

# Asymptotic normality of quadratic forms of martingale differences

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**Abstract** We establish the asymptotic normality of a quadratic form  $Q_n$  in martingale difference random variables  $\eta_t$  when the weight matrix  $A$  of the quadratic form has an asymptotically vanishing diagonal. Such a result has numerous potential applications in time series analysis. While for i.i.d. random variables  $\eta_t$ , asymptotic normality holds under condition  $\|A\|_{sp} = o(\|A\|)$ , where  $\|A\|_{sp}$  and  $\|A\|$  are the spectral and Euclidean norms of the matrix  $A$ , respectively, finding corresponding sufficient conditions in the case of martingale differences  $\eta_t$  has been an important open problem. We provide such sufficient conditions in this paper.

**Keywords** Asymptotic normality · Quadratic form · Martingale differences

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### 1 Main results

We study here quadratic forms

$$Q_n = \sum_{t,k=1}^n a_{n;tk} \eta_t \eta_k \tag{1.1}$$

where  $\{\eta_k\}$  is a stationary ergodic martingale difference (m.d.) sequence with respect to some natural filtration  $\mathcal{F}_t$ , with moments

$$E \eta_k = 0, \quad E \eta_k^2 = 1 \quad \text{and} \quad E \eta_k^4 < \infty.$$

The real-valued coefficients  $a_{n;tk}$  in (1.1) are entries of a symmetric matrix  $A_n = (a_{n;tk})_{t,k=1,\dots,n}$ . We denote by

$$\|A_n\| = \left( \sum_{t,k=1}^n a_{n;tk}^2 \right)^{1/2}$$

the Euclidean norm and by

$$\|A_n\|_{sp} = \max_{\|x\|=1} \|A_n x\|$$

the spectral norm of the matrix  $A_n$ . For convenience, we set  $a_{n;tk} = 0$  for  $t \leq 0, t > n$  or  $k \leq 0, k > n$ .

The asymptotic normality property of the quadratic form  $Q_n$  has been well investigated when the random variables  $\eta_j$  are i.i.d. If  $A_n$  has vanishing diagonal:  $a_{n;tt} = 0$  for all  $t$ , then asymptotic normality is implied by the condition

$$\|A_n\|_{sp} = o(\|A_n\|), \tag{1.2}$$

see [Rotar \(1973\)](#), [De Jong \(1987\)](#), [Guttorp and Lockhart \(1988\)](#), [Mikosch \(1991\)](#) and [Bhansali et al. \(2007a\)](#).

The aim of this paper is to extend these results to the m.d. noise  $\eta_j$ . We will need the following additional assumptions on the m.d. noise  $\eta_t$ :

$$E \left( \eta_j^2 | \mathcal{F}_{j-1} \right) \geq c > 0, \quad (\exists c > 0). \tag{1.3}$$

The assumption (1.3) bounds the conditional variance of  $\eta_j$  away from zero. We also assume that  $A_n$  has an asymptotically “vanishing” diagonal in the sense:

$$\sum_{t=1}^n |a_{n;tt}| = o(\|A_n\|), \quad n \rightarrow \infty. \tag{1.4}$$

Relation (1.4) implies

$$\sum_{t=1}^n a_{n;tt}^2 = o(\|A_n\|^2), \quad n \rightarrow \infty. \tag{1.5}$$

The following theorem shows that in case of m.d. noise  $\{\eta_k\}$ , the condition

$$\|A_n\|_{sp} / \|A_n\| \rightarrow 0$$

above needs to be strengthened by including the assumptions (1.8) and (1.9) on the weights  $a_{n;ts}$ . Its proof is based on the martingale central limit theorem.

**Theorem 1.1** Let  $Q_n$  be as in (1.1), where  $\{\eta_j\}$  is a stationary ergodic m.d. noise such that  $E\eta_j^4 < \infty$  and (1.3) hold. Suppose that the  $a_{n;ts}$ 's are such that, as  $n \rightarrow \infty$ ,

$$\|A_n\|_{sp}/\|A_n\| \rightarrow 0. \tag{1.6}$$

Then there exist  $c_1, c_2 > 0$  such that

$$c_1\|A_n\|^2 \leq \text{Var}(Q_n) \leq c_2\|A_n\|^2, \quad n \geq 1. \tag{1.7}$$

If in addition,

$$\sum_{t,s=1:|t-s|\geq L}^n a_{n;ts}^2 = o(\|A_n\|^2), \quad n \rightarrow \infty, \quad L \rightarrow \infty, \tag{1.8}$$

and

$$\sum_{t=k+2}^n (a_{n;t,t-k} - a_{n;t-1,t-1-k})^2 = o(\|A_n\|^2), \quad \forall k \geq 1 \tag{1.9}$$

then the following normal convergence holds:

$$(\text{Var}(Q_n))^{-1/2}(Q_n - EQ_n) \xrightarrow{d} N(0, 1). \tag{1.10}$$

As usual, " $\xrightarrow{d} N(0, 1)$ " denotes convergence in distribution to a normal random variable with mean zero and variance one.

