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The q-cosine Fourier transform and the q-heat equation

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Abstract The aim of this work is to establish in great detail The q-Fourier analysis related to the q-cosine. The wise reader will note that the considered q-cosine coincides with the one given by T.H. Koornwinder and S.F. Swarttouw. Through the q-cosine product formula, we define and analyze the properties of the q-even translation and the q-convolution. Adopting the Titchmarsh approach, we study the q-cosine Fourier transform and its inverse formula.

The second theme of this paper is an application of the q-Fourier analysis developed earlier. We extend the heat representation theory inaugurated by P.C. Rosenbloom and D.V. Widder to the q-analogue. We construct the q-solution source, the q-heat polynomials and solve the q-analytic Cauchy problem.

Keywords Basic orthogonal polynomials and functions · Basic hypergeometric integrals

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1 Introduction

During the last years, an intensive work was founded about the so-called q-basic theory. Taking account of the well-known Ramanujan works shown at the beginning of this century by Jackson ([9, 10]), many authors such as Askey, Gasper, Ismail, Rogers, Andrew, Koornwinder, and others (see references) have recently developed this topic.

The present article is devoted to the study of the q-analogue of the Fourier transforms and to showing how it plays a central role in solving the q-heat equation associated to the second q-derivative operator. The method used here differs from those given by T.H. Koornwinder and R.F. Swarttouw, who discovered a q-analogue of Hankel's Fourier–Bessel via some q-analogue orthogonality relations. We note that Ph. Feinsilver [4] gave a q-Harmonic Analysis for a q-Laplace transform with inversion formula.

Without entering into a dilemma through the analysis presented here, it seems that the point of view of T.H. Koornwinder and R.F. Swarttouw [12] is more suitable for harmonic analysis. We take as definition of the q-cosine the one given by the previous authors with a simple change and we prefer to write it as a series of functions denoted as $b_n(x;q^2)$. This q-cosine appears as an eigenfunction of the operator Δ_q. Owing to a nice paper [12], we give a product formula written with the q-Jackson integral and we study the q-translation and the q-convolution. Next we define the q-analogue of the cosine Fourier transform with the purpose to find the transformation inverse. To this end, we prove the equivalent of the so-called Riemann–Lebesgue Lemma and discover that the Titchmarsh approach holds [15].

A motivation behind this work is to state some result about the q-heat equation associated to Δ_q operator. We attempt to extend the heat representation theory studied in many cases ([5, 7, 14], etc.). We define the q-heat polynomials and find that they are linked to the q-Hermite polynomials [13] and constitute with the q-associated functions a biorthogonal system. We conclude by solving the q-analytic Cauchy problem related to the q-heat equation.

2 Notations and preliminaries

We begin by recalling some q-elements of quantum analysis adapting the notation used in the book of Gasper and Rahman [6]. Let a and q be real numbers such that $0 < q < 1$, the q-shift factorial is defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right), \quad n = 1,2,\dots,\infty. \tag{1}$$

A basic hypergeometric series is

$${}_r\varphi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,z) = \sum_{k=0}^{\infty}\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s,q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k.$$

A function f is q-regular at zero if $\lim_{n\to\infty} f(xq^n) = f(0)$ exists and is independent of x.

The q-derivative $D_q f$ of a function f is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0. \tag{2}$$

The q-derivative at zero is defined by

$$D_q f(0) = \lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n},$$

if it exists and does not depend on x.

We introduce the set

$$\mathbb{R}_q = \{q^k; k \in \mathbb{Z}\}.$$

The q-integral of Jackson is defined by

$$\int_0^a f(x)\, d_q x = (1-q)a \sum_{k=0}^{\infty} f(aq^k) q^k,$$

$$\int_0^\infty f(x)\, d_q x = (1-q) \sum_{k=-\infty}^{\infty} f(q^k) q^k.$$

The q-integration by parts is given for suitable functions f and g by

$$\int_0^\infty f(x) D_q g(x)\, d_q x = \big[f(x)g(x)\big]_0^\infty - \int_0^\infty f(x) D_q g(q^{-1}x)\, d_q x. \tag{3}$$

The q-analogue of the Gamma function is defined as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \tag{4}$$

which tends to $\Gamma(x)$ when q tends to 1^-.

3 q-Trigonometric functions

We define the q-cosine as

$$\cos(x;q^2) = {}_1\phi_1\big(0;q;q^2,(1-q)^2x^2\big) = \sum_{n=0}^{\infty} (-1)^n b_n(x;q^2), \tag{5}$$

where we have put

$$b_n(x;q^2) = b_n(1;q^2)x^{2n} = q^{n(n-1)} \frac{(1-q)^{2n}}{(q;q)_{2n}} x^{2n}. \tag{6}$$

In the same way, the q-sine is given by

$$\sin\big(x;q^2\big) = (1-q)x\,{}_1\phi_1\big(0;q^3;q^2,(1-q)^2x^2\big) = \sum_{n=0}^{\infty}(-1)^n c_n\big(x;q^2\big),$$

with

$$c_n\big(x;q^2\big) = c_n\big(1;q^2\big)x^{2n+1} = \frac{q^{n(n-1)}(1-q)^{2n+1}}{(q;q)_{2n+1}}x^{2n+1}.$$

