \text{ for all } K \in \mathcal{K} \text{ iff } \phi/\psi \text{ is admissible in } \Lambda.$$

We thus know of a frame $K \in \mathcal{K}$ and a model $V : K \to \mathsf{P}(X)$ such that $V, K \Vdash \top$ yet $V, K \nVdash \chi$. As the rule (3) was assumed to be admissible in Λ, we know that this rule is valid on K. By Theorem 2.7, it follows that K has the binary offspring property.

Combining Theorem 3.2 and Theorem 3.6, we know that K is either not of width 2 or K is not a Scott frame. Through Lemma 3.3 or Lemma 3.7, we can now infer that K is infinite, *quod non*. ∎

The theorems immediately entail the corollary below. Here BB_n refers to the logic of bounded branching degree at most n, also known as the $(n-1)$th Gabbay–de Jongh logic. As an example, BB_2 is the logic of finite binary trees.

COROLLARY 4.2. *Neither* IPC *nor any of the logics* BB_n *for* $n \geq 2$ *have the finite model property for admissibility.*

Interestingly, much of Theorem 4.1 can be retrieved solely using results from the seventies. The notion of projectivity is as in [2].

THEOREM 4.3. *A finite, subdirectly irreducible Heyting algebra is projective if and only if it satisfies the rule* (3) *for* $n = 2$.

PROOF. Immediate via [2, Theorem 4.10] and [4, Theorem 2]. ∎

THEOREM 4.4. ([1, Corollary 3.10]) *The equational class generated by the finite, projective Heyting algebras is axiomatized by the equation:*[3]

$$\big((y \to x) \vee (x \to y \vee z) \vee z \to y\big) = \top. \tag{10}$$

In the following, we make liberal use of the duality between finite Kripke frames and finite Heyting algebras, as given in [6].

COROLLARY 4.5. *Any class of finite, rooted frames that is sound with respect to all admissible rules of* IPC *satisfies the axiom* (10)*, and is thus not complete with respect to admissibility.*

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[3] The actual equation used by Baker equals, up to renaming, the following:

$$(x \to y) \vee (y \to x) \vee (x \to y \vee z) \vee \big((y \vee z) \to x\big) \vee \big((y \vee z) \to y\big) = \top.$$

After the anonymous referee pointed out that is was redundant, we arrived at the equation given here in a conversation with Dick de Jongh.

PROOF. Suppose $\mathcal{K}$ is a class of finite, rooted frames that is sound with respect to IPC. Through the duality, this corresponds to a class of finite, subdirectly irreducible Heyting algebras. By Theorem 4.3, all elements of $\mathcal{K}$ are finite, projective Heyting algebras. Consequently, per Theorem 4.4, $\mathcal{K}$ contains no model on which (10) is invalid. Yet this is not a theorem of IPC, proving the desired result. ∎

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J. P. GOUDSMIT
Utrecht University
Utrecht, The Netherlands
jeroen@jeroengoudsmit.com