Theorem 1.1 plays an important instrumental role in establishing asymptotic properties of various estimation and testing procedures in parametric and non-parametric time series analysis where the object of interest can be written as a quadratic form

$$Q_{n,X} = \sum_{t,s=1}^n e_n(t-s)X_tX_s$$

of a linear (moving-average) process

$$X_t = \sum_{j=0}^{\infty} a_j\eta_{t-j}$$

of uncorrelated noise  $\eta_t$  and the weights  $e_n(s)$  may depend on  $n$ . In the case of i.i.d. noise  $\eta_t$ , the asymptotic normality for  $Q_{n,X}$  is established by approximating it by a simpler quadratic form

$$Q_{n,\eta} = \sum_{t,s=1}^n b_n(t-s)\eta_t\eta_s$$

with some different weights  $b_n(t)$  and then deriving the asymptotic normality for  $Q_{n,\eta}$ , as in Bhansali et al. (2007b). For example, one sets

$$b_n(t) = \int_{-\pi}^{\pi} u_n(x)f(x)e^{itx}dx$$

where  $f(x)$  is the spectral density of the sequence  $X_t$ , and where  $u_n(x)$  is some convenient function related to  $e_n(t)$ , typically such that

$$e_n(t) = \int_{-\pi}^{\pi} u_n(x)e^{itx}dx.$$

In general, obtaining simple asymptotic normality conditions for  $Q_{n,X}$  is a hard theoretical problem but of great practical importance, which for an i.i.d. noise  $\eta_t$  was solved in [Bhansali et al. \(2007b\)](#). In addition, in Sect. 6.2 in [Giraitis et al. \(2012\)](#) one considers discreet frequencies and shows that a sum

$$S_n = \sum_{j=1}^{n/2} b_{nj} I(u_j)$$

of weighted periodograms

$$I(u_j) = (2\pi n)^{-1} \left| \sum_{k=1}^n e^{iku_j} X_k \right|^2$$

of the sequence  $X_t$  at Fourier frequencies  $u_j$  can be also effectively approximated by a quadratic form  $Q_{n,\eta}$ . This allows, by theorem like [Theorem 1.1](#), to establish the asymptotic normality for such sums  $S_n$ . However, assumption of i.i.d. noise is restrictive and may be not satisfied in practical applications and in some theoretical, i.e. ARCH, settings. In a subsequent paper we will derive corresponding normal approximation results for  $Q_{n,X}$  and  $S_n$  when  $\eta_t$  is a martingale difference process.

The following [Corollary 1.1](#) displays situations where the conditions of [Theorem 1.1](#) are easily satisfied. For a Toeplitz matrix  $A_n$ , that is with entries

$$a_{n;ts} = b_n(t - s),$$

the assumption [\(1.9\)](#) is clearly satisfied, since

$$a_{n;t,t-k} - a_{n;t-1,t-1-k} = b_n(k) - b_n(k) = 0.$$

The following lemma provides a useful bound that can be used to prove [\(1.6\)](#).

**Lemma 1.1** *Suppose that*

$$b_n(t) = \int_{-\pi}^{\pi} e^{itx} g_n(x) dx, \quad t = 0, 1, \dots,$$

where  $g_n(x)$ ,  $|x| \leq \pi$  is an even real function. If there exists

$$0 \leq \alpha < 1/2$$

and a sequence of constants  $k_n > 0$  such that

$$|g_n(x)| \leq k_n |x|^{-\alpha}, \quad |x| \leq \pi,$$

then

$$\|A_n\|_{sp} \leq C k_n n^\alpha, \quad n \geq 1. \tag{1.11}$$

For the proof see [Theorem 2.2\(i\)](#) in [Bhansali et al. \(2007a\)](#).