These q-trigonometric functions differ and should not be confused with the functions $\cos_q$ and $\sin_q$ considered in [6, p. 23]; but coincide with the one given in [12] and [15] with a minor change of variable. Furthermore, we have

Proposition 3.1 *The following statements hold*:

1.

$$b_n\big(0,q^2\big) = \delta_{n,0}, \qquad \Delta_q b_n\big(x;q^2\big) = b_{n-1}\big(x;q^2\big), \quad n \geq 1;$$

2.

$$\big|b_n\big(x;q^2\big)\big| \leq \frac{x^{2n}}{(2n)!},$$

where

$$\Delta_q u(x) = \big(D_q^2 u\big)\big(q^{-1}x\big). \tag{7}$$

Proof We only prove Part 2 since Part 1 is deduced from the definition of Δ_q.

The coefficients $b_n(1;q^2)$, defined by (6), can be written as

$$\begin{aligned} b_n\big(1;q^2\big) &= \prod_{j=0}^{n-1}\frac{q^j-q^{j+1}}{1-q^{2j+1}}\frac{q^j-q^{j+1}}{1-q^{2j+2}} \\ &= \prod_{j=0}^{n-1}\frac{e^{-jt}-e^{-(j+1)t}}{1-e^{-(2j+1)t}}\cdot\frac{e^{-jt}-e^{-(j+1)t}}{1-e^{-(2j+2)t}}, \end{aligned}$$

where we have put $q=e^{-t}$, $t>0$.

Since the functions

$$f(t) = \frac{e^{-jt}-e^{-(j+1)t}}{1-e^{-(j+1)t}} \quad \text{and} \quad g(t) = \frac{e^{-jt}-e^{-(j+1)t}}{1-e^{-(2j+2)t}},$$

decrease on $]0,\infty[$, we obtain

$$b_n\big(1;q^2\big) \leq \frac{1}{(2n)!}.$$

□

As a consequence of the previous proposition, we can show that for $\lambda \in \mathbb{C}$ the function

$$\cos\big(\lambda x; q^2\big) = \sum_0^{\infty} (-1)^n b_n\big(x; q^2\big)\lambda^{2n},$$

is the unique analytic solution of the q-differential equation

$$\Delta_q u(x) = -\lambda^2 u(x), \tag{8}$$

with

$$u(0, q) = 1, \qquad (D_q u)(0) = 0. \tag{9}$$

Proposition 3.2 *For $x \in \mathbb{R}_q$ and $\frac{\mathrm{Log}(1-q)}{\mathrm{Log}(q)} \in \mathbb{Z}$, we have*

1.
$$\big|\cos\big(x, q^2\big)\big| \le \frac{1}{(q; q^2)^2_\infty};$$

2.
$$\lim_{x\to\infty} \cos\big(x, q^2\big) = 0;$$

3.
$$\big|\sin\big(x, q^2\big)\big| \le \frac{1}{(q; q^2)^2_\infty};$$

4.
$$\lim_{x\to\infty} \sin\big(x, q^2\big) = 0.$$

Proof To prove Parts 1 and 2, we use the properties of ${}_1\phi_1$ given in [12] and their connection to the q-cosine. We obtain

$$\big|\cos\big(q^{1+n}; q^2\big)\big| \le \frac{1}{(q; q^2)^2_\infty} \begin{cases} 1 & \text{if } n \ge 0, \\ q^{n^2} & \text{if } n \le 0. \end{cases} \tag{10}$$

hence Parts 1 and 2 follow. A similar argument shows Parts 3 and 4. □

Now we try to find a product formula for the q-cosine functions. We begin by proving the following result.

Proposition 3.3 *For reals x and y, $y \ne 0$, we have*

$$\cos\big(x, q^2\big)\cos\big(y, q^2\big) = \sum_{k=0}^{\infty} q^k \left(\frac{x}{y}\right)^{2k} \sum_{s=-k}^{s=k} (-1)^{k-s} \frac{q^{\binom{k-s}{2}}}{(q; q)_{k-s}(q; q)_{k+s}} \cos\big(q^s y, q^2\big). \tag{11}$$

Note that this formula can be expressed in terms of ${}_1\varphi_1$ as follows

$$\cos(x,q^2)\cos(y,q^2)=\sum_{s=-\infty}^{\infty}q^s\left(\frac{x}{y}\right)^{2s}\frac{(q^{1+2s};q)_\infty}{(q;q)_\infty}$$
$$\times{}_1\varphi_1\left(0;q^{1+2s};q^2,q\frac{x^2}{y^2}\right)\cos(q^sy,q^2). \qquad (12)$$

Proof To show (11) and (12), we begin by expanding the q-cosines in series absolutely and uniformly convergent on every compact of $\mathbb{R}$. From the product rule of series and the fact that

$$\frac{1}{(q;q)_{2n-2k}}=\frac{(q^{2n-2k+1},q)_\infty}{(q;q)_\infty}=0,\quad k>n,$$

we obtain for $y\neq 0$

$$\cos(x;q^2)\cos(y;q^2)=\sum_{k=0}^{\infty}\frac{q^{2k^2}}{(q;q)_{2k}}\left(\frac{x}{y}\right)^{2k}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n^2-n}}{(q;q)_{2n-2k}}q^{-2nk}y^{2n}.$$