Suppose now, in addition, that  $g_n(x) \equiv g(x)$ ,  $n \geq 1$  and  $|g(x)| \leq C|x|^{-\alpha}$ ,  $|x| \leq \pi$ . Then

$$\int_{-\pi}^{\pi} g^2(x) dx < \infty, \quad b_n(t) = b(t) \quad \text{and} \quad \sum_{t=-\infty}^{\infty} b^2(t) < \infty$$

and, in addition,  $k_n = 1$  in [\(1.11\)](#). Hence

$$\|A\|^2 = \sum_{t,s=1}^n b^2(t-s) = \sum_{k=-n}^n b^2(k)(n-|k|) \sim n \sum_{t=-\infty}^{\infty} b^2(t) \quad \text{as } n \rightarrow \infty$$

and

$$\|A_n\|_{sp} \leq Cn^\alpha = o(n^{1/2}) = o(\|A\|)$$

which implies (1.6). Moreover,

$$\sum_{t,s=1:|t-s|\geq L}^n a_{n;ts}^2 = \sum_{t,s=1:|t-s|\geq L}^n b^2(t-s) \leq n \sum_{|k|\geq L} b^2(|k|).$$

Since  $\sum_{|k|\geq L} b^2(|k|) \rightarrow 0$  as  $L \rightarrow \infty$ , we obtain (1.8). This together with Theorem 1.1 implies the following corollary.

**Corollary 1.1** *Let*

$$Q_n = \sum_{t,k=1}^n b(t-k)\eta_t\eta_k,$$

where  $b(t) = b(-t)$ ,  $b(0) = 0$  are real weights and  $\{\eta_j\}$  is a stationary ergodic m.d. noise such that  $E\eta_j^4 < \infty$  and (1.3) hold.

(i) *If  $\sum_{t=0}^\infty |b(t)| < \infty$ , then*

$$\exists c_1, c_2 > 0 : c_1n \leq \text{Var}(Q_n) \leq c_2n, \quad n \geq 1, \tag{1.12}$$

$$(\text{Var}(Q_n))^{-1/2}(Q_n - EQ_n) \xrightarrow{d} N(0, 1). \tag{1.13}$$

(ii) *If  $b(t) = \int_{-\pi}^\pi e^{itx} g(x)dx$ ,  $t = 0, 1, \dots$ , where  $g(x)$ ,  $|x| \leq \pi$  is an even real function such that for some  $0 \leq \alpha < 1/2$  and  $C > 0$ ,*

$$|g(x)| \leq C|x|^{-\alpha}, \quad |x| \leq \pi \tag{1.14}$$

*then (1.12) and (1.13) hold.*

Next we consider two quadratic forms

$$\begin{aligned} Q_n^{(1)} &= \sum_{t,s=1}^n a_{n;ts}^{(1)} \eta_t \eta_s, \quad \text{and} \\ Q_n^{(2)} &= \sum_{t,s=1}^n a_{n;ts}^{(2)} \eta_t \eta_s, \end{aligned} \tag{1.15}$$

with corresponding matrices  $A_n^{(1)}$ ,  $A_n^{(2)}$  and a m.d. sequence  $\eta_t$  which satisfy the assumptions of Theorem 1.1, so that

$$\left(\text{Var}(Q_n^{(i)})\right)^{-1/2} \left(Q_n^{(i)} - EQ_n^{(i)}\right) \xrightarrow{d} N(0, 1), \quad i = 1, 2.$$

The next corollary provides additional sufficient condition that implies asymptotic normality of their sum.

**Corollary 1.2** *Suppose that the quadratic forms  $Q_n^{(1)}$ ,  $Q_n^{(2)}$  in (1.15) satisfy the assumptions of Theorem 1.1. Set*

$$A_n = A_n^{(1)} + A_n^{(2)}.$$

If in addition

$$\lim_{n \rightarrow \infty} \left\| A_n^{(1)} \right\|^{-1} \left\| A_n^{(2)} \right\|^{-1} \text{tr} \left( A_n^{(1)} A_n^{(2)} \right) = 0 \tag{1.16}$$

then the quadratic form  $Q_n := Q_n^{(1)} + Q_n^{(2)}$  satisfies

$$\exists c_1, c_2 > 0 : c_1 \left( \left\| A_n^{(1)} \right\| + \left\| A_n^{(2)} \right\| \right) \leq \text{Var}(Q_n) \leq c_2 \left( \left\| A_n^{(1)} \right\| + \left\| A_n^{(2)} \right\| \right), \quad n \geq 1,$$

and

$$\left( \text{Var}(Q_n) \right)^{-1/2} (Q_n - E Q_n) \xrightarrow{d} N(0, 1).$$

*Proof* We have  $Q_n = \sum_{t,s=1}^n a_{n;ts} \eta_t \eta_s$  where  $a_{n;ts} = a_{n;ts}^{(1)} + a_{n;ts}^{(2)}$ . Thus, to prove the corollary, it suffices to show that  $A_n$  satisfies assumptions of Theorem 1.1. This easily follows from the fact that both  $A_n^{(1)}$  and  $A_n^{(2)}$  satisfy assumptions of Theorem 1.1, and the property

$$\|A_n\|^2 = \left( \left\| A_n^{(1)} \right\|^2 + \left\| A_n^{(2)} \right\|^2 \right) (1 + o(1)).$$

The latter follows from

$$\|A_n\|^2 = \|A_n^{(1)}\|^2 + \|A_n^{(2)}\|^2 + 2\text{tr} \left( A_n^{(1)} A_n^{(2)} \right)$$

because the matrices  $A_n^{(1)}$  and  $A_n^{(2)}$  are symmetric so the cross term

$$2 \sum_{t,s} a_{n;ts}^{(1)} a_{n;ts}^{(2)} = 2 \sum_{t,s} a_{n;ts}^{(1)} a_{n;st}^{(2)} = 2\text{tr} \left( A_n^{(1)} A_n^{(2)} \right).$$

Hence

$$\|A_n\|^2 = \left( \left\| A_n^{(1)} \right\|^2 + \left\| A_n^{(2)} \right\|^2 \right) (1 + r_n)$$

where

$$r_n = 2\text{tr} \left( A_n^{(1)} A_n^{(2)} \right) / \left( \left\| A_n^{(1)} \right\|^2 + \left\| A_n^{(2)} \right\|^2 \right).$$

Since  $\|A_n^{(1)}\|^2 + \|A_n^{(2)}\|^2 \geq 2\|A_n^{(1)}\| \|A_n^{(2)}\|$  we get  $r_n = o(1)$  by (1.16). □

Corollary 1.2 indicates that we need the additional condition (1.16) in order to obtain the asymptotic normality of  $Q_n$ . It does not imply that in this case the components  $Q_n^{(1)}$  and  $Q_n^{(2)}$  are asymptotically uncorrelated and hence asymptotically independent. We conjecture that  $Q_n^{(1)}$  and  $Q_n^{(2)}$  will be asymptotically independent in the case when  $\eta_t$  is an i.i.d. noise.

## 2 Proof of Theorem 1.1

In the proof of Theorem 1.1 we shall use the following result.

**Lemma 2.1** (Dalla et al. (2014), Lemma 10).

(i) Let

$$T_n = \sum_{j \in Z} c_{nj} V_j,$$

where  $\{V_j\}$ ,  $j \in Z = \{\dots, -1, 0, 1, \dots\}$  is a stationary ergodic sequence,  $E|V_1| < \infty$ , and  $c_{nj}$  are real numbers such that for some  $0 < \alpha_n < \infty$ ,  $n \geq 1$ ,

$$\sum_{j \in Z} |c_{nj}| = O(\alpha_n), \quad \sum_{j \in Z} |c_{nj} - c_{n,j-1}| = o(\alpha_n). \tag{2.1}$$

Then

$$E|T_n - ET_n| = o(\alpha_n).$$

In particular, if  $\alpha_n = 1$ , then

$$T_n = ET_n + o_p(1).$$

(ii) If the m.d. sequence  $\eta_t$  satisfies  $\max_t E|\eta_t|^p < \infty$ , for some  $p \geq 2$ , then

$$E \left| \sum_{j \in Z} d_j \eta_j \right|^p \leq C \left( \sum_{j \in Z} d_j^2 \right)^{p/2}, \tag{2.2}$$

for any  $d_j$ 's such that  $\sum_{j \in Z} d_j^2 < \infty$ , where  $C < \infty$  does not depend on  $d_j$ 's.

For the convenience of the reader we provide the proof of the following lemma.

**Lemma 2.2** One has

$$\max_{t=1, \dots, n} \sum_{s=1}^n a_{n;ts}^2 \leq \|A_n\|_{sp}^2, \quad \max_{t,s=1, \dots, n} |a_{n;ts}| \leq \|A_n\|_{sp}. \tag{2.3}$$

*Proof* We drop the index  $n$  and let  $A = (a_{ts})$ . The second inequality  $|a_{ts}| \leq \|A_n\|_{sp}$  follows from the first since

$$\max_{t,s} a_{ts}^2 \leq \max_t \sum_{s=1}^n a_{ts}^2 \leq \|A_n\|_{sp}^2.$$

Turning to the first inequality, we have  $\|A_n\|_{sp}^2 = \sup_{\|x\|=1} \|Ax\|^2$  where  $x = (x_1, \dots, x_n)'$  and

$$\|Ax\|^2 = \left\| \sum_{s=1}^n a_{1s}x_s, \dots, \sum_{s=1}^n a_{ns}x_s \right\|^2 = \left( \sum_{s=1}^n a_{1s}x_s \right)^2 + \dots + \left( \sum_{s=1}^n a_{ns}x_s \right)^2.$$