On the other hand, we have

$$\frac{1}{(q;q)_{2n-2k}}=\frac{q^{-k(2k-1)+2nk}}{(q;q)_{2n}}\sum_{s=-k}^{s=k}(-1)^{k-s}\frac{q^{\binom{k-s}{2}}}{(q;q)_{k-s}(q;q)_{k+s}}q^{2ns}.$$

We deduce (11) after the interchange of summation order. To prove (12), we write

$$\cos(x;q^2)\cos(y;q^2)=I+J,$$

with

$$I=\sum_{s=0}^{\infty}\cos(q^sy;q^2)\sum_{k\geq s}q^k\left(\frac{x}{y}\right)^{2k}\frac{(-1)^{k-s}q^{\frac{(k-s)(k-s-1)}{2}}}{(q;q)_{k+s}(q;q)_{k-s}},$$

$$J=\sum_{s=-\infty}^{-1}\cos(q^sy;q^2)\sum_{k\geq -s}q^k\left(\frac{x}{y}\right)^{2k}\frac{(-1)^{k-s}q^{\frac{(k-s)(k-s-1)}{2}}}{(q;q)_{k-s}(q;q)_{k+s}}.$$

In I, we make the change $k-s$ into k and use the equality

$$(q;q)_{k+2s}=(q;q)_{2s}(q^{1+2s};q)_k,$$

to obtain

$$I=\sum_{s=0}^{\infty}q^s\left(\frac{x}{y}\right)^{2s}\frac{(q^{2s+1};q)_\infty}{(q;q)_\infty}{}_1\phi_1(0;q^{1+2s};q,q(q^2/y^2))\cos(q^sy;q^2).$$

Now we make the change $k+s$ into k in J and use the equalities

$$(q;q)_{k-2s}=(q;q)_{-2s}(q^{1-2s};q)_k,\quad -s\geq 1,$$

$$\frac{(k-2s)(k-2s-1)}{2}=\frac{(k-2)(k-3)}{2}-2sk+2s^2-1,$$

and

$$\begin{aligned}&\big(q^{1-2s};q\big)_\infty {}_1\phi_1\big(0;q^{1-2s};q,q^{1-2s}x^2/y^2\big)\\&\quad=q^{s(2s-1)}q^{1-2s}\big(x^2/y^2\big)^{2s}\big(q^{1+2s};q\big)_\infty\ {}_1\phi_1\big(0;q^{1+2s};q,qx^2/y^2\big).\end{aligned}$$

This identity is easily deduced from [11]. Then we obtain

$$J=\sum_{s=-\infty}^{-1}q^s\big(x^2/y^2\big)^{2s}\frac{(q^{1+2s};q)_\infty}{(q;q)_\infty}{}_1\phi_1\big(0;q^{1+2s};q,qx^2/y^2\big)\cos\big(q^sy;q^2\big).$$

We add these sums to find that (12) holds. □

Remark 3.4 (1) If we replace y by q^y, x by q^x, and assume the proposition the hypothesis, we obtain from (12) that the following integral representation holds

$$\begin{aligned}&\cos\big(q^x;q^2\big)\cos\big(q^y;q^2\big)\\&\quad=\frac{(q^{2(x-y)+1};q)_\infty}{(q;q)_\infty}\int_0^\infty u^{2(x-y)}{}_1\phi_1\big(0;u^{2(x-y)+1};q,qu^2\big)\cos\big(q^yu;q^2\big)\,d_qu.\end{aligned}$$

(2) The product formula (11) leads to

$$\cos\big(x;q^2\big)\cos\big(y;q^2\big)=\sum_{n=0}^{\infty}b_n\big(x;q^2\big)\Delta_q^n\cos\big(y;q^2\big).\tag{13}$$

4 q-Translation and q-convolution

We define, for x and y in $\mathbb{R}_q$, the measure

$$d_q\mu_{(x,y)}=\sum_{s=-\infty}^{\infty}\mathcal{D}\big(x,y;q^s\big)q^s\delta_{yq^s},\tag{14}$$

where δ_u denotes the unit mass supported at u, and

$$\mathcal{D}\big(x,y;q^s\big)=\left(\frac{x}{y}\right)^{2s}\frac{(q(\frac{x}{y})^2;q)_\infty}{(q;q)_\infty}{}_1\phi_1\left(0;q\left(\frac{x}{y}\right)^2;q,q^{1+2s}\right).\tag{15}$$

Proposition 4.1 (1) *For x and y in $\mathbb{R}_q$, we have*

$$d_q\mu_{(x,y)}=d_q\mu_{(y,x)}.$$

(2) $d_q\mu_{(x,y)}$ *is of bounded variation.*

(3)

$$\int d_q\mu_{(x,y)}(t) = 1.$$

Proof For $n, m \in \mathbb{Z}$, the relation (2.3) from [12] leads to

$$\mathcal{D}\big(q^n, q^m; q^s\big) = \mathcal{D}\big(q^m, q^n; q^{s+m-n}\big).$$

We obtain Part 1 after the change $s-n+m$ by s.