Set  $y = (0, \dots, 0, 1, 0, \dots, 0)'$  where 1 is at the  $t_0$  position. Note that  $\|y\| = 1$ . Then

$$\|A_n\|_{sp}^2 \geq \|Ay\|^2 = a_{1t_0}^2 + \dots + a_{nt_0}^2 = \sum_{s=1}^n a_{st_0}^2 = \sum_{s=1}^n a_{t_0s}^2$$

since  $A$  is symmetric. Hence

$$\|A_n\|_{sp}^2 \geq \max_{t_0=1, \dots, n} \sum_{s=1}^n a_{t_0s}^2.$$

□

*Proof of Theorem 1.1* Using (1.6), the second claim of (2.3) implies

$$\max_{1 \leq k, u \leq L} |a_{n;ku}| = o(\|A\|), \quad \forall L \geq 1 \text{ fixed.} \tag{2.4}$$

Relation (2.4) implies that no single  $a_{n;ku}$  dominates.

• *Proof of (1.7)* Below we write  $a_{ts}$  instead of  $a_{n;ts}$ . Let

$$z_{nt} = 2\eta_t \sum_{s=1}^{t-1} a_{ts}\eta_s \quad \text{and} \quad z'_t = a_{tt} (\eta_t^2 - E\eta_t^2). \tag{2.5}$$

Then

$$Q_n - EQ_n = \sum_{t=2}^n z_{nt} + \sum_{t=1}^n z'_{nt} = S_n + S'_n. \tag{2.6}$$

Observe that  $E\eta_t\eta_s = 0$  for  $t > s$  and hence  $ES_n = 0$  since  $\eta_s$  is a m.d. sequence. In addition,

$$ES_n^2 = 4 \sum_{t=2}^n E \left[ \eta_t^2 \left( \sum_{s=1}^{t-1} a_{ts}\eta_s \right)^2 \right]. \tag{2.7}$$

Using  $E\eta_t^4 \leq C$  and (1.4),

$$E|S'_n| \leq C \sum_{t=1}^n |a_{tt}| = o(\|A_n\|), \quad ES_n^2 \leq C \left( \sum_{t=1}^n |a_{tt}| \right)^2 = o(\|A_n\|^2). \tag{2.8}$$

Now we show that

$$c_1 \|A_n\|^2 \leq ES_n^2 \leq c_2 \|A_n\|^2.$$

The lower bound follows by using (1.3) and (1.5) because of the fact that  $c > 0$ :

$$\begin{aligned} ES_n^2 &= 4 \sum_{t=2}^n E \left[ \eta_t^2 \left( \sum_{s=1}^{t-1} a_{ts}\eta_s \right)^2 \right] = 4 \sum_{t=2}^n E \left[ E[\eta_t^2 | \mathcal{F}_{t-1}] \left( \sum_{s=1}^{t-1} a_{ts}\eta_s \right)^2 \right] \\ &\geq 4c \sum_{t=2}^n E \left( \sum_{s=1}^{t-1} a_{ts}\eta_s \right)^2 = 4c \sum_{t=2}^n \sum_{s=1}^{t-1} a_{ts}^2 \\ &= 2c \sum_{t,s=1}^n a_{ts}^2 - 2c \sum_{t=1}^n a_{tt}^2 = 2\|A\|^2 - o(\|A\|^2) \geq \|A\|^2, \end{aligned} \tag{2.9}$$

for large  $n$ .

To prove the upper bound, notice that

$$\begin{aligned} ES_n^2 &= 4 \sum_{t=2}^n E \left[ \eta_t^2 \left( \sum_{s=1}^{t-1} a_{ts}\eta_s \right)^2 \right] \\ &\leq 4 \sum_{t=2}^n (E\eta_t^4)^{1/2} \left( E \left( \sum_{s=1}^{t-1} a_{ts}\eta_s \right)^4 \right)^{1/2} \leq C \sum_{t,s=1}^n a_{ts}^2 = C\|A\|^2 \end{aligned} \tag{2.10}$$



by (2.2) and assumption  $E\eta_t^4 = E\eta_1^4 < \infty$ . To obtain (1.7), note that

$$\text{Var}(Q_n) \leq 2ES_n^2 + 2ES_n'^2 \leq C\|A\|^2 + o(\|A\|^2) \leq 2C\|A\|^2$$

by (2.8) and (2.10). In addition, (2.6)–(2.10) imply

$$\text{Var}(Q_n) = (ES_n^2)(1 + o(1)), \quad n \rightarrow \infty. \tag{2.11}$$

Indeed, by (2.6),

$$\begin{aligned} |\text{Var}(Q_n) - \text{Var}(S_n)| &= |\text{Var}(S_n') + 2\text{Cov}(S_n, S_n')| \leq \text{Var}(S_n') + 2(\text{Var}(S_n)\text{Var}(S_n'))^{1/2} \\ &= o(\|A\|^2) + (O(\|A\|^2)o(\|A\|^2))^{1/2} = o(\|A\|^2) \end{aligned}$$

so that  $\text{Var}(Q_n) = \text{Var}(S_n) + o(\|A\|^2)$  and by (2.9) we have  $ES_n^2 \geq \|A\|^2$ , which leads to (2.11).

• *Proof of (1.10)* We now prove the asymptotic normality of  $Q_n$ . Let  $B_n^2 = \text{Var}(Q_n)$ ,  $X_{nt} = B_n^{-1}z_{nt}$  and  $X_t' = B_n^{-1}z_{nt}'$ . Then, by (2.6)

$$B_n^{-1}(Q_n - EQ_n) = \sum_{t=2}^n X_{nt} + \sum_{t=1}^n X_{nt}'. \tag{2.12}$$

Observe that by (1.7) and (2.8),  $E|\sum_{t=1}^n X_t'| = B_n^{-1}E|\sum_{s=1}^n z_{nt}'| \leq C\|A_n\|^{-1} \sum_{t=1}^n |a_{tt}| = o(1)$ . Therefore, to prove (1.10) it remains to show that

$$\sum_{t=2}^n X_{nt} \xrightarrow{d} N(0, 1). \tag{2.13}$$

Since  $X_{nt}$  is a m.d. sequence, then by Theorem 3.2 of Hall and Heyde (1980), it suffices to show

$$(a) E \max_{1 \leq j \leq n} X_{nj}^2 \rightarrow 0, \quad (b) \max_{1 \leq j \leq n} |X_{nj}| \rightarrow_p 0, \quad (c) \sum_{j=1}^n X_{nj}^2 \rightarrow_p 1. \tag{2.14}$$

•• To verify (a) and (b), it suffices to show that for any  $\varepsilon > 0$ ,

$$\sum_{j=1}^n EX_{nj}^2 I(|X_{nj}| \geq \varepsilon) \rightarrow 0, \tag{2.15}$$

which clearly implies (a), while (b) follows from (2.15) noting that

$$P\left(\max_{1 \leq j \leq n} |X_{nj}| \geq \varepsilon\right) \leq \varepsilon^{-2} \sum_{j=1}^n EX_{nj}^2 I(|X_{nj}| \geq \varepsilon) \rightarrow 0.$$

To prove (2.15), let  $K > 0$  be large. We consider two cases:  $\eta_t^2 \leq K$  and  $\eta_t^2 > K$ . Then,

$$\begin{aligned} EX_{nt}^2 I(X_{nt}^2 \geq \varepsilon) I(\eta_t^2 \leq K) &\leq \varepsilon^{-1} EX_{nt}^4 I(\eta_t^2 \leq K) \leq \varepsilon^{-1} 2^4 K^2 B_n^{-4} E\left(\sum_{s=1}^{t-1} a_{ts} \eta_s\right)^4 \\ &\leq C\varepsilon^{-1} K^2 B_n^{-4} \left(\sum_{s=1}^{t-1} a_{ts}^2\right)^2 \leq C\varepsilon^{-1} K^2 B_n^{-4} \|A\|_{sp}^2 \sum_{s=1}^{t-1} a_{ts}^2 \end{aligned}$$

by (2.2) and (2.3). Recall that by (1.7),  $B_n^{-2} \leq C\|A\|^{-2}$ . Thus, for any  $\varepsilon > 0$  and  $K > 0$ ,

$$\begin{aligned} \sum_{t=2}^n EX_{nt}^2 I(X_{nt}^2 \geq \varepsilon) I(\eta_t^2 \leq K) &\leq C\varepsilon^{-1} K^2 B_n^{-4} \|A\|_{sp}^2 \sum_{t=2}^n \sum_{s=1}^{t-1} a_{ts}^2 \\ &\leq C\varepsilon^{-1} K^2 (\|A\|_{sp}/\|A\|)^2 \rightarrow 0 \end{aligned} \tag{2.16}$$

by (1.6) as  $n \rightarrow \infty$  for any finite  $K$ .