To prove Part 2, we suppose $|\frac{x}{y}| \leq 1$; from the formulas (2.4) in [12] we have

$$|d_q\mu_{(x,y)}|_{var} \leq \left(\frac{|y|^2 + q|x|^2}{|y|^2 - q|x|^2}\right)\frac{(q|\frac{x}{y}|^2; -q, q)_\infty}{(q,q)_\infty}. \tag{16}$$

Finally, from (2.8) in [12], we can show that Part 3 is true. □

We introduce the q-translation which generalizes the even translation given by $\frac{1}{2}(\delta_{x+y} + \delta_{x-y})$.

Let f be a function with support in $\mathbb{R}_q$, the q-translation is defined for x and y in $\mathbb{R}_q$ by

$$T_{x,q} f(y) = \int_0^\infty f(t)\, d_q\mu_{(x,y)}(t). \tag{17}$$

From the previous proposition and the q-product formula (12), we have

Proposition 4.2 *Let f be a function with compact support in $\mathbb{R}_q$. We have*

(i)

$$T_{q,y}\cos\big(x; q^2\big) = \cos\big(x; q^2\big)\cos\big(y; q^2\big).$$

(ii)

$$T_{q,y} f(x) = T_{q,x} f(y),$$
$$T_{q,0} f = f.$$

(iii)

$$\Delta_q T_{q,y} f = T_{q,y}\Delta_q f,$$
$$\Delta_{q,;y} T_{q,y} f = T_{q,y}\Delta_{q,y} f.$$

(iv) *The function $u(x,y) = T_{q,y} f(x)$ is a solution of the problem*

$$\Delta_{q,x} u(x,y) = \Delta_{q,y} u(x,y),$$
$$u(x,0) = f(x).$$

From the relation

$$\Delta_q^n(f)(x) = \frac{q^{(2-n)n}(q;q)_{2n}}{(1-q)^{2n}} \sum_{k=-n}^{n} (-1)^{n-k} \frac{q^{\binom{n-k}{2}}}{(q;q)_{n-k}(q;q)_{n+k}} f\big(q^k x\big),$$

we can write the q-translation of a function f as

$$T_{q,y} f(x) = \sum_{n=0}^{\infty} b_n\big(y, q^2\big) \Delta_{q,x}^n f(x), \tag{18}$$

and have in the limit when q tends to 1^- the classical even translation cited before.

Now we denote by $L_q^1(\mathbb{R}_q)$ the space of functions f defined on $\mathbb{R}_q$ such that

$$\|f\|_{1,q} = \int_{-\infty}^{\infty} \big|f(t)\big| \, d_q t < \infty.$$

Then we are able to define the q-convolution by

$$f \star_q g(x) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{\infty} T_{x,q} f(y) g(y) \, d_q y, \tag{19}$$

where f and g are two functions in $L_q^1(\mathbb{R}_q)$. We can show that this space is an algebra.

5 q-Analogue of Fourier-cosine

In this section, we suppose $\frac{\mathrm{Log}(1-q)}{\mathrm{Log}(q)} \in \mathbb{Z}$. The q-analogue of Fourier transform is defined for $\lambda \in \mathbb{R}_q$ by

$$\mathcal{F}(f)(\lambda) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{\infty} f(t) \cos\big(\lambda t; q^2\big) \, d_q t, \tag{20}$$

where f is a function in $L_q^1(\mathbb{R}_q)$.

This definition is the same (after a minor change) as that given by T.H. Koornwinder and R.F. Swarttouw (see [12]).

Proposition 5.1 *For $f, g \in L_q^1(\mathbb{R}_q)$, the following properties hold*:

(1)

$$\big|\mathcal{F}_q(f)(\lambda)\big| \le \frac{1}{[q(1-q)]^{\frac{1}{2}}(q;q)_\infty} \|f\|_{1,q}, \quad \lambda \in \mathbb{R}_q; \tag{21}$$

(2)

$$\mathcal{F}_q(\mathcal{T}_{q,x} f)(\lambda) = \cos\big(\lambda x; q^2\big) \mathcal{F}_q(f)(\lambda), \quad \lambda \in \mathbb{R}_q; \tag{22}$$

(3)

$$\mathcal{F}_q(f \star_q g) = \mathcal{F}_q(f)\mathcal{F}_q(g).$$

Proof Part 1. The inequality (21) follows from Proposition 3.2 and the identity

$$\left(q;q^2\right)_\infty \left(q^2;q^2\right)_\infty = (q;q)_\infty.$$

Part 2 is a direct consequence of the q-product formula (12).

Part 3 is obtained after the exchange of the integration order and taking into account the invariability of the q-integral by the q-translation. □

Now we focus our attention on the inversion of the linear map $\mathcal{F}_q$. We proceed by looking at the q-analogue of the Riemman–Lebesgue Lemma, the localization theorem, and we show that the Titchmarsh approach holds in the q-theory.