We now focus on the case  $\eta_t^2 \geq K$ . Since  $E\eta_t^4 < \infty$  and, by stationarity of  $\eta_t$ ,  $\delta_K := E\eta_1^4 I(\eta_1^2 > K) \rightarrow 0$  as  $K \rightarrow \infty$ , this implies

$$\begin{aligned} EX_{nt}^2 I(X_{nt}^2 \geq \varepsilon) I(\eta_t^2 > K) &\leq EX_{nt}^2 I(\eta_t^2 > K) \leq B_n^{-2} 2^2 E \left[ \eta_t^2 I(\eta_t^2 > K) \left( \sum_{s=1}^{t-1} a_{ts} \eta_s \right)^2 \right] \\ &\leq C\|A\|^{-2} \delta_K^{1/2} \left( E \left( \sum_{s=1}^{t-1} a_{ts} \eta_s \right)^4 \right)^{1/2} \leq C\|A\|^{-2} \delta_K^{1/2} \sum_{s=1}^{t-1} a_{ts}^2 \end{aligned}$$

by (2.2). Hence,

$$\begin{aligned} \sum_{t=2}^n EX_{nt}^2 I(X_{nt}^2 \geq \varepsilon) I(\eta_t^2 > K) &\leq C\delta_K^{1/2} \|A\|^{-2} \sum_{t=2}^n \sum_{s=1}^{t-1} a_{ts}^2 \\ &\leq C\delta_K^{1/2} \rightarrow 0, \quad K \rightarrow \infty. \end{aligned} \tag{2.17}$$

Since (2.16) holds for any fixed  $K$  as  $n \rightarrow \infty$ , and since (2.17) holds as  $K \rightarrow \infty$  uniformly in  $n$ , we get (2.15).

•• The verification of (c) in (2.14) is particularly delicate. We want to show that  $s_n \rightarrow_p 1$ . Recall that  $x_{nt} = B^{-1}z_{nt}$  where  $z_{nt}$  is defined in (2.5). We shall decompose  $s_n = \sum_{s=1}^n X_{ns}^2$  into two parts involving  $L > 1$ . Write

$$s_n = 4B_n^{-2} \sum_{t=1}^n \eta_t^2 \left( \sum_{s=1}^{t-1} a_{ts} \eta_s \right)^2 = s_{n,1} + s_{n,2}, \tag{2.18}$$

where

$$s_{n,1} := 4B_n^{-2} \sum_{t=1}^n \eta_t^2 \left( \sum_{s=t-L}^{t-1} a_{ts} \eta_s \right)^2, \quad s_{n,2} := s_n - s_{n,1}.$$

Then,

$$s_n = Es_n + (s_{n,1} - Es_{n,1}) + (s_{n,2} - Es_{n,2}).$$

We show that as  $n \rightarrow \infty$ ,

$$\begin{aligned} (i) \quad &Es_n \rightarrow 1; \quad (ii) \quad s_{n,1} - Es_{n,1} \rightarrow_p 0, \quad \forall L \geq 1; \\ (iii) \quad &E|s_{n,2}| \rightarrow 0, \quad n \rightarrow \infty, \quad L \rightarrow \infty \end{aligned} \tag{2.19}$$

which proves (2.14)(c) since  $E|s_n| \rightarrow 0$  implies  $s_n \rightarrow_p 0$  as  $n \rightarrow \infty$  and  $L \rightarrow \infty$ .

••• The claim (2.19)(i) follows from (2.11),

$$(ES_n^2) / \text{Var}(Q_n) = B_n^{-2} ES_n^2 \rightarrow 1,$$

noting that  $B_n^{-2} E S_n^2 = E s_n$ , which holds by definition of  $s_n$  and (2.7).

••• To show (2.19)(ii), open up the squares, set  $s = t - k$  and  $s' = t - u$ , to get

$$s_{n,1} - E s_{n,1} = 4 \sum_{k,u=1}^L \left\{ B_n^{-2} \sum_{t=1}^n a_{t,t-k} a_{t,t-u} \left[ \eta_t^2 \eta_{t-k} \eta_{t-u} - E \eta_t^2 \eta_{t-k} \eta_{t-u} \right] \right\} = 4 \sum_{k,u=1}^L g_{n,ku}.$$

It suffices to verify that for any fixed  $k$  and  $u$ ,  $g_{n,ku} = o_p(1)$ . Setting

$$c_{nt} := B_n^{-2} a_{t,t-k} a_{t,t-u}$$

and

$$V_t := \eta_t^2 \eta_{t-k} \eta_{t-u} - E \eta_t^2 \eta_{t-k} \eta_{t-u},$$

write

$$g_{n,ku} = \sum_{t=1}^n c_{nt} V_t.$$

Since the noise  $\{\eta_t\}$  is stationary ergodic and such that  $E \eta_1^4 < \infty$ , by Theorem 3.5.8 in Stout (1974), the process  $\{V_j\}$  is stationary and ergodic, and  $E|V_1| < \infty$ . Because of the centering,  $E g_{n,ku} = 0$ . Thus, by Lemma 2.1(i), to prove  $g_{n,ku} = o_p(1)$ , it remains to show that  $c_{nt}$ 's satisfy (2.1) with  $\alpha_n = 1$ . Observe that

$$\sum_{t \in Z} |c_{nt}| = B_n^{-2} \sum_{t=1}^n |a_{t,t-k} a_{t,t-u}| \leq 2 B_n^{-2} \sum_{t,s=1}^n a_{t,s}^2 = 2 B_n^{-2} \|A\|^2 \leq C, \quad n \rightarrow \infty$$

by (1.7). On the other hand,

$$\begin{aligned} \sum_{t \in Z} |c_{nt} - c_{n,t-1}| &= B_n^{-2} \sum_{t=1}^{n+1} |a_{t,t-k} a_{t,t-u} - a_{t-1,t-1-k} a_{t-1,t-1-u}| \\ &\leq B_n^{-2} \sum_{t=1}^{n+1} \{ |a_{t,t-k} - a_{t-1,t-1-k}| |a_{t,t-u}| + |a_{t-1,t-1-k}| |a_{t,t-u} - a_{t-1,t-1-u}| \} \\ &\leq B_n^{-2} \left\{ \left( \sum_{t=1}^{n+1} (a_{t,t-k} - a_{t-1,t-1-k})^2 \right)^{1/2} + \left( \sum_{t=1}^{n+1} (a_{t,t-u} - a_{t-1,t-1-u})^2 \right)^{1/2} \right\} \\ &\quad \times \left( \sum_{t,s=1}^n a_{t,s}^2 \right)^{1/2} \\ &= o(B_n^{-2} \|A\|^2) = o(1), \end{aligned}$$

by (1.9), (2.3) and (1.7). Hence (2.1) holds. By Lemma 2.1(i) we conclude that  $g_{n,ku} = o_p(1)$  and, thus,  $s_{n,1} - E s_{n,1} = o_p(1)$ . Hence (2.19)(ii) holds.

••• To verify  $E|s_{n,2}| \rightarrow 0$  in (2.19)(iii), write

$$s_{n,2} = s_n - s_{n,1} = 4 B_n^{-2} \sum_{t=1}^n \eta_t^2 \left[ \left( \sum_{s=1}^{t-1} a_{ts} \eta_s \right)^2 - \left( \sum_{s=t-L}^{t-1} a_{ts} \eta_s \right)^2 \right].$$

We use the identity  $a^2 - b^2 = (a - b)^2 + 2(a - b)b$ , to obtain

$$\begin{aligned}
 |s_{n,2}| &= 4B_n^{-2} \left| \sum_{t=1}^n \eta_t^2 \left\{ \left( \sum_{s=1}^{t-1} a_{ts} \eta_s \right)^2 - \left( \sum_{s=t-L}^{t-1} a_{ts} \eta_s \right)^2 \right\} \right| \\
 &= 4B_n^{-2} \left| \sum_{t=1}^n \eta_t^2 \left\{ \left( \sum_{s=1}^{t-L-1} a_{ts} \eta_s \right)^2 + 2 \left( \sum_{s=1}^{t-L-1} a_{ts} \eta_s \right) \left( \sum_{s=t-L}^{t-1} a_{ts} \eta_s \right) \right\} \right| \\
 &\leq 4q_{n,1} + 4 \left( B_n^{-2} \sum_{t=1}^n \eta_t^2 \left( \sum_{s=1}^{t-L-1} a_{ts} \eta_s \right)^2 \right)^{1/2} \\
 &\quad \times \left( 4B_n^{-2} \sum_{t=1}^n \eta_t^2 \left( \sum_{s=t-L}^{t-1} a_{ts} \eta_s \right)^2 \right)^{1/2} \leq 4 \left( q_{n,1} + q_{n,1}^{1/2} s_{n,1}^{1/2} \right),
 \end{aligned}$$

where

$$q_{n,1} := B_n^{-2} \sum_{t=1}^n \eta_t^2 \left( \sum_{s=1}^{t-L-1} a_{ts} \eta_s \right)^2.$$

Hence,  $E|s_{n,2}| \leq 4Eq_{n,1} + 4(Eq_{n,1}Es_{n,1})^{1/2}$ . To bound  $Eq_{n,1}$ , we argue partly as in (2.10):

$$Eq_{n,1} \leq C \|A_n\|^{-2} \sum_{t=1}^n \sum_{s=1}^{t-L-1} a_{ts}^2 \rightarrow 0, \quad n \rightarrow \infty, \quad L \rightarrow \infty$$

by (1.8). We also have

$$Es_{n,1} \leq C \|A_n\|^{-2} \sum_{t=1}^n \sum_{s=t-L}^{t-1} a_{ts}^2 \leq C.$$

Hence  $E|s_{n,2}| \rightarrow 0$  as  $n \rightarrow \infty$  and  $L \rightarrow \infty$ . This completes the proof of (2.19)(iii) and the theorem. □

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