Proposition 5.2 *Let f be a function in $L_q^1(\mathbb{R}_q)$, then*

$$\lim_{\lambda \longrightarrow \infty} \mathcal{F}_q(f)(\lambda) = 0, \quad \lambda \in \mathbb{R}_q.$$

Proof To prove this, first we have from Proposition 3.2

$$\left|f(x)\cos\left(\lambda x;q^2\right)\right| \leq \frac{1}{(q;q^2)_\infty^2}\left|f(x)\right| \in L_q^1(\mathbb{R}_q), \quad x, \lambda \in \mathbb{R}_q.$$

And for $\lambda \in \mathbb{R}_q$ we have

$$\lim_{\lambda \to \infty} f(x)\cos\left(\lambda x;q^2\right) = 0, \quad \lambda \in \mathbb{R}_q,$$

so the result is true. □

Proposition 5.3 *We have the identity*

$$\int_0^\infty \frac{\sin(x;q^2)}{x}\,d_q x = \frac{\Gamma_{q^2}^2(\frac{1}{2})}{1+q^{-1}}.$$

Proof This is a consequence of (2.8) in [12]. □

Proposition 5.4 *Let $f:(0,\infty) \to \mathbb{C}$ satisfy the conditions*:

(1) $f \in L_q^1(\mathbb{R}_q)$,
(2) *For $a \in \mathbb{R}_q$, there exists $C(a) > 0$ such that*

$$\left|f\left(aq^k\right) - f(0)\right| \leq C(a)q^k, \quad k = 0, 1, 2, \ldots.$$

Then

$$\lim_{\lambda \to +\infty} \int_0^\infty f(x)\frac{\sin(\lambda x;q^2)}{x}\,d_q x = \frac{\Gamma_{q^2}^2(\frac{1}{2})}{1+q^{-1}} f(0).$$

Proof Indeed, the first hypothesis shows that for an arbitrary $\varepsilon > 0$ we have for large q^{-N}, $N = 0, 1, \ldots,$ that

$$\int_{q^{-N}}^{\infty} \left| \frac{f(x)}{x} \right| d_q x \le \frac{\varepsilon}{2} \big(q; q^2\big)_{\infty}^2$$

and

$$\begin{aligned} &\left| \int_0^{\infty} f(x) \frac{\sin(\lambda x; q^2)}{x} d_q x - f(0) \int_0^{q^{-N}} f(x) \frac{\sin(\lambda x; q^2)}{x} d_q x \right| \\ &\quad \le \frac{\varepsilon}{2} + \int_0^{q^{-N}} \frac{f(x) - f(0)}{x} \sin\big(\lambda x; q^2\big)\, d_q x. \end{aligned}$$

The second hypothesis and Proposition 3.2 show that

$$\left| \frac{f(q^{k-N}) - f(0)}{q^{k-N}} \sin\big(\lambda q^{k-N}; q^2\big) \right| \le \frac{C(N)}{q^{-N}(q, q^2)_{\infty}^2}.$$

Since from Proposition 3.2 we have that $\sin(\lambda x; q^2)$ tends to zero as λ tends to ∞, the proposition is then a direct consequence. □

Theorem 5.5 (The q-cosine Fourier integral theorem) *If $f \in L_q^1(\mathbb{R}_q)$ is such that for $a \in \mathbb{R}_q$ there exist positive constants $C(a)$ such that*

$$\big|T_{x,q} f\big(aq^k\big) - f\big(q^k\big)\big| \le C(a) q^k, \quad k = 0, 1, \ldots, \tag{23}$$

then

$$\begin{aligned} &\frac{(1+q)^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^{\infty} d_q\xi \int_0^{\infty} f(t) \cos\big(\xi t; q^2\big) \cos\big(\xi x; q^2\big)\, d_q t = f(x), \\ &x \in L_q^1(\mathbb{R}_q). \end{aligned} \tag{24}$$

6 q-Heat equation and q-heat polynomials

In this section, the two q-analogues of the elementary exponential functions are crucial and they are defined by

$$\begin{aligned} E\big(x; q^2\big) &= \big(-\big(1-q^2\big)x, q^2\big)_{\infty} \\ &= \sum_0^{\infty} \frac{(1-q^2)^n}{(q^2; q^2)_{\infty}} q^{n(n-1)} x^n, \quad x \in \mathbb{R}, \end{aligned} \tag{25}$$

and

$$e\big(x; q^2\big) = \frac{1}{((1-q^2)x; q^2)_{\infty}} = \sum_0^{\infty} \frac{(1-q^2)^n}{(q^2; q^2)_n} x^n, \quad |x| < \frac{1}{1-q^2}. \tag{26}$$

These functions satisfy the identity

$$e(x;q^2)E(-x;q^2)=1,$$

and have as limit, when q tends to 1^-, the classical exponential function.

Now we purpose to give the q-analogue of the heat equation associated to the second derivative operator (even in x)

$$\frac{\delta^2 u}{\delta x^2}=\frac{\delta u}{\delta t},\quad x\in\mathbb{R},\ t>0. \tag{27}$$

We consider as q-heat equation associated to the second q-derivative operator the partial q-difference equation

$$(\Delta_{q,x}u)(x,t)=(D_{q^2,t}u)(x,t). \tag{28}$$

We take as the initial condition

$$u(x,0)=f(x),\quad f\in L^1_q(\mathbb{R}_q). \tag{29}$$

6.1 q-Solution source

To find the solution source related to the q-heat equation, we apply the Fourier method with the adapted q-Fourier cosine studied before.

Putting

$$U(\lambda,t)=\mathcal{F}\big(u(x,t)\big)(\lambda),$$

Eq. (28) becomes

$$D_{q^2,t}U(\lambda,qt)=-\lambda^2U(\lambda,t),$$

and, taking into account conditions (29), we obtain

$$U(\lambda,t)=\mathcal{F}(f)(\lambda)e\big(-\lambda^2t;q^2\big).$$

The problem consists in finding the function which has $e(-\lambda^2t;q^2)$ as its q-Fourier-cosine transform. For this end, we need the following lemma.

Lemma 6.1 *For* $n=0,1,2,\dots$ *and* $t>0$, *we have*

$$\int_0^\infty e\Big(-\frac{\lambda^2}{qt(1+q)^2},q^2\Big)b_n\big(\lambda;q^2\big)\,d_q\lambda$$
$$=(1-q)\frac{(q^2,-\frac{1+q}{1-q}q^2t,-\frac{1-q}{1+q}\frac{1}{t},q^2)_\infty}{(q,-\frac{1-q}{1+q}\frac{1}{qt},-\frac{1+q}{1-q}q^3t;q^2)_\infty}\frac{(1-q^2)^n}{(q^2,q^2)_n}t^n.$$

Proof From (26) we find

$$\int_0^\infty e\Big(-\frac{\lambda^2}{qt(1+q)^2},q^2\Big)\lambda^{2n}\,d_q\lambda=(1-q)\sum_{-\infty}^{\infty}\frac{q^{(2n+1)k}}{(-\frac{1-q}{1+q}\frac{q^{2k}}{qt},q^2)_\infty}.$$

Secondly, the use of the well-known Ramanujan [8] identity

$$\sum_{-\infty}^{\infty} \frac{z^k}{(bq^k, q)_\infty} = \frac{(bz, q/bz, q, q)_\infty}{(b, z, q/b, q)_\infty}, \quad b \neq 0,$$

leads to the result after minor computation. □

Proposition 6.2

$$\frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^\infty e\left(-\frac{\lambda^2}{qt(1+q)^2}, q^2\right) \cos\left(\lambda x, q^2\right) d_q\lambda = A\left(t, q^2\right) e\left(-tx^2, q^2\right),$$

where

$$A\left(t, q^2\right) = \left[(1-q)q^{-1}\right]^{1/2} \frac{(-\frac{1+q}{1-q}q^2 t, -\frac{1-q}{1+q}\frac{1}{t}, q^2)_\infty}{(-\frac{1-q}{1+q}\frac{1}{qt}, -\frac{1+q}{1-q}q^3 t; q^2)_\infty}. \tag{30}$$

As an immediate consequence we are now able to define the q-source solution associated to the q-heat equation (28) by

$$G\left(x, t, q^2\right) = \left(A\left(t, q^2\right)\right)^{-1} e\left(-\frac{x^2}{qt(1+q)^2}; q^2\right). \tag{31}$$

In the same manner as in the classical heat equation theory, we put

$$G\left(x, y, t; q^2\right) = T_{y,q} G\left(x, t; q^2\right), \tag{32}$$

with $T_{y,q}$ being the q-translation studied in Sect. 4.

Through this approach we show that the solution of the q-Cauchy problem (28) and (29) can been written in the form of

$$u(x,t) = \left(G\left(\cdot, t; q^2\right) \star_q f\right)(x) = \int_0^\infty G\left(x, y, t; q^2\right) f(y)\, d_q y. \tag{33}$$

It is natural to ask how other properties such as the positivity of $G(x,t;q^2)$ and the existence of the q-semigroup can be established.

6.2 q-Heat polynomials

Proposition 6.3 *It is easy to see that, for $x \in \mathbb{R}$ and $t > 0$, the analytic function*

$$\lambda \to e\left(-\lambda^2 t; q^2\right) \cos\left(\lambda x; q^2\right),$$

is a solution of (28) *and it has the expansion*

$$e\left(-\lambda^2 t, q^2\right) \cos\left(\lambda x, q^2\right) = \sum_{n=0}^{\infty} (-1)^n v_{2n}(x,t,q) \lambda^{2n},$$

where

$$v_{2n}(x,t,q)=\sum_{k=0}^{n}b_k\big(x,q^2\big)\frac{(1-q^2)^{n-k}}{(q^2;q^2)_{n-k}}t^{n-k}, \tag{34}$$

with the functions b_n *being given by* (6).

From Proposition 3.1 *we deduce immediately the following properties*:

$$\begin{aligned}\Delta_{q,x}v_{2n}(x,t,q)&=D_{q^2,t}v_{2n}(x,t,q),\quad n\geq 0,\\ v_{2n}(x,0,q)&=b_n\big(x,q^2\big),\\ v_{2n}(x,t,q)&\geq 0,\quad \textit{if } t\geq 0.\end{aligned}$$

We note that formula (34) *can be inverted*:

$$b_n\big(x;q^2\big)=\sum_{k=0}^{n}(-1)^{n-k}v_{2k}(x,t;q)q^{(n-k)(n-k-1)}\frac{(1-q^2)^{n-k}}{(q^2;q^2)_{n-k}}t^{n-k}. \tag{35}$$

Proposition 6.4 *The* q*-heat polynomials* (34) *possess the* q*-integral representation*

(1)

$$v_{2n}(x,t;q)=\int_0^\infty G\big(x,y,t,q^2\big)b_n\big(y;q^2\big)\,d_qy. \tag{36}$$

(2)

$$b_n\big(x;q^2\big)=\int_0^\infty G\big(x,y,t,q^2\big)v_{2n}\big(q^{-1/2}y,t;q^{-1}\big)\,d_qy. \tag{37}$$

Proof We have

$$\begin{aligned}\int_0^\infty G\big(x,y,t,q^2\big)b_n\big(y;q^2\big)\,d_qy&=\int_0^\infty T_{q,x}G\big(y,t,q^2\big)b_n\big(y;q^2\big)\,d_qy\\ &=\int_0^\infty G\big(y,t,q^2\big)T_{q,x}b_n\big(y;q^2\big)\,d_qy\\ &=\sum_{k=0}^{n}b_k\big(x;q^2\big)\int_0^\infty G\big(y,t,q^2\big)b_{n-k}\big(y;q^2\big)\,d_qy\\ &=v_{2n}(x,t;q)\end{aligned}$$

and

$$\begin{aligned}&\int_0^\infty G\big(x,y,t,q^2\big)v_{2n}\big(q^{-1/2}y,-t;q^{-1}\big)\,d_qy\\ &\quad=\sum_{k=0}^{n}(-1)^{n-k}\frac{q^{(n-k)(n-k-1)}(1-q^2)^{n-k}}{(q^2;q^2)_{n-k}}t^{n-k}\int_0^\infty G\big(x,y,t,q^2\big)b_k\big(y,q^2\big)\,d_qy\end{aligned}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} q^{(n-k)(n-k-1)} \frac{(1-q^2)^{n-k}}{(q^2;q^2)_{n-k}} t^{n-k} v_{2k}(x,t;q)$$

$$= b_n\big(x;q^2\big).$$ □

In [14], the authors defined the so-called associated functions by the Appell transform. We extend this notion by defining for $t > 0$ the q-associated functions of v_{2n} by

$$w_{2n}(x,t;q) = (-1)^n \Delta_{q,y}^n G\big(x,y,t;q^2\big)\big|_{y=0}. \tag{38}$$

It is easy to see that

$$w_{2n}(x,t;q) = \frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)} \int_0^\infty e\big(-t\lambda^2, q^2\big)\lambda^{2n} \cos\big(\lambda x, q^2\big)\, d_q\lambda. \tag{39}$$

Proposition 6.5 (Biorthogonality) *For $t > 0$ and $n, m \in \mathbb{N}$, we have*

$$\int_0^\infty w_{2m}(x,t;q) v_{2n}\big(q^{1/2}x, -t;q\big)\, d_q x = (-1)^m \delta_{n,m}.$$

Proof By (37), we have

$$\Delta_q^m b_n\big(x;q^2\big) = \int_0^\infty \Delta_q^m G\big(x,y,t,q^2\big) v_{2n}\big(q^{-1/2}y, t; q^{-1}\big)\, d_q y.$$

Putting $x = 0$, we obtain

$$\int_0^\infty w_{2m}(y,t;q) v_{2n}\big(q^{-1/2}y, t; q^{-1}\big)\, d_q y = (-1)^m \delta_{n,m}.$$

□

6.3 Convergence of $\sum_{n\geq 0} \alpha_n v_{2n}(x,t;q)$

Now we establish the following estimates that will be needed later

Lemma 6.6 *For $n = 0, 1, \ldots$ and $0 < \frac{x_0^2}{t_0} < +\infty$, we have*

$$\big|v_{2n}(x_0, t_0, q)\big| \geq \frac{(1-q^2)^n}{(q^2;q^2)_n} |t_0|^n \geq \frac{|t_0|^n}{n!}.$$

Proof Indeed, the first inequality is a consequence of $b_0(1;q^2) = 1$ and the hypothesis, and the second follows from

$$\frac{1}{n!} \leq \frac{(1-q^2)^n}{(q^2;q^2)_n}.$$ □

Corollary 6.7 *For $n = 0, 1, \dots$ and $0 < \frac{x_0^2}{t_0} < +\infty$, we have*

$$\left|v_{2n}(x_0, t_0, q)\right| \geq C n^{-\frac{1}{2}} \left(\frac{|t_0| e}{n}\right)^n,$$

where C is a constant depending on x_0 and t_0.

Lemma 6.8 *For $n = 0, 1, \dots, \delta > 0$, and $|\frac{x^2}{\delta(1+q)}| < 1$, we have*

$$\frac{(1-q^2)^n}{(q^2;q^2)_n}\left|v_{2n}\big(|x|, |t|, q\big)\right| \leq q^{-n(n-1)} \frac{(\delta + |t|)^n}{n!} e\left(\frac{x^2}{\delta(1+q)}; q\right). \tag{40}$$

Proof To show (40), we note that

$$(q;q)_{2k} = \big(q, q^2; q^2\big)_k,$$

and

$$\big(q;q^2\big)_k \geq (q;q)_k.$$

For $\delta > 0$, and by using the fact that

$$\frac{(1-q)^k}{(q;q)_k} \frac{|x|^2}{(\delta(1+q))^k} \leq q^{-\binom{k}{2}} \exp\left(\frac{|x|^2}{\delta(1+q)}\right),$$

we obtain

$$\begin{aligned} v_{2n}\big(|x|, |t|; q\big) &\leq \frac{(1-q^2)^n}{(q^2;q^2)_n} \sum_{k=0}^{n} q^{k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \frac{(1-q)^k}{(q;q)_k} \frac{|x|^{2k}}{(1+q)^k} |t|^{n-k} \\ &\leq q^{-\binom{n}{2}} \delta^n \left(-\frac{|t|}{\delta}; q^2\right)_n e\left(\frac{|x|^2}{\delta(1+q)}; q\right). \end{aligned}$$

The inequalities

$$\left(-\frac{|t|}{\delta}; q^2\right)_n \leq \left(\frac{|t|}{\delta} + 1\right)^n,$$

and

$$q^{\binom{n}{2}} n! \leq \frac{(q;q)_n}{(1-q)^n} \leq n!,$$

give the result. □

By the Stirling formula, we obtain

Corollary 6.9 *For $n = 0, 1, \dots, \delta > 0$, and $|\frac{x^2}{\delta(1+q)}| < 1$, we have*

$$v_{2n}\big(|x|,|t|,q\big) \le Kq^{-n(n-1)}\left(\big(\delta+|t|\big)\frac{n}{e}\right)^n, \tag{41}$$

where K is a constant depending δ.

Theorem 6.10 *Let (α_n) be a sequence of real or complex numbers such that*

$$\overline{\lim_{n\to\infty}} \frac{n}{e} q^{-2(n-1)} |\alpha_n|^{1/n} = \frac{1}{\sigma} < +\infty.$$

Then the series

$$\sum_{n\ge 0} \alpha_n v_{2n}(x,t;q),$$

converges in the strip

$$S_\sigma = \big\{(x,t), x\in\mathbb{R},\ |t|<\sigma\big\}, \tag{42}$$

and converges uniformly in any region of this strip.

To prove the theorem, we adopt the same approach as in [14] by taking account of the q-equivalent estimation (41).

Remark If we write $u(x,t)$ as the sum of the previous series, then this function satisfies the q-heat equation (28) and

$$u(x,0) = \sum_{n=0}^{\infty} \alpha_n b_n\big(x;q^2\big),$$

where the $b_n(x;q^2)$ is given by (6).

6.4 Analytic Cauchy problem related to the q-heat equation

Lemma 6.11 *Under the hypothesis of Theorem 6.10 and putting*

$$u(x,t) = \sum_{n\ge 0} \alpha_n v_{2n}(x,t;q), \tag{43}$$

$u(x;t)$ is an analytic function of two variables x and t in the strip S_σ given by (42) and satisfies the q-heat equation (28). Furthermore, the coefficients α_n are given by

$$\alpha_n = \Delta_q^n u(x,t)\big|_{(x,t)=(0,0)}. \tag{44}$$

Proof To show this, we note that the theorem gives that $u(x,t)$ is analytic in the whole strip S_σ. Now for a fixed integer p the series

$$\sum_{n\ge 0} \alpha_{n+p} v_{2n}(x,t;q)$$

converges uniformly in any compact region of S_σ. To prove (44), it suffices to see that for integers n and p we have

$$\big(\Delta^n_{q,x} v_{2p}(x,t;q)\big)\big|_{(0,0)} = \delta_{n,p},$$

where $\delta_{n,p}$ is the Kronecker symbol. □

Finally the following statement is established.

Theorem 6.12 *Under the hypothesis of Lemma 6.11, the function $u(x,t)$ given by (43) has the q-Maclaurin expansion*

$$u(x,t) = \sum_{m,p\geq 0} \beta_{m,p} \frac{(1-q^2)^m}{(q^2;q^2)_m} x^{2p} t^m,$$

where

$$\beta_{m,p} = \alpha_{m+p} b_p\big(1,q^2\big). \tag{45}$$

If for $x \in \mathbb{R}$ and $|t| < \sigma$ then function

$$u(x,t) = \sum_{m,p} \beta_{m,p} \frac{(1-q^2)^m}{(q^2;q^2)_m} x^{2p} t^m,$$

satisfies the q-heat equation (28) with the coefficients $\beta_{m,p}$ given by (44), then $u(x,t)$ can be extended to an analytic function in the strip S_σ and we have

$$u(x,t) = \sum_{n\geq 0} \alpha_n v_{2n}(x,t;q).$$